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Article

A Generalization of Routh–Hurwitz Stability Criterion for Fractional-Order Systems with Order $\alpha \in (1, 2)$

Jing Yang [1], Xiaorong Hou [1,*] and Yajun Li [2]

[1] School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China
[2] State Grid Xinyang Electric Power Supply Company, Xinyang 464099, China
* Correspondence: houxr@uestc.edu.cn

Abstract: Based on the generalized Routh–Hurwitz criterion, we propose a sufficient and necessary criterion for testing the stability of fractional-order linear systems with order $\alpha \in (1, 2)$, called the fractional-order Routh–Hurwitz criterion. Compared with the existing criterion, ours involves fewer and simpler expressions, which is significant for analyzing the robust stability of high-dimensional uncertain systems. All these expressions are explicit ones about the coefficients of the characteristic polynomial of the system matrix, so the stable parameter region of fractional-order systems can be described directly. Some examples show the effectiveness of our method.

Keywords: fractional-order systems; generalized Routh–Hurwitz criterion; robust stability; high-dimensional systems

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1. Introduction

Routh–Hurwitz criterion is a sufficient and necessary condition for testing whether all roots of a real polynomial have negative real parts. This criterion can analyze the asymptotic stability of linear time–invariant systems by the coefficients of the characteristic polynomial of the system matrix. For integer–order systems, this criterion is equivalent to a stability criterion, called the Routh–Hurwitz stability criterion. Based on the classical Routh–Hurwitz, some stability conditions were given for low-dimensional fractional-order systems with order $\alpha \in (0, 1]$ in 2006 [1]. Recently, these results were extended to the case of order $\alpha \in (0, 2)$ [2]. All these stability conditions that are suitable for fractional-order systems are only sufficient and necessary for dimensions $n = 2, 3$, but just sufficient or necessary for dimensions $n \geq 4$. The classical Routh–Hurwitz criterion is unsuitable for directly analyzing general n-dimensional fractional-order systems.

Some researchers proposed an idea that transforms an n-dimensional fractional-order system into a $2n$-dimensional integer-order system, then the stability of the fractional-order system with order $\alpha \in (1, 2)$ [3] can be tested by analyzing the corresponding higher-dimensional integer-order system. The classical Routh–Hurwitz criterion and related results are valid for analyzing the corresponding higher-dimensional integer-order system. This equivalent transformation has a beautiful form and is the theoretical basis for some stability methods of fractional-order uncertain systems. For example, based on this equivalent transformation, the classical μ-analysis method can be used to discuss the robust stability of fractional-order systems with order $\alpha \in (1, 2)$ [4]. All robust stability results can be obtained with these methods, but the higher-dimensional transformation has a high computational complexity, especially for multi-parameter systems.

For reducing complicated calculations, some parameter space algorithms were established for determining the stable parameter region of fractional-order systems with multi-parameter by transforming characteristic polynomials to the corresponding parameter polynomials. Based on the cylindrical algebraic decomposition technique, the robust stability of high-dimensional cases can be discussed through a visual representation of the parameter space [5,6]. However, if the number of uncertain parameters is more than three, these parameter space algorithms may fail.

Due to the advantage of fractional calculus for describing the memory and genetic characteristics more suitable, fractional-order systems have been modeled and discussed in various research fields. Among these discussions, we see that the case of $\alpha \in (1,2)$ appears in several dynamical problems used in physical and engineering applications [7–10]. For that complex situation, its stability analysis as the basis of the dynamical problems is still an open topic. This paper considers the problems of stability and robust stability on n-dimensional fractional-order linear systems with order $\alpha \in (1,2)$. Based on the generalized Routh–Hurwitz criterion, a sufficient and necessary criterion for testing the stability of fractional-order linear systems with order $\alpha \in (1,2)$, called the fractional-order Routh–Hurwitz criterion, is proposed. Our criterion involves fewer explicit expressions than the exiting method [3] and describes stability analysis results of fractional-order systems with uncertain parameters more easily than existing methods [4–6,11,12].

2. Preliminaries

Since the Laplace transform of a Liouville–Caputo fractional-order derivative involves the initial values of integer-order derivatives with clear physical interpretations [13–15], the Liouville–Caputo (LC) definition of a fractional-order derivative is used in this paper,

$$ {}^{LC}D^{\alpha}f(t) = \frac{1}{\Gamma(\alpha - m)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{1}$$

where m is an integer satisfying $m-1 < \alpha \leq m$ and $\Gamma(\cdot)$ is the gamma function.

In this paper, D^{α} is short for ${}^{LC}D^{\alpha}$.

Consider the following n-dimensional fractional-order linear time-invariant system:

$$ D^{\alpha}x = Ax, \tag{2}$$

where $\alpha \in (1,2)$ is the fractional order, $A \in \mathbb{R}^{n\times n}$, $x = (x_1, x_2, \cdots, x_n)^T$ is a state vector. The initial conditions for system (2) are $x_i(0) = x_{i0}, x_i'(0) = x_{i0}'$.

Lemma 1 ([16–18])**.** *System (2) is asymptotically stable if and only if* $\left|arg(\lambda_j)\right| > \frac{\alpha\pi}{2}$*, where* $\lambda_j (j = 1,2,\cdots,n)$ *are the eigenvalues of matrix* A *and* $arg(\cdot)$ *denotes the argument of a complex number.*

Let

$$\begin{aligned} SR &:= \left\{\gamma \in \mathbb{C} \,|\, |arg(\gamma)| > \tfrac{\alpha\pi}{2}\right\}, \\ UR &:= \left\{\gamma \in \mathbb{C} \,|\, |arg(\gamma)| < \tfrac{\alpha\pi}{2}\right\}, \\ CL &:= \left\{\gamma \in \mathbb{C} \,|\, |arg(\gamma)| = \tfrac{\alpha\pi}{2}\right\}, \end{aligned} \tag{3}$$

be called the stable region, the unstable region and the critical line of system (2), respectively (as shown in Figure 1).

Suppose the characteristic polynomial of the system matrix A is

$$ P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n. \tag{4}$$

From Lemma 1, we know that system (2) is asymptotically stable if and only if all roots of $P(\lambda)$ are in the stable region SR.

Remark 1. *Since the stable region SR of system (2) is a classical study region in integer-order systems [19], researchers have analyzed the stability of system (2) by finding an integer-order linear time-invariant system with integer-order that has equivalently the same stability property as of the fractional-order system [3]. This idea is valid, but the corresponding integer-order system has a higher-dimensional system matrix. We need to analyze a higher-dimensional polynomial if we test the stability of the corresponding integer-order system by the classical Routh–Hurwitz criterion.*

For example, consider the general two-dimensional fractional-order system as follows:

$$D^{\alpha}x = Ax, \tag{5}$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $x = (x_1, x_2)^T$. *The initial conditions are* $x_i(0) = x_{i0}, x_i'(0) = x_{i0}'$, $i = 1, 2$.

System (5) is asymptotically stable if and only if the following four-dimensional integer-order system is asymptotically stable [3],

$$\dot{\tilde{x}} = \begin{bmatrix} Asin\left(\frac{\alpha\pi}{2}\right) & Acos\left(\frac{\alpha\pi}{2}\right) \\ -Acos\left(\frac{\alpha\pi}{2}\right) & Asin\left(\frac{\alpha\pi}{2}\right) \end{bmatrix}\tilde{x}. \tag{6}$$

The characteristic polynomial of system (6) is

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4, \tag{7}$$

where

$$\begin{aligned}
a_1 &= -2(a_{11} + a_{22})sin\left(\frac{\alpha\pi}{2}\right), \\
a_2 &= (a_{11}^2 + 4a_{11}a_{22} - 2a_{12}a_{21} + a_{22}^2)sin^2\left(\frac{\alpha\pi}{2}\right) + (a_{11}^2 + 2a_{12}a_{21} + a_{22}^2)cos^2\left(\frac{\alpha\pi}{2}\right), \\
a_3 &= -2(a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21})sin\left(\frac{\alpha\pi}{2}\right), \\
a_4 &= (a_{11}a_{12} - a_{12}a_{21})^2.
\end{aligned} \tag{8}$$

The integer-order Hurwitz matrix H_r *of* $P(\lambda)$ *in Equation (7) is as follows:*

$$H_r = \begin{bmatrix} a_1 & a_3 & 0 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & 1 & a_2 & a_4 \end{bmatrix}. \tag{9}$$

Based on the classical Routh–Hurwitz criterion, we need to check $\Delta_i > 0 (i = 1, 2, 3, 4)$ *to determine the stability of two-dimensional fractional-order system (5), where* $\Delta_i (i = 1, 2, 3, 4)$ *is the ith order leading principal minor of* H_r *and*

$$\begin{aligned}
\Delta_1 &= a_1, \\
\Delta_2 &= a_1a_2 - a_3, \\
\Delta_3 &= -a_1^2a_4 + a_1a_2a_3 - a_3^2, \\
\Delta_4 &= -a_4(a_1^2a_4 - a_1a_2a_3 + a_3^2).
\end{aligned} \tag{10}$$

The above classical Routh–Hurwitz criterion needs to calculate four leading principal minors that are complex, although fractional-order system (5) is just a two-dimensional system.

Let $f(z)$ be a complex coefficient polynomial satisfying:

$$f(iz) = b_0z^n + b_1z^{n-1} + \cdots + b_n + i\left(a_0z^n + a_1z^{n-1} + \cdots + a_n\right), a_0 \neq 0, \tag{11}$$

where $a_j (j = 0, 1, \cdots, n)$ and $b_j (j = 0, 1, \cdots, n)$ are real numbers.

The $2n \times 2n$ generalized Hurwitz matrix H_f is constructed from $f(z)$ as follows:

$$H_f = \begin{bmatrix} a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ b_0 & b_1 & \cdots & b_n & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & a_{n-1} & a_n & \cdots & 0 \\ 0 & b_0 & \cdots & b_{n-1} & b_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_0 & \cdots & a_{n-1} & a_n \\ 0 & \cdots & 0 & b_0 & \cdots & b_{n-1} & b_n \end{bmatrix}. \tag{12}$$

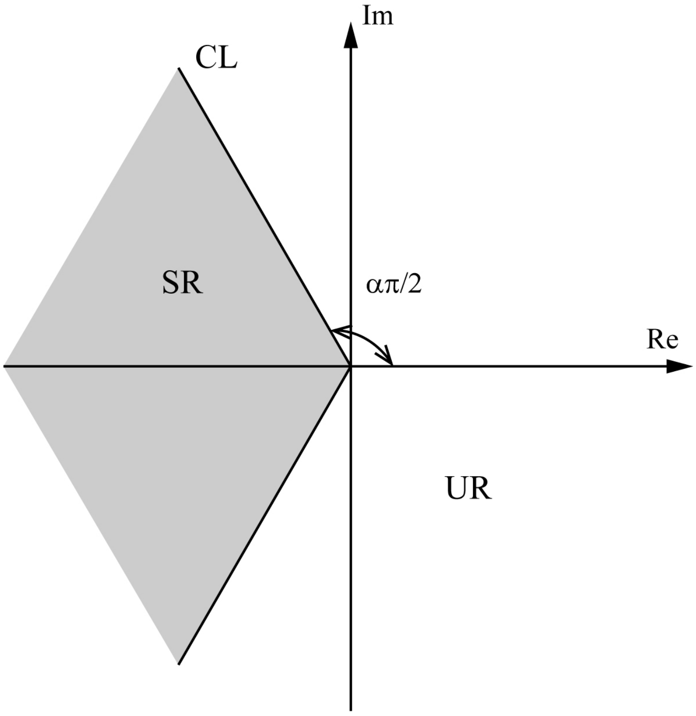

Figure 1. The stable region SR, the unstable region UR and the critical line CL of system (2).

Lemma 2 ([20] The Generalized Routh–Hurwitz Criterion)**.** *All roots of $f(z)$ have negative real parts if and only if $\sigma_k > 0 (k = 1, 2, \cdots, n)$, where $\sigma_k (k = 1, 2, \cdots, n)$ is the 2kth order leading principal minor of H_f.*

The $2k$th order leading principal minors $\sigma_k (k = 1, 2, \cdots, n)$ of H_f are given by

$$\sigma_1 = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}, \sigma_2 = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \end{vmatrix}, \cdots, \sigma_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ b_0 & b_1 & \cdots & b_n & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & a_{n-1} & a_n & \cdots & 0 \\ 0 & b_0 & \cdots & b_{n-1} & b_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_0 & \cdots & a_{n-1} & a_n \\ 0 & \cdots & 0 & b_0 & \cdots & b_{n-1} & b_n \end{vmatrix}. \tag{13}$$

From the generalized Routh–Hurwitz criterion, we can test whether all roots of an n-dimensional complex coefficient polynomial have negative real parts by n leading principal minors.

In this paper, based on the generalized Routh–Hurwitz criterion, we propose a sufficient and necessary criterion, called the fractional-order Routh–Hurwitz criterion. This criterion can directly analyze the stability and robust stability of n-dimensional fractional-order systems with order $\alpha \in (1, 2)$ by these corresponding complex coefficient polynomials, which can reduce the complicated calculations caused by the higher-dimensional transformation.

3. Main Results

From the characteristic polynomial $P(\lambda)$ of fractional-order system (2), we have the corresponding complex coefficient polynomial:

$$f\left(r \cdot e^{i\frac{\alpha\pi}{2}}\right) = \sum_{j=0}^{n} a_j \cdot cos\left(\frac{(n-j)\cdot\alpha\pi}{2}\right) \cdot r^{n-j} + i \cdot \left(\sum_{j=0}^{n} a_j \cdot sin\left(\frac{(n-j)\cdot\alpha\pi}{2}\right) \cdot r^{n-j}\right), \tag{14}$$

where $a_0 = 1$.

From Equations (11) and (12), the fractional-order Hurwitz matrix is defined, which is constructed from the complex coefficient polynomial in Equation (14).

Definition 1 (The Fractional-Order Hurwitz Matrix). *For system (2), the $2n \times 2n$ fractional-order Hurwitz matrix H_α of $P(\lambda)$ is defined as follows:*

$$H_\alpha = \begin{bmatrix} sin\left(\frac{n\cdot\alpha\pi}{2}\right) & a_1 sin\left(\frac{(n-1)\cdot\alpha\pi}{2}\right) & \cdots & 0 & \cdots & \cdots & 0 \\ cos\left(\frac{n\cdot\alpha\pi}{2}\right) & a_1 cos\left(\frac{(n-1)\cdot\alpha\pi}{2}\right) & \cdots & a_n & \cdots & \cdots & 0 \\ 0 & sin\left(\frac{n\cdot\alpha\pi}{2}\right) & \cdots & a_{n-1} sin\left(\frac{\alpha\pi}{2}\right) & \cdots & \cdots & 0 \\ 0 & cos\left(\frac{n\cdot\alpha\pi}{2}\right) & \cdots & a_{n-1} cos\left(\frac{\alpha\pi}{2}\right) & a_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & sin\left(\frac{n\cdot\alpha\pi}{2}\right) & \cdots & a_{n-1} sin\left(\frac{\alpha\pi}{2}\right) & 0 \\ 0 & \cdots & \cdots & cos\left(\frac{n\cdot\alpha\pi}{2}\right) & \cdots & a_{n-1} cos\left(\frac{\alpha\pi}{2}\right) & a_n \end{bmatrix}. \tag{15}$$

Theorem 1 (The Fractional-Order Routh–Hurwitz Criterion). *System (2) is asymptotically stable if and only if $\Sigma_p > 0 (p = 1, 2, \cdots, n)$, where $\Sigma_p (p = 1, 2, \cdots, n)$ is the $2p$th order leading principal minor of H_α.*

Proof. The coordinate system xy counterclockwise turns through angle $\theta = \frac{(\alpha-1)\pi}{2}$ as the coordinate system $x'y'$ (as shown in Figure 2).

For system (2), in the new coordinate system $x'y'$, $P(\lambda)$ can be expressed as

$$g(r) = f\left(r \cdot e^{i\frac{\alpha-1}{2}\pi}\right), \tag{16}$$

thus

$$g(ir) = f\left(r \cdot e^{i\frac{\alpha\pi}{2}}\right). \tag{17}$$

Since $P(\lambda)$ is a real polynomial whose roots are symmetrical about the real axis in the coordinate system xy, its roots are in the stable region SR if and only if they are in the left half plane of the coordinate system $x'y'$. Based on the above analysis, according to Lemma 2 and Lemma 1, we have Theorem 1. □

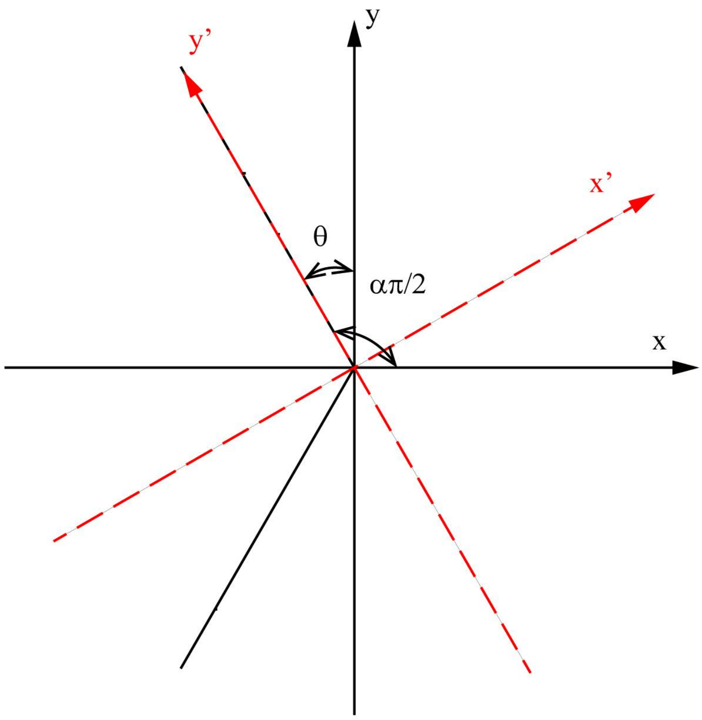

Figure 2. Rotate the coordinate system.

The linearization of a fractional-order nonlinear system $D^\alpha x = f(x)$ around an equilibrium point can be given as $D^\alpha x = Jx$, where J is the Jacobian matrix of $D^\alpha x = f(x)$ around the equilibrium point. The linearized system $D^\alpha x = Jx$ is locally asymptotically stable if for each eigenvalue λ of J, $|arg(\lambda)| > \frac{\alpha\pi}{2}$ [21]. The fractional-order Routh–Hurwitz criterion of system (2) can be used to analyze the local stability of fractional-order nonlinear systems.

Remark 2. *Consider the same two-dimensional fractional-order system (5) as in Remark 1, the characteristic polynomial of matrix A is*

$$P(\lambda) = \lambda^2 - (a_{11} + a_{22})x + a_{11}a_{22} - a_{12}a_{21}. \tag{18}$$

The corresponding fractional-order Hurwitz matrix H_α is as follows:

$$H_\alpha = \begin{bmatrix} sin(\alpha\pi) & -(a_{11}+a_{22})sin(\frac{\alpha\pi}{2}) & 0 & 0 \\ cos(\alpha\pi) & -(a_{11}+a_{22})cos(\frac{\alpha\pi}{2}) & a_{11}a_{22}-a_{12}a_{21} & 0 \\ 0 & sin(\alpha\pi) & -(a_{11}+a_{22})sin(\frac{\alpha\pi}{2}) & 0 \\ 0 & cos(\alpha\pi) & -(a_{11}+a_{22})cos(\frac{\alpha\pi}{2}) & a_{11}a_{22}-a_{12}a_{21} \end{bmatrix}. \tag{19}$$

Based on the fractional-order Routh–Hurwitz criterion in Theorem 1, we only need to check two even-order leading principal minors $\Sigma_p > 0 (p = 1, 2)$ of H_α to determine the stability of system (5), where

$$\begin{aligned} \Sigma_1 &= -(a_{11}+a_{22})sin\left(\frac{\alpha\pi}{2}\right), \\ \Sigma_2 &= (a_{22}a_{11}-a_{21}a_{12})sin^2\left(\frac{\alpha\pi}{2}\right)\left((a_{11}+a_{22})^2 - 4(a_{22}a_{11}-a_{21}a_{12})cos^2\left(\frac{\alpha\pi}{2}\right)\right). \end{aligned} \tag{20}$$

In this paper, we consider the case of $\alpha \in (1, 2)$, so $\Sigma_1 > 0, \Sigma_2 > 0$ is equivalent to

$$\begin{aligned} \widetilde{\Sigma}_1 &= -(a_{11}+a_{22}) > 0, \\ \widetilde{\Sigma}_2 &= (a_{22}a_{11}-a_{21}a_{12})\left((a_{11}+a_{22})^2 - 4(a_{22}a_{11}-a_{21}a_{12})cos^2\left(\frac{\alpha\pi}{2}\right)\right) > 0. \end{aligned} \tag{21}$$

Compared with the existing method, the fractional-order Routh–Hurwitz criterion involves smaller numbers of leading principal minors. Each Σ_p in Equation (20) is simpler than Δ_i in Equation (10), $i = 2p$. Our method has less computational complexity than the existing method [3]. Especially for high-dimensional fractional-order systems, the advantage of a low computational complexity is significant.

The fractional-order Routh–Hurwitz criterion can be used to analyze the robust stability of fractional-order uncertain systems.

Consider a fractional-order uncertain system as follows:

$$D^{\alpha} x = A(\beta) x, \tag{22}$$

where $(\alpha, \beta) = (\alpha, \beta_1, \beta_2, \cdots, \beta_i)$ are uncertain parameters, $x = (x_1, x_2, \cdots, x_n)^T$ is the state vector. The initial conditions for system (22) are $x_i(0) = x_{i0}, x_i'(0) = x_{i0}'$.

The characteristic polynomial of matrix $A(\beta)$ is

$$f(\lambda; \alpha, \beta) = \lambda^n + a_1(\alpha, \beta)\lambda^{n-1} + \cdots + a_n(\alpha, \beta). \tag{23}$$

Since all expressions in our method are explicit ones about the coefficients of the characteristic polynomial of the system matrix, Theorem 1 is also effective for analyzing the robust stability of system (22).

System (22) is of certain parameters for given parameter (α, β). We call the parameter (α, β) a stable parameter if the corresponding system is asymptotically stable. The set of all stable parameters is called the stable parameter region, denoted by $SR(\alpha, \beta)$. According to Theorem 1, the stable parameter region $SR(\alpha, \beta)$ of system (22) is the set of the solutions of $\{\Sigma_p > 0, p = 1, 2, \cdots, n\}$. The stable parameter region $SR(\alpha, \beta)$ can be described directly by the explicit expressions.

4. Illustrative Examples

Example 1. *Consider the following fractional-order uncertain system with dimension $n = 4$:*

$$D^{1.5} x(t) = (\beta_1 A_1 + \beta_2 A_2) x(t), x(0) = x_0, x'(0) = x_0', \tag{24}$$

where $\beta_1 + \beta_2 = 1, \beta_1, \beta_2 \geq 0$, $A_1 = \Gamma - \varepsilon b * c$, $A_2 = \Gamma + \varepsilon b * c$,

$$\Gamma = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ -10 & -10 & -20 & -6 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, c' = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

Let $\beta_2 = 1 - \beta_1$, the characteristic polynomial of system matrix is:

$$f(\lambda) = \lambda^4 + 12\lambda^3 + 67\lambda^2 + (6\beta_1\varepsilon - 3\varepsilon + 142)\lambda + 12\beta_1\varepsilon - 6\varepsilon + 96, \tag{25}$$

based on Corollary A3, the solutions of the set of inequalities are shown in Figure 3, where different color line mean the corresponding inequality. The stable parameter region $SR(\alpha, \beta)$ of system (24) is marked. For system (24), since β_1 satisfies $\beta_1 \in [0, 1)$, which means the set of inequalities has solutions for all $\beta_1 \in [0, 1)$, the maximum value of ε is 7.274.

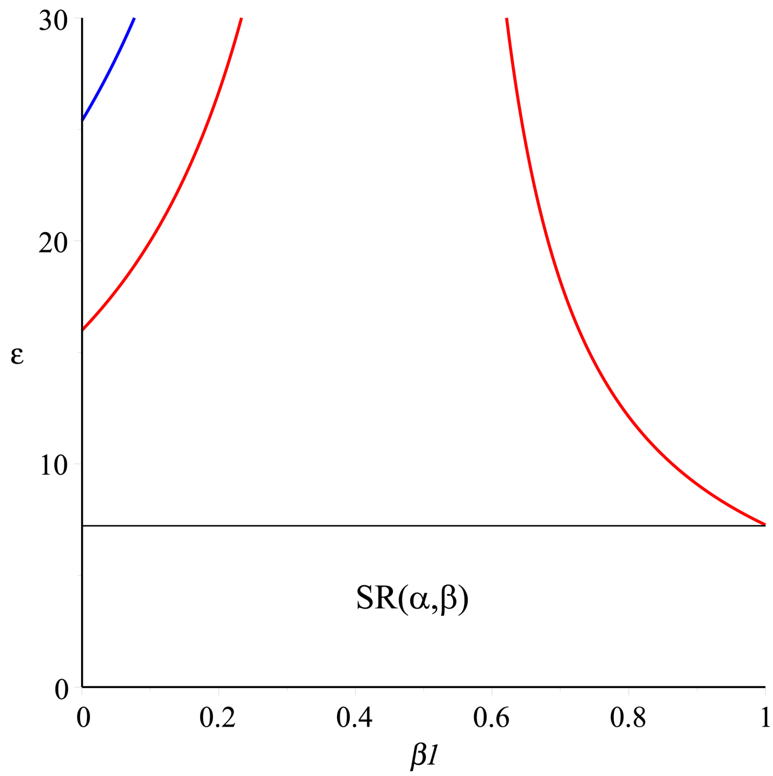

Figure 3. The solutions of the set of inequalities.

Remark 3. *The above example showed the effectiveness of our method for analyzing fractional-order systems with $n = 4$. For a general n, based on the existing methods [3,22], we need to use the idea of testing $\Delta_i > 0, i = 1, 2, \cdots, 2n$ to determine the stability of system (22). Furthermore, as a comparison in this case, using our method, we only need to test whether $\Sigma_p > 0, p = 1, 2, \cdots, n$ to analyze the stability. The number of leading principal minors of our method is smaller than that of existing methods.*

Example 2. *Consider the following fractional-order hyperchaotic system with $n = 4$:*

$$\begin{cases} D^{\alpha}x_1 = a(x_2 - x_1) \\ D^{\alpha}x_2 = cx_1 - x_1x_3 - x_2 + ex_4 \\ D^{\alpha}x_3 = x_1^4 + x_2^2 - bx_3 \\ D^{\alpha}x_4 = -dx_2 \end{cases}, \tag{26}$$

where a, b, c, d, e are uncertain parameters, $\alpha \in (1, 2)$ is an uncertain fractional order and the initial conditions are $x_i(0) = x_{i0}, x_i{}'(0) = x_{i0}{}'$.

System (26) has only one equilibrium point $(0, 0, 0, 0)$ and the Jacobian matrix J at the equilibrium point $(0, 0, 0, 0)$ is

$$J = \begin{bmatrix} -a & a & 0 & 0 \\ c & -1 & 0 & e \\ 0 & 0 & -b & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of the Jacobian matrix is:

$$f(\lambda) = \lambda^4 + (a + b + 1)\lambda^3 + (ab - ac + de + a + b)\lambda^2 + (ab - abc + ade + bde)\lambda + abde, \tag{27}$$

based on Corollary A3, the stable parameter region $SR(\alpha, \beta)$ of system (26) is the set of the solutions of the four inequalities, where

$$\begin{aligned} a_1 &= a + b + 1, \\ a_2 &= ab - ac + de + a + b, \\ a_3 &= ab - abc + ade + bde, \\ a_4 &= abde. \end{aligned} \tag{28}$$

Remark 4. *We have proposed a parameter space algorithm for analyzing the robust stability of fractional-order uncertain systems with $\alpha \in (0, 2)$ [5,6], with which the stable parameter region can be obtained through a visual representation of the parameter space if the systems have fewer parameters. If the number of uncertain parameters is more than three [23], it is difficult to show the stable parameter region with a figure. The explicit expressions of the stable parameter region $SR(\alpha, \beta)$ by the fractional-order Routh–Hurwitz criterion in Theorem 1 are still valid for the multi-parameter situation.*

5. Conclusions

In this paper, we give a fractional-order Routh-Hurwitz criterion for analyzing the stability and robust stability of fractional-order linear systems with $\alpha \in (1, 2)$. This criterion directly tests complex coefficient polynomials corresponding to fractional-order systems without the higher-dimensional transformation, so the stable parameter region of fractional-order uncertain systems can be described with fewer explicit expressions about the coefficients of the characteristic polynomial. Our method is suitable for some complex cases such as systems with uncertain order and uncertain other parameters. Due to the advantage of explicit expressions, one of our future works is finding a way to directly give multi-parameter results of other complex dynamic problems based on our stability results.

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Appendix A

For system (2), suppose the characteristic polynomial of A is $f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$. Since systems with dimensions $n = 2, 3, 4$ are often used, based on Theorem 1, we have the following corollaries. In the following, always set $cos^2(\frac{\alpha\pi}{2}) = s$.

Corollary A1. *In the case of $n = 2$, system (2) is asymptotically stable if and only if*

$$a_1 > 0, \quad a_2\left(a_1^2 - 4a_2 s\right) > 0 \tag{A1}$$

Example A1. *Consider the following fractional-order uncertain system:*

$$D^{1.5}x(t) = (A_0 + \beta_1 A_1 + \beta_2 A_2)x(t), x(0) = x_0, x'(0) = x'_0, \tag{A2}$$

where

$$A_0 = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

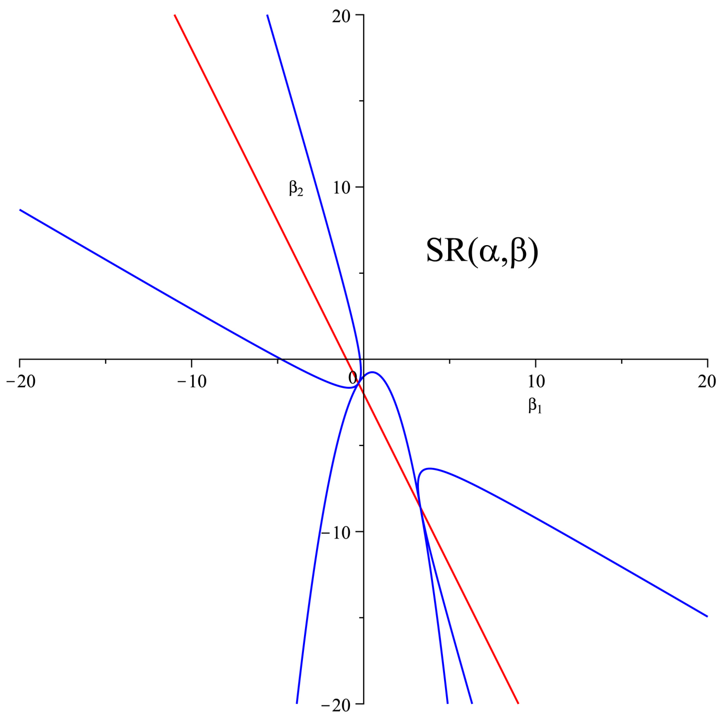

Figure A1. The solutions of the set of inequalities (A4).

The characteristic polynomial of $A_0 + \beta_1 A_1 + \beta_2 A_2$ *is*

$$f(\lambda) = \lambda^2 + (2\beta_1 + 2 + \beta_2)\lambda + \beta_1^2 + \beta_2 - \beta_1 + 1, \tag{A3}$$

Based on Corollary A1, we know that system (A2) is asymptotically stable if and only if

$$\begin{aligned} &2\beta_1 + 2 + \beta_2 > 0, \\ &(\beta_1^2 + \beta_2 - \beta_1 + 1)\Big((2\beta_1 + 2 + \beta_2)^2 - 2(\beta_1^2 + \beta_2 - \beta_1 + 1)\Big) > 0. \end{aligned} \tag{A4}$$

Parameters that satisfy the set of inequalities (A4) are stable parameters of system (A2). The solutions of the set of inequalities (A4) are shown in Figure A1, where different color line mean the corresponding inequality, in which the stable parameter region is marked.

Remark A1. *System (A2) has been considered [4,5]. Using existing methods, the robustness bound of a single parameter can be obtained, or the stable parameter region can be determined by taking points to test the stability of some corresponding systems. All expressions in our results are explicit ones about the coefficients of the characteristic polynomial of the system matrix, the relationship among multiple parameters can be described and the stable parameter region can be solved directly.*

Corollary A2. *In the case of* $n = 3$*, system (2) is asymptotically stable if and only if*

$$\begin{aligned} &a_1 > 0, \quad \Big(4a_1a_3 - 4a_2^2\Big)s + a_1^2a_2 - a_1a_3 > 0, \\ &a_3 \cdot (64a_3^2s^3 - \Big(16a_1a_2a_3 + 48a_3^2\Big)s^2 + (4a_1^3a_3 - 4a_1a_2a_3 + 4a_2^3 + 12a_3^2)s \\ &- a_1^2a_2^2 + 2a_1a_2a_3 - a_3^2) > 0. \end{aligned} \tag{A5}$$

Example A2. *Consider a fractional-order system with $n = 3$, where $\alpha \in (1, 2)$ and $\beta \in (0, 10)$ are uncertain parameters. Suppose the characteristic equation of the system matrix is:*

$$\lambda^3 + (\beta - \alpha)\lambda^2 + 2\beta\lambda + 4 = 0. \tag{A6}$$

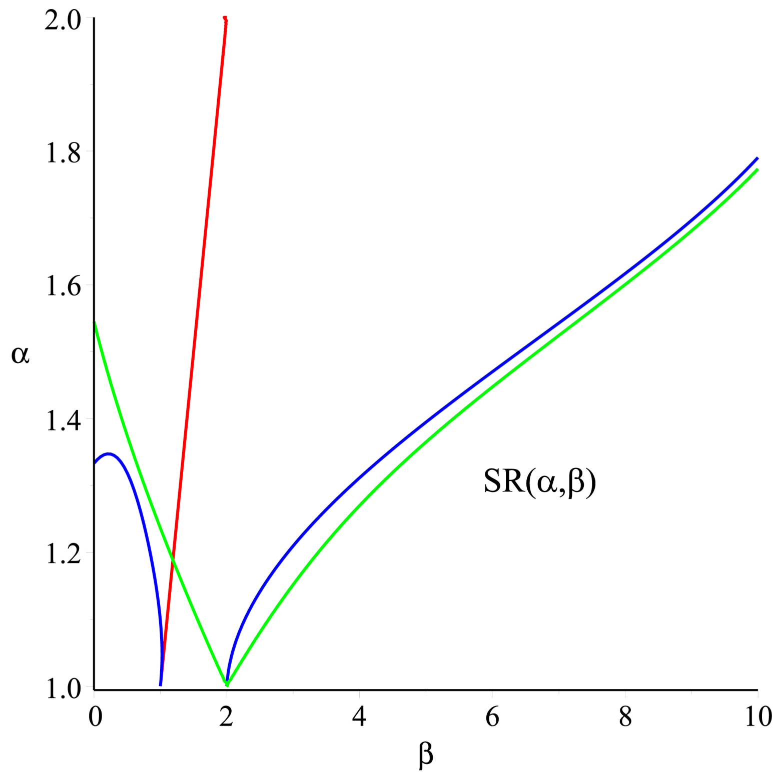

Figure A2. The solutions of the set of inequalities (A7).

Based on Corollary A2, we know that the system is asymptotically stable if and only if

$$\begin{aligned} &\beta - \alpha > 0, \\ &\left(16(\beta - \alpha) - 16\beta^2\right)s + 2(\beta - \alpha)^2\beta - 4\beta + 4\alpha > 0, \\ &-4(1024s^3 - (128(\beta - \alpha)\beta + 768)s^2 + (16(\beta - \alpha)^3 - 32(\beta - \alpha)\beta \\ &+ 32\beta^3 + 192)s - 4(\beta - \alpha)^2\beta^2 + 16(\beta - \alpha)\beta - 16) > 0. \end{aligned} \tag{A7}$$

The solutions of the set of inequalities (A7) are shown in Figure A2, where different color line mean the corresponding inequality, in which the stable parameter region is marked.

Remark A2. *For fractional-order systems with uncertain order, the existing methods cannot describe the relationship between the order parameter and stability directly [6,11,12]. Using our method, systems with uncertain order and uncertain other parameters can be analyzed easily, and all results are explicit expressions of the coefficients of the characteristic polynomial of the system matrix.*

Corollary A3. *In the case of $n = 4$, system (2) is asymptotically stable if and only if*

$$\begin{aligned}
&a_1 > 0, \quad \left(4a_1a_3 - 4a_2^2\right)s + a_1^2a_2 - a_1a_3 > 0,\\
&\left(64a_1a_4^2 - 128a_2a_3a_4 + 64a_3^3\right)s^3 - (16a_1^2a_3a_4 + 16a_1a_2^2a_4 + 16a_1a_2a_3^2 - 64a_1a_4^2\\
&+ 96a_2a_3a_4 - 48a_3^3)s^2 + (4a_1^3a_2a_4 - 4a_1^3a_3^2 + 8a_1^2a_3a_4 + 4a_1a_2^2a_4 + 4a_1a_2a_3^2 - 4a_2^3a_3\\
&- 16a_1a_4^2 + 16a_2a_3a_4 - 12a_3^3)s - a_1^3a_2a_4 + a_1^2a_2^2a_3 + a_1^2a_3a_4 2a_1a_2a_3^2 + a_3^3 > 0,\\
&a_4 \cdot (4096a_4^3s^6 + (-1024a_1a_3a_4^2 - 8192a_4^3)s^5 + (256a_1^2a_2a_4^2 + 1536a_1a_3a_4^2 - 512a_2^2a_4^2\\
&+ 256a_2a_3^2a_4 + 6144a_4^3)s^4 + (-64a_1^4a_4^2 - 64a_1^2a_2a_4^2 - 64a_1a_2^2a_3a_4 - 1024a_1a_3a_4^2\\
&+ 512a_2^2a_4^2 - 64a_2a_3^2a_4 - 64a_3^4 - 2048a_4^3)s^3 + (48a_1^4a_4^2 + 16a_1^3a_2a_3a_4 - 64a_1^2a_2a_4^2\\
&- 16a_1^2a_3^2a_4 - 32a_1a_2^2a_3a_4 + 16a_1a_2a_3^3 + 16a_2^4a_4 + 384a_1a_3a_4^2 - 128a_2^2a_4^2 - 64a_2a_3^2a_4\\
&+ 48a_3^4 + 256a_4^3)s^2 + (-12a_1^4a_4^2 + 4a_1^3a_2a_3a_4 - 4a_1^3a_3^3 - 4a_1^2a_2^3a_4 + 16a_1^2a_2a_4^2 + 8a_1^2a_3^2a_4\\
&+ 16a_1a_2^2a_3a_4 + 4a_1a_2a_3^3 - 4a_2^3a_3^2 - 64a_1a_3a_4^2 + 16a_2a_3^2a_4 - 12a_3^4)s + a_1^4a_4^2 - 2a_1^3a_2a_3a_4\\
&+ a_1^2a_2^2a_3^2 + 2a_1^2a_3^2a_4 - 2a_1a_2a_3^3 + a_3^4) > 0.
\end{aligned} \tag{A8}$$

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