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Article

Fractional COVID-19 Modeling and Analysis on Successive Optimal Control Policies

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Abstract: A fractional-order coronavirus disease of 2019 (COVID-19) model is constructed of five compartments in the Caputo-Fabrizio sense. The main aim of the paper is to study the effects of successive optimal control policies in different susceptible classes; a susceptible unaware class where awareness control is observed, a susceptible aware class where vaccine control is observed, and a susceptible vaccinated class where optimal vaccination control is observed. These control policies are considered awareness and actions toward vaccination and non-pharmaceuticals to control infection. Equilibrium points are calculated, which subsequently leads to the computation of the basic reproduction ratio. The existence and uniqueness properties of the model are established. The optimal control problem is constructed and subsequently analyzed. Numerical simulations are carried out and the significance of the fractional-order from the biological point of view is established. The results showed that applying various control functions will lead to a decrease in the infected population, and it is evident that introducing the three control measures together causes a drastic decrease in the infected population.

Keywords: mathematical model; fractional-order; Caputo-Fabrizio; optimal control; existence and uniqueness; basic reproduction ratio

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1. Introduction

Toward the end of December 2019, a deadly disease called COVID-19 resurfaced around the world. It destabilized many sectors, including transport, economies, education systems, sports, entertainment and many others. Many people die from the pandemic while many have been infected and battling with their lives. The behavior spread patterns and much other biological information about the COVID-19 outbreak is still not completely known. Many research works have been dedicated to finding new and adequate vaccines for the disease. Many items, such as ventilators, have been used to help infected individuals in many countries. The main target is to reduce the number of infected individuals and subsequently deaths due to the pandemic, which is why many countries adopt non-pharmaceutical measures, such as lockdowns, airport closures, use of sanitizers, and social distancing. Many studies from theoretical to practical points of view about the pandemic have been carried out [1–12].

While 75% of infected individuals recover without falling seriously sick, most of the infected individuals recover naturally [13]. Throat infection, chest pain, runny nose or nasal congestion, loss of smell and taste, vomiting, diarrhea and nausea are some of the symptoms of COVID-19. In most cases, these symptoms appear slowly. Older age suffers major complications compared to younger age. In general, an infected person takes two days to two weeks to show symptoms of the disease [14]. Mostly mild cases take two weeks to recover, whereas critical cases take three to six weeks to recover [15].

Now that the COVID-19 vaccine is available and the non-pharmaceutical interventions to prevent the spread of the disease such as quarantine, social distancing, self-isolation, and use of personal protective equipment (such as face mask, hand globes, overall gown, etc.) regular hand washing using sanitizer, avoid having contact with the person showing the symptoms, reporting any suspected case, and compliance with orientation exercises are also available, there is need for the increase of awareness level among people. This will help in total compliance and the subsequent eradication of the disease.

Since the inception of the pandemic in 2019, it has caused millions of infections and thousands of deaths. It also caused a predicament in the socio-economic growth of the entire world. Hence, there is an urgent need to clearly understand the transmission dynamics of the disease. This leads to the need to develop mathematical models that study the dynamics of the disease and the impact of control measures on curtailing the spread of the disease.

Because of the hereditary properties and provision of a good description of the memory, fractional order derivatives and fractional integrals play an important role in the study of mathematical modeling. This is why many researchers about real-life phenomena use fractional order differential equations [16–19]. The Caputo-Fabrizio (CF) fractional derivative fractional-order derivative was developed in 2015. This fractional-order derivative is based on an exponential kernel and the details of the operator can be found in [20]. Many problems used the Caputo-Fabrizio derivative to model problems in various fields [21–23], also used in modeling the COVID-19 pandemic in [24–27]. However, the Caputo-Fabrizio fractional derivative gives less noise than the Riemann–Liouville derivative [28]. Hence, in this research, the Caputo-Fabrizio fractional derivative was chosen.

Epidemiology, aeronautic engineering, economics and finance, robotics and many other fields use optimal control as an effective mathematical tool to optimize the control problems that arise in the fields [29].

Most mathematical models of COVID-19 that studied control in the literature did not consider time-dependent control strategies, which are the most realistic approaches [30–38]. However, very little research in this direction does exist, such as in [39–44] and this type of strategy can be used to suggest or design epidemic control programs [16–23,28,45,46]. Many studies consider different parameters such as geolocation in different countries, as in [26] for India, [47] Japan and [48] for Saudi Arabia. The global and local dynamics of COVID-19 may be completely characterized by mathematical models operating under fractional order derivatives. In addition, models of this type that make use of fractional calculus are superior in terms of their ability to precisely and accurately represent observed occurrences [49–57]. The researchers utilize models to track the evolution epidemic over a period of time, such as SEIR [58], which considers four compartments: Susceptible Exposed, Infected and Recovered.

In [59–61], researchers developed models and applied optimal controls for vaccination or restriction methods. In [62], we conclude that, regardless of control measures and vaccination process, COVID-19 is affected by environmental and seasonal factors.

The main contribution of this paper is to study the effect of successive optimal control policies in different susceptible classes; susceptible unaware class where awareness control is observed, susceptible aware class where vaccine control is observed and susceptible vaccinated class where optimal vaccination control is observed. Briefly, using awareness with vaccination in modeling and optimally controlling the COVID-19 epidemic has been investigated.

The paper is organized as follows: Introduction is given in chapter one, formulation of the model is given in chapter two, analysis of the model is given in chapter three, construction and analysis of optimal control problem is given in chapter four, numerical simulation is given in chapter five and finally, the conclusion is given in chapter six.

2. Formulation of the Model

The model consists of a system of fractional order differential equations in the Caputo-Fabrizio sense with five compartments. The compartments are: $U\_s(t)$, $A\_s(t)$, $V\_s(t)$, $I(t)$ and $R(t)$ stands for susceptible unaware compartment, susceptible aware compartment, susceptible vaccinated compartment, infected compartment, and recovered compartment, respectively. The model is given below:

$$\begin{gathered}
{}_{0}^{CF}D_t^{\alpha}U_s(t) = \pi^{\alpha} - \beta_1^{\alpha}U_s(t)I(t) - \mu^{\alpha}U_s(t), \\
{}_{0}^{CF}D_t^{\alpha}A_s(t) = -\beta_2^{\alpha}A_s(t)I(t) - \mu^{\alpha}A_s(t) \\
{}_{0}^{CF}D_t^{\alpha}V_s(t) = -\beta_3^{\alpha}V_s(t)I(t) - \mu^{\alpha}V_s(t) \\
{}_{0}^{CF}D_t^{\alpha}I(t) = \beta_1^{\alpha}U_s(t)I(t) + \beta_2^{\alpha}A_s(t)I(t) + \beta_3^{\alpha}V_s(t)I(t) - (\mu^{\alpha} + \gamma^{\alpha} + \delta^{\alpha})I(t) \\
{}_{0}^{CF}D_t^{\alpha}R(t) = \delta^{\alpha}I(t) - \mu^{\alpha}R(t)
\end{gathered}$$

with the following initial conditions:

$$U_s(0) = a_1, A_s(0) = a_2, V_s(0) = a_3, I(0) = a_4 \ and \ R(0) = a_5$$

The meaning of the parameters involved in the model is given in Table 1.

Table 1. Meaning of Parameters.

Parameter	Meaning
π	Recruitment rate
β_1	The transmission rate of COVID-19 in a susceptible unaware compartment
$\beta_2 < \beta_1$	The transmission rate of COVID-19 in a susceptible aware compartment
$\beta_3 < \beta_2 < \beta_1$	The transmission rate of COVID-19 in a susceptible vaccinated compartment
μ	Natural death rate
γ	Recovery rate
δ	Disease induced death rate
$0 < \alpha < 1$	Fraction order

3. Analysis of the Model

Here, equilibrium, basic reproduction number, existence and uniqueness analysis of the solution of the model are carried out.

Equilibria and basic reproduction number

The equilibrium solutions are obtained by equating the equations in the model to zero and solving the system simultaneously. We obtain five equilibrium solutions:

i. Disease-free equilibrium (E_0)

$$E_0 = \left\{U_s^0,\ A_s^0, V_s^0, I^0, R^0\right\} = \left\{\frac{\pi^{\alpha}}{\mu^{\alpha}}, 0, 0, 0, 0\right\}.$$

ii. Endemic with respect to U_s only (E_1)

$$\begin{aligned}
&E_1 = \left\{U_s^1,\ I^1, R^1\right\} \\
&= \left\{\frac{\mu^{\alpha}+\gamma^{\alpha}+\delta^{\alpha}}{\beta_1^{\alpha}}, \frac{\pi^{\alpha}\beta_1^{\alpha}-\mu^{\alpha}(\mu^{\alpha}+\gamma^{\alpha}+\delta^{\alpha})}{\beta_1^{\alpha}(\mu^{\alpha}+\gamma^{\alpha}+\delta^{\alpha})}, \frac{\delta^{\alpha}\left[\pi^{\alpha}\beta_1^{\alpha}-\mu^{\alpha}(\mu^{\alpha}+\gamma^{\alpha}+\delta^{\alpha})\right]}{\mu^{\alpha}\beta_1^{\alpha}(\mu^{\alpha}+\gamma^{\alpha}+\delta^{\alpha})}\right\}
\end{aligned}$$

iii. Endemic with respect to A_s only (E_2)
This equilibrium point does not exist, as we have:

$$I^2 = \frac{-\mu^{\alpha}}{\beta_1^{\alpha}}$$

which is not biologically meaningful, as we do not have a negative population.

iv. Endemic with respect to V_s only (E_3)

This equilibrium point does not exist, as we have:

$$I^3 = \frac{-\mu^\alpha}{\beta_2^\alpha}$$

which is not biologically meaningful, as we do not have a negative population.

v. Endemic with respect to U_s, A_s and $V_s(E_4)$

This equilibrium point does not exist, as we have:

$$I^4 = \frac{-\mu^\alpha}{\beta_1^\alpha} or I^3 = \frac{-\mu^\alpha}{\beta_2^\alpha}$$

which is not biologically meaningful, as we do not have a negative population.

Hence, the only feasible endemic equilibrium solution is E_1.

Now E_1 only exists if

$$\frac{\pi^\alpha \beta_1^\alpha - \mu^\alpha(\mu^\alpha + \gamma^\alpha + \delta^\alpha)}{\beta_1^\alpha(\mu^\alpha + \gamma^\alpha + \delta^\alpha)} > 0$$

This implies:

$$\frac{\pi^\alpha \beta_1^\alpha}{\mu^\alpha(\mu^\alpha + \gamma^\alpha + \delta^\alpha)} > 1$$

Let,

$$\frac{\pi^\alpha \beta_1^\alpha}{\mu^\alpha(\mu^\alpha + \gamma^\alpha + \delta^\alpha)} = R_0$$

where R_0 is the basic reproduction ratio.

Existence and uniqueness of a solution to the model

In this section, a fixed-point result is applied to check the existence and uniqueness of the solution of the model. Let the system be rewritten as:

$$
\begin{aligned}
{}_0^{CF}D_t^\alpha U_s(t) &= F_1(t, U_s)\\
{}_0^{CF}D_t^\alpha A_s(t) &= F_2(t, A_s)\\
{}_0^{CF}D_t^\alpha V_s(t) &= F_3(t, U_s)\\
{}_0^{CF}D_t^\alpha I(t) &= F_4(t, I)\\
{}_0^{CF}D_t^\alpha R(t) &= F_5(t, R)
\end{aligned}
$$

Applying the Caputo-Fabrizio operator, the system becomes:

$$U_s(t) - U_s(0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F_1(t, U_s) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t F_1(\eta, U_s) d\eta$$

$$A_s(t) - A_s(0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F_2(t, A_s) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t F_2(\eta, A_s) d\eta$$

$$V_s(t) - V_s(0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F_3(t, V_s) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t F_3(\eta, V_s) d\eta$$

$$I(t) - I(0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F_4(t, I) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t F_4(\eta, I) d\eta$$

$$R(t) - R(0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} F_5(t, R) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t F_5(\eta, R) d\eta$$

Now, we need to prove $F_1, \ldots, F_5$ satisfy Lipschitz continuity and contraction. See the theorem below:

Theorem 1: *F_1 is Lipschitz and if*

$$0 \leq \beta_1^\alpha h_1 + \mu^\alpha < 1$$

it is a contraction.

Proof of Theorem 1:

$$\begin{aligned}
&\| F_1(t, U_s) - F_1(t, U_{s1}) \| \\
&\quad = \| \pi^\alpha - \beta_1^\alpha U_s(t) I(t) - \mu^\alpha U_s(t) - \pi^\alpha - \beta_1^\alpha U_{s1}(t) I(t) \\
&\quad - \mu^\alpha U_{s1}(t) \| \\
&= \| -\beta_1^\alpha I(t)(U_s(t) - U_{s1}(t)) - \mu^\alpha (U_s(t) - U_{s1}(t)) \| \\
&\leq \beta_1^\alpha \| I(t) \| \| U_s(t) - U_{s1}(t) \| + \mu^\alpha \| U_s(t) - U_{s1}(t) \| \\
&\leq (\beta_1^\alpha h_1 + \mu^\alpha) \| U_s(t) - U_{s1}(t) \| \\
&\leq L_1 \| U_s(t) - U_{s1}(t) \|
\end{aligned}$$

where

$$L_1 = \beta_1^\alpha h_1 + \mu^\alpha and h_1 \geq \| I(t) \|$$

In the same way, we show the Lipschitz continuity and contraction for $F_2, \ldots, F_5$, where we obtain $L_2, \ldots, L_5$ respectively as their Lipschitz constants. □

In recursive form, let

$$\begin{aligned}
q_{1n}(t) = U_{s_n}(t) - U_{s_{n-1}}(t) \\
&= \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left(F_1\left(t, U_{s_{n-1}}\right) - F_1\left(t, U_{s_{n-2}}\right)\right) \\
&+ \tfrac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left(F_1\left(\eta, U_{s_{n-1}}\right) - F_1\left(\eta, U_{s_{n-2}}\right)\right) d\eta
\end{aligned}$$

$$\begin{aligned}
q_{2n}(t) = A_{s_n}(t) - A_{s_{n-1}}(t) \\
&= \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left(F_2\left(t, A_{s_{n-1}}\right) - F_2\left(t, A_{s_{n-2}}\right)\right) \\
&+ \tfrac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left(F_2\left(\eta, A_{s_{n-1}}\right) - F_2\left(\eta, A_{s_{n-2}}\right)\right) d\eta
\end{aligned}$$

$$\begin{aligned}
q_{4n}(t) = I_n(t) - I_{n-1}(t) \\
&= \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_4(t, I_{n-1}) - F_4(t, I_{n-2})) \\
&+ \tfrac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_4(\eta, I_{n-1}) - F_4(\eta, I_{n-2})) d\eta
\end{aligned}$$

$$\begin{aligned}
q_{5n}(t) = R_n(t) - R_{n-1}(t) \\
&= \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} (F_5(t, R_{n-1}) - F_5(t, R_{n-2})) \\
&+ \tfrac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t (F_5(\eta, R_{n-1}) - F_5(\eta, R_{n-2})) d\eta
\end{aligned}$$

with initial conditions:

$$U_s{}^0(t) = U_s(0), A_s{}^0(t) = A_s(0), V_s{}^0(t) = V_s(0), I_0(0) = I(0) \; and \; R_0(0) = R(0)$$

Taking the norm of q_{1n}, we have:

$$\begin{aligned}\| q_{1n}(t) \| =& \| U_{s_n}(t) - U_{s_{n-1}}(t) \| \\ &= \| \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(F_1\left(t, U_{s_{n-1}}\right) - F_1\left(t, U_{s_{n-2}}\right)\right) \\ &+ \tfrac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left(F_1\left(\eta, U_{s_{n-1}}\right) - F_1\left(\eta, U_{s_{n-2}}\right)\right) d\eta \|\end{aligned}$$

Applying triangular inequality, we have:

$$\begin{aligned}\| q_{1n}(t) \| =& \| U_{s_n}(t) - U_{s_{n-1}}(t) \| \\ &= \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \| F_1\left(t, U_{s_{n-1}}\right) - F_1\left(t, U_{s_{n-2}}\right) \| \\ &+ \tfrac{2\alpha}{(2-\alpha)M(\alpha)} \| \int_0^t \left(F_1\left(\eta, U_{s_{n-1}}\right) - F_1\left(\eta, U_{s_{n-2}}\right)\right) d\eta \|\end{aligned}$$

$$\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 \| U_{s_{n-1}} - U_{s_{n-2}} \| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_1 \int_0^t \| U_{s_{n-1}} - U_{s_{n-2}} \| d\eta$$

This implies:

$$\| q_{1n}(t) \| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_1 \| q_{1n-1}(t) \| + \frac{2\alpha}{(2-\alpha)M(\alpha)} L_1 \int_0^t \| q_{1n-1}(t) \| d\eta$$

Similarly,

$$\begin{aligned}\| q_{2n}(t) \| &\leq \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_2 \| q_{2n-1}(t) \| + \tfrac{2\alpha}{(2-\alpha)M(\alpha)} L_2 \int_0^t \| q_{2n-1}(t) \| d\eta \\ \| q_{3n}(t) \| &\leq \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_3 \| q_{3n-1}(t) \| + \tfrac{2\alpha}{(2-\alpha)M(\alpha)} L_3 \int_0^t \| q_{3n-1}(t) \| d\eta \\ \| q_{4n}(t) \| &\leq \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_4 \| q_{4n-1}(t) \| + \tfrac{2\alpha}{(2-\alpha)M(\alpha)} L_4 \int_0^t \| q_{4n-1}(t) \| d\eta \\ \| q_{5n}(t) \| &\leq \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_5 \| q_{5n-1}(t) \| + \tfrac{2\alpha}{(2-\alpha)M(\alpha)} L_5 \int_0^t \| q_{5n-1}(t) \| d\eta\end{aligned}$$

Subsequently, we have:

$$\begin{aligned}U_{s_n}(t) = \sum_{i=1}^{n} q_{1i}(t),\ A_{s_n}(t) = \sum_{i=1}^{n} q_{2i}(t),\ V_{s_n}(t) = \sum_{i=1}^{n} q_{3i}(t),\ I_n(t) \\ = \sum_{i=1}^{n} q_{4i}(t),\ R_n(t) = \sum_{i=1}^{n} q_{5i}(t)\end{aligned}$$

To show the existence of the solution, we prove the following theorem:

Theorem 2: *The solution exists if exist t_1 exists such that the following inequality is true:*

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} L_i + \frac{2\alpha t_1}{(2-\alpha)M(\alpha)} L_i < 1, \quad i = 1, \ldots, 5$$

Proof of Theorem 2: Recursively, we have

$$\| q_{1n}(t) \| \leq \| U_{s_n}(0) \| \left[\tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_1 + \tfrac{2\alpha}{(2-\alpha)M(\alpha)}L_1\right]^n$$
$$\| q_{2n}(t) \| \leq \| A_{s_n}(0) \| \left[\tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_2 + \tfrac{2\alpha}{(2-\alpha)M(\alpha)}L_2\right]^n$$
$$\| q_{3n}(t) \| \leq \| V_{s_n}(0) \| \left[\tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_3 + \tfrac{2\alpha}{(2-\alpha)M(\alpha)}L_3\right]^n$$
$$\| q_{4n}(t) \| \leq \| I_n(0) \| \left[\tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_4 + \tfrac{2\alpha}{(2-\alpha)M(\alpha)}L_4\right]^n$$
$$\| q_{5n}(t) \| \leq \| R_n(0) \| \left[\tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_5 + \tfrac{2\alpha}{(2-\alpha)M(\alpha)}L_5\right]^n$$

Hence, solutions exist and are continuous. To show that the functions above construct the solutions, consider:

$$U_s(t) - U_s(0) = U_{s_n}(t) - K_{1_n}(t)$$
$$A_s(t) - A_s(0) = A_{s_n}(t) - K_{2_n}(t)$$
$$V_s(t) - V_s(0) = V_{s_n}(t) - K_{3_n}(t)$$
$$I(t) - I(0) = I_n(t) - K_{4_n}(t)$$
$$R(t) - R(0) = R_n(t) - K_{5_n}(t)$$

Hence,

$$\| K_{1_n}(t) \| = \| \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(F_1(t, U_{s_{n-1}}) - F_1(t, U_{s_{n-2}})\right) + \tfrac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left(F_1(\eta, U_{s_{n-1}}) - F_1(\eta, U_{s_{n-2}})\right)d\eta \|$$

$$\leq \tfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \| F_1(t, U_{s_{n-1}}) - F_1(t, U_{s_{n-2}}) \| + \tfrac{2\alpha}{(2-\alpha)M(\alpha)} \| \int_0^t \left(F_1(\eta, U_{s_{n-1}}) - F_1(\eta, U_{s_{n-2}})\right)d\eta \|$$

$$\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_1 \| U_s - U_{s_{n-1}} \| + \frac{2\alpha}{(2-\alpha)M(\alpha)}L_1 \| U_s - U_{s_{n-1}} \| t$$

Carrying out the procedure, we get

$$\| K_{1_n}(t) \| \leq \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha t}{(2-\alpha)M(\alpha)}\right]^{n+1} {L_1}^{n+1}k$$

At $t = t_1$, we get

$$\| K_{1_n}(t) \| \leq \left[\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha t_1}{(2-\alpha)M(\alpha)}\right]^{n+1} {L_1}^{n+1}k$$

Taking the limit as $n \to \infty$, we get

$$\| K_{1_n}(t) \| \to 0$$

Similarly, we get

$$\| K_{2_n}(t) \|, \| K_{3_n}(t) \|, \| K_{4_n}(t) \|, \| K_{5_n}(t) \| \to 0$$

Finally, to show uniqueness, assume that some solutions exist, say, ${U_s}^1(t), {A_s}^1(t)$, ${V_s}^1(t), I^1(t)$ and $R^1(t)$, then

$$\| U_s(t) - {U_s}^1(t) \| \left(1 - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_1 - \frac{2\alpha t}{(2-\alpha)M(\alpha)}L_1\right) \leq 0$$

The following theorem completes the result. □

Theorem 3: *If*

$$\left(1-\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_1-\frac{2\alpha t}{(2-\alpha)M(\alpha)}L_1\right)>0$$

then the solution is unique.

Proof of Theorem 3: Consider

$$\| U_s(t)-{U_s}^1(t) \| \left(1-\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_1-\frac{2\alpha t}{(2-\alpha)M(\alpha)}L_1\right)\leq 0$$

Since,

$$\left(1-\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L_1-\frac{2\alpha t}{(2-\alpha)M(\alpha)}L_1\right)>0$$

then

$$\| U_s(t)-{U_s}^1(t) \|=0$$

This implies:

$$U_s(t)={U_s}^1(t)$$

This applies to the remaining functions. □

4. Optimal Control Analysis

In this chapter, we give details of the formation of the optimal control problem, together with the analysis of the control function.

Formation of Optimal Control Problems

The dynamics of the control system can be described by the following system of fractional-order differential equations in the Caputo-Fabrizio sense:

$$\begin{gathered}
{}_0^{CF}D_t^{\alpha}U_S(t)=\pi^{\alpha}-\beta_1^{\alpha}U_SI-\mu^{\alpha}U_S-\theta u_1U_S+\in A_S\\
{}_0^{CF}D_t^{\alpha}A_S(t)=\beta_2^{\alpha}A_SI-\mu^{\alpha}A_S-\in A_S-\varnothing u_2A_S+\rho u_3V_S\\
{}_0^{CF}D_t^{\alpha}V_S(t)=\varnothing u_2A_S-\beta_3^{\alpha}V_SI-\mu^{\alpha}V_S-\rho u_3V_S\\
{}_0^{CF}D_t^{\alpha}I(t)=\beta_1^{\alpha}U_SI+\beta_2^{\alpha}A_SI+\beta_3^{\alpha}V_SI-(\mu^{\alpha}+\gamma^{\alpha}+\delta^{\alpha})I\\
{}_0^{CF}D_t^{\alpha}R(t)=\delta^{\alpha}I-\mu^{\alpha}R
\end{gathered} \quad (1)$$

where

$$\begin{gathered}
u_1=\text{AwarenesscampaignaboutCOVID-19}\\
u_2=\text{vaccinationfortheawareclass}\\
u_3=\text{takingoptimalvaccine}
\end{gathered}$$

The objective function to be minimized can be given as:

$$J(u_1\ ,\ u_2,u_3)=\int_0^{t_f}(aU_S+bA_S+cV_S+du_1^2\ +eu_2^2+fu_3^2\ )dt \quad (2)$$

The objective here is minimizing U_S , A_S and V_S at the same time to minimize the cost of the three controls u_1, u_2 and u_3. Hence, we need to get the optimal control u_1^* , u_2^* and u_3^* such that:

$$J(u_1^*\ ,u_2^*\ ,u_3^*\ )=\min_{u_1,\ u_2}\{J(u_1\ ,\ u_2,u_3)|u_1,\ u_2,u_3\in\Omega\} \quad (3)$$

The set of control as:

$$\Omega = \left\{ (u_1,\ u_2, u_3) \middle| u_i : \left[0,\ t_f\right] \to [0,\ \infty) Lebesguemeasurable,\ i = 1,2,3 \right\}$$

The expenses of minimizing U_S is represented by the term aU_S, that of minimizing A_S is represented by bA_S, while minimizing V_S is represented by cV_S. Likewise, all the expenses associated with the control u_1 is represented by du_1^2 , all the expenses associated with the control u_2 are represented by eu_2^2 and also all the expenses associated with the control u_3 is represented by fu_3^2 The sufficient conditions required for the optimal control to be fulfilled can be found by using the most popular PMP. The said principle can be used to turn Equations (1) and (3) into a point-wise minimizing problem of the Hamiltonian H for $(\ u_1,\ u_2 u_3)$ stated as follows:

$$\begin{gathered} H = aU_S + bA_S + cV_S + du_1^2 + eu_2^2 + fu_3^2 + \lambda_{U_S}\{\pi^\alpha - \beta_1^\alpha U_S I - \mu^\alpha U_S - \\ \theta u_1 U_S + \in A_S\} + \lambda_{A_S}\{\beta_2^\alpha A_S I - \mu^\alpha A_S - \in A_S - \varnothing u_2 A_S + \rho u_3 V_S\} + \lambda_{V_S}\{\varnothing u_2 A_S - \\ \beta_3^\alpha V_S I - \mu^\alpha V_S - \rho u_3 V_S\} + \lambda_I\{\beta_1^\alpha U_S I + \beta_2^\alpha A_S I + \beta_3^\alpha V_S I - (\mu^\alpha + \gamma^\alpha + \delta^\alpha) I\} + \\ \lambda_R\{\delta^\alpha I - \mu^\alpha R\} \end{gathered} \tag{4}$$

where, λ_{U_S} , λ_{A_S} , λ_{V_S} , λ_I , *and* λ_R are the adjoint variables or co-state variables.

$$\begin{gathered} -\frac{d\lambda_{U_S}}{dt} = \frac{\partial H}{\partial U_S} = a + \lambda_{U_S}\{-\beta_1^\alpha I - \mu^\alpha - \theta u_1\} + \lambda_I \beta_1^\alpha I \\ -\frac{d\lambda_{A_S}}{dt} = \frac{\partial H}{\partial A_S} = b + \lambda_{U_S} \in + \lambda_{A_S}\{\beta_2^\alpha I - \mu^\alpha - \in - \varnothing u_2\} + \lambda_I \beta_2^\alpha I \\ -\frac{d\lambda_{V_S}}{dt} = \frac{\partial H}{\partial V_S} = c + \lambda_{A_S}\rho u_3 + \lambda_{V_S}\{-\beta_3^\alpha I - \mu^\alpha - \rho u_3\} + \lambda_I \beta_3^\alpha I \\ -\frac{d\lambda_I}{dt} = \frac{\partial H}{\partial I} = -\lambda_{U_S}\beta_1^\alpha U_S + \lambda_{A_S}\beta_2^\alpha A_S + \lambda_{V_S}\beta_3^\alpha V_S + \lambda_I\{\beta_1^\alpha U_S + \beta_2^\alpha A_S + \beta_3^\alpha V_S - \\ (\mu^\alpha + \gamma^\alpha + \delta^\alpha)\} \\ -\frac{d\lambda_R}{dt} = \frac{\partial H}{\partial R} = -\lambda_R \mu^\alpha R \end{gathered} \tag{5}$$

The transversality conditions are:

$$\lambda_{U_S}\left(t_f\right) = \lambda_{A_S}\left(t_f\right) = \lambda_{V_S}\left(t_f\right) = \lambda_I\left(t_f\right) = \lambda_R\left(t_f\right) = 0$$

for $0 < u_i < 1$, for $i = 1, 2, 3$,

From the interior of the controls, we have:

$$\begin{gathered} \frac{\partial H}{\partial u_1} = 2du_1 - \lambda_{U_S}\theta U_S = 0 \\ \frac{\partial H}{\partial u_2} = 2eu_2 - \lambda_{A_S}\varnothing A_S + \lambda_{V_S}\varnothing A_S = 0 \\ \frac{\partial H}{\partial u_3} = 2fu_3 + \lambda_{A_S}\rho V_S - \lambda_{V_S}\rho V_S = 0 \end{gathered} \tag{6}$$

from where:

$$\begin{gathered} u_1 = \frac{1}{2d}\lambda_{U_S}\theta U_S \\ u_2 = \frac{1}{2e}\varnothing A_S\left[\lambda_{A_S} - \lambda_{V_S}\right] \\ u_3 = \frac{1}{2f}\rho V_S\left[\lambda_{V_S} - \lambda_{A_S}\right] \end{gathered} \tag{7}$$

Existence of optimal solutions

We give the following theorem for the existence of optimal controls:

Theorem 4: *The control values (u_1^* , u_2^* , u_3^*) which can minimize* $(u_1\ ,\ u_2, u_3)$ *over U are given by*

$$u_1^* = max\left\{0, min\left[1, \frac{1}{2d}\lambda_{U_S}\theta U_S\right]\right\}$$
$$u_2^* = max\left\{0, min\left[1, \frac{1}{2e}\varnothing A_S\left[\lambda_{A_S} - \lambda_{V_S}\right]\right]\right\} \tag{8}$$
$$u_3^* = max\left\{0, min\left[1, \frac{1}{2f}\rho V_S\left[\lambda_{V_S} - \lambda_{A_S}\right]\right]\right\}$$

where, λ_{U_S} , λ_{A_S} , λ_{V_S} , λ_I , *and* λ_R *are, co-state variables that satisfy (1–8) as well as the transversality conditions that follow* $\lambda_{U_S}\left(t_f\right) = \lambda_{A_S}\left(t_f\right) = \lambda_{V_S}\left(t_f\right) = \lambda_I\left(t_f\right) = \lambda_R\left(t_f\right) = 0$ *and*

$$u_1^* = \begin{cases} 0, \ if u_1 \leq 0, \\ u_1, \quad if \quad 0 < u_1 < 1, \\ 1, \quad if u_1 \geq 0, \end{cases}$$
$$u_2^* = \begin{cases} 0, \quad if u_2 \leq 0, \\ u_2, \quad if \quad 0 < u_2 < 1, \\ 1, \quad if u_2 \geq 0. \end{cases} \tag{9}$$
$$u_3^* = \begin{cases} 0, \quad if u_3 \leq 0, \\ u_3, \quad if \quad 0 < u_3 < 1. \\ 1, \quad if u_3 \geq 0. \end{cases}$$

Proof of Theorem 4: To prove the existence of the optimal control solution, we use the convexity of the integrand of J to controls u_1, u_2 and u_3 for the boundedness of the solutions of the state and the Lipschitz property of the system of the state concerning the variables of the state. Hence, we apply the PMP and obtain the following:

$${}_{0}^{C_F}D_t^\alpha \lambda_{U_S}(t) = \frac{\partial H}{\partial U_S}; {}_{0}^{C_F}D_t^\alpha \lambda_{A_S}(t) = \frac{\partial H}{\partial A_S} \tag{10}$$
$${}_{0}^{C_F}D_t^\alpha \lambda_{V_S}(t) = \frac{\partial H}{\partial V_S}; {}_{0}^{C_F}D_t^\alpha \lambda_I(t) = \frac{\partial H}{\partial I}; {}_{0}^{C_F}D_t^\alpha \lambda_R(t) = \frac{\partial H}{\partial R}$$

with,

$$\lambda_{U_S}\left(t_f\right) = \lambda_{A_S}\left(t_f\right) = \lambda_{V_S}\left(t_f\right) = \lambda_I\left(t_f\right) = \lambda_R\left(t_f\right) = 0$$

The conditions for optimality can be obtained after differentiating the Hamiltonian H with respect to u_1, u_2 and u_3:

$$\frac{\partial H}{\partial u_1} = 0\,; \frac{\partial H}{\partial u_2} = 0; \frac{\partial H}{\partial u_3} = 0. \tag{11}$$

The adjoint systems (4) and (5) come from the solution of (1), and the optimal controls (7) can be obtained from (8). The optimal system comprises the controlled system (1) and its initial conditions, the system of adjoint (4), and the conditions for transversality. □

5. Numerical Simulation and Discussion

In this chapter, numerical simulations are carried out. Variable and parameter values are given as

$$\pi = 1,\ \beta_1 = 0.0007, \beta_2 = 0.00007, \beta_3 = 0.000007, \mu = 0.02,\ \gamma = 0.2,$$
$$\delta = 0.01,\ \theta = 0.002, \varnothing = 0.0012,\ \mathrm{p} = 0.001$$

Figure 1 depicts the dynamics of the model. It can be seen that without any control, the susceptible unaware population, susceptible aware population and susceptible vaccinated populations all go to extinction, whereas infected and recovered populations proliferate. This clearly shows the need for the application of various control measures to control the pandemic.

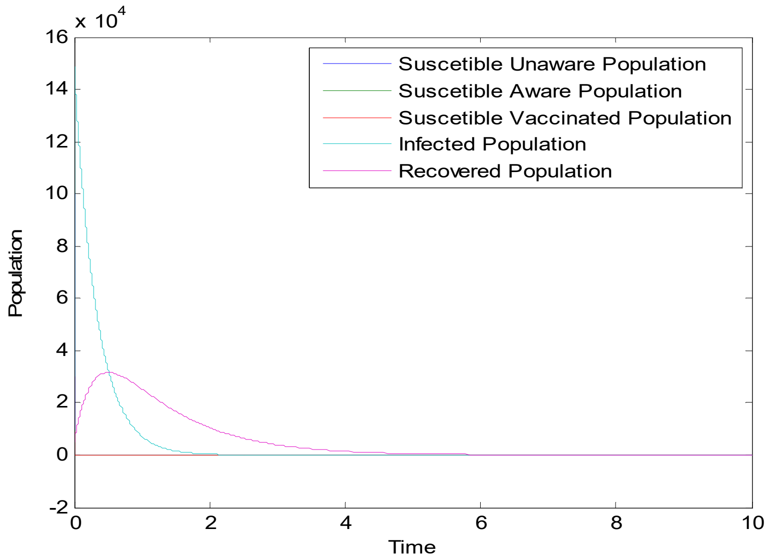

Figure 1. Dynamics of the model.

Figure 2 shows the influence of the variation in the fractional-order α on the biological behavior of the infected population. It is clear from this figure that the population has a decreasing effect when α is decreased from 1 to 0.2. Hence, the memory effect can be seen clearly.

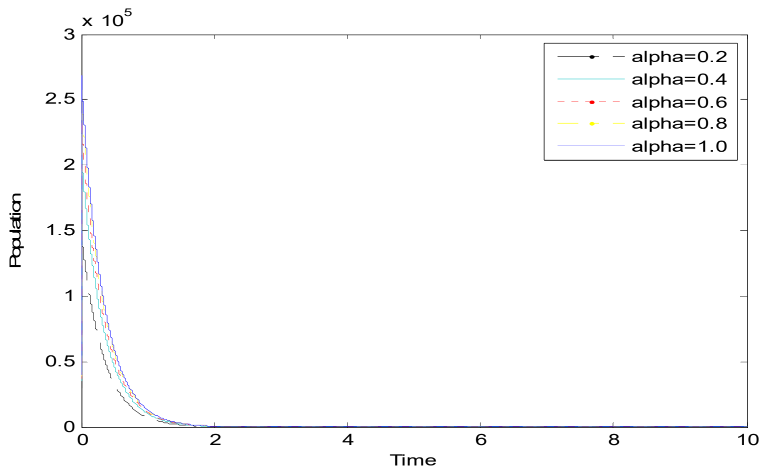

Figure 2. Dynamics of the infected population for various values of α.

Figures 3–5 compare the effect of controls $u_1, u_2 \& u_3$ respectively on the dynamics of the infected population. It is clear that when any control is observed, the population of infected individuals is reduced. This is a positive effect and hence there is a need for compliance with the control measures.

Figures 6–8 compare the effect of two controls, i.e., $u_1 \& u_2$, $u_1 \& u_3$, and $u_2 \& u_3$ respectively on the dynamics of the infected population. It is clear that when two controls are applied, a drastic change in the population of infected individuals is seen more than in the application of a single control. Hence, to control the pandemic, there is a need for the application of more than one control measure. However, the economic implications of combining more than one control measure must be taken into consideration.

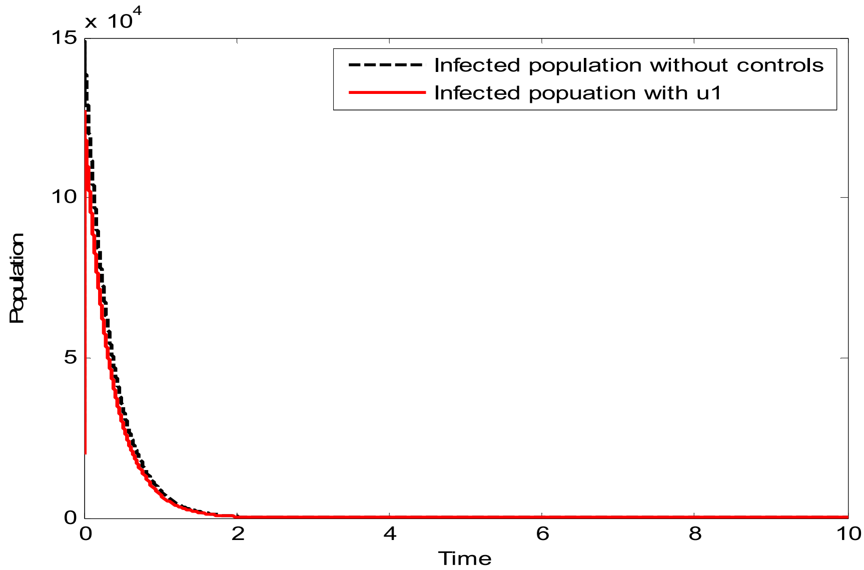

Figure 3. Comparing the dynamics of the infected population without control and with control u_1.

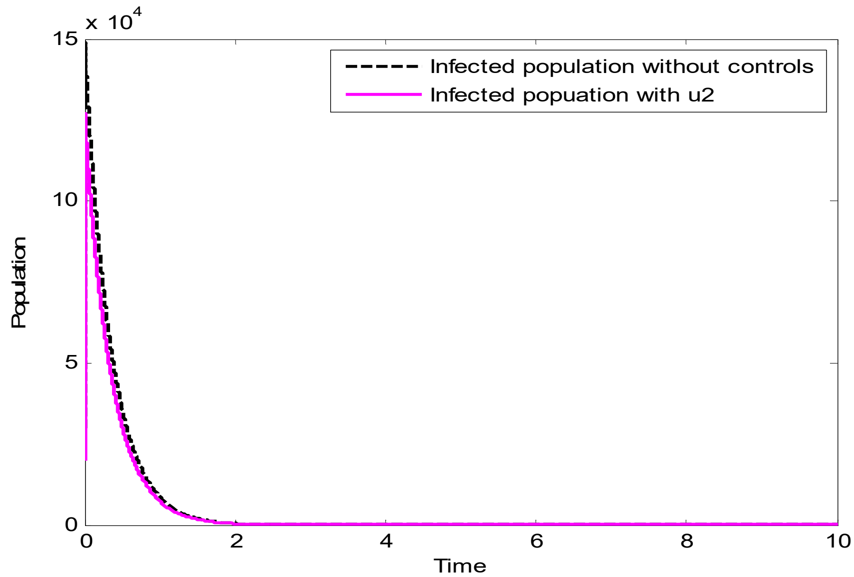

Figure 4. Comparing the dynamics of the infected population without control and with control u_2.

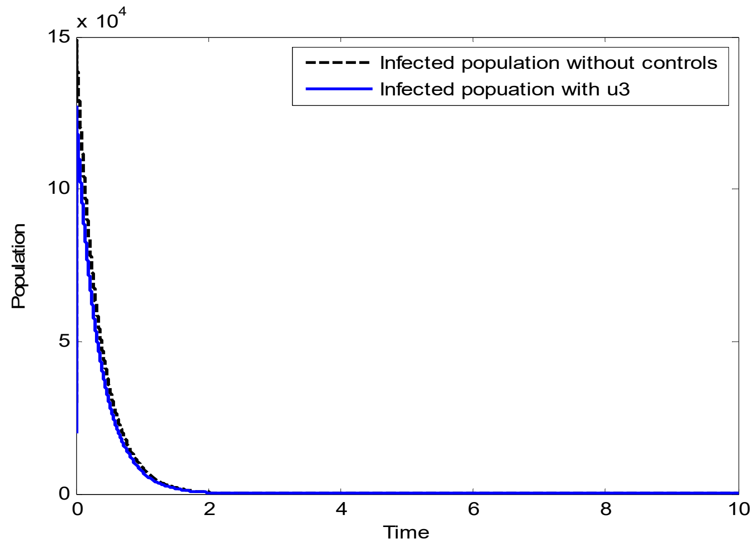

Figure 5. Comparing the dynamics of the infected population without control and with control u_3.

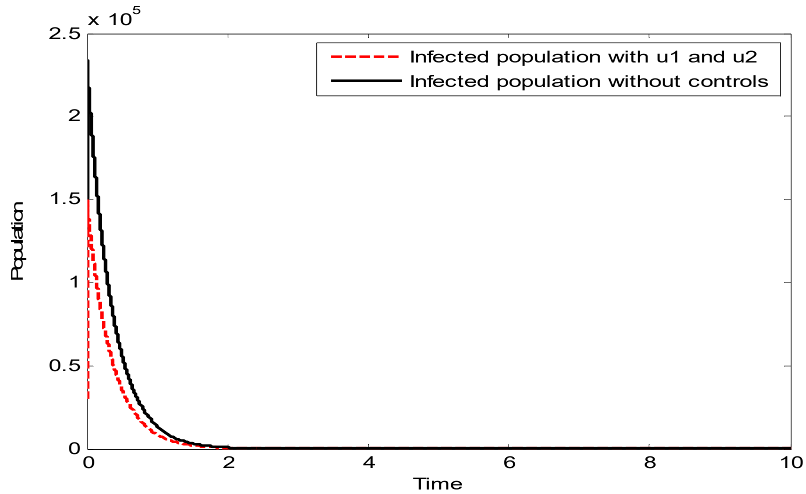

Figure 6. Comparing the dynamics of the infected population without control and with control u_1&u_2.

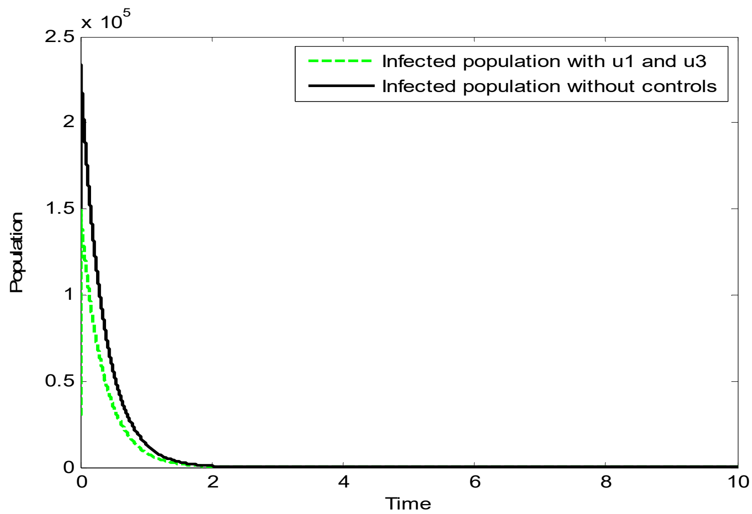

Figure 7. Comparing the dynamics of the infected population without control and with control u_1&u_3.

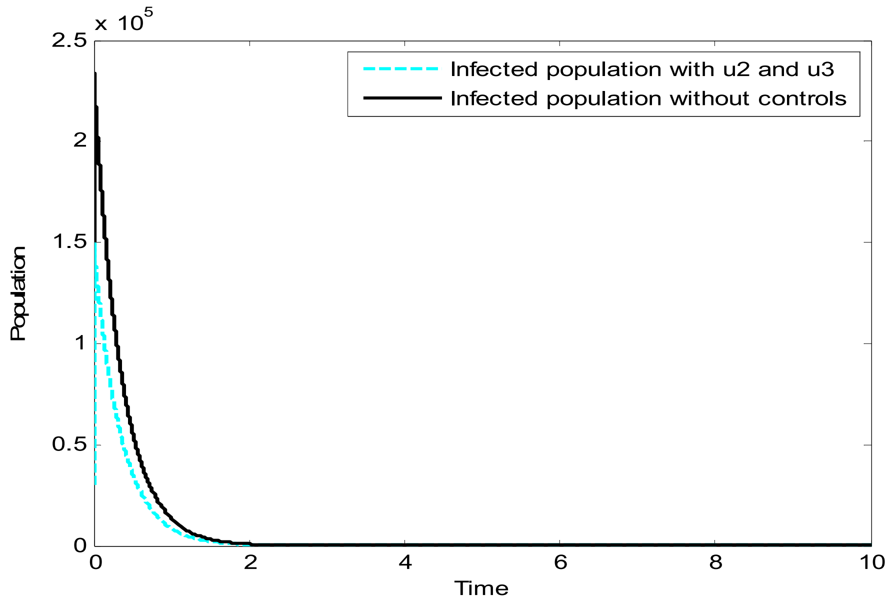

Figure 8. Comparing the dynamics of the infected population without control and with control $u_2 \& u_3$.

Figure 9 compares the effects of the three controls, i.e., u_1, $u_2 \& u_3$ on the dynamics of the infected population. The application of all the control measures in the partitioned susceptible population leads to the desired outcome. This effect is clearly seen. Hence, to obtain the desired result, there is a need for awareness, and not only vaccinating the susceptible population but also making sure that a full dosage is given.

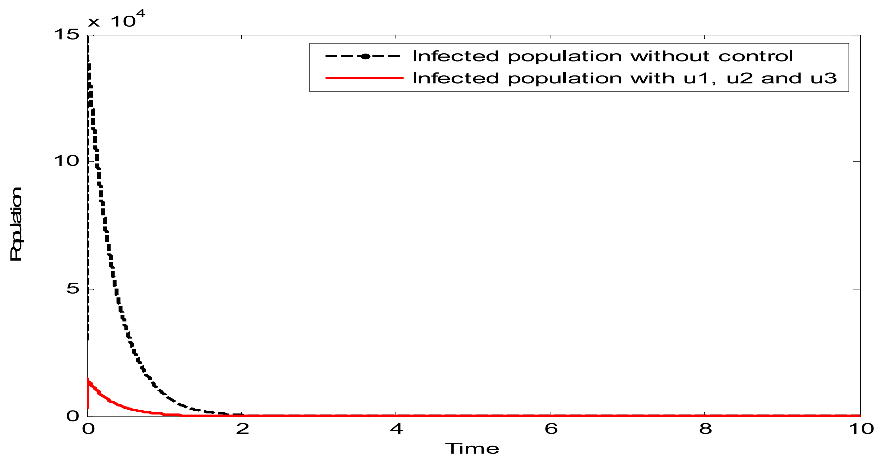

Figure 9. Comparing the dynamics of the infected population without control and with control u_1, $u_2 \& u_3$.

These results show the significant impact of awareness of COVID-19 and the vaccination process. Other models investigate the optimal control of vaccinations or the restriction measures applied to susceptible classes, which do not reflect social awareness about infections.

6. Summary and Conclusions

In this paper, Caputo-Fabrizio's sense is used to develop the fractional-order COVID-19 model, which consists of five compartments: susceptible unaware compartment, susceptible aware compartment, susceptible vaccinated compartment, infected compartment, and recovered compartment. Three types of susceptible classes are studied in this paper: a susceptible unaware class where awareness control is observed, a susceptible aware class where vaccine control is observed, and a susceptible vaccinated class where optimal vaccination control is observed. Calculation of equilibrium points leads to the determination of the basic reproduction ratio. The model's properties of existence and uniqueness are confirmed. In addition, the optimal control problem was developed, and consequently, the existence of an optimal solution was achieved. The biological significance of fractional order is established by the use of numerical simulations, which are conducted. By utilizing a variety of control functions, it is evident that combining the three control methods has a significant impact on decreasing the number of infected individuals. This study incorporates both vaccination and awareness into consideration of the COVID-19 epidemic. For further studies, it is suggested to utilize the environmental conditions as in [62] with the awareness of susceptible class to see the impact of optimal control on such a model.

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