



## Article

# Stability of Generalized Proportional Caputo Fractional Differential Equations by Lyapunov Functions

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**Abstract:** In this paper, nonlinear nonautonomous equations with the generalized proportional Caputo fractional derivative (GPFD) are considered. Some stability properties are studied by the help of the Lyapunov functions and their GPFDs. A scalar nonlinear fractional differential equation with the GPFD is considered as a comparison equation, and some comparison results are proven. Sufficient conditions for stability and asymptotic stability were obtained. Examples illustrating the results and ideas in this paper are also provided.

**Keywords:** generalized proportional Caputo fractional derivative; fractional differential equations; stability; asymptotic stability; Lyapunov functions

**MSC:** 34A08; 34A34; 34D20



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## 1. Introduction

The stability properties of solutions are an important branch in the qualitative theory of differential equations. One of the most effective and applicable methods for investigation of the stability properties of solutions is the Lyapunov approach [1]. Various types of stability for Caputo fractional differential equations by the Caputo fractional derivative of Lyapunov functions have been presented and discussed (see, for example, [2–7]). Recently, in 2017 [8], the generalized proportional Caputo fractional derivative (GPFD) was introduced and applied to various problems. This derivative is a generalization of the Caputo fractional derivative. Note that the GPFD is similar to the so-called tempered fractional derivative. For some results concerning the GPFD and the differential equations with the GPFD, as well as its applications, we refer the reader to [9–13]. However, the study of the stability properties of the solutions of fractional differential equations with the GPFD is at its initial stage (see, for example, [14]). Note that, in contrast to ordinary derivatives, fractional derivatives depend significantly on the initial time point  $t_0$ , which is equal to the lower limit of the derivative ([15–20]). Therefore, any change of the initial time leads to a change of the fractional derivative and the corresponding fractional differential equation. For this reason, we study only stability and asymptotic stability instead of the uniform ones.

In this paper, we used appropriate Lyapunov functions and their GPFD among the solutions of the appropriate equation. Comparison results with the scalar fractional differential equations with the GPFD and Lyapunov functions were obtained. Several sufficient conditions for stability and asymptotic stability are presented. Some examples illustrate the ideas and results in this paper.

## 2. Notes on Fractional Calculus

Let  $u : [a, T] \rightarrow \mathbb{R}$  (if  $T = \infty$ , then the interval is half-open). The generalized proportional fractional integral is defined by (as long as all integrals are well defined; see [10]):

$$({}_a I^{\alpha, \rho} u)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{\alpha-1} u(s) ds, \quad t \in (a, T], \quad \alpha \geq 0, \quad \rho \in (0, 1], \quad (1)$$

and the generalized Caputo proportional fractional derivative (GPFDD) is defined by (as long as all integrals are well defined; see [10]):

$$\begin{aligned} ({}_a^C D^{\alpha, \rho} u)(t) &= ({}_a I^{1-\alpha, \rho} (D^{1, \rho} u))(t) \\ &= \frac{1}{\rho^{1-\alpha} \Gamma(1-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{-\alpha} (D^{1, \rho} u)(s) ds, \\ &\text{for } t \in (a, T], \quad \alpha \in (0, 1), \quad \rho \in (0, 1], \end{aligned} \quad (2)$$

where  $(D^{1, \rho} u)(t) = (D^\rho u)(t) = (1-\rho)u(t) + \rho u'(t)$ .

**Remark 1.** Note that in the case  $\rho = 1$ , the GPFDD is reduced to the Caputo fractional derivative:  $({}_a^C D^{\alpha, 1} u)(t) = {}_a^C D^\alpha u(t)$ .

**Remark 2.** The GPFDD given by (2) could be generalized for any function  $u \in C([a, b], \mathbb{R}^n)$  via a componentwise approach.

**Lemma 1** (Theorem 5.3 [10]). For  $\rho \in (0, 1]$  and  $\alpha \in (0, 1)$ , we have:

$$({}_a I^{\alpha, \rho} ({}_a^C D^{\alpha, \rho} u))(t) = u(t) - u(a) e^{\frac{\rho-1}{\rho}(t-a)} \quad (3)$$

**Lemma 2** (Theorem 5.2 [10]). For  $\rho \in (0, 1]$  and  $\alpha \in (0, 1)$ , we have:

$$({}_a I^{\alpha, \rho} e^{\frac{\rho-1}{\rho}t} (t-a)^{\beta-1})(\tau) = \frac{\Gamma(\beta)}{\rho^\alpha \Gamma(\beta+\alpha)} e^{\frac{\rho-1}{\rho}\tau} (\tau-a)^{\beta-1+\alpha} \quad \beta > 0. \quad (4)$$

**Remark 3.** If  $\rho \in (0, 1)$ , then  $({}_a^C D^{\alpha, \rho} c)(t) \neq 0$ , where  $c$  is a nonzero constant.

**Remark 4.** The relation:

$$({}_a^C D^{\alpha, \rho} e^{\frac{\rho-1}{\rho}(\cdot)})(t) = 0 \quad \text{for } t > a \quad (5)$$

is known from [10], Remark 3.2.

We used the result given in Example 5.7 of [10] (with necessary slight corrections).

Consider the initial-value problem for the scalar linear fractional differential equation with the GPFDD:

$$({}_a^C D^{\alpha, \rho} y)(t) = \lambda y(t), \quad y(a) = y_0, \quad t \in [a, T], \quad \alpha \in (0, 1), \quad \rho \in (0, 1] \quad (6)$$

with  $y_0 \in \mathbb{R}$ .

**Lemma 3.** The initial-value problem (6) has a solution:

$$y(t) = y_0 e^{\frac{\rho-1}{\rho}(t-a)} E_\alpha \left( \lambda \left( \frac{t-a}{\rho} \right)^\alpha \right), \quad (7)$$

where  $E_\alpha(t)$  is the Mittag-Leffler function.

**Lemma 4** ([14]). Let the function  $u \in C^1([a, T], \mathbb{R})$ ,  $T \leq \infty$  and  $\alpha \in (0, 1)$ ,  $\rho \in (0, 1]$  be two reals. Then,

$$({}_a^C D^{\alpha, \rho} u^2)(t) \leq 2u(t)({}_a^C D^{\alpha, \rho} u)(t), \quad t \in (a, T]. \quad (8)$$

### 3. Statement of the Problem

Consider the following nonlinear system of fractional differential equations with the GPFE:

$$({}_{t_0}^C D^{\alpha, \rho} x)(t) = f(t, x(t)), \quad \text{for } t > t_0, \quad \alpha \in (0, 1), \quad \rho \in (0, 1] \quad (9)$$

with the initial condition:

$$x(t_0) = x_0, \quad (10)$$

where  $t_0 > 0$ ,  $x_0 \in \mathbb{R}^n$ ,  $f \in C([t_0, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$ .

In our paper, we assumed that  $f(t, 0) \equiv 0$ , so the zero is a solution of (9).

We denote the solution of (9), (10) by  $x(t; t_0, x_0)$ . We assumed in the paper the initial-value problem (9), (10) has a solution defined for  $t \geq t_0$  for any initial value  $x_0 \in \mathbb{R}^n$ . Some existence results were given in [21,22].

**Definition 1.** The zero solution of (9) is:

- Stable if for every  $\epsilon > 0$ , there exist  $\delta = \delta(\epsilon) > 0$  such that for any  $x_0 \in \mathbb{R}^n$ , the inequality  $\|x_0\| < \delta$  implies  $\|x(t; t_0, x_0)\| < \epsilon$  for  $t \geq t_0$ ;
- Attractive if there exists  $B > 0$  such that for any  $x_0 \in \mathbb{R}^n$  with  $\|x_0\| < B$  and for every  $\epsilon > 0$ , there exists  $\tilde{T} = \tilde{T}(\epsilon) > 0$  such that  $\|x(t; t_0, x_0)\| < \epsilon$  for  $t \geq t_0 + \tilde{T}$ ;
- Asymptotically stable if the zero solution is stable and attractive.

Define the following set:

$$\mathcal{K} = \{a \in C([0, \infty), [0, \infty)) : a \text{ is strictly increasing and } a(0) = 0\}.$$

Consider the comparison scalar fractional differential equation with the GPFE:

$$({}_{t_0}^C D^{\alpha, \rho} y)(t) = h(t, y(t)), \quad t > t_0, \quad (11)$$

where  $h : [t_0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $h(t, 0) \equiv 0$ . We assumed there exists a small enough number  $L > 0$  such that the equation  $({}_{t_0}^C D^{\alpha, \rho} y)(t) = h(t, y(t)) + \eta e^{\frac{\rho-1}{\rho}(t-s)}$ ,  $\eta \in (0, L]$  with  $y(t_0) = y_0$  has a solution  $y(t; t_0, y_0, \eta)$  where  $y_0 \in \mathbb{R}$ .

**Example 1.** Let us consider the scalar fractional differential equation:

$$({}_{t_0}^C D^{\alpha, \rho} y)(t) = -by, \quad y(t_0) = y_0, \quad (12)$$

where  $y_0 \in \mathbb{R}$ ,  $b > 0$ .

According to Lemma 3, the solution of (12) is:

$$y(t) = y_0 e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-b(\frac{t-t_0}{\rho})^\alpha). \quad (13)$$

From (13) and the inequality  $0 < E_\alpha(-b(t-t_0)^\alpha) < 1$ ,  $t \geq t_0$ , we obtain:

$$|y(t)| \leq |y_0|. \quad (14)$$

Inequality (14) proves that the zero solution of (12) is stable.

We define a class of Lyapunov functions.

**Definition 2.** Let  $V(t, x) \in C([t_0, T] \times \Delta, \mathbb{R}_+)$  be locally Lipschitzian with respect to its second argument and  $V(t, 0) \equiv 0$ , where  $t_0, T \in \mathbb{R}_+ : T > t_0, \Delta \subset \mathbb{R}^n, 0 \in \Delta$ . Then, we say  $V \in \Lambda([t_0, T], \Delta)$ .

We used the generalized proportional Caputo fractional derivative of the Lyapunov function for any solution of the system of fractional equations. Note that in the case of Caputo fractional derivatives, some authors [3,6,7] have used a similar approach.

**4. Comparison Results**

**Lemma 5.** Let  $u \in C([t_0, T], \mathbb{R})$ , and there exists a point  $t^* \in (t_0, T]$ , such that  $u(t^*) = 0$  and  $u(t) < 0$  for  $t_0 \leq t < t^*$ . Then, if the generalized proportional Caputo fractional derivative of  $u$  exists at  $t^*$ , then the inequality  $({}^C D^{\alpha, \rho} u)(t)|_{t=t^*} > 0$  holds.

**Proof.** From the definition of the GPDF and integration by parts, we obtain:

$$\begin{aligned}
 ({}^C D^{\alpha, \rho} u)(t)|_{t=t^*} &= \frac{1}{\rho^{1-\alpha} \Gamma(1-\alpha)} \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t^*-s)} \frac{(1-\rho)u(s) + \rho u'(s)}{(t^*-s)^\alpha} ds \\
 &= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_{t_0}^{t^*} \left\{ \frac{1-\rho}{\rho} e^{\frac{\rho-1}{\rho}(t^*-s)} \frac{u(s)}{(t^*-s)^\alpha} + e^{\frac{\rho-1}{\rho}(t-s)} \frac{u'(s)}{(t^*-s)^\alpha} \right\} ds \\
 &= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \left\{ \int_{t_0}^{t^*} \frac{d}{ds} \left( e^{\frac{\rho-1}{\rho}(t^*-s)} \right) \frac{u(s)}{(t^*-s)^\alpha} ds \right. \\
 &\quad \left. + e^{\frac{\rho-1}{\rho}(t^*-s)} \left[ \frac{d}{ds} \left( \frac{u(s)}{(t^*-s)^\alpha} \right) - \frac{\alpha u(s)}{(t^*-s)^{\alpha+1}} \right] ds \right\} \\
 &= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_{t_0}^{t^*} \frac{d}{ds} \left( e^{\frac{\rho-1}{\rho}(t^*-s)} \frac{u(s)}{(t^*-s)^\alpha} \right) ds \\
 &\quad - \frac{\alpha \rho^\alpha}{\Gamma(1-\alpha)} \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t^*-s)} \frac{u(s)}{(t^*-s)^{\alpha+1}} ds \\
 &= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \lim_{s \rightarrow t^{*-}} e^{\frac{\rho-1}{\rho}(t^*-s)} \frac{u(s)}{(t^*-s)^\alpha} - e^{\frac{\rho-1}{\rho}(t^*-t_0)} \frac{u(t_0)}{(t^*-t_0)^\alpha} \\
 &\quad + \frac{\rho^\alpha}{\Gamma(-\alpha)} \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t^*-s)} \frac{u(s)}{(t^*-s)^{1+\alpha}} ds.
 \end{aligned}
 \tag{15}$$

Using  $\Gamma(-\alpha) < 0$  for  $\alpha \in (0, 1)$ ,  $u(t) < 0$  for  $t_0 \leq t < t^*$ , and L'Hôpital's rule applied to:

$$\lim_{s \rightarrow t^{*-}} \frac{e^{\frac{\rho-1}{\rho}(t^*-s)} u(s)}{(t^*-s)^\alpha} = \lim_{s \rightarrow t^{*-}} [e^{\frac{\rho-1}{\rho}(t^*-s)}]' u(s) + u'(s) e^{\frac{\rho-1}{\rho}(t^*-s)} \alpha (t^*-s)^{1-\alpha} = 0,$$

we obtain:

$$({}^C D^{\alpha, \rho} u)(t)|_{t=t^*} = -e^{\frac{\rho-1}{\rho}(t^*-t_0)} \frac{u(t_0)}{(t^*-t_0)^\alpha} + \frac{\rho^\alpha}{\Gamma(-\alpha)} \int_{t_0}^{t^*} e^{\frac{\rho-1}{\rho}(t^*-s)} \frac{u(s)}{(t^*-s)^{\alpha+1}} ds > 0. \tag{16}$$

□

We now obtain a comparison result.

**Lemma 6.** (Comparison result). Assume:

1. The function  $\tilde{x}(\cdot) = x(\cdot; t_0, x_0) \in \Delta$  is a solution of (9) defined on  $[t_0, T]$ , where  $\Delta \subset \mathbb{R}^n, 0 \in \Delta$ , and  $T : t_0 < T \leq \infty$  is a given constant,  $x_0 \in \Delta$ ;
2. The function  $h \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ ;

3. The function  $V \in \Lambda([t_0, T], \Delta)$ ; the generalized proportional Caputo fractional derivative exists for any point  $t \in (t_0, T)$ , and the inequality:

$$({}^C_{t_0}D^{\alpha,\rho}V(\cdot, \tilde{x}(\cdot)))(t) \leq h(t, V(t, \tilde{x}(t))), \quad t \in (t_0, T]$$

holds;

4. The function  $y^*(t) = y(t; t_0, y_0)$  is the maximal solution of (11) on  $[t_0, T]$ . Then, the inequality  $V(t_0, x_0) \leq y_0$  implies  $V(t, \tilde{x}(t)) \leq y^*(t)$  for  $t \in [t_0, T]$ .

**Proof.** Consider:

$$({}^C_{t_0}D^{\alpha,\rho}y)(t) = h(t, y(t)) + \eta e^{\frac{\rho-1}{\rho}(t-t_0)}, \quad t \in [t_0, T], \quad y(t_0) = y_0 + \eta, \quad (17)$$

where  $\eta \leq L$ , as described after (11).

Take the generalized proportional integral operator on both sides of Equation (17), apply Lemma 2 with  $\beta = 1$  and Lemma 1, and obtain for the solution:  $y(t, \eta)$  of (17):

$$\begin{aligned} y(t, \eta) &= y(t_0)e^{\frac{\rho-1}{\rho}(t-t_0)} + ({}_{t_0}I^{\alpha,\rho}(h(\cdot, y(\cdot, \eta)) + \eta))(t) \\ &= (y_0 + \eta)e^{\frac{\rho-1}{\rho}(t-t_0)} + ({}_{t_0}I^{\alpha,\rho}(h(\cdot, y(\cdot, \eta))))(t) + \frac{\eta}{\rho^\alpha \Gamma(1 + \alpha)} e^{\frac{\rho-1}{\rho}(t-t_0)} (t - t_0)^\alpha \\ &= (y_0 + \eta)e^{\frac{\rho-1}{\rho}(t-t_0)} + \frac{\eta}{\rho^\alpha \Gamma(1 + \alpha)} e^{\frac{\rho-1}{\rho}(t-t_0)} (t - t_0)^\alpha \\ &\quad + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_0}^t e^{\frac{\rho-1}{\rho}(t-s)} (t - s)^{\alpha-1} h(s, y(s, \eta)) ds \quad \text{for } t \in [t_0, T]. \end{aligned} \quad (18)$$

Consider the function  $\mu(t) \in C([t_0, T], \mathbb{R}_+)$  defined by  $\mu(t) = V(t, \tilde{x}(t))$ . We prove that:

$$\mu(t) < y(t, \eta) \quad \text{for } t \in [t_0, T]. \quad (19)$$

The inequality (19) holds for  $t = t_0$  because  $\mu(t_0) = V(t_0, x_0) \leq y_0 < y_0 + \eta = y(t_0, \eta)$ . Assume that Inequality (19) is not true. Then, there exists a point  $\tau$  such that  $\mu(\tau) = y(\tau, \eta)$ ,  $\mu(t) < y(t, \eta)$  for  $t \in [t_0, \tau)$ . Now, Lemma 6 (applied to  $\mu(t) - y(t, \eta)$ ) yields  $({}^C_{t_0}D^{\alpha,\rho}\mu(\cdot) - y(\cdot, \eta))(t)|_{t=\tau} > 0$ , i.e.,

$$({}^C_{t_0}D^{\alpha,\rho}\mu)(t)|_{t=\tau} > ({}^C_{t_0}D^{\alpha,\rho}y(\cdot, \eta))(t)|_{t=\tau} = h(\tau, y(\tau, \eta)) + \eta > h(\tau, \mu(\tau)). \quad (20)$$

From Condition 3 with  $t = \tau$ , the inequality  $({}^C_{t_0}D^{\alpha,\rho}V)(\cdot, \tilde{x}(\cdot))(t) = ({}^C_{t_0}D^{\alpha,\rho}\mu)(t)|_{t=\tau} \leq h(\tau, V(\tau, \tilde{x}(\tau))) = h(\tau, \mu(\tau))$  holds. The obtained contradiction proves (19).

We now show that if  $\eta_2 < \eta_1$ , then:

$$y(t, \eta_2) < y(t, \eta_1) \quad \text{for } t \in [t_0, T]. \quad (21)$$

Inequality (21) holds for  $t = t_0$ . Assume that Inequality (21) is not true. Then, there exists a point  $t^* : y(t^*, \eta_2) = y(t^*, \eta_1)$  and  $y(t, \eta_2) < y(t, \eta_1)$  for  $t \in [t_0, t^*)$ . From Lemma 6 (applied to  $y(t, \eta_2) - y(t, \eta_1)$ ), we obtain  $({}^C_{t_0}D^{\alpha,\rho}(y(\cdot, \eta_2) - y(\cdot, \eta_1)))(t)|_{t=t^*} > 0$ . However:

$$\begin{aligned} &({}^C_{t_0}D^{\alpha,\rho}(y(\cdot, \eta_2) - y(\cdot, \eta_1)))(t)|_{t=t^*} \\ &= ({}^C_{t_0}D^{\alpha,\rho}(y(\cdot, \eta_2)))(t)|_{t=t^*} - ({}^C_{t_0}D^{\alpha,\rho}(y(\cdot, \eta_1)))(t)|_{t=t^*} \\ &= h(t^*, y(t^*, \eta_2)) + \eta_2 e^{\frac{\rho-1}{\rho}(t-t_0)} - [h(t^*, y(t^*, \eta_1)) + \eta_1 e^{\frac{\rho-1}{\rho}(t-t_0)}] \\ &= h(t^*, y(t^*, \eta_1)) + \eta_2 e^{\frac{\rho-1}{\rho}(t-t_0)} - [h(t^*, y(t^*, \eta_1)) + \eta_1 e^{\frac{\rho-1}{\rho}(t-t_0)}] \\ &= (\eta_2 - \eta_1) e^{\frac{\rho-1}{\rho}(t-t_0)} < 0. \end{aligned} \quad (22)$$

The obtained contradiction proves Inequality (21).

From (19) and (21), it follows that the family of solutions  $\{y(t, \eta)\}, t \in [t_0, T]$  of (17) is uniformly bounded, i.e, there exists  $K > 0$  with  $|y(t, \eta)| \leq K$  for  $(t, \eta) \in [t_0, T] \times [0, L]$ . Denote  $M = \sup\{|h(t, x)| : (t, x) \in [t_0, T] \times [-K, K]\}$ .

Let  $\{\eta_j\}_{j=0}^\infty : \eta_j < \eta_{j-1} \leq \eta_0 \leq L$  and  $\lim_{j \rightarrow \infty} \eta_j = 0$ . Then, for  $\{y(t; \eta_j)\}_{j=0}^\infty$ , we obtain:

$$y(t, \eta_j) = (y_0 + \eta_j)e^{\frac{\rho-1}{\rho}(t-t_0)} + \frac{\eta_j}{\rho^\alpha \Gamma(1+\alpha)} e^{\frac{\rho-1}{\rho}(t-t_0)} (t-t_0)^\alpha + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_0}^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{\alpha-1} h(s, y(s, \eta_j)) ds \quad \text{for } t \in [t_0, T]. \tag{23}$$

Now, for  $t_1, t_2 \in [t_0, T], t_1 < t_2$ , from (23), we have:

$$\begin{aligned} |y(t_2, \eta_j) - y(t_1, \eta_j)| &= \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_0}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)} (t_2-s)^{\alpha-1} h(s, y(s, \eta_j)) ds \\ &\quad - \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_0}^{t_1} e^{\frac{\rho-1}{\rho}(t_1-s)} (t_1-s)^{\alpha-1} h(s, y(s, \eta_j)) ds \\ &\leq \frac{1}{\rho^\alpha \Gamma(\alpha)} \left| \int_{t_0}^{t_1} \left( e^{\frac{\rho-1}{\rho}(t_2-s)} (t_2-s)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(t_1-s)} (t_1-s)^{\alpha-1} \right) h(s, y(s, \eta_j)) ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)} (t_2-s)^{\alpha-1} h(s, y(s, \eta_j)) ds \right| \\ &\leq \frac{M}{\rho^\alpha \Gamma(\alpha)} \int_{t_0}^{t_1} \left| e^{\frac{\rho-1}{\rho}(t_2-s)} (t_2-s)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(t_1-s)} (t_1-s)^{\alpha-1} \right| ds \\ &\quad + \frac{M}{\rho^\alpha \Gamma(\alpha)} \int_{t_1}^{t_2} e^{\frac{\rho-1}{\rho}(t_2-s)} (t_2-s)^{\alpha-1} ds \\ &\leq \frac{M}{\rho^\alpha \Gamma(\alpha)} \left[ \int_{t_0}^{t_1} \left| e^{\frac{\rho-1}{\rho}(t_2-s)} (t_2-s)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(t_1-s)} (t_1-s)^{\alpha-1} \right| ds + \frac{(t_2-t_1)^\alpha}{\alpha} \right] \end{aligned} \tag{24}$$

and thus,  $y(t_2, \eta_j) - y(t_1, \eta_j) \rightarrow 0$  as  $t_1 \rightarrow t_2$ .

Therefore, the family of functions  $\{u(t; \eta_j)\}$  is equicontinuous on  $[t_0, T]$ . According to the Arzela–Ascoli theorem, there exists a subsequence  $\{y(t; \eta_{j_k})\}$  and a  $Y \in C[t_0, T]$  with  $\lim_{k \rightarrow \infty} y(t; \eta_{j_k}) = Y(t)$  for  $t \in [t_0, T]$ . Taking the limit in (23) as  $k \rightarrow \infty$ , we obtain that the function  $Y(t)$  satisfies:

$$Y(t) = y_0 e^{\frac{\rho-1}{\rho}(t-t_0)} + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_0}^t e^{\frac{\rho-1}{\rho}(t-s)} (t-s)^{\alpha-1} h(s, Y(s)) ds \quad \text{for } t \in [t_0, T], \tag{25}$$

Therefore,  $Y(t)$  is a solution of (11) for  $t \in [t_0, T]$ , and according to Condition 4, we have  $Y(t) \leq y^*(t)$  on  $[t_0, T]$ . From (19), we obtain  $\mu(t) \leq Y(t) \leq y^*(t)$  on  $[t_0, T]$ .  $\square$

**Corollary 1.** Let Condition 1 of Lemma 6 be satisfied and  $V \in \Lambda([t_0, T], \Delta)$  be such that the generalized proportional Caputo fractional derivative exists for any point  $t \in [t_0, T]$ , and the inequality:

$$({}^C D^{\alpha, \rho} V(\cdot, \tilde{x})) (t) \leq 0$$

holds.

Then,  $V(t, \tilde{x}(t)) \leq V(t_0, x_0) e^{\frac{\rho-1}{\rho}(t-t_0)} \leq V(t_0, x_0)$  for  $t \in [t_0, T]$ .

**Proof.** The proof follows from Remark 4  $({}^C D^{\alpha, \rho} e^{\frac{\rho-1}{\rho}(\cdot)})(t) = 0$ , i.e., the solution of (11) with  $h(t, u) \equiv 0$  and initial condition  $y_0 = V(t_0, x_0)$  is  $y(t) = V(t_0, x_0) e^{\frac{\rho-1}{\rho}(t-t_0)}$ .  $\square$

**Corollary 2.** Let Condition 1 of Lemma 6 be satisfied and the function  $V \in \Lambda([t_0, T], \Delta)$  be such that the generalized proportional Caputo fractional derivative exists for any point  $t \in [t_0, T]$ , and the inequality:

$$({}^C_{t_0} D^{\alpha, \rho} V(\cdot, \tilde{x}(\cdot)))(t) \leq -\beta V(t, \tilde{x}(t))$$

holds where  $\beta > 0$ .

Then:

$$V(t, \tilde{x}(t)) \leq V(t_0, x_0) e^{\frac{\rho-1}{\rho}(t-t_0)} E_{\alpha}(-\beta(\frac{t-t_0}{\rho})^{\alpha}), \quad t \in [t_0, T]. \quad (26)$$

**Proof.** The proof follows from Lemma 3 about the solution of (11) with  $h(t, u) \equiv -\beta u$  and initial condition  $y_0 = V(t_0, x_0)$ .  $\square$

**Remark 5.** The results of Lemma 6, Corollaries 1 and 2 are true for  $T = \infty$ .

## 5. Main Result

We study the stability properties of the nonlinear generalized proportional Caputo fractional differential Equation (9).

**Theorem 1 (Stability).** Assume:

1. The function  $h \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$ ,  $h(t, 0) \equiv 0$ ;
2. There exists a function  $V \in \Lambda([t_0, \infty), \mathbb{R}^n)$ ,  $V(t, 0) = 0$  such that:
  - (i) for any initial value  $x_0 \in \mathbb{R}^n$ , the generalized proportional Caputo fractional derivative exists and the inequality:

$$({}^C_{t_0} D^{\alpha, \rho} V(\cdot, x(\cdot)))(t) \leq h(t, V(t, x(t))), \quad t > t_0 \quad (27)$$

holds where  $x(t) = x(t; t_0, x_0)$  is the solution of (9), (10);

- (ii)  $b(\|x\|) \leq V(t, x)$  for  $t \geq t_0$ ,  $x \in \mathbb{R}^n$ , where  $b \in \mathcal{K}$ ;
3. The zero solution of the scalar (11) is stable.  
Then, the zero solution of (9) is stable.

**Proof.** Let  $\epsilon > 0$ . According to Condition 3, there exists  $\delta_1 = \delta_1(\epsilon) > 0$  such that the inequality  $|y_0| < \delta_1$  implies:

$$|y(t; t_0, y_0)| < b(\epsilon), \quad t \geq t_0, \quad (28)$$

where  $y(t; t_0, y_0)$  is any solution of (11). Since  $V(t_0, 0) = 0$ , there exists  $\delta_2 = \delta_2(t_0, \delta_1) > 0$  such that  $V(t_0, x) < \delta_1$  for  $\|x\| < \delta_2$ . Choose  $x_0 \in \mathbb{R}^n$ :  $\|x_0\| < \delta_2$ . Then,  $V(t_0, x_0) < \delta_1$ , and let  $\tilde{x}(t) = x(t; t_0, x_0)$ ,  $t \geq t_0$  be the corresponding solution of (9) and (10). According to Condition 2(i), the inequality  $({}^C_{t_0} D^{\alpha, \rho} V(\cdot, \tilde{x}(\cdot)))(t) \leq h(t, V(t, \tilde{x}(t))), t \geq t_0$  holds.

Let  $y_0^* = V(t_0, x_0)$ . Then,  $y_0^* < \delta_1$ , and Inequality (28) holds for the solution  $y(t; t_0, y_0^*)$  of (11). From Lemma 6, Remark 5, and Inequality (28), we have:

$$V(t, \tilde{x}(t)) \leq \bar{y}(t) < b(\epsilon), \quad t \geq t_0;$$

here,  $\bar{y}(t) = \bar{y}(t; t_0, y_0^*)$  is the maximal solution of (11). Then, from Condition 2(ii), we obtain:

$$b(\|\tilde{x}(t)\|) \leq V(t, \tilde{x}(t)) < b(\epsilon), \quad t \geq t_0$$

so the result follows.  $\square$

**Corollary 3.** Assume  $V \in \Lambda([t_0, \infty), \mathbb{R}^n)$ ,  $V(t, 0) = 0$ , such that:

- (i) For  $x_0 \in \mathbb{R}^n$ , the generalized proportional Caputo fractional derivative exists and the inequality:

$$({}^C_{t_0} D^{\alpha, \rho} V(\cdot, x(\cdot)))(t) \leq 0, \quad t \geq t_0 \quad (29)$$

- holds where  $x(t) = x(t; t_0, x_0)$  is the solution of (9) and (10);
- (ii)  $b(\|x\|) \leq V(t, x)$  for  $t \geq t_0$ ,  $x \in \mathbb{R}^n$ , where  $b \in \mathcal{K}$ .

Then, the zero solution of (9) is stable.

The stability property is also satisfied if the conditions for the Lyapunov functions are satisfied on a ball:

$$S_\lambda = \{x \in \mathbb{R}^n : \|x\| \leq \lambda\}.$$

**Theorem 2 (Stability).** Assume:

1. The function  $h \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$ ,  $h(t, 0) \equiv 0$ ;
2. There exists a constants  $\lambda > 0$  and a function  $V \in \Lambda([t_0, \infty), S_\lambda)$ ,  $V(t, 0) = 0$  and:
  - (i) For any initial value  $x_0 \in S_\lambda$  and  $x(t) = x(t; t_0, x_0) \in S_\lambda$ ,  $t \geq t_0$ , the generalized proportional Caputo fractional derivative exists and the inequality:

$$({}^C_{t_0} D^{\alpha, \rho} V(\cdot, x(\cdot)))(t) \leq h(t, V(t, x(t))), \quad t \geq t_0 \quad (30)$$

holds;

- (ii)  $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$  for  $t \geq t_0$ ,  $x \in S_\lambda$  where  $a, b \in \mathcal{K}$ ;
3. The zero solution of the scalar (11) is stable.  
Then, the zero solution of (9) is stable.

**Proof.** Let  $\epsilon \in (0, \lambda]$ . Then, there exists  $\delta_1 = \delta_1(\epsilon) > 0$  such that the inequality  $|y_0| < \delta_1$  implies:

$$|y(t; t_0, y_0)| < b(\epsilon), \quad t \geq t_0, \quad (31)$$

where  $y(t; t_0, 1_0)$  is any solution of (11).

Let  $\delta_1 = \min\{\epsilon, b(\epsilon)\}$ .

Since  $a \in \mathcal{K}$ , there exists  $\delta_2 = \delta_2(\delta_1) > 0$  such that if  $\zeta < \delta_2$ , then  $a(\zeta) < \delta_1$ . Let  $\delta = \min\{\epsilon, \delta_2\}$ .

Choose  $x_0 : \|x_0\| < \delta$ , and let  $\tilde{x}(t) = x(t; t_0, x_0)$ ,  $t \geq t_0$  be the corresponding solution of (9) and (10).

We prove:

$$\|\tilde{x}(t)\| < \epsilon, \quad t \geq t_0. \quad (32)$$

Assume this is false. Then, there exists a point  $\tau > t_0$  such that  $\|\tilde{x}(t)\| < \epsilon$ ,  $t \in [t_0, \tau)$ , and  $\|\tilde{x}(\tau)\| = \epsilon$ , i.e.,  $\tilde{x}(\tau) \in S_\lambda$ ,  $t \in [t_0, \tau]$ .

Let  $y_0^* = V(t_0, x_0)$ . Then  $y_0^* < a(\|x_0\|)$ ,  $a(\delta_2) < \delta_1$ , and Inequality (31) holds for the solution  $y(t; t_0, y_0^*)$  of the scalar FrDE (11). Then, from Lemma 6, we have:

$$V(t, \tilde{x}(t)) \leq \bar{y}(t) < b(\epsilon), \quad t \in [t_0, \tau];$$

here,  $\bar{y}(t) = \bar{y}(t; t_0, y_0^*)$  is the maximal solution of (11). Then, from Condition 2(ii), we obtain:

$$b(\epsilon) = b(\|\tilde{x}(\tau)\|) \leq V(\tau, \tilde{x}(\tau)) < b(\epsilon).$$

The obtained contradiction proves Inequality (32).  $\square$

**Theorem 3 (Asymptotic stability).** Let  $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$  and:

- (i) For any initial value  $x_0 \in S_\lambda$  and any solution  $x(t) = x(t; t_0, x_0) \in S_\lambda$ ,  $t \geq t_0$  of (9), (10), the generalized proportional Caputo fractional derivative exists and the inequality:

$$({}^C_{t_0} D^{\alpha, \rho} V(\cdot, x(\cdot)))(t) \leq -cV(t, x(t)), \quad t \geq t_0$$

holds where  $c > 0$ ;

- (ii)  $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$  for  $t \geq t_0$ ,  $x \in S_\lambda$  where  $a, b \in \mathcal{K}$ .  
Then, the zero solution of (9) is asymptotically stable.

**Proof.** According to Theorem 2, the zero solution of (9) is stable. Thus, there exists a constant  $A = A(\lambda) \in (0, \lambda)$  such that the inequality  $\|\tilde{x}_0\| < A$  implies:

$$\|x(t; t_0, \tilde{x}_0)\| < \lambda \quad \text{for } t \geq t_0, \tag{33}$$

where  $x(t; t_0, \tilde{x}_0)$  is a solution of (9) and (10).

Let  $\beta \in (0, A] : a(\beta) \leq b(A)$ , and choose  $x_0 \in \mathbb{R}^n : \|x_0\| < \beta$ . Consider the solution  $\tilde{x}(t) = x(t; t_0, x_0)$  of (9) with initial condition  $x(t_0) = x_0$ . Then,  $b(\|x_0\|) \leq a(\|x_0\|) < a(B) < b(A)$ , i.e.,  $\|x_0\| < A$ , and therefore,  $\tilde{x}(t) \in S_\lambda, t \geq t_0$ .

According to Corollary 2, we obtain:

$$V(t, \tilde{x}(t)) \leq V(t_0, x_0) e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-c(\frac{t-t_0}{\rho})^\alpha), \quad t \geq t_0. \tag{34}$$

Choose an arbitrary number  $\varepsilon \in (0, B)$ . Consider the function  $\eta(t) = e^{\frac{\rho-1}{\rho}t} E_\alpha(-c(\frac{t}{\rho})^\alpha)$ , which is decreasing. Thus, there exists a number  $T > 0$  such that  $\eta(t) < \frac{b(\varepsilon)}{a(B)}$  for  $t > T$ . Thus,

$$\begin{aligned} b(\|\tilde{x}(t)\|) &\leq V(t, \tilde{x}(t)) \leq V(t_0, x_0) e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-c(\frac{t-t_0}{\rho})^\alpha) \\ &\leq a(\|x_0\|) e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-c(\frac{t-t_0}{\rho})^\alpha) < b(\varepsilon), \quad t \geq T. \end{aligned}$$

Therefore,  $\|\tilde{x}(t)\| < \varepsilon, t \geq t_0 + T$  holds, and the zero solution of (9), (10) is attractive.  $\square$

### 6. Applications

**Example 2.** Consider the following system of fractional differential equations with the GPDF:

$$\begin{aligned} ({}^C_0 D^{\alpha, \rho} x_1)(t) &= -(1-\rho) \frac{(1+t^2)}{(t^2+2)} x_1(t) - g_2(t)x_2(t), \\ ({}^C_0 D^{\alpha, \rho} x_2)(t) &= -(1-\rho) \frac{(1+t^2)}{(t^2+2)} x_2(t) + g_2(t)x_1(t) \quad \text{for } t \geq 0 \end{aligned} \tag{35}$$

with initial condition:

$$x_1(0) = x_1^0 \quad \text{and} \quad x_2(0) = x_2^0,$$

where  $x_1, x_2 \in \mathbb{R}, \alpha \in (0, 1), \rho \in (0, 1]$ , and  $g_2 \in C(\mathbb{R}_+, \mathbb{R})$  is an arbitrary function.

Note that  $(1-\rho) \frac{(1+t^2)}{(t^2+2)} \in [0.5(1-\rho), (1-\rho)]$  for  $t \geq 0$ .

Consider  $V(t, x_1, x_2) = x_1^2 + x_2^2$  for  $t \in \mathbb{R}_+, x = (x_1, x_2) \in \mathbb{R}^2$ .

For any  $x_1, x_2 \in \mathbb{R} : x = (x_1, x_2)$ , apply Lemma 4, and obtain:

$$\begin{aligned} ({}^C_{t_0} D^{\alpha, \rho} V(\cdot, x_1(\cdot), x_2(\cdot)))(t) &= ({}^C_{t_0} D^{\alpha, \rho} x_1^2(\cdot))(t) + ({}^C_{t_0} D^{\alpha, \rho} x_2^2(\cdot))(t) \\ &\leq 2x_1(t)({}^C_{t_0} D^{\alpha, \rho} x_1(\cdot))(t) + 2x_2(t)({}^C_{t_0} D^{\alpha, \rho} x_2(\cdot))(t) \\ &= -2(1-\rho) \frac{(1+t^2)}{(t^2+2)} V(t, x_1, x_2) \leq -(1-\rho)V(t, x_1, x_2). \end{aligned} \tag{36}$$

Case 1. Let  $\rho = 1$  (Caputo fractional differential equation). From (36), we obtain the inequality  $({}^C_{t_0} D^{\alpha, \rho} V(\cdot, x_1(\cdot), x_2(\cdot)))(t) \leq 0$ , and according to Corollary 3, the zero solution of (35) is stable.

Case 2. Let  $\rho \in (0, 1)$ , and from (36), according to Theorem 3, the zero solution of (35) is asymptotically stable.

Therefore, the stability properties depend on the parameter  $\rho$  of the generalized proportional fractional derivative.

Consider the corresponding to (35) system with ordinary derivatives:

$$\begin{aligned}
 x_1'(t) &= -(1 - \rho) \frac{(1 + t^2)}{(t^2 + 2)}(t)x_1 - g_2(t)x_2, \\
 x_2'(t) &= -(1 - \rho) \frac{(1 + t^2)}{(t^2 + 2)}(t)x_2 + g_2(t)x_1, \quad \text{for } t \geq 0, \rho \in (0, 1), \\
 x_1(0) &= x_1^0 \quad \text{and} \quad x_2(0) = x_2^0.
 \end{aligned}
 \tag{37}$$

Case 3. Let  $\rho = 1$ . Then, the solutions are not stable (see Figure 1) (compare with Case 1).

Case 4. Let  $\rho = 0.9$ . The zero solution is asymptotically stable (see Figure 2 with  $x_1(0) = x_2(0) = 0.52$  and  $g_2(t) \equiv -1$ ).

Therefore, the type of the derivative (ordinary derivative, or Caputo fractional derivative, or the GPDF) in the differential equation has a significant influence on the behavior of the solutions.

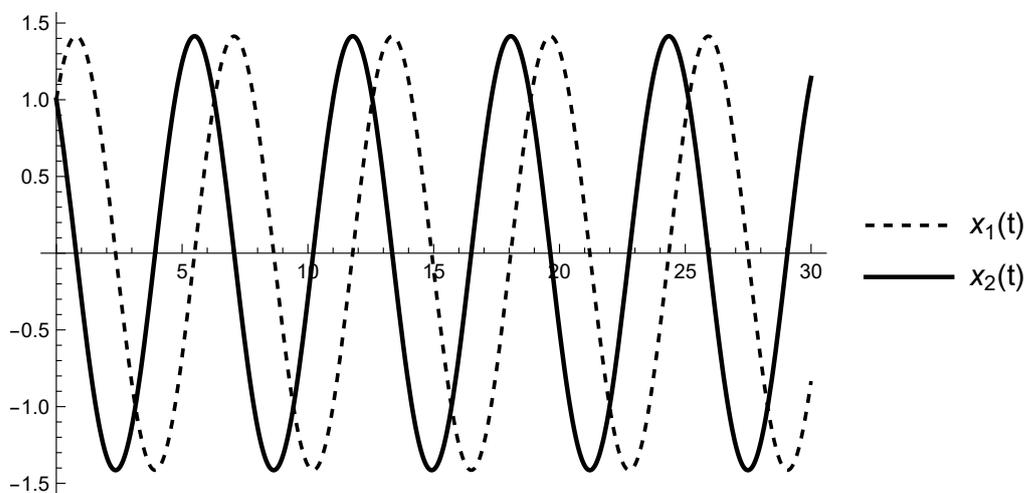


Figure 1. Graph of the solutions (37) for  $\rho = 1, g_2(t) \equiv -1$ .

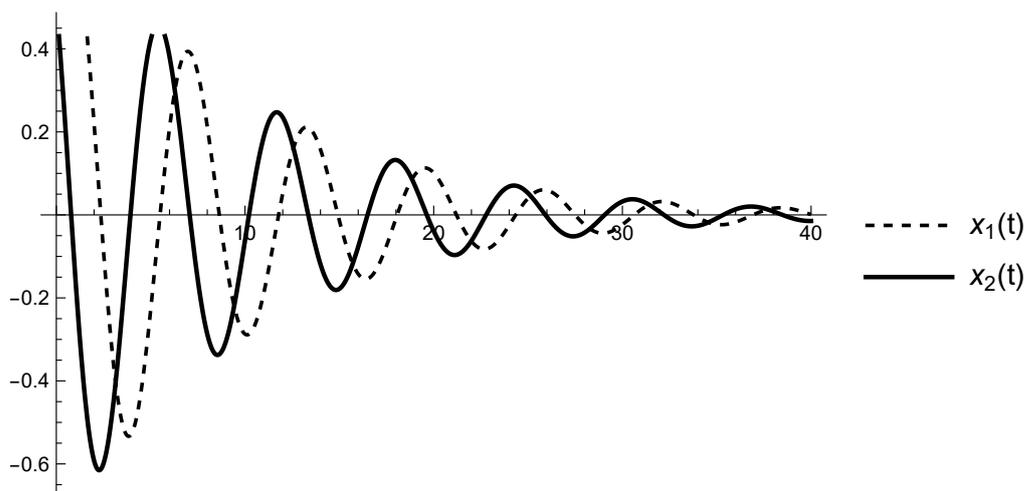


Figure 2. Graph of the solutions (37) for  $\rho = 0.9, g_2(t) \equiv -1$ .

**Example 3.** Consider the following system of nonlinear fractional differential equations with the GPDF:

$$\begin{aligned}
 ({}^C_0 D^{\alpha, \rho} x_1)(t) &= a \sin(x_1(t)) - g(t)x - 2(t), \\
 ({}^C_0 D^{\alpha, \rho} x_2)(t) &= g(t)x_1(t) + a \sin(x_2(t)) \quad \text{for } t \geq 0, \alpha \in (0, 1), \rho \in (0, 1),
 \end{aligned}
 \tag{38}$$

with initial condition,

$$x_1(0) = x_1^0 \quad \text{and} \quad x_2(0) = x_2^0,$$

where  $x_1, x_2 \in \mathbb{R}$ ,  $g \in C([0, \infty), \mathbb{R})$  is an arbitrary function and  $a > 0$ :  $a < 0.5(1 - \rho)^\alpha$ .

Consider  $V(t, x_1, x_2) = x_1^2 + x_2^2$  for  $t \in \mathbb{R}_+$ ,  $x_1, x_2 \in \mathbb{R}$ .

For any  $x_1, x_2 \in \mathbb{R}$ , apply Lemma 4, and obtain:

$$\begin{aligned} &({}^C_{t_0} D^{\alpha, \rho} V(\cdot, x_1(\cdot), x_2(\cdot)))(t) = ({}^C_{t_0} D^{\alpha, \rho} x_1^2(\cdot))(t) + ({}^C_{t_0} D^{\alpha, \rho} x_2^2(\cdot))(t) \\ &\leq 2x_1(t)({}^C_{t_0} D^{\alpha, \rho} x_1(\cdot))(t) + 2x_2(t)({}^C_{t_0} D^{\alpha, \rho} x_2(\cdot))(t) \\ &= 2ax_1(t) \sin(x_1(t)) - 2g(t)x_1(t)x_2(t) + 2x_2(t)g(t)x_1(t) + 2ax_2(t) \sin(x_2(t)) \\ &\leq 2a(x_1^2(t) + x_2^2(t)) = 2aV(t, x_1, x_2). \end{aligned} \quad (39)$$

Consider the scalar fractional Equation (11) with  $h(t, y) = 2ay$ ,  $y \in \mathbb{R}$ , the solution of which according to Lemma 3 with  $\lambda = 2a$  is given by  $y(t) = y_0 e^{\frac{\rho-1}{\rho}t} E_\alpha(2a(\frac{t}{\rho})^\alpha)$ . According to Theorem 1.2 [23], for every  $\alpha \in (0, 1)$ , the function,  $\frac{e^t}{\alpha} - E_\alpha(t^\alpha)$  is completely monotonic and  $E_\alpha(t^\alpha) < \frac{e^t}{\alpha}$ ,  $t \geq 0$ . Therefore,  $E_\alpha(2a(\frac{t}{\rho})^\alpha) = E_\alpha((\frac{\sqrt[\alpha]{2at}}{\rho})^\alpha) \leq \frac{e^{\frac{\sqrt[\alpha]{2at}}{\rho}}}{\alpha}$  and  $|y(t)| \leq |y_0| e^{\frac{\rho-1}{\rho}t} \frac{e^{\frac{\sqrt[\alpha]{2at}}{\rho}}}{\alpha} = \frac{|y_0|}{\alpha} e^{\frac{\rho-1+\sqrt[\alpha]{2a}}{\rho}t}$ .

Thus, the zero solution of the considered particular case of the scalar (11) is stable since  $\rho - 1 + \sqrt[\alpha]{2a} < 0$ . According to Theorem 1, the zero solution of the nonlinear system (38) is stable.

## 7. Conclusions

A system of nonlinear equations with the GPDFD was considered. The stability properties of the zero solution were studied using Lyapunov functions and their GPDFDs. First, some comparison results with scalar nonlinear fractional differential equations with the GPDFD and Lyapunov functions were established and several sufficient conditions for stability and asymptotic stability were obtained and illustrated with examples. In the future, we hope to study other types of stability for nonlinear fractional differential equations with the GPDFD and apply them to some new models.

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