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# Existence of Solutions to a System of Riemann-Liouville Fractional Differential Equations with Coupled Riemann-Stieltjes Integrals Boundary Conditions

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**Abstract:** A general system of fractional differential equations with coupled fractional Stieltjes integrals and a Riemann–Liouville fractional integral in boundary conditions is studied in the context of pattern formation. We need to transform the fractional differential system into the corresponding integral operator to obtain the existence and uniqueness of solutions for the system. The contraction mapping principle in Banach space and the alternative theorem of Leray–Schauder are applied. Finally, we give two applications to illustrate our theoretical results.

**Keywords:** coupled system; Riemann–Liouville fractional derivative; contraction mapping principle in Banach space; alternative theorem of Leray–Schauder

**MSC:** 34A08, 26A33

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## 1. Introduction

A general system of fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha_1}(D_{0^+}^{\beta_1}x(t)) + f(t,x(t),y(t)) = 0, t \in [0,1], \\ D_{0^+}^{\alpha_2}(D_{0^+}^{\beta_2}y(t)) + g(t,x(t),y(t)) = 0, t \in [0,1], \end{cases} \tag{1}$$

supplemented with coupled nonlocal integral boundary conditions are considered.

$$\begin{cases} D_{0^+}^{\beta_1}x(0) = 0, x(0) = 0, x(1) = \gamma_1 I_{0^+}^{\delta_1}y(\xi) + \sum\limits_{i=1}^{p} \int_0^1 y(\tau)d\mathcal{H}_i(\tau), \\ D_{0^+}^{\beta_2}y(0) = 0, y(0) = 0, y(1) = \gamma_2 I_{0^+}^{\delta_2}x(\eta) + \sum\limits_{j=1}^{q} \int_0^1 x(\tau)d\mathcal{K}_j(\tau), \end{cases} \tag{2}$$

where $\alpha_1$ is in the interval $(0,1)$, $\beta_1$ is in the interval $(1,2)$, $\alpha_2$ is in the interval $(0,1]$, $\beta_2$ is in the interval $(1,2]$, $p,q \in N$, and $\gamma_1,\gamma_2,\delta_1,\delta_2 > 0$, $0 < \xi,\eta < 1$ $\mathcal{K}_j(t), j = 1,\cdots,q$, $\mathcal{H}_i(t), i = 1,\cdots,p$ are bounded variation functions. Both function $f$ and function $g$ are nonlinear.

Coupled boundary conditions appear in the study of reaction-diffusion equations [1], heat equations [2] and mathematical biology [3]. Boundary value problems with coupled boundary conditions constitute a very interesting and important class of problems. Recently, much attention has been focused on the study of the existence of solutions for boundary value problems with coupled boundary conditions, see [4–13].

In [14], Tudorache and Luca investigated the systems of Riemann–Liouville fractional differential equations with coupled integral boundary conditions.

$$\begin{cases} D_{0^+}^{\alpha}x(t) + f(t,x(t),y(t),I_{0^+}^{\theta_1}x(t),I_{0^+}^{\sigma_1}y(t)) = 0, t \in (0,1), \\ D_{0^+}^{\beta}y(t) + g(t,x(t),y(t),I_{0^+}^{\theta_2}x(t),I_{0^+}^{\sigma_2}y(t)) = 0, t \in (0,1), \end{cases}$$

$$\begin{cases} x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, D_{0^+}^{\gamma_0} x(1) = \sum_{i=1}^{p} \int_0^1 D_{0^+}^{\gamma_i} y(t) d\mathcal{H}_i(t), \\ y(0) = y'(0) = \cdots = y^{(m-2)}(0) = 0, D_{0^+}^{\delta_0} y(1) = \sum_{i=1}^{q} \int_0^1 D_{0^+}^{\delta_i} x(t) d\mathcal{K}_i(t), \end{cases}$$

where $\sigma_1, \theta_1, \theta_2, \sigma_2 > 0$, $f$ and $g$ are functions that are nonlinear. The contraction mapping principle in Banach space, the alternative theorem of Leray–Schauder and Krasnosel'skii-type theorem are adopted.

In [15], Bashir Ahmad and Rodica Luca considered the system of fractional integro-differential equations

$$\begin{cases} ({}^cD^{\alpha} + \lambda {}^cD^{\alpha-1})u(t) = f(t, u(t), v(t), {}^cD^{p_1}v(t), I^{q_1}v(t)), t \in (0,1), \\ ({}^cD^{\beta} + \mu {}^cD^{\beta-1})v(t) = g(t, u(t), v(t), {}^cD^{p_2}u(t), I^{q_2}u(t)), t \in (0,1), \end{cases}$$

with the coupled boundary conditions

$$\begin{cases} u(0) = u'(0) = u''(0) = 0, u(1) = \int_0^1 u(s) d\mathcal{H}_1(s) + \int_0^1 v(s) d\mathcal{H}_2(s), \\ v(0) = v'(0) = v''(0) = 0, v(1) = \int_0^1 u(s) d\mathcal{K}_1(s) + \int_0^1 v(s) d\mathcal{K}_2(s). \end{cases}$$

On the other hand, boundary value problems with Riemann–Liouville fractional integral boundary conditions have attracted much attention.

In [16], Laadjal, M. Al-Mdallal and Jarad discussed the coupled system of fractional Langevin equations

$$\begin{cases} {}^cD^{\alpha_1}({}^cD^{\beta_1} + \lambda)\psi_1(t) = f(t, \psi_1(t), \psi_2(t)), t \in J, 0 < \alpha_1 \leq 1 < \beta_1 \leq 2, \\ {}^cD^{\alpha_2}({}^cD^{\beta_2} + k)\psi_2(t) = g(t, \psi_1(t), \psi_2(t)), t \in J, 0 < \alpha_2 \leq 1 < \beta_2 \leq 2, \end{cases}$$

with nonlocal nonseparated boundary conditions

$$\begin{cases} \psi_1(0) = a_0, \psi_2(0) = b_0, \psi_1'(0) = \psi_2'(0) = 0, \\ \psi_1(\xi) = a({}^cD^p\psi_2)(\mu_1), \xi \in (0,1], \mu_1 \in J, 0 < p < \beta_2, \\ \psi_2(\eta) = b(I^q\psi_1)(\mu_2), \eta \in (0,1], \mu_2 \in J, q \geq 0. \end{cases}$$

In [17], Zhang, Li and Lu considered the fractional differential system with Riemann–Liouville fractional integral boundary conditions

$$\begin{cases} D_{0^+}^{\alpha_1} u(t) = f_1(t, u(t), v(t), D_{0^+}^{\rho_1} u(t), D_{0^+}^{\rho_2} v(t)), t \in (0,1), \\ D_{0^+}^{\alpha_2} v(t) = f_2(t, u(t), v(t), D_{0^+}^{\rho_1} u(t), D_{0^+}^{\rho_2} v(t)), t \in (0,1), \end{cases}$$

$$\begin{cases} u(0) = u'(0) = 0, \ v(0) = v'(0) = 0, \\ u(1) = \gamma_1 I_{0^+}^{\beta_1} u(\eta_1), \ v(1) = \gamma_2 I_{0^+}^{\beta_2} v(\eta_2). \end{cases}$$

However, boundary value problems with fractional Stieltjes integrals and Riemann–Liouville fractional integrals in boundary conditions have not been discussed until now. Now, in this paper, we shall investigate the existence and uniqueness of the solutions for the system (1), (2). As far as the authors know, the contraction mapping principle in Banach space and the alternative theorem of Leray–Schauder type have not been developed for boundary value problems with fractional Stieltjes integrals and Riemann–Liouville fractional integrals in boundary conditions, so it is interesting and important to discuss the (1), (2).

The organization of this paper is as follows. In Section 2, we present some useful basics definitions and lemmas. Section 3 gives the uniqueness and existence of solutions for the system. At the end of the paper, two examples that illustrate our results are given.

## 2. Preliminary

For convenience, we first present some useful basics lemmas of fractional calculus [18] in this part.

**Definition 1** ([18])**.** *For a function $k : (0, +\infty) \to R$,*

$$I_{0+}^{\beta} k(\tau) = \frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau - s)^{\beta - 1} k(s) ds,$$

*is defined as the $\beta(\beta > 0)$ order Riemann–Liouville fractional integral of the function $k$.*

**Definition 2** ([18])**.** *For a function $k : (0, +\infty) \to R$,*

$$D_{0+}^{\beta} k(\tau) = \frac{1}{\Gamma(n - \beta)} (\frac{d}{d\tau})^n \int_0^{\tau} (\tau - s)^{n - \beta - 1} k(s) ds,$$

*is defined as the $\beta(\beta > 0)$ order Riemann–Liouville fractional derivative of the function $k$, in this place $n = [\beta] + 1$.*

**Lemma 1** ([18])**.** *Assume that $v \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\beta > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then,*

$$I_{0+}^{\beta} D_{0+}^{\beta} v(\tau) = v(\tau) + c_1 \tau^{\beta - 1} + c_2 \tau^{\beta - 2} + \cdots + c_N \tau^{\beta - N},$$

*for some $c_i \in R, i = 1, 2, \cdots, N$, where $N$ is the smallest integer greater than or equal to $\beta$.*

**Lemma 2** ([19])**.** *Let $T : X \to X$ be continuous and compact. Denote $M(T) = \{u \in X : u = mT(u)$ for some $0 < m < 1\}$. Then, one of the following conclusions is true:*

*(i)* *$M(T)$ is an unbounded set;*

*(ii)* *there exists $x \in X$ satisfying $Tx = x$.*

*We denote by*

$$\Delta_1 = \frac{\Gamma(\beta_2)}{\Gamma(\beta_2 + \delta_1)} \gamma_1 \xi^{\beta_2 + \delta_1 - 1} + \sum_{i=1}^{p} \int_0^1 \tau^{\beta_2 - 1} d\mathcal{H}_i(\tau),$$

$$\Delta_2 = \frac{\Gamma(\beta_1)}{\Gamma(\beta_1 + \delta_2)} \gamma_2 \eta^{\beta_1 + \delta_2 - 1} + \sum_{j=1}^{q} \int_0^1 \tau^{\beta_1 - 1} d\mathcal{K}_j(\tau).$$

**Lemma 3.** *Suppose $x, y \in C[0, 1], \Delta = 1 - \Delta_1 \Delta_2 \neq 0$, $\beta_1, \beta_2 \in (1, 2]$, $\alpha_1, \alpha_2 \in (0, 1]$, $p, q \in N$, $(\overline{\alpha}_1 := \alpha_1 + \beta_1 + \delta_2, \overline{\alpha}_2 := \alpha_2 + \beta_2 + \delta_1)$, $\gamma_1, \gamma_2, \delta_1, \delta_2 > 0$, $0 < \xi, \eta < 1$, $\mathcal{K}_j(t), j = 1, \cdots, q$, $\mathcal{H}_i(t), i = 1, \cdots, p$, are bounded variation functions, $h, k$ are continuous on the interval $(0, 1)$, furthermore, $h, k$ are integrable on the interval $(0, 1)$. Then, the functional expressions*

$$\begin{aligned}x(t) = \; & -\tfrac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t (t-s)^{\alpha_1+\beta_1-1}h(s)ds \\ & +\tfrac{t^{\beta_1-1}}{\Delta}\Bigg[\tfrac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1 (1-s)^{\alpha_1+\beta_1-1}h(s)ds - \tfrac{\gamma_1}{\Gamma(\overline{\alpha}_2)}\int_0^{\xi}(\xi-s)^{\overline{\alpha}_2-1}k(s)ds \\ & -\tfrac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{p}\int_0^1\left(\int_0^{\tau}(\tau-s)^{\alpha_2+\beta_2-1}k(s)ds\right)d\mathcal{H}_i(\tau) \\ & +\Delta_1\Bigg(\tfrac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}k(s)ds \\ & -\tfrac{\gamma_2}{\Gamma(\overline{\alpha}_1)}\int_0^{\eta}(\eta-s)^{\overline{\alpha}_1-1}h(s)ds \\ & -\tfrac{1}{\Gamma(\alpha_1+\beta_1)}\sum_{j=1}^{q}\int_0^1\left(\int_0^{\tau}(\tau-s)^{\alpha_1+\beta_1-1}h(s)ds\right)d\mathcal{K}_j(\tau)\Bigg)\Bigg],\end{aligned} \tag{3}$$

$$\begin{aligned}y(t) = & -\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t (t-s)^{\alpha_2+\beta_2-1}k(s)ds \\ & +\frac{t^{\beta_2-1}}{\Delta}\Bigg[\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}k(s)ds - \frac{\gamma_2}{\Gamma(\overline{\alpha}_1)}\int_0^{\eta}(\eta-s)^{\overline{\alpha}_1-1}h(s)ds \\ & -\frac{1}{\Gamma(\alpha_1+\beta_1)}\sum_{j=1}^{q}\int_0^1\left(\int_0^{\tau}(\tau-s)^{\alpha_1+\beta_1-1}h(s)ds\right)d\mathcal{K}_j(\tau) \\ & +\Delta_2\Bigg(\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}h(s)ds \\ & -\frac{\gamma_1}{\Gamma(\overline{\alpha}_2)}\int_0^{\xi}(\xi-s)^{\overline{\alpha}_2-1}k(s)ds \\ & -\frac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{p}\int_0^1\left(\int_0^{\tau}(\tau-s)^{\alpha_2+\beta_2-1}k(s)ds\right)d\mathcal{H}_i(\tau)\Bigg)\Bigg].\end{aligned} \tag{4}$$

*is the solution of the system*

$$\begin{cases} D_{0^+}^{\alpha_1}(D_{0^+}^{\beta_1}x(t)) + h(t) = 0, t\in(0,1), \\ D_{0^+}^{\alpha_2}(D_{0^+}^{\beta_2}y(t)) + k(t) = 0, t\in(0,1). \end{cases} \tag{5}$$

*Furthermore,* $(x(t), y(t))$ *satisfies the equation condition (2).*

**Proof.** By Lemma 1, the solutions for the systems (2), (5) are give by

$$x(t) = -I_{0^+}^{\alpha_1+\beta_1}h(t) + c_1 t^{\beta_1-1}, \tag{6}$$

$$y(t) = -I_{0^+}^{\alpha_2+\beta_2}k(t) + d_1 t^{\beta_2-1}, \tag{7}$$

where $c_1, d_1 \in R$. From the boundary conditions $x(1) = \gamma_1 I_{0^+}^{\delta_1}y(\xi) + \sum_{i=1}^{p}\int_0^1 y(\tau)d\mathcal{H}_i(\tau)$ and $y(1) = \gamma_2 I_{0^+}^{\delta_2}x(\eta) + \sum_{j=1}^{q}\int_0^1 x(\tau)d\mathcal{K}_j(\tau)$, we get

$$\begin{aligned}-I_{0^+}^{\alpha_1+\beta_1}h(1) + c_1 = & -\gamma_1 I_{0^+}^{\overline{\alpha}_2}k(\xi) + \gamma_1 d_1\frac{\Gamma(\beta_2)}{\Gamma(\beta_2+\delta_1)}\xi^{\beta_2+\delta_1-1} \\ & + \sum_{i=1}^{p}\int_0^1\left(d_1\tau^{\beta_2-1} - I_{0^+}^{\alpha_2+\beta_2}k(\tau)\right)d\mathcal{H}_i(\tau),\end{aligned}$$

$$-I_{0^+}^{\alpha_2+\beta_2}k(1)+d_1=-\gamma_2 I_{0^+}^{\overline{\alpha}_1}h(\eta)+\gamma_2 c_1\frac{\Gamma(\beta_1)}{\Gamma(\beta_1+\delta_2)}\eta^{\beta_1+\delta_2-1}$$
$$+\sum_{j=1}^{q}\int_0^1\left(c_1\tau^{\beta_1-1}-I_{0^+}^{\alpha_1+\beta_1}h(\tau)\right)d\mathcal{K}_j(\tau).$$

Solving the above system, we find that

$$c_1=\frac{1}{\Delta}\left(I_{0^+}^{\alpha_1+\beta_1}h(1)-\gamma_1 I_{0^+}^{\overline{\alpha}_2}k(\xi)-\sum_{i=1}^{p}\int_0^1 I^{\alpha_2+\beta_2}k(\tau)d\mathcal{H}_i(\tau)\right)$$
$$+\frac{\Delta_1}{\Delta}\left(I_{0^+}^{\alpha_2+\beta_2}k(1)-\gamma_2 I_{0^+}^{\overline{\alpha}_1}h(\eta)-\sum_{j=1}^{q}\int_0^1 I_{0^+}^{\alpha_1+\beta_1}h(\tau)d\mathcal{K}_j(\tau)\right),$$

$$d_1=\frac{1}{\Delta}\left(I_{0^+}^{\alpha_2+\beta_2}k(1)-\gamma_2 I_{0^+}^{\overline{\alpha}_1}h(\eta)-\sum_{j=1}^{q}\int_0^1 I_{0^+}^{\alpha_1+\beta_1}h(\tau)d\mathcal{K}_j(\tau)\right)$$
$$+\frac{\Delta_2}{\Delta}\left(I_{0^+}^{\alpha_1+\beta_1}h(1)-\gamma_1 I_{0^+}^{\overline{\alpha}_2}k(\xi)-\sum_{i=1}^{p}\int_0^1 I^{\alpha_2+\beta_2}k(\tau)d\mathcal{H}_i(\tau)\right).$$

Substituting the values of $c_1, d_1$ in (6) and (7), we get the integral functional expressions (3) and (4). The conclusion can be obtained. □

The Banach space $E=C[0,1]$ is defined with the norm $\|\omega\|=\max\limits_{0\le\tau\le1}|\omega(\tau)|$.

Let $Y=E\times E$. So, the space $Y=\{(x,y):(x,y)\in Y\}$ with the norm $\|(x,y)\|_Y=\|x\|+\|y\|$ is Banach space. The operator expression $T:Y\to Y$ is defined by $T(x,y)(t)=(T_1(x,y)(t),T_2(x,y)(t))$, where

$$\begin{aligned}T_1(x,y)(t)=&\frac{t^{\beta_1-1}}{\Delta}\Bigg[-\frac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{p}\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s))ds\right)d\mathcal{H}_i(\tau)\\&-\frac{\gamma_1}{\Gamma(\overline{\alpha}_2)}\int_0^\xi(\xi-s)^{\overline{\alpha}_2-1}g(s,x(s),y(s))ds\\&+\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s))ds\\&+\Delta_1\Bigg(-\frac{1}{\Gamma(\alpha_1+\beta_1)}\sum_{j=1}^{q}\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s))ds\right)d\mathcal{K}_j(\tau)\\&-\frac{\gamma_2}{\Gamma(\overline{\alpha}_1)}\int_0^\eta(\eta-s)^{\overline{\alpha}_1-1}f(s,x(s),y(s))ds\\&+\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s))ds\Bigg)\Bigg]\\&-\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^t(t-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s))ds,\end{aligned}\tag{8}$$

$$\begin{aligned}
T_2(x,y)(t) =& \frac{t^{\beta_2-1}}{\Delta}\Bigg[-\frac{1}{\Gamma(\alpha_1+\beta_1)}\sum_{j=1}^{q}\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s))ds\right)d\mathcal{K}_j(\tau)\\
&-\frac{\gamma_2}{\Gamma(\overline{\alpha}_1)}\int_0^\eta(\eta-s)^{\overline{\alpha}_1-1}f(s,x(s),y(s))ds\\
&+\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s))ds\\
&+\Delta_2\Bigg(-\frac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{p}\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s))ds\right)d\mathcal{H}_i(\tau)\\
&-\frac{\gamma_1}{\Gamma(\overline{\alpha}_2)}\int_0^\xi(\xi-s)^{\overline{\alpha}_2-1}g(s,x(s),y(s))ds\\
&+\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s))ds\Bigg)\Bigg]\\
&-\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}g(s,x(s),y(s))ds.
\end{aligned} \tag{9}$$

Note that the couple fixed point of the integral operator $T$ happens to satisfy the system (1) and the boundary condition (2).

## 3. Main Result

Now we present the main conclusions of the system (1), (2). The tools we used include the contraction mapping principle in Banach space and the alternative theorem of Leray–Schauder type.

We give the following notation:

$$\begin{aligned}
M_1 =& \frac{|\Delta_1|}{|\Delta|\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\left|\int_0^1\tau^{\alpha_1+\beta_1}d\mathcal{K}_j(\tau)\right|+\frac{|\Delta_1|\gamma_2\eta^{\overline{\alpha}_1}}{|\Delta|\Gamma(\overline{\alpha}_1+1)}+\frac{1}{|\Delta|\Gamma(\alpha_1+\beta_1+1)}\\
&+\frac{1}{\Gamma(\alpha_1+\beta_1+1)},\\
M_2 =& \frac{1}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\sum_{i=1}^{p}\left|\int_0^1\tau^{\alpha_2+\beta_2}d\mathcal{H}_i(\tau)\right|+\frac{\gamma_1\xi^{\overline{\alpha}_2}}{|\Delta|\Gamma(\overline{\alpha}_2+1)}+\frac{|\Delta_1|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)},\\
M_3 =& \frac{|\Delta_2|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\sum_{i=1}^{p}\left|\int_0^1\tau^{\alpha_2+\beta_2}d\mathcal{H}_i(\tau)\right|+\frac{|\Delta_2|\gamma_1\xi^{\overline{\alpha}_2}}{|\Delta|\Gamma(\overline{\alpha}_2+1)}+\frac{1}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\\
&+\frac{1}{\Gamma(\alpha_2+\beta_2+1)},\\
M_4 =& \frac{1}{|\Delta|\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\left|\int_0^1\tau^{\alpha_1+\beta_1}d\mathcal{K}_j(\tau)\right|+\frac{\gamma_2\eta^{\overline{\alpha}_1}}{|\Delta|\Gamma(\overline{\alpha}_1+1)}+\frac{|\Delta_2|}{|\Delta|\Gamma(\alpha_1+\beta_1+1)},\\
M_5 =& M_1-\frac{1}{\Gamma(\alpha_1+\beta_1+1)},\quad M_6=M_3-\frac{1}{\Gamma(\alpha_2+\beta_2+1)}.
\end{aligned}$$

Additionally, the following assumptions hold:

**Hypothesis 1 ($\mathbf{H_1}$).** *By continuity of function $f$, there exist real constants $a_i(i=0,1,2)$ that satisfy*

$$|f(t,u,v)|\le a_0+a_1|u|+a_2|v|.$$

*By continuity of function $g$, there exist real constants $b_i(i=0,1,2)$ that satisfy*

$$|g(t,u,v)|\le b_0+b_1|u|+b_2|v|$$

*for all $(t,u,v)\in[0,1]\times R\times R$.*

**Hypothesis 2 ($\mathbf{H_2}$).** *There exist positive constants $K$ that satisfy*

$$K(|u-\overline{u}|+|v-\overline{v}|)\geq |f(t,u,v)-f(t,\overline{u},\overline{v})|,$$

*there exist positive constants $L$ that satisfy*

$$L(|u-\overline{u}|+|v-\overline{v}|)\geq |g(t,u,v)-g(t,\overline{u},\overline{v})|,$$

*for all $(t,u,u),(t,\overline{u},\overline{v})\in[0,1]\times R\times R$.*

**Hypothesis 3 ($\mathbf{H_3}$).** *There exist positive constants $F_0$ such that $F_0=\sup\limits_{t\in J}|f(t,0,0)|$, and there exist positive constants $G_0$ such that $G_0=\sup\limits_{t\in J}|g(t,0,0)|$.*

**Theorem 1.** *Suppose that conditions $(H_2)$ and $(H_3)$ are satisfied. Moreover,*

$$K(M_1+M_4)+L(M_2+M_3)<1,$$

*then there is a unique solution for system (1), (2).*

**Proof.** We consider a real constant $R>0$ such that

$$\frac{(M_1+M_4)F_0+(M_2+M_3)G_0}{1-[K(M_1+M_4)+L(M_2+M_3)]}\leq R.$$

Let $B_R=\{(x,y)\in Y,\|(x,y)\|_Y\leq R\}$. We prove that $T$ mapping $B_R$ to $B_R$. From $(H_2)$ and $(H_3)$, we deduce that the following holds:

$$\begin{aligned}|f(t,x(t),y(t)| &\leq |f(t,0,0)|+|f(t,x(t),y(t)-f(t,0,0)|\\ &\leq F_0+K(|x|+|y|)\\ &\leq F_0+K(\|x\|+\|y\|)\\ &=F_0+K\|(x,y)\|.\end{aligned}$$

Similarly, we have $|g(t,x(t),y(t)|\leq G_0+L\|(x,y)\|$.

For all $(x,y)$ in $B_R$, we obtain

$$\begin{aligned}|T_1(x,y)(t)| \leq{}& \frac{t^{\beta_1-1}}{|\Delta|}\Bigg[\frac{LR+G_0}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{p}\left|\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_2+\beta_2-1}ds\right)d\mathcal{H}_i(\tau)\right|\\ &+\frac{\gamma_1(LR+G_0)}{\Gamma(\overline{\alpha}_2)}\int_0^\xi(\xi-s)^{\overline{\alpha}_2-1}ds+\frac{KR+F_0}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}ds\\ &+|\Delta_1|\left(\frac{KR+F_0}{\Gamma(\alpha_1+\beta_1)}\sum_{j=1}^{q}\left|\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_1+\beta_1-1}ds\right)d\mathcal{K}_j(\tau)\right|\right.\\ &\left.+\frac{\gamma_2(KR+F_0)}{\Gamma(\overline{\alpha}_1)}\int_0^\eta(\eta-s)^{\overline{\alpha}_1-1}ds+\frac{LR+G_0}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}ds\right)\Bigg]\\ &+\frac{KR+F_0}{\Gamma(\alpha_1+\beta_1)}\int_0^t(t-s)^{\alpha_1+\beta_1-1}ds\\ \leq{}& \frac{1}{|\Delta|}\Bigg[\frac{LR+G_0}{\Gamma(\alpha_2+\beta_2+1)}\times\sum_{i=1}^{p}\left|\int_0^1\tau^{\alpha_2+\beta_2}d\mathcal{H}_i(\tau)\right|+\frac{\gamma_1(LR+G_0)\xi^{\overline{\alpha}_2}}{\Gamma(\overline{\alpha}_2+1)}\\ &+\frac{KR+F_0}{\Gamma(\alpha_1+\beta_1+1)}\quad+|\Delta_1|\left(\frac{KR+F_0}{\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\left|\int_0^1\tau^{\alpha_1+\beta_1}d\mathcal{K}_j(\tau)\right|\right.\\ &\left.+\frac{\gamma_2(KR+F_0)\eta^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}+\frac{LR+G_0}{\Gamma(\alpha_2+\beta_2+1)}\right)\Bigg]+\frac{KR+F_0}{\Gamma(\alpha_1+\beta_1+1)}\end{aligned}$$

$$
\begin{aligned}
&= (KR+F_0)\left(\frac{|\Delta_1|}{|\Delta|\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\left|\int_0^1 \tau^{\alpha_1+\beta_1}d\mathcal{K}_j(\tau)\right| + \frac{|\Delta_1|\gamma_2\eta^{\overline{\alpha}_1}}{|\Delta|\Gamma(\overline{\alpha}_1+1)}\right.\\
&\left.+\frac{1}{|\Delta|\Gamma(\alpha_1+\beta_1+1)}+\frac{1}{\Gamma(\alpha_1+\beta_1+1)}\right)\\
&+(LR+G_0)\left(\frac{1}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\sum_{i=1}^{p}\left|\int_0^1 \tau^{\alpha_2+\beta_2}d\mathcal{H}_i(\tau)\right|\right.\\
&\left.+\frac{\gamma_1\xi^{\overline{\alpha}_2}}{|\Delta|\Gamma(\overline{\alpha}_2+1)}+\frac{|\Delta_1|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\right)\\
&= (KR+F_0)M_1+(LR+G_0)M_2.
\end{aligned} \tag{10}
$$

Let us continue with the calculations:

$$
\begin{aligned}
|T_2(x,y)(t)| \quad &\le \frac{t^{\beta_2-1}}{|\Delta|}\left[\frac{KR+F_0}{\Gamma(\alpha_1+\beta_1)}\sum_{j=1}^{q}\left|\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_1+\beta_1-1}ds\right)d\mathcal{K}_j(\tau)\right|\right.\\
&+\frac{\gamma_2(KR+F_0)}{\Gamma(\overline{\alpha}_1)}\int_0^\eta(\eta-s)^{\overline{\alpha}_1-1}ds+\frac{LR+G_0}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}ds\\
&+|\Delta_2|\left(\frac{LR+G_0}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{p}\left|\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_2+\beta_2-1}ds\right)d\mathcal{H}_i(\tau)\right|\right.\\
&\left.\left.+\frac{\gamma_1(LR+G_0)}{\Gamma(\overline{\alpha}_2)}\int_0^\xi(\xi-s)^{\overline{\alpha}_2-1}ds+\frac{KR+F_0}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}ds\right)\right]\\
&+\frac{LR+G_0}{\Gamma(\alpha_2+\beta_2)}\int_0^t(t-s)^{\alpha_2+\beta_2-1}ds\\
&\le \frac{1}{|\Delta|}\left[\frac{KR+F_0}{\Gamma(\alpha_1+\beta_1+1)}\times\sum_{j=1}^{q}\left|\int_0^1\tau^{\alpha_1+\beta_1}d\mathcal{K}_j(\tau)\right|+\frac{\gamma_2(KR+F_0)\eta^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)}\right.\\
&+\frac{LR+G_0}{\Gamma(\alpha_2+\beta_2+1)}\quad+|\Delta_2|\left(\frac{LR+G_0}{\Gamma(\alpha_2+\beta_2+1)}\sum_{i=1}^{p}\left|\int_0^1\tau^{\alpha_2+\beta_2}d\mathcal{H}_i(\tau)\right|\right.\\
&\left.\left.+\frac{\gamma_1(LR+G_0)\xi^{\overline{\alpha}_2}}{\Gamma(\overline{\alpha}_2+1)}+\frac{KR+F_0}{\Gamma(\alpha_2+\beta_2+1)}\right)\right]+\frac{LR+G_0}{\Gamma(\alpha_2+\beta_2+1)}\\
&=(LR+G_0)\left(\frac{|\Delta_2|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\sum_{i=1}^{p}\left|\int_0^1\tau^{\alpha_2+\beta_2}d\mathcal{H}_i(\tau)\right|+\frac{|\Delta_2|\gamma_1\xi^{\overline{\alpha}_2}}{|\Delta|\Gamma(\overline{\alpha}_2+1)}\right.\\
&\left.+\frac{1}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}+\frac{1}{\Gamma(\alpha_2+\beta_2+1)}\right)\\
&+(KR+F_0)\left(\frac{1}{|\Delta|\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\left|\int_0^1\tau^{\alpha_1+\beta_1}d\mathcal{K}_j(\tau)\right|\right.\\
&\left.+\frac{\gamma_2\eta^{\overline{\alpha}_1}}{|\Delta|\Gamma(\overline{\alpha}_1+1)}+\frac{|\Delta_2|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\right)\\
&=(LR+G_0)M_3+(KR+F_0)M_4.
\end{aligned} \tag{11}
$$

Consequently,

$$\|T(x,y)\| \le (KR+F_0)M_1+(LR+G_0)M_2+(LR+G_0)M_3+(KR+F_0)M_4 \le R.$$

Hence, $T(B_R) \subseteq R$.

Now we will prove that $T$ is a contraction operator. Choose $(x,y),(\overline{x},\overline{y})$ in $Y$. For all $t\in[0,1]$, we find

$$
\begin{aligned}
|T_1(x,y)(t) - T_1(\overline{x},\overline{y})(t)| \leq\ & \frac{t^{\beta_1-1}}{|\Delta|}\Bigg[\frac{L(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\alpha_2+\beta_2+1)}\sum_{i=1}^{p}\left|\int_0^1 \tau^{\alpha_2+\beta_2} d\mathcal{H}_i(\tau)\right| \\
&+\frac{\gamma_1 L(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\overline{\alpha}_2+1)}\xi^{\overline{\alpha}_2} \\
&+\frac{K(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\alpha_1+\beta_1+1)} \\
&+|\Delta_1|\Bigg(\frac{K(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\left|\int_0^1 \tau^{\alpha_1+\beta_1} d\mathcal{K}_j(\tau)\right| \\
&+\frac{\gamma_2 K(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\overline{\alpha}_1+1)}\eta^{\overline{\alpha}_1} \\
&+\frac{L(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\alpha_2+\beta_2+1)}\Bigg)\Bigg] + \frac{K(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\alpha_1+\beta_1+1)}t^{\alpha_1+\beta_1} \\
\leq\ & K\Bigg(\frac{1}{|\Delta|\Gamma(\alpha_1+\beta_1+1)} + \frac{|\Delta_1|}{|\Delta|\Gamma(\alpha_1+\beta_1+1)} \\
&\sum_{j=1}^{q}\left|\int_0^1 \tau^{\alpha_1+\beta_1} d\mathcal{K}_j(\tau)\right| \\
&+\frac{|\Delta_1|\gamma_2\eta^{\overline{\alpha}_1}}{|\Delta|\Gamma(\overline{\alpha}_1+1)} + \frac{1}{\Gamma(\alpha_1+\beta_1+1)}\Bigg)\|(x,y)-(\overline{x},\overline{y})\| \\
&+L\Bigg(\frac{1}{|\Delta|\Gamma(\alpha_2+\beta_2+1)} \times \sum_{i=1}^{p}\left|\int_0^1 \tau^{\alpha_2+\beta_2} d\mathcal{H}_i(\tau)\right. \\
&+\frac{\gamma_1\xi^{\overline{\alpha}_2}}{|\Delta|\Gamma(\overline{\alpha}_2+1)} + \left.\frac{|\Delta_1|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\right|\Bigg)\|(x,y)-(\overline{x},\overline{y})\| \\
=\ & (M_1K + M_2L)\|(x,y)-(\overline{x},\overline{y})\|.
\end{aligned} \tag{12}
$$

Let us continue with the calculations:

$$
\begin{aligned}
|T_2(x,y)(t) - T_2(\overline{x},\overline{y})(t)| \leq\ & \frac{t^{\beta_2-1}}{|\Delta|}\Bigg[\frac{K(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\left|\int_0^1 \tau^{\alpha_1+\beta_1} d\mathcal{K}_j(\tau)\right| \\
&+\frac{\gamma_2 K(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\overline{\alpha}_1+1)}\eta^{\overline{\alpha}_1} \\
&+\frac{L(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\alpha_2+\beta_2+1)} \\
&+|\Delta_2|\Bigg(\frac{L(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\alpha_2+\beta_2+1)}\sum_{i=1}^{p}\left|\int_0^1 \tau^{\alpha_2+\beta_2} d\mathcal{H}_i(\tau)\right| \\
&+\frac{\gamma_1 L(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\overline{\alpha}_2+1)}\xi^{\overline{\alpha}_2} \\
&+\frac{K(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\alpha_1+\beta_1+1)}\Bigg)\Bigg] + \frac{L(\|x-\overline{x}\|+\|y-\overline{y}\|)}{\Gamma(\alpha_2+\beta_2+1)}t^{\alpha_2+\beta_2} \\
\leq\ & L\Bigg(\frac{1}{|\Delta|\Gamma(\alpha_2+\beta_2+1)} + \frac{|\Delta_2|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)} \\
&\sum_{i=1}^{p}\left|\int_0^1 \tau^{\alpha_2+\beta_2} d\mathcal{H}_i(\tau)\right| \\
&+\frac{|\Delta_2|\gamma_1\xi^{\overline{\alpha}_2}}{|\Delta|\Gamma(\overline{\alpha}_2+1)} + \frac{1}{\Gamma(\alpha_2+\beta_2+1)}\Bigg)\|(x,y)-(\overline{x},\overline{y})\| \\
&+K\Bigg(\frac{1}{|\Delta|\Gamma(\alpha_1+\beta_1+1)} \times \sum_{j=1}^{q}\left|\int_0^1 \tau^{\alpha_1+\beta_1} d\mathcal{K}_j(\tau)\right. \\
&+\frac{\gamma_2\eta^{\overline{\alpha}_1}}{|\Delta|\Gamma(\overline{\alpha}_1+1)} + \left.\frac{|\Delta_2|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\right|\Bigg)\|(x,y)-(\overline{x},\overline{y})\| \\
=\ & (M_1L + M_2K)\|(x,y)-(\overline{x},\overline{y})\|.
\end{aligned} \tag{13}
$$

Consequently,

$$
\|T(x,y)(t) - T(\overline{x},\overline{y})(t)\| \leq [(M_1+M_4)K + (M_2+M_3)L]\|(x,y)-(\overline{x},\overline{y})\|.
$$

Using contraction mapping principle in Banach space, there is a unique function that satisfies $Tu = u$, which happens to be the solution of the system (1), (2). □

**Theorem 2.** *Suppose that condition* $(H_1)$ *is satisfied. If* $\rho := \max\{M_7, M_8\} < 1$, *where* $M_7 = a_1(M_1 + M_4) + b_1(M_2 + M_3)$ *and* $M_8 = a_2(M_1 + M_4) + b_2(M_2 + M_3)$, *then at least one couple functions* $(x(t), y(t))$ *satisfy the system (1), (2).*

**Proof.** By continuity of functions $f$ and $g$, the operators $T_1$ and $T_2$ are continuous, this means the operator $T$ is also continuous. We choose an arbitrarily bounded open subset $\Omega$ from $E$. There exist $\overline{K} > 0$ and $\overline{L} > 0$ that satisfy $|f(t, x(t), y(t)| \le \overline{K}$ and $|g(t, x(t), y(t)| \le \overline{L}$ for all $t$ in the [0,1] and $(x, y)$ in $\Omega$. Thus, by the proof of Theorem 1, we have

$$|T_1(x,y)(t)| \le \overline{K}M_1 + \overline{L}M_2, \quad |T_2(x,y)(t)| \le \overline{L}M_3 + \overline{K}M_4$$

for all $t$ in the [0,1] and $(x, y)$ in $\Omega$. Then, we obtain

$$\|T(x,y)\| \le \overline{K}(M_1 + M_4) + \overline{L}(M_2 + M_3), \quad \forall (x,y) \in \Omega.$$

So, we get the boundedness of $T(\Omega)$.

Take $(x, y) \in \Omega$ and $0 \le t_1 < t_2 \le 1$, one has

$$\begin{aligned}
&|T_1(x,y)(t_2) - T_1(x,y)(t_1)| \\
&\le \frac{t_2^{\beta_1-1} - t_1^{\beta_1-1}}{|\Delta|}\Bigg[\frac{1}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{p}\left|\int_0^1\left(\int_0^\tau (\tau-s)^{\alpha_2+\beta_2-1}|g(s,x(s),y(s))|ds\right)d\mathcal{H}_i(\tau)\right| \\
&+\frac{\gamma_1}{\Gamma(\overline{\alpha}_2)}\int_0^\xi(\xi-s)^{\overline{\alpha}_2-1}|g(s,x(s),y(s))|ds \\
&+\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}|f(s,x(s),y(s))|ds \\
&+|\Delta_1|\Bigg(\frac{1}{\Gamma(\alpha_1+\beta_1)}\sum_{j=1}^{q}\left|\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_1+\beta_1-1}|f(s,x(s),y(s))|ds\right)d\mathcal{K}_j(\tau)\right| \\
&+\frac{\gamma_2}{\Gamma(\overline{\alpha}_1)}\int_0^\eta(\eta-s)^{\overline{\alpha}_1-1}|f(s,x(s),y(s))|ds \\
&+\frac{1}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}|g(s,x(s),y(s))|ds\Bigg)\Bigg] \\
&+\Bigg|-\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^{t_2}(t_2-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s))ds \\
&+\frac{1}{\Gamma(\alpha_1+\beta_1)}\int_0^{t_1}(t_1-s)^{\alpha_1+\beta_1-1}f(s,x(s),y(s))ds\Bigg| \\
&\le \frac{t_2^{\beta_1-1} - t_1^{\beta_1-1}}{|\Delta|}\Bigg[\frac{\overline{L}}{\Gamma(\alpha_2+\beta_2)}\sum_{i=1}^{p}\left|\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_2+\beta_2-1}ds\right)d\mathcal{H}_i(\tau)\right| \\
&+\frac{\gamma_1\overline{L}}{\Gamma(\overline{\alpha}_2)}\int_0^\xi(\xi-s)^{\overline{\alpha}_2-1}ds + \frac{\overline{K}}{\Gamma(\alpha_1+\beta_1)}\int_0^1(1-s)^{\alpha_1+\beta_1-1}ds \\
&+|\Delta_1|\Bigg(\frac{\overline{K}}{\Gamma(\alpha_1+\beta_1)}\sum_{j=1}^{q}\left|\int_0^1\left(\int_0^\tau(\tau-s)^{\alpha_1+\beta_1-1}ds\right)d\mathcal{K}_j(\tau)\right| + \frac{\gamma_2\overline{K}}{\Gamma(\overline{\alpha}_1)}\int_0^\eta(\eta-s)^{\overline{\alpha}_1-1}ds \\
&+\frac{\overline{L}}{\Gamma(\alpha_2+\beta_2)}\int_0^1(1-s)^{\alpha_2+\beta_2-1}ds\Bigg)\Bigg] \\
&+\frac{\overline{K}}{\Gamma(\alpha_1+\beta_1)}\int_0^{t_1}[(t_2-s)^{\alpha_1+\beta_1-1}-(t_1-s)^{\alpha_1+\beta_1-1}]ds \\
&+\frac{\overline{K}}{\Gamma(\alpha_1+\beta_1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha_1+\beta_1-1}ds \\
&\le \frac{\overline{K}}{\Gamma(\alpha_1+\beta_1+1)}(t_2^{\alpha_1+\beta_1} - t_1^{\alpha_1+\beta_1}) + \overline{K}(t_2^{\beta_1-1} - t_1^{\beta_1-1})\Bigg(\frac{1}{|\Delta|\Gamma(\alpha_1+\beta_1+1)} \\
&+\frac{|\Delta_1|\gamma_2\eta^{\overline{\alpha}_1}}{|\Delta|\Gamma(\overline{\alpha}_1+1)} \\
&+\frac{|\Delta_1|}{|\Delta|\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\left|\int_0^1\tau^{\alpha_1+\beta_1}d\mathcal{K}_j(\tau)\right|\Bigg) + \overline{L}(t_2^{\beta_1-1} - t_1^{\beta_1-1}) \\
&\times\left(\frac{\gamma_1\xi^{\overline{\alpha}_2}}{|\Delta|\Gamma(\overline{\alpha}_2+1)} + \frac{|\Delta_1|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)} + \frac{1}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\sum_{i=1}^{p}\left|\int_0^1\tau^{\alpha_2+\beta_2}d\mathcal{H}_i(\tau)\right|\right) \\
&= \frac{\overline{K}}{\Gamma(\alpha_1+\beta_1+1)}(t_2^{\alpha_1+\beta_1} - t_1^{\alpha_1+\beta_1}) + (\overline{K}M_5 + \overline{L}M_2)(t_2^{\beta_1-1} - t_1^{\beta_1-1}).
\end{aligned} \tag{14}$$

We find result

$T_1(x,y)(t_2) \to T_1(x,y)(t_1)$ when $t_2 \to t_1$, for arbitrary $(x,y) \in \Omega$.

Similarly, for $(x,y) \in \Omega, 0 \le t_1 < t_2 \le 1$,

$$
\begin{aligned}
&|T_2(x,y)(t_2) - T_2(x,y)(t_1)| \\
&\le \frac{\overline{L}}{\Gamma(\alpha_2+\beta_2+1)}(t_2^{\alpha_2+\beta_2} - t_1^{\alpha_2+\beta_2}) + \overline{L}(t_2^{\beta_2-1} - t_1^{\beta_2-1})\bigg(\frac{1}{|\Delta|\Gamma(\alpha_2+\beta_2+1)} \\
&+ \frac{|\Delta_2|\gamma_1\xi^{\overline{\alpha}_2}}{|\Delta|\Gamma(\overline{\alpha}_2+1)} + \frac{|\Delta_2|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)}\sum_{i=1}^{p}\bigg|\int_0^1 \tau^{\alpha_2+\beta_2}d\mathcal{H}_i(\tau)\bigg|\bigg) + \overline{K}(t_2^{\beta_2-1} - t_1^{\beta_2-1}) \\
&\times \bigg(\frac{\gamma_2\eta^{\overline{\alpha}_1}}{|\Delta|\Gamma(\overline{\alpha}_1+1)} + \frac{|\Delta_2|}{|\Delta|\Gamma(\alpha_1+\beta_1+1)} + \frac{1}{|\Delta|\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\bigg|\int_0^1 \tau^{\alpha_1+\beta_1}d\mathcal{K}_j(\tau)\bigg|\bigg) \\
&= \frac{\overline{L}}{\Gamma(\alpha_2+\beta_2+1)}(t_2^{\alpha_2+\beta_2} - t_1^{\alpha_2+\beta_2}) + (\overline{L}M_6 + \overline{K}M_4)(t_2^{\beta_2-1} - t_1^{\beta_2-1}).
\end{aligned}
\tag{15}
$$

So we obtain

$T_2(x,y)(t_2) \to T_2(x,y)(t_1)$ when $t_2 \to t_1$, for arbitrary $(x,y) \in \Omega$.

The conclusion that $T : B_R \to B_R$ is continuous and compact can be deduced from the Arzela–Ascoli theorem.

Finally, we will give the fact $M(T) = \{(x,y) \in E \times E : (x,y) = mT(x,y)$ for some $0 < m < 1\}$ is bounded. Let $(x,y)$ in $M(T)$ and for any $t$ on [0,1], we have $mT(x,y) = (mT_1(x,y), mT_2(x,y))$.

By $(H_1)$, we have

$$
\begin{aligned}
|x(t)| &\le |T_1(x,y)(t)| \\
&\le \tfrac{1}{|\Delta|}\bigg[\tfrac{b_0+b_1\|x\|+b_2\|y\|}{\Gamma(\alpha_2+\beta_2+1)}\sum_{i=1}^{p}\bigg|\int_0^1 \tau^{\alpha_2+\beta_2}d\mathcal{H}_i(\tau)\bigg| + \tfrac{\gamma_1(b_0+b_1\|x\|+b_2\|y\|)\xi^{\overline{\alpha}_2}}{\Gamma(\overline{\alpha}_2+1)} \\
&+ \tfrac{a_0+a_1\|x\|+a_2\|y\|}{\Gamma(\alpha_1+\beta_1+1)} + |\Delta_1|\bigg(\tfrac{a_0+a_1\|x\|+a_2\|y\|}{\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\bigg|\int_0^1 \tau^{\alpha_1+\beta_1}d\mathcal{K}_j(\tau)\bigg| \\
&+ \tfrac{\gamma_2(a_0+a_1\|x\|+a_2\|y\|)\eta^{\overline{\alpha}_1}}{\Gamma(\overline{\alpha}_1+1)} + \tfrac{b_0+b_1\|x\|+b_2\|y\|}{\Gamma(\alpha_2+\beta_2+1)}\bigg)\bigg] \\
&+ \tfrac{a_0+a_1\|x\|+a_2\|y\|}{\Gamma(\alpha_1+\beta_1+1)} \\
&= (a_0 + a_1\|x\| + a_2\|y\|)\bigg(\tfrac{|\Delta_1|}{|\Delta|\Gamma(\alpha_1+\beta_1+1)}\sum_{j=1}^{q}\bigg|\int_0^1 \tau^{\alpha_1+\beta_1}d\mathcal{K}_j(\tau)\bigg| \\
&+ \tfrac{1}{|\Delta|\Gamma(\alpha_1+\beta_1+1)} + \tfrac{|\Delta_1|\gamma_2\eta^{\overline{\alpha}_1}}{|\Delta|\Gamma(\overline{\alpha}_1+1)} + \tfrac{1}{\Gamma(\alpha_1+\beta_1+1)}\bigg) \\
&+ (b_0 + b_1\|x\| + b_2\|y\|)\bigg(\tfrac{\gamma_1\xi^{\overline{\alpha}_2}}{|\Delta|\Gamma(\overline{\alpha}_2+1)} + \tfrac{|\Delta_1|}{|\Delta|\Gamma(\alpha_2+\beta_2+1)} \\
&+ \tfrac{1}{|\Delta|\Gamma(\alpha_2+\beta_2+1)} \times \sum_{i=1}^{p}\bigg|\int_0^1 \tau^{\alpha_2+\beta_2}d\mathcal{H}_i(\tau)\bigg|\bigg).
\end{aligned}
\tag{16}
$$

So we deduce

$$
\|x\| \le (a_0 + a_1\|x\| + a_2\|y\|)M_1 + (b_0 + b_1\|x\| + b_2\|y\|)M_2. \tag{17}
$$

Using the same proof process, we get

$$
\|y\| \le (a_0 + a_1\|x\| + a_2\|y\|)M_4 + (b_0 + b_1\|x\| + b_2\|y\|)M_3. \tag{18}
$$

By (17) and (18), we have

$$
\begin{aligned}
\|(x,y)\| = \|x\| + \|y\| &\le a_0(M_1 + M_4) + b_0(M_2 + M_3) \\
&+ \Big[a_1(M_1 + M_4) + b_1(M_2 + M_3)\Big]\|x\| + \Big[a_2(M_1 + M_4) + b_2(M_2 + M_3)\Big]\|y\| \\
&= a_0(M_1 + M_4) + b_0(M_2 + M_3) + M_7\|x\| + M_8\|y\| \\
&\le a_0(M_1 + M_4) + b_0(M_2 + M_3) + \rho\|(x,y)\|.
\end{aligned}
$$

For $\rho < 1$, we obtain

$$\|(x,y)\| \leq \frac{a_0(M_1+M_4)+b_0(M_2+M_3)}{1-\rho}, \quad \forall (x,y) \in M(T).$$

Hence, we prove $M(T)$ is a bounded set.

By using the alternative theorem of Leray–Schauder, there exists $x \in X$ that satisfy $Tx = x$, therefore, coupled function $(x,y)$ satisfy system (1) and integral boundary condition (2). □

## 4. Example

Let $\alpha_1 = \frac{1}{3}$, $\mathcal{H}_1(t) = 2t, t \in [0,1]$, $\alpha_2 = \frac{5}{6}$, $\beta_1 = \frac{5}{4}$, $\mathcal{K}_1 = t, t \in [0,1]$, $\beta_2 = \frac{7}{5}$, $p = 2$, $q = 1$, $\gamma_1 = 2$, $\gamma_2 = 3$, $\delta_1 = \frac{3}{7}$, $\delta_2 = \frac{8}{5}$, $\xi = \frac{1}{5}$, $\eta = \frac{1}{3}$, $\mathcal{H}_2(t) = \{0, t \in [0, \frac{1}{4}); 3, t \in [\frac{1}{4}, 1]\}$.

We consider the following specific fractional order systems

$$\begin{cases} D_{0^+}^{\frac{1}{3}}(D_{0^+}^{\frac{5}{4}}x(t)) + f(t,x(t),y(t)) = 0, t \in [0,1], \\ D_{0^+}^{\frac{5}{6}}(D_{0^+}^{\frac{7}{5}}y(t)) + g(t,x(t),y(t)) = 0, t \in [0,1], \end{cases} \tag{19}$$

supplemented with the condition

$$\begin{cases} D_{0^+}^{\frac{5}{4}}x(0) = 0, x(0) = 0, x(1) = 2I_{0^+}^{\frac{3}{7}}y(\frac{1}{5}) + 2\int_0^1 y(t)dt + 3y(\frac{1}{4}), \\ D_{0^+}^{\frac{7}{5}}y(0) = 0, y(0) = 0, y(1) = 3I_{0^+}^{\frac{8}{5}}x(\frac{1}{3}) + \int_0^1 x(t)dt. \end{cases} \tag{20}$$

We obtain $\Delta \approx -3.2945773664941695 \neq 0$. By calculation, we have $M_4 \approx 0.33982021146365704$, $M_3 \approx 0.6299976999210883$, $M_2 \approx 0.5353700439729107$, $M_1 \approx 1.2401800473948743$.

**Example 1.** *We choose*

$$f(t,u_1,v_1) = \frac{1}{\sqrt{t^3+3}} + \frac{t}{8}u_1 - \frac{1}{5}\sin v_1,$$
$$g(t,u_1,v_1) = \frac{t}{t^2+12} - \frac{t}{4}\arctan u_1 + \frac{|v_1|}{15+|v_1|},$$

*for all t on [0,1], $u_1, v_1$ in R. Then, we get the following estimates*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq \frac{1}{5}(|u_1-u_2| + |v_1-v_2|).$$

*Thus, $K = \frac{1}{5}$, moreover,*

$$|g(t,u_1,v_1) - g(t,u_2,v_2)| \leq \frac{1}{4}(|u_1-u_2| + |v_1-v_2|).$$

*Thus, $L = \frac{1}{4}$. Hence, $K(M_1+M_4) + L(M_2+M_3) \approx 0.6073419877452061 < 1$. So the condition $(H_2)$ holds, and by Theorem 1, there is a couple function $(x(t), y(t))$ satisfies the systems (19) and (20).*

**Example 2.** *We choose*

$$f(t,u,v) = \frac{t+1}{5} - \frac{1}{t+8}\sin u + \frac{1}{12}v,$$
$$g(t,u,v) = \frac{e^{-t}}{t^2+3} + \frac{5}{8}\arctan u + \frac{1}{6}v,$$

*for all t on [0,1], $u_1, v_1$ in R. Then, we get the following estimates*

$$|f(t,u,v)| \leq \frac{2}{5} + \frac{1}{8}|u| + \frac{1}{12}|v|,$$
$$|g(t,u,v)| \leq \frac{1}{3} + \frac{5}{8}|u| + \frac{1}{6}|v|,$$

*for all t on [0,1], $u_1, v_1$ in R. Since the assumption $(H_1)$, we get $a_0 = \frac{2}{5}$, $a_1 = \frac{1}{8}$, $a_2 = \frac{1}{12}$, $b_0 = \frac{1}{3}$, $b_1 = \frac{5}{8}$ and $b_2 = \frac{1}{6}$. Thus, we obtain $M_7 \approx 0.9258548722910658$, $M_8 \approx 0.3258946455538774$, and $\rho = \max\{M_7, M_8\} = M_7 < 1$. Hence, by Theorem 2, we conclude that problem (19) and (20) have at least one solution $(x(t), y(t))$, $t \in [0,1]$.*

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