



Article **Fixed Point Results for F-Contractions in Cone Metric Spaces over Topological Modules and Applications to Integral Equations**

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Abstract: In this paper, the concept of *F*-contraction was generalized for cone metric spaces over topological left modules and some fixed point results were obtained for self-mappings satisfying a contractive condition of this type. Some applications of the main result to the study of the existence and uniqueness of the solutions for certain types of integral equations were presented in the last part of the article, one of them being a fractional integral equation.

Keywords: *F*-contractions; fixed point theorems; integral equations; topological left modules; cone metric spaces

1. Introduction

The concept of cone metric space was introduced by Huang and Zhang [1] as a generalization of a metric space, proving that Banach's contraction theorem remains valid in this context. Afterwards, many authors have obtained fixed point results on cone metric spaces: Radenović and Rhoades [2], Rezapour and Hamlbarani [3], Kadelburg et al. [4], Du [5] and the references therein. Further, Liu and Xu [6] introduced the concept of cone metric space over a Banach algebra and proved some fixed point theorems for Lipschitz mappings. Later on, Xu and Radenović [7] extended the results of Liu and Xu [6], without the assumption of normality of the cone involved. Also, generalizations of Banach's theorem have been obtained in other directions. Wardowski [8] defined the class of F-contractions and proved a fixed point result as a extension of Banach contraction principle. After this, Wardowski and Van Dung [9] introduced the concept of *F*-weak contraction and obtained a new fixed point theorem. Cosentino and Vetro [10] obtained new fixed point theorems of Hardy-Rogers type for F-contractions in ordered metric spaces. Other results concerning F-contractions have been obtained by: Piri and Kumam [11], Minak et al. [12], Ahmad et al. [13], Kadelburg and Radenović [14], Dey et al. [15], Wardowski [16], Alfaqih et al. [17], Karapinar et al. [18] and the references therein.

Regarding the fractional differentiation and integration, many models have been proposed in the literature: the Riemann–Liouville fractional model [19], the Caputo model [20], the Atangana–Baleanu (or AB) fractional model [21,22], the generalised proportional fractional (or GPF) model [23], the Prabhakar fractional model [24,25], and others. Fernandez et al. [26] proposed a unified model of fractional calculus by using a general operator which includes many types of fractional operators. They consider some fractional differential equations and solve a general Cauchy problem in this new framework. Important results on nonlinear fractional differential equations were obtained by Agarwal et al. [27,28], Soradi-Zeid et al. [29], Almeida [30], Khan et al. [31], Keten et al. [32] and the references therein.

In this article, we generalize the notion of cone metric space for topological left module and we define the concept of *F*-contraction on this new space. Next, we obtain some fixed point results for self-mappings satisfying a contractive condition of this type. In the last part of the article, some applications of the main result to the study of the existence and



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Copyright: © 2021 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). uniqueness of the solutions for integral equations were presented, one of them being an integral equation of fractional type.

2. Methods

We will start by presenting some notions relating to ordered topological rings and topological left modules. For more details see the papers Warner [33].

Definition 1. A ring $(\mathcal{A}, \oplus, \odot)$ is called a topological ring if the set \mathcal{A} is endowed with a topology $\tau_{\mathcal{A}}$ such that the mappings $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \ominus : \mathcal{A} \to \mathcal{A}, \odot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ are continuous, where $\mathcal{A} \times \mathcal{A}$ is considered with respect to the product topology.

We denote by $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}})$ *a topological ring.*

Definition 2. Let $(\mathcal{A}, \oplus, \odot)$ be a ring and let \leq be a partial order relation on \mathcal{A} . \mathcal{A} is called a partially ordered ring if it fulfills the following conditions:

- (*i*) *if* $a, b, c \in A$, $a \leq b$, then $a \oplus c \leq b \oplus c$;
- (ii) if $a, b \in A$, $0 \leq a, 0 \leq b$, then $0 \leq a \odot b$.

We denote by $(\mathcal{A}, \oplus, \odot, \preceq)$ a partially ordered ring. The set $\mathcal{A}_+ = \{a \in \mathcal{A} \mid 0_{\mathcal{A}} \preceq a\}$ is named the positive cone of \mathcal{A} . A partially ordered topological ring and is denoted by $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$.

Definition 3. Let $(\mathcal{A}, \oplus, \odot)$ be a ring. A left \mathcal{A} -module consists of an abelian group (E, +) and a mapping $\cdot : \mathcal{A} \times E \to E$ satisfying the following properties:

- (1) *if* $a \in A$, $x, y \in E$, then $a \cdot (x + y) = a \cdot x + a \cdot y$;
- (2) *if* $a, b \in A$, $x \in E$, then $(a \oplus b) \cdot x = a \cdot x + b \cdot x$;
- (3) *if* $a, b \in A$, $x \in E$, then $(a \odot b) \cdot x = a \cdot (b \cdot x)$;
- (4) *if* $x \in E$, then $1_A \cdot x = x$.

A left A-module is denoted by $(E, +, \cdot)$.

Definition 4. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}})$ be a topological ring. A left A-module $(E, +, \cdot)$ is called a topological left A-module if on E is specified a topology τ_E such that the mappings $+ : E \times E \rightarrow E$, $- : E \rightarrow E, \cdot : \mathcal{A} \times E \rightarrow E$ are continuous, where $E \times E$ and $\mathcal{A} \times E$ are considered with respect to the product topologies, respectively.

A topological left A-module is denoted by $(E, +, \cdot, \tau_E)$.

A topological left A-module, $(E, +, \cdot, \tau_E)$, where τ_E is a Hausdorff topology, is called a Hausdorff topological left A-module.

3. Results

Definition 5. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered topological ring and $(E, +, \cdot, \tau_E)$ a topological left \mathcal{A} -module. A cone is a non-empty subset P of E satisfying the conditions:

- (1) $P \neq \{0_E\}$ and $P \cap (-P) = \{0_E\};$
- (2) P is closed;
- (3) $a \cdot x + b \cdot y \in P$ for all $a, b \in A_+, x, y \in P$.
 - The cone P is solid if $int(P) \neq \emptyset$. We define the set $P^* = P \setminus \{0_E\}$.

Lemma 1. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered topological ring, with $1_{\mathcal{A}} \in \mathcal{A}_+$, $(E, +, \cdot, \tau_E)$ a topological left \mathcal{A} -module and P a solid cone of E. Then, the following statements are valid:

- (i) $P + P \subseteq P$;
- (ii) the relation \leq_P over E, defined by $x \leq_P y$ if and only if $y x \in P$, is a partial order relation on E;
- (iii) if $x \leq_P y$ and $y \ll_P z$, then $x \ll_P z$, for all $x, y, z \in E$;
- (iv) if $x \leq_P y$ and $u \leq_P v$, then $x + u \leq_P y + v$, for all $x, y, u, v \in E$.

Lemma 2. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered topological ring, $(E, +, \cdot, \tau_E)$ a topological left \mathcal{A} -module and P a solid cone of E. Then, the following statements are valid:

- (i) $int(P) \cap (-P) = \begin{cases} \emptyset & , \ 0_E \notin int(P) \\ \{0_E\} & , \ 0_E \in int(P); \end{cases}$
- (*ii*) $int(P) \cap Fr(P) = \emptyset$;
- (iii) assuming that τ_A is a Hausdorff topology, 0_A is an accumulation point of A_+ and $a \cdot int(P) \subseteq int(P)$ for all $a \in A_+^*$, if $x \in P$, $x \ll_P \varepsilon + \varepsilon$ for all $\varepsilon \in int(P)$, then $x = 0_E$;
- (iv) if $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset E, x, y \in E, x_n \xrightarrow{n} x, y_n \xrightarrow{n} y$, and there is a number $N \in \mathbb{N}$ such that $x_n \leq_P y_n$ for all $n \geq N$, then $x \leq_P y$.

Definition 6. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered topological ring, $(E, +, \cdot, \tau_E)$ a topological left \mathcal{A} -module, P a solid cone of E and X a non-empty set. A cone metric on X is an application $d : X \times X \to E$ which fulfills the conditions:

- (1) $0_E \leq_P d(x, y)$ for all $x, y \in X$;
- (2) $d(x, y) = 0_E$ if and only if x = y;
- (3) d(x,y) = d(y,x) for all $x, y \in X$;
- (4) $d(x,y) \leq_P d(x,z) + d(z,y)$ for all $x, y, z \in X$.

(X, d) is called a cone metric space over the topological left A-module.

Definition 7. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered topological ring, $(E, +, \cdot, \tau_E)$ a topological left \mathcal{A} -module, P a solid cone of E and (X, d) a cone metric space over the topological left \mathcal{A} -module.

(1) A sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is called convergent to a point $x \in X$ if it has the property: for every $\varepsilon \gg_P 0_E$ there is a number $n(\varepsilon) \in \mathbb{N}$ such that for all $n \ge n(\varepsilon)$ we have $d(x_n, x) \ll_P \varepsilon$; we

also say that $(x_n)_{n \in \mathbb{N}} \subset X$ converges to $x \in X$ and we denote by $x_n \xrightarrow{d} x$;

- (2) A sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is named a Cauchy sequence if it satisfies the condition: for every $\varepsilon \gg_P 0_E$ there is a number $n(\varepsilon) \in \mathbb{N}$ such that for all $m, n \ge n(\varepsilon)$ we have $d(x_m, x_n) \ll_P \varepsilon$;
- (3) The cone metric space (X, d) is called complete if: any Cauchy sequence of points in X is convergent in X.

Lemma 3. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered Hausdorff topological ring, with $1_{\mathcal{A}} \in \mathcal{A}_+$, $0_{\mathcal{A}}$ an accumulation point of \mathcal{A}_+ , $(E, +, \cdot, \tau_E)$ a topological left \mathcal{A} -module, P a solid cone of E, with $a \cdot int(P) \subseteq int(P)$ for all $a \in \mathcal{A}_+^*$, and (X, d) a cone metric space over the topological left \mathcal{A} -module. If the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is convergent in X, then it has a unique limit.

Definition 8. Let (E, τ_E) be a topological space. A subset *S* of *E* is called sequentially compact if any sequence in *S* has a convergent subsequence in *S*.

Lemma 4. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered topological ring, with $1_{\mathcal{A}} \in \mathcal{A}_+$, $(E, +, \cdot, \tau_E)$ a Hausdorff topological left \mathcal{A} -module, P a solid cone of E and (X, d) a cone metric space over the topological left \mathcal{A} -module. If $(x_n)_{n \in \mathbb{N}} \subset X$ is not a Cauchy sequence and there is a sequentially compact subset $S_x \subseteq P$ having the property $\{d(x_m, x_n) \mid m, n \in \mathbb{N}^*, m < n\} \subseteq S_x$, then the following statements are valid:

(*i*) there exist $\varepsilon_0 \gg_P 0_E$ and two subsequences $(x_{m(k)})_{k \in \mathbb{N}}$, $(x_{n(k)})_{k \in \mathbb{N}}$, where $m, n : \mathbb{N} \to \mathbb{N}$ are strictly increasing functions, with k < m(k) < n(k) for all $k \in \mathbb{N}$, such that

$$d(x_{m(k)}, x_{n(k)}) \not\ll_P \varepsilon_0, \ d(x_{m(k)}, x_{n(k)-1}) \ll_P \varepsilon_0 \text{ for all } k \in \mathbb{N};$$

(ii) moreover, if $\lim_{n \to +\infty} d(x_{n-1}, x_n) = 0_E$, there exists a point $x \in Fr(P)$ such that

$$\lim_{k \to +\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0 - x, \lim_{k \to +\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon_0 - x \text{ and } \varepsilon_0 - x \in P.$$

Proof. (i) Since $(x_n)_{n \in \mathbb{N}}$ is not a Cauchy sequence, we deduce that there is $\varepsilon_0 \gg_P 0_E$ and two subsequences $(x_{m_1(k)})_{k \in \mathbb{N}}$, $(x_{\tilde{n}_1(k)})_{k \in \mathbb{N}}$, where $m_1, \tilde{n}_1 : \mathbb{N} \to \mathbb{N}$ are strictly increasing functions, with $k < m_1(k) < \tilde{n}_1(k)$ for all $k \in \mathbb{N}$, such that $d(x_{m_1(k)}, x_{\tilde{n}_1(k)}) \not\ll_P \varepsilon_0$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ we choose $n_1(k)$ to be the smallest integer $\tilde{n}_1(k)$ which verifies $m_1(k) < n_1(k)$ and $d(x_{m_1(k)}, x_{n_1(k)}) \not\ll_P \varepsilon_0$, thus $d(x_{m_1(k)}, x_{n_1(k)-1}) \ll_P \varepsilon_0$. Hence, there exist $\varepsilon_0 \gg_P 0_E$ and two subsequences $(x_{m_1(k)})_{k \in \mathbb{N}}$, $(x_{n_1(k)})_{k \in \mathbb{N}}$, where $m_1, n_1 : \mathbb{N} \to \mathbb{N}$ are strictly increasing functions, with $k < m_1(k) < n_1(k)$ for all $k \in \mathbb{N}$, such that

$$d(x_{m_1(k)}, x_{n_1(k)}) \not\ll_P \varepsilon_0, \ d(x_{m_1(k)}, x_{n_1(k)-1}) \ll_P \varepsilon_0 \text{ for all } k \in \mathbb{N}.$$

$$\tag{1}$$

As S_x is sequentially compact, any sequence in S_x has a convergent subsequence in S_x . According to the hypothesis we have $\{d(x_m, x_n) \mid m, n \in \mathbb{N}^*, m < n\} \subseteq S_x$. Therefore, for the sequence $(d(x_{m_1(k)}, x_{n_1(k)}))_{k \in \mathbb{N}} \subset S_x$ there is a subsequence $(d(x_{m_1(p(k))}, x_{n_1(p(k))}))_{k \in \mathbb{N}}$, where $p : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, with $k < p(k) < m_1(p(k)) < n_1(p(k))$ for all $k \in \mathbb{N}$, such that

$$\lim_{k \to +\infty} d(x_{m_1(p(k))}, x_{n_1(p(k))}) = l_1 \in S_x \subseteq P.$$
⁽²⁾

Similarly, for the sequence $(d(x_{m_1(p(k))-1}, x_{n_1(p(k))-1}))_{k \in \mathbb{N}} \subset S_x$ there exists a subsequence $(d(x_{m_1(p(q(k)))-1}, x_{n_1(p(q(k)))-1}))_{k \in \mathbb{N}}$, where $q : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, with $k < q(k) < p(q(k)) < m_1(p(q(k))) < n_1(p(q(k)))$ for all $k \in \mathbb{N}$, such that

$$\lim_{k \to +\infty} d(x_{m_1(p(q(k)))-1}, x_{n_1(p(q(k)))-1}) = l_2 \in S_x \subseteq P.$$
(3)

Applying the statement (1) for $k := p(q(k)), k \in \mathbb{N}$, it follows that there exist $\varepsilon_0 \gg_P 0_E$ and two subsequences $(x_{m(k)})_{k \in \mathbb{N}}$, $(x_{n(k)})_{k \in \mathbb{N}}$, where $m, n : \mathbb{N} \to \mathbb{N}$, $m(k) = m_1(p(q(k)))$, $n(k) = n_1(p(q(k)))$ are strictly increasing functions, with k < m(k) < n(k) for all $k \in \mathbb{N}$, such that $d(x_{m(k)}, x_{n(k)}) \not\ll_P \varepsilon_0$, $d(x_{m(k)}, x_{n(k)-1}) \ll_P \varepsilon_0$ for all $k \in \mathbb{N}$.

(ii) As τ_E is a Hausdorff topology, every convergent sequence in *E* has a unique limit in *E*.

Considering (i) we get $d(x_{m(k)}, x_{n(k)}) \not\ll_P \varepsilon_0$ for all $k \in \mathbb{N}$, hence $\varepsilon_0 - d(x_{m(k)}, x_{n(k)}) \in c(int(P))$ for all $k \in \mathbb{N}$. Since $(d(x_{m(k)}, x_{n(k)}))_{k \in \mathbb{N}}$ represents the sequence $(d(x_{m_1(p(q(k)))}, x_{n_1(p(q(k)))}))_{k \in \mathbb{N}}$ which is a subsequence of $(d(x_{m_1(p(k))}, x_{n_1(p(k))}))_{k \in \mathbb{N}}$ and taking into account that $\lim_{k \to +\infty} d(x_{m_1(p(k))}, x_{n_1(p(k))}) = l_1$ (according to the relation (2)), we deduce $\lim_{k \to +\infty} d(x_{m(k)}, x_{n(k)}) = l_1$. We find $\lim_{k \to +\infty} (\varepsilon_0 - d(x_{m(k)}, x_{n(k)})) = \varepsilon_0 - l_1$. Therefore, $\varepsilon_0 - l_1 \in \overline{c(int(P))} = c(int(P))(int(P)$ is an open set, hence c(int(P)) is a closed set). Taking into account the properties of the cone metric d, the statement (i) and using Lemma 1 (iii) we obtain

$$d(x_{m(k)}, x_{n(k)}) \leq_P d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \ll_P$$

 $\varepsilon_0 + d(x_{n(k)-1}, x_{n(k)}) \text{ for all } k \in \mathbb{N},$

thus

$$\varepsilon_0 + d(x_{n(k)-1}, x_{n(k)}) - d(x_{m(k)}, x_{n(k)}) \in int(P)$$
 for all $k \in \mathbb{N}$

As $(d(x_{n(k)-1}, x_{n(k)}))_{k \in \mathbb{N}}$ is a subsequence of $(d(x_{n-1}, x_n))_{n \in \mathbb{N}}$ and $\lim_{n \to +\infty} d(x_{n-1}, x_n) = 0_E$ we deduce $\lim_{k \to +\infty} d(x_{n(k)-1}, x_{n(k)}) = 0_E$; also, $\lim_{k \to +\infty} d(x_{m(k)}, x_{n(k)}) = l_1$. Therefore, $\lim_{k \to +\infty} (\varepsilon_0 + d(x_{n(k)-1}, x_{n(k)}) - d(x_{m(k)}, x_{n(k)})) = \varepsilon_0 - l_1$. It follows that $\varepsilon_0 - l_1 \in \overline{int(P)} \subseteq \overline{P} = P$, hence $\varepsilon_0 - l_1 \in c(int(P)) \cap P = P \cap c(int(P)) = \overline{P} \cap c(int(P)) = Fr(P)$. Consequently, there exists a point $x \in Fr(P)$ such that $\varepsilon_0 - l_1 = x$, thus $l_1 = \varepsilon_0 - x$, so $\lim_{k \to +\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0 - x$.

The properties of the cone metric *d* leads to

$$d(x_{m(k)-1}, x_{n(k)-1}) \leq_P d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1})$$
for all $k \in \mathbb{N}$ (4)

and

 $d(x_{m(k)}, x_{n(k)}) \leq_P d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})$ for all $k \in \mathbb{N}$. (5)

Since $(d(x_{m(k)-1}, x_{n(k)-1}))_{k \in \mathbb{N}}$ represents the sequence $(d(x_{m_1(p(q(k)))-1}, x_{n_1(p(q(k)))-1}))_{k \in \mathbb{N}}$ and $\lim_{k \to +\infty} d(x_{m_1(p(q(k)))-1}, x_{n_1(p(q(k)))-1})) = l_2$ (according to the relation (3)), we find $\lim_{k \to +\infty} d(x_{m(k)-1}, x_{n(k)-1}) = l_2$. Also, $(d(x_{m(k)-1}, x_{m(k)}))_{k \in \mathbb{N}}$, $(d(x_{n(k)-1}, x_{n(k)}))_{k \in \mathbb{N}}$ are subsequences of $(d(x_{n-1}, x_n))_{n \in \mathbb{N}}$ and $\lim_{n \to +\infty} d(x_{n-1}, x_n) = 0_E$, hence $\lim_{k \to +\infty} d(x_{m(k)-1}, x_{m(k)}) = \lim_{k \to +\infty} d(x_{n(k)-1}, x_{n(k)}) = 0_E$; moreover, $\lim_{k \to +\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0 - x$. Passing to the limit for the sequences in relations (4), (5), we obtain $l_2 \leq_P \varepsilon_0 - x$ and $\varepsilon_0 - x \leq_P l_2$, thus $l_2 = \varepsilon_0 - x$, so $\lim_{k \to +\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon_0 - x$.

As $(d(x_{m(k)}, x_{n(k)}))_{k \in \mathbb{N}} \subset P$ and $\lim_{k \to +\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0 - x$, it follows that $\varepsilon_0 - x \in \overline{P} = P$ (*P* being a closed set).

Consequently, we proved that there exists a point $x \in Fr(P)$ such that $\lim_{k \to +\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0 - x$, $\lim_{k \to +\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon_0 - x$ and $\varepsilon_0 - x \in P$. \Box

In the following, we define the operation

$$*: \mathbb{N} \times E \to E, n * t = \begin{cases} 0_E , n = 0\\ \underbrace{t + \dots + t}_n, n \ge 1. \end{cases}$$

Definition 9. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered topological ring, $(E, +, \cdot, \tau_E)$ a topological left \mathcal{A} -module and P a solid cone of E. A sequence $(u_n)_{n \in \mathbb{N}} \subset E$ is called:

- (1) an increasing sequence if $u_{n-1} \leq_P u_n$ for all $n \in \mathbb{N}^*$;
- (2) *a decreasing sequence if* $u_{n-1} \ge_P u_n$ *for all* $n \in \mathbb{N}^*$.

Definition 10. Let (E, τ_E) be a topological space, *S* a subset of *E* and *F* : *S* \rightarrow *E* a function.

- (1) *F* is sequentially continuous at a point $x \in S$ if: for every sequence $(x_n)_{n \in \mathbb{N}} \subset S$ convergent to *x*, the sequence $(F(x_n))_{n \in \mathbb{N}} \subset E$ is convergent to F(x);
- (2) *F* is named sequentially continuous on *S* if: it is sequentially continuous at every point $x \in S$.

Definition 11. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered topological ring, $(E, +, \cdot, \tau_E)$ a topological left \mathcal{A} -module and P a solid cone of E. We consider the set \mathcal{F}' of all functions $F : P^* \to E$ which fulfill the conditions:

- (1) *if* $u \in P^*$, $v \in int(P)$, then for every $\varepsilon \gg_P 0_E$ there is a number $n(\varepsilon, u, v) \in \mathbb{N}$ such that for all $n \ge n(\varepsilon, u, v)$ we have $F(u) n * v \ll_P -\varepsilon$;
- (2) *if* $(u_n)_{n \in \mathbb{N}} \subset P^*$ *is a decreasing sequence and for every* $\varepsilon \gg_P 0_E$ *there is a number* $n(\varepsilon) \in \mathbb{N}$ *such that for all* $n \ge n(\varepsilon)$ *we have* $F(u_n) \ll_P -\varepsilon$ *, then* $u_n \xrightarrow{n} 0_E$;
- (3) if $u, v \in P^*$, $F(u) \ll_P F(v)$, then $u \leq_P v$;
- (4) *F* is sequentially continuous on P^* .

Definition 12. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered topological ring, $(E, +, \cdot, \tau_E)$ a topological left \mathcal{A} -module, P a solid cone of E and (X, d) a cone metric space over the topological left \mathcal{A} -module. An F-contraction corresponding to the function class \mathcal{F}' is a mapping $T : X \to X$ for which there exist $\tau \gg_P 0_E$ and a function $F \in \mathcal{F}'$ such that

$$\tau + F(d(Tx, Ty)) \le_P F(d(x, y)), \text{ for all } x, y \in X, Tx \neq Ty.$$
(6)

Remark 1. The condition $Tx \neq Ty$ from the property (6) implies $x \neq y$. Hence, $d(x,y) \in P^*$ whenever $d(Tx, Ty) \in P^*$. Therefore, the function $F \in \mathcal{F}'$ is defined for every $x, y \in X$ satisfying the condition $Tx \neq Ty$.

Theorem 1. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered Hausdorff topological ring, with $1_{\mathcal{A}} \in \mathcal{A}_+$, $0_{\mathcal{A}}$ an accumulation point of \mathcal{A}_+ , $(E, +, \cdot, \tau_E)$ a Hausdorff topological left \mathcal{A} -module, P a solid cone of E, with $a \cdot int(P) \subseteq int(P)$ for all $a \in \mathcal{A}_+^*$, $0_E \notin int(P)$, and (X, d) a cone metric space over the topological left \mathcal{A} -module. We suppose that (X, d) is complete, $T : X \to X$ is an F-contraction corresponding to the function class \mathcal{F}' and for every $x_0 \in X$ we consider the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}^*$. If for any $x_0 \in X$ for which $(x_n)_{n\in\mathbb{N}} \subset X$ is not a Cauchy sequence, there is a sequentially compact subset $S_{x_0} \subseteq P$ having the property $\{d(x_m, x_n) \mid m, n \in \mathbb{N}^*, m < n\} \subseteq S_{x_0}$, then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to x^* .

Proof. Let us suppose that *T* has two distinct fixed points, so there exist $x_1, x_2 \in X$ such that $Tx_1 = x_1 \neq x_2 = Tx_2$. Because $T : X \to X$ is an *F*-contraction corresponding to the function class \mathcal{F}' we deduce that there exist $\tau \gg_P 0_E$ and a function $F \in \mathcal{F}'$ such that the property (6) is verified. Applying this property we get $\tau + F(d(Tx_1, Tx_2)) \leq_P F(d(x_1, x_2))$, thus $\tau + F(d(x_1, x_2)) \leq_P F(d(x_1, x_2)))$, so $\tau \leq_P 0_E$, hence $\tau \in (-P)$. It follows that $\tau \in int(P) \cap (-P)$. On the other hand, $int(P) \cap (-P) = \emptyset$ (according to Lemma 2 (i)), consequently $\tau \in \emptyset$, which is in contradiction with $\tau \gg_P 0_E$. Therefore, *T* has at most one fixed point.

We choose $x_0 \in X$ be an arbitrary element and let us define the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ by the recurrence relation $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}^*$. To prove the existence of a fixed point of the operator T we distinguish the following cases:

I: there is $N \in \mathbb{N}^*$ such that $x_N = x_{N-1}$. We deduce $Tx_{N-1} = x_{N-1}$, thus x_{N-1} is a fixed point of *T*. Considering that *T* has at most one fixed point, we get $x^* := x_{N-1} \in X$ is the unique fixed point of *T*. Moreover, $x_n = x_{N-1}$ for all $n \ge N$, which means that the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ converges to x^* .

II: $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}^*$. Let us consider $n \in \mathbb{N}^*$ be an arbitrary element. Using the property (6) we find $\tau + F(d(Tx_{k-1}, Tx_k)) \leq_P F(d(x_{k-1}, x_k)), k = \overline{1, n}$, thus $\tau + F(d(x_k, x_{k+1})) \leq_P F(d(x_{k-1}, x_k)), k = \overline{1, n}$. Following Lemma 1 (iv) we deduce $n * \tau + \sum_{k=1}^n F(d(x_k, x_{k+1})) \leq_P \sum_{k=1}^n F(d(x_{k-1}, x_k))$, thus $n * \tau + F(d(x_n, x_{n+1})) \leq_P F(d(x_0, x_1))$. Therefore, we get

$$F(d(x_n, x_{n+1})) \leq_P F(d(x_0, x_1)) - n * \tau \text{ for all } n \in \mathbb{N}^*.$$

$$\tag{7}$$

On the other hand, $F \in \mathcal{F}'$, hence from Definition 11 (1) we obtain for every $\varepsilon \gg_P 0_E$ there is a number $n(\varepsilon, d(x_0, x_1), \tau) \in \mathbb{N}$ such that for all $n \ge n(\varepsilon, d(x_0, x_1), \tau)$ we have $F(d(x_0, x_1)) - n * \tau \ll_P -\varepsilon$. Consequently, from the previous affirmation and using the property (7), via Lemma 1 (iii) we find:

for every $\varepsilon \gg_P 0_E$ there is a number $n(\varepsilon, d(x_0, x_1), \tau) \in \mathbb{N}$ such that for all $n \ge n(\varepsilon, d(x_0, x_1), \tau)$ we have

$$F(d(x_n, x_{n+1})) \ll_P -\varepsilon.$$
(8)

Let us consider $n \in \mathbb{N}^*$ be an arbitrary number. Using the property (6) we get $\tau + F(d(Tx_{n-1}, Tx_n)) \leq_P F(d(x_{n-1}, x_n))$, so $F(d(x_n, x_{n+1})) \leq_P F(d(x_{n-1}, x_n)) - \tau$. Since $\tau \gg_P 0_E$ it follows that $F(d(x_n, x_{n+1})) \ll_P F(d(x_{n-1}, x_n))$. As $F \in \mathcal{F}'$, from Definition 11 (3) we obtain $d(x_n, x_{n+1}) \leq_P d(x_{n-1}, x_n)$, thus $(d(x_n, x_{n+1}))_{n \in \mathbb{N}}$ is a decreasing sequence. Considering Definition 11 (2) and the property (8), we deduce $d(x_n, x_{n+1}) \xrightarrow{n} 0_E$, so $d(x_{n-1}, x_n) \xrightarrow{n} 0_E$. In the following, we prove that $(x_n)_{n \in \mathbb{N}} \subset X$ is a Cauchy sequence. Let us suppose that $(x_n)_{n \in \mathbb{N}} \subset X$ is not a Cauchy sequence. Since for $x_0 \in X$ for which $(x_n)_{n \in \mathbb{N}} \subset X$ is not a Cauchy sequence, there is a sequentially compact subset $S_{x_0} \subseteq P$ having the property $\{d(x_m, x_n) \mid m, n \in \mathbb{N}^*, m < n\} \subseteq S_{x_0}$ and considering $d(x_{n-1}, x_n) \xrightarrow{n} 0_E$, using Lemma 4 (i), (ii) we deduce there exist $\varepsilon_0 \gg_P 0_E$, two subsequences $(x_m(k))_{k \in \mathbb{N}}$,

 $(x_{n(k)})_{k \in \mathbb{N}}$, where $m, n : \mathbb{N} \to \mathbb{N}$ are strictly increasing functions, with k < m(k) < n(k) for all $k \in \mathbb{N}$, and a point $x \in Fr(P)$ such that

$$\lim_{k \to +\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0 - x, \lim_{k \to +\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon_0 - x \text{ and } \varepsilon_0 - x \in P.$$
(9)

As $\varepsilon_0 \in int(P)$, $x \in Fr(P)$ and according to Lemma 2 (ii), $int(P) \cap Fr(P) = \emptyset$, we get $\varepsilon_0 - x \neq 0_E$. Considering relation (9), it follows that there is a number $K \in \mathbb{N}$ such that $d(x_{m(k)}, x_{n(k)}) \neq 0_E$ for all $k \ge K$, hence $x_{m(k)} \neq x_{n(k)}$ for all $k \ge K$. Using the property (6) we find $\tau + F(d(Tx_{m(k)-1}, Tx_{n(k)-1})) \le P F(d(x_{m(k)-1}, x_{n(k)-1}))$ for all $k \ge K$, hence

$$\tau + F(d(x_{m(k)}, x_{n(k)})) \le_P F(d(x_{m(k)-1}, x_{n(k)-1})) \text{ for all } k \ge K.$$
(10)

On the other hand, as $F \in \mathcal{F}'$, from Definition 11 (4) we have F is sequentially continuous on P^* , hence F is sequentially continuous at $\varepsilon_0 - x \in P^*$. Considering the property (9) we obtain

$$\lim_{k \to +\infty} F(d(x_{m(k)}, x_{n(k)})) = F(\varepsilon_0 - x), \lim_{k \to +\infty} F(d(x_{m(k)-1}, x_{n(k)-1})) = F(\varepsilon_0 - x).$$
(11)

Passing to the limit in inequality (10) and considering the relation (11), we deduce $\tau + F(\varepsilon_0 - x) \leq_P F(\varepsilon_0 - x)$, thus $\tau \leq_P 0_E$, so $\tau \in (-P)$. It follows that $\tau \in int(P) \cap (-P)$. On the other hand, $int(P) \cap (-P) = \emptyset$ (according to Lemma 2 (i)), therefore $\tau \in \emptyset$ which is in contradiction with $\tau \gg_P 0_E$. Consequently, $(x_n)_{n \in \mathbb{N}} \subset X$ is a Cauchy sequence. Since (X, d) is a complete cone metric space, we deduce that there exists an element $x^* \in X$ such that the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ converges to $x^* \in X$.

Further, we show that x^* is a fixed point of *T*. For this, we consider the set $U = \{n \in \mathbb{N} \mid x_n = Tx^*\}$ and we distinguish the following subcases:

II.1: *U* is an infinite set. We can choose a subsequence $(x_{m(k)})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, where $m : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, with k < m(k) for all $k \in \mathbb{N}$, such that $(x_{m(k)})_{k\in\mathbb{N}}$ converges to Tx^* . However, we show that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x^* . Considering Lemma 3 we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ has a unique limit, hence $Tx^* = x^*$, so x^* is a fixed point of *T*.

II.2: *H* is a finite set. We find that there exists a number $N \in \mathbb{N}$ such that $x_n \neq Tx^*$ for all $n \geq N$. Let us consider $n \geq N$ be an arbitrary element. Using the property (6) we obtain $\tau + F(d(Tx_{n-1}, Tx^*)) \leq_P F(d(x_{n-1}, x^*))$, thus $F(d(x_n, Tx^*)) \leq_P F(d(x_{n-1}, x^*)) - \tau$. Since $\tau \gg_P 0_E$ it follows that $F(d(x_n, Tx^*)) \ll_P F(d(x_{n-1}, x^*))$. As $F \in \mathcal{F}'$, from Definition 11 (3) we obtain

$$d(x_n, Tx^*) \leq_P d(x_{n-1}, x^*) \text{ for all } n \geq N.$$
(12)

Further, the properties of the cone metric *d* lead to

$$d(x^{\star}, Tx^{\star}) \leq_P d(x^{\star}, x_n) + d(x_n, Tx^{\star}) = d(x_n, x^{\star}) + d(x_n, Tx^{\star}) \text{ for all } n \in \mathbb{N}.$$
(13)

Taking into account the relations (12), (13) we get

$$d(x^{\star}, Tx^{\star}) \leq_{P} d(x_{n}, x^{\star}) + d(x_{n-1}, x^{\star}) \text{ for all } n \geq N.$$
(14)

We choose $\varepsilon \in int(P)$ be an arbitrary element. Since $(x_n)_{n \in \mathbb{N}} \subset X$ converges to $x^* \in X$, we deduce that there is a number $n(\varepsilon) \in \mathbb{N}$ such that for all $n \ge n(\varepsilon)$ we have $d(x_n, x^*) \ll_P \varepsilon$. It follows that

$$d(x_n, x^*) + d(x_{n-1}, x^*) \ll_P \varepsilon + \varepsilon \text{ for all } n \ge n(\varepsilon) + 1.$$
(15)

Let us consider $n \ge \max\{n(\varepsilon) + 1, N\}$ to be a natural number. Using the relations (14), (15) and taking into account Lemma 1 (iii) we obtain

$$d(x^{\star}, Tx^{\star}) \ll_P \varepsilon + \varepsilon.$$

Therefore, $d(x^*, Tx^*) \in P$ and $d(x^*, Tx^*) \ll_P \varepsilon + \varepsilon$ for all $\varepsilon \in int(P)$. Applying Lemma 2 (iii) it follows that $d(x^*, Tx^*) = 0_E$, thus $x^* = Tx^*$, hence x^* is a fixed point of *T*.

Consequently, in both subcases II.1 and II.2 we showed that $x^* \in X$ is a fixed point of *T*. Considering that *T* has at most one fixed point, we obtain that $x^* \in X$ is the unique fixed point of *T*. Also, we proved that $(x_n)_{n \in \mathbb{N}}$ converges to x^* . \Box

Corollary 1. Let $(\mathcal{A}, \oplus, \odot, \tau_{\mathcal{A}}, \preceq)$ be a partially ordered Hausdorff topological ring, with $1_{\mathcal{A}} \in \mathcal{A}_+$, $0_{\mathcal{A}}$ an accumulation point of \mathcal{A}_+ , $(E, +, \cdot, \tau_E)$ a Hausdorff topological left \mathcal{A} -module, P a solid cone of E, with $a \cdot int(P) \subseteq int(P)$ for all $a \in \mathcal{A}_+^*$, $0_E \notin int(P)$, and (X, d) a cone metric space over the topological left \mathcal{A} -module. We suppose that (X, d) is complete, $T : X \to X$ is an F-contraction corresponding to the function class \mathcal{F}' and for every $x_0 \in X$ we consider the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}^*$. If there is a sequentially compact subset $S \subseteq P$ having the property $\{d(Tx, Ty) \mid x, y \in X\} \subseteq S$, then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to x^* .

Proof. Let $x_0 \in X$ for which $(x_n)_{n \in \mathbb{N}} \subset X$ is not a Cauchy sequence, be an arbitrary element. Because for every $m, n \in \mathbb{N}^*, m < n$, we have $x_m = Tx_{m-1}, x_n = Tx_{n-1}$, it follows that $\{d(x_m, x_n) \mid m, n \in \mathbb{N}^*, m < n\} \subseteq \{d(Tx, Ty) \mid x, y \in X\}$. According to the hypothesis, there is a sequentially compact subset $S \subseteq P$ having the property $\{d(Tx, Ty) \mid x, y \in X\} \subseteq S$, hence $\{d(x_m, x_n) \mid m, n \in \mathbb{N}^*, m < n\} \subseteq S$. Therefore, for any $x_0 \in X$ for which $(x_n)_{n \in \mathbb{N}} \subset X$ is not a Cauchy sequence, there is a sequentially compact subset $S_{x_0} := S \subseteq P$ having the property $\{d(x_m, x_n) \mid m, n \in \mathbb{N}^*, m < n\} \subseteq S_{x_0}$. Consequently, the hypotheses of Theorem 1 are fulfilled, hence *T* has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ converges to x^* . \Box

In the following, we determine some conditions for the existence and uniqueness of a solution of the following integral equation

$$x(t) = \int_{a}^{t} K(t, s, x(s)) ds + f(t), \ t \in [a, b],$$
(16)

where $x : [a, b] \to \mathbb{R}^n$, $K : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, $f : [a, b] \to \mathbb{R}^n$, by using Corollary 1.

Theorem 2. If the functions $K : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$, $f : [a,b] \to \mathbb{R}^n$ are continuous and there is $\tau > 0$ such that

$$|K_i(t,s,x(s)) - K_i(t,s,y(s))| \le \frac{1}{b-a} \frac{|x_i(t) - y_i(t)|}{\tau d(x,y) + 1},$$
(17)

for all $t, s \in [a, b]$, $x, y \in C([a, b], \mathbb{R}^n)$, $i = \overline{1, n}$, then the integral Equation (16) has a unique solution in $C([a, b], \mathbb{R}^n)$.

Proof. Let $(\mathbb{R}, +, \cdot, \tau_{\mathbb{R}}, \leq)$ be the partially ordered Hausdorff topological ring and $(\mathbb{R}, +, \cdot, \tau_{\mathbb{R}})$ the Hausdorff topological left \mathbb{R} -module, where $\tau_{\mathbb{R}}$ is the Euclidean topology. We consider the solid cone $P = \mathbb{R}_+$, with $int(P) = \mathbb{R}^*_+$. We define the set $C([a, b], \mathbb{R}^n) = \{x : [a, b] \to \mathbb{R}^n \mid x \text{ is continuous on } [a, b]\}$ and the mapping

$$d: C([a,b],\mathbb{R}^n) \times C([a,b],\mathbb{R}^n) \to \mathbb{R}_+, \ d(x,y) = \|x-y\|, \tag{18}$$

where $\|\cdot\| : C([a,b],\mathbb{R}^n) \to \mathbb{R}_+, \|x\| = \max_{i=\overline{1,n}} \max_{t\in[a,b]} (|x_i(t)|e^{-\tau(t-a)})$. We remark that

 $(C([a, b], \mathbb{R}^n), d)$ is a complete cone metric space over the topological left \mathbb{R} -module. Next, we consider the function

$$F: \mathbb{R}^*_+ \to \mathbb{R}, \ F(x) = -\frac{1}{x}, \tag{19}$$

which belongs to the class \mathcal{F}' .

Since the functions $K : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $f : [a, b] \to \mathbb{R}^n$ are continuous, we can define the operator $T : C([a, b], \mathbb{R}^n) \to C([a, b], \mathbb{R}^n)$,

$$T(x)(t) = \int_{a}^{t} K(t, s, x(s))ds + f(t), \ t \in [a, b].$$
(20)

A function $x \in C([a, b], \mathbb{R}^n)$ is a solution of the integral Equation (16) if and only if it is a fixed point of the operator *T*.

Let $t, s \in [a, b]$, $x, y \in C([a, b], \mathbb{R}^n)$, $i = \overline{1, n}$, be arbitrary elements.

The definition of the operator *T* leads to

$$|T_i(x)(t) - T_i(y)(t)|$$

$$= \left| \int_a^t K_i(t,s,x(s)) ds + f_i(t) - \int_a^t K_i(t,s,y(s)) ds - f_i(t) \right|$$

$$= \left| \int_{a}^{t} (K_{i}(t,s,x(s)) - K_{i}(t,s,y(s))) ds \right| \leq \int_{a}^{t} |K_{i}(t,s,x(s)) - K_{i}(t,s,y(s))| ds.$$

Considering the hypothesis we find

$$|K_i(t,s,x(s)) - K_i(t,s,y(s))| \le \frac{1}{b-a} \frac{|x_i(t) - y_i(t)|}{\tau d(x,y) + 1}$$

$$= \frac{1}{b-a} \frac{|x_i(t) - y_i(t)|e^{-\tau(t-a)}e^{\tau(t-a)}}{\tau d(x,y) + 1} \le \frac{1}{b-a} \frac{\max_{t \in [a,b]} (|x_i(t) - y_i(t)|e^{-\tau(t-a)})e^{\tau(t-a)}}{\tau d(x,y) + 1}$$
$$\le \frac{1}{b-a} \frac{\max_{i=1,n} \max_{t \in [a,b]} (|x_i(t) - y_i(t)|e^{-\tau(t-a)})e^{\tau(t-a)}}{\tau d(x,y) + 1}$$
$$= \frac{1}{b-a} \frac{||x - y||e^{\tau(t-a)}}{\tau d(x,y) + 1} = \frac{1}{b-a} \frac{d(x,y)e^{\tau(t-a)}}{\tau d(x,y) + 1},$$

thus

$$\int_{a}^{t} |K_{i}(t,s,x(s)) - K_{i}(t,s,y(s))| ds \leq \int_{a}^{t} \frac{1}{b-a} \frac{d(x,y)e^{\tau(t-a)}}{\tau d(x,y)+1} ds$$
$$= \frac{1}{b-a} \frac{d(x,y)e^{\tau(t-a)}}{\tau d(x,y)+1} \int_{a}^{t} ds = \frac{1}{b-a} \frac{d(x,y)e^{\tau(t-a)}}{\tau d(x,y)+1} s \Big|_{a}^{t}$$

$$=\frac{1}{b-a}\frac{d(x,y)e^{\tau(t-a)}}{\tau d(x,y)+1}(t-a) \le \frac{1}{b-a}\frac{d(x,y)e^{\tau(t-a)}}{\tau d(x,y)+1}(b-a) = \frac{d(x,y)e^{\tau(t-a)}}{\tau d(x,y)+1}$$

From the previous relations we deduce that

$$|T_i(x)(t) - T_i(y)(t)| \le \frac{d(x,y)e^{\tau(t-a)}}{\tau d(x,y) + 1},$$

hence

$$|T_i(x)(t) - T_i(y)(t)|e^{-\tau(t-a)} \le \frac{d(x,y)}{\tau d(x,y) + 1},$$

so

$$\max_{i=\overline{1,n}} \max_{t\in[a,b]} (|T_i(x)(t) - T_i(y)(t)|e^{-\tau(t-a)}) \le \frac{d(x,y)}{\tau d(x,y) + 1},$$

thus

$$||T(x) - T(y)|| \le \frac{d(x,y)}{\tau d(x,y) + 1},$$

i.e.,

$$d(T(x), T(y)) \le \frac{d(x, y)}{\tau d(x, y) + 1}.$$
(21)

It follows that, for every $x, y \in C([a, b], \mathbb{R}^n)$, $T(x) \neq T(y)$, we have

$$\frac{\tau d(x,y)+1}{d(x,y)} \leq \frac{1}{d(T(x),T(y))}$$

thus

$$-\frac{1}{d(T(x),T(y))} \leq -\frac{1}{d(x,y)}.$$

Considering the relation (19), we find

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$$\tau + F(d(T(x), T(y))) \le F(d(x, y)), \text{ for all } x, y \in C([a, b], \mathbb{R}^n), T(x) \neq T(y),$$

hence the operator $T : C([a, b], \mathbb{R}^n) \to C([a, b], \mathbb{R}^n)$ is an *F*-contraction corresponding to the function class \mathcal{F}' .

Moreover, from the relation (21) we deduce

$$d(T(x), T(y)) < \frac{1}{\tau}$$
, for every $x, y \in C([a, b], \mathbb{R}^n)$,

thus there is a sequentially compact subset $S := [0, \frac{1}{\tau}] \subseteq \mathbb{R}_+$ having the property $\{d(Tx, Ty) \mid x, y \in X\} \subseteq S$.

Consequently, the hypotheses of Corollary 1 are satisfied, hence the operator *T* has a unique fixed point $x^* \in C([a, b], \mathbb{R}^n)$. Further, for every $x_0 \in C([a, b], \mathbb{R}^n)$ the sequence $(x_n)_{n \in \mathbb{N}} \subset C([a, b], \mathbb{R}^n)$, defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}^*$, converges to x^* . It follows that the integral Equation (16) has a unique solution in $C([a, b], \mathbb{R}^n)$. \Box

In the sequel, we will apply Corollary 1 to a fractional integral equation. Following the study of Fernandez et al. [26] we consider [a, b] a real interval, $\alpha \ge 0$, $\beta \ge 0$, R > 0 satisfying $R > (b - a)^{\beta}$ and A a real analytic function on the interval (-R, R), defined by the locally uniformly convergent power series

$$A(x) = \sum_{k=0}^{+\infty} a_k(\alpha, \beta) x^k,$$
(22)

where $a_k(\alpha, \beta)$, $k \in \mathbb{N}$, are real coefficients.

Next, we determine some conditions for the existence and uniqueness of a solution of the following fractional integral equation

$$x(t) = \int_{a}^{t} (t-s)^{\alpha+\gamma-1} A((t-s)^{\beta}) g(s, x(s)) ds + f(t), \ t \in [a, b],$$
(23)

where $\gamma \ge 1$, $x : [a, b] \to \mathbb{R}$, $g : [a, b] \times \mathbb{R} \to \mathbb{R}$, $f : [a, b] \to \mathbb{R}$, by using Corollary 1.

Considering the fractional integral operator ${}^{A}I_{a+}^{\alpha,\beta}: L^{1}[a.b] \to L^{1}[a.b],$

$${}^{A}I_{a+}^{\alpha,\beta}g(t) := \int_{a}^{t} (t-s)^{\alpha-1}A((t-s)^{\beta})g(s)ds, \ t \in [a,b],$$
(24)

defined by Fernandez et al. [26], the above fractional integral equation can be rewritten as

$$x(t) = {}^{A} I_{a+}^{\alpha+\gamma,\rho} g(t, x(t)) + f(t), \ t \in [a, b].$$
(25)

Theorem 3. *If the functions* $g : [a,b] \times \mathbb{R} \to \mathbb{R}$, $f : [a,b] \to \mathbb{R}$ *are continuous and there is* $\tau > 0$ *such that*

$$|(t-s)^{\alpha+\gamma-1}A((t-s)^{\beta})(g(s,x(s)) - g(s,y(s)))| \le \frac{1}{b-a} \frac{|x(t) - y(t)|}{\tau\rho(x,y) + 1},$$
(26)

for all $t, s \in [a, b], s \le t, x, y \in C([a, b], \mathbb{R})$, then the fractional integral Equation (23) has a unique solution in $C([a, b], \mathbb{R})$.

Proof. We consider $(\mathbb{R}, +, \cdot, \tau_{\mathbb{R}}, \leq)$ the partially ordered Hausdorff topological ring and $(\mathbb{R}, +, \cdot, \tau_{\mathbb{R}})$ the Hausdorff topological left \mathbb{R} -module, where $\tau_{\mathbb{R}}$ is the Euclidean topology. Let $P = \mathbb{R}_+$ be the solid cone, with $int(P) = \mathbb{R}^*_+$. We define the set $C([a, b], \mathbb{R}) = \{x : [a, b] \to \mathbb{R} \mid x \text{ is continuous on } [a, b]\}$ and the mapping

$$\rho: C([a,b],\mathbb{R}) \times C([a,b],\mathbb{R}) \to \mathbb{R}_+, \ \rho(x,y) = \max_{t \in [a,b]} (|x(t) - y(t)|e^{-\tau(t-a)}).$$
(27)

We remark that $(C([a, b], \mathbb{R}), \rho)$ is a complete cone metric space over the topological left \mathbb{R} -module. Moreover, the function

$$F: \mathbb{R}^*_+ \to \mathbb{R}, \ F(x) = -\frac{1}{x'}, \tag{28}$$

belongs to the class \mathcal{F}' .

As the functions $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are continuous, we can define the operator $T : C([a, b], \mathbb{R}) \to C([a, b], \mathbb{R})$,

$$T(x)(t) = \int_{a}^{t} (t-s)^{\alpha+\gamma-1} A((t-s)^{\beta}) g(s,x(s)) ds + f(t), \ t \in [a,b].$$
(29)

A function $x \in C([a, b], \mathbb{R})$ is a solution of the integral Equation (23) if and only if it is a fixed point of the operator *T*.

Considering the relation (26) we can easily prove that

$$\rho(T(x), T(y)) \le \frac{\rho(x, y)}{\tau \rho(x, y) + 1}, \text{ for all } x, y \in C([a, b], \mathbb{R}).$$
(30)

Therefore,

$$\tau - \frac{1}{\rho(T(x), T(y))} \le -\frac{1}{\rho(x, y)}, \text{ for all } x, y \in C([a, b], \mathbb{R}), \ T(x) \neq T(y).$$

and taking into account the relation (28) we deduce

$$\tau + F(\rho(T(x), T(y))) \le F(\rho(x, y)), \text{ for all } x, y \in C([a, b], \mathbb{R}), T(x) \neq T(y),$$

hence the operator $T : C([a, b], \mathbb{R}) \to C([a, b], \mathbb{R})$ is an *F*-contraction corresponding to the function class \mathcal{F}' .

Moreover, from the relation (30) we obtain

$$\rho(T(x), T(y)) < \frac{1}{\tau}, \text{ for every } x, y \in C([a, b], \mathbb{R}),$$

thus there exists a sequentially compact subset $S := [0, \frac{1}{\tau}] \subseteq \mathbb{R}_+$ having the property $\{\rho(Tx, Ty) \mid x, y \in X\} \subseteq S$.

Consequently, the hypotheses of Corollary 1 are satisfied, hence the operator *T* has a unique fixed point $x^* \in C([a, b], \mathbb{R})$. Further, for every $x_0 \in C([a, b], \mathbb{R})$ the sequence $(x_n)_{n \in \mathbb{N}} \subset C([a, b], \mathbb{R})$, defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}^*$, converges to x^* . It follows that, the integral Equation (23) has a unique solution in $C([a, b], \mathbb{R})$. \Box

Remark 2. A sufficient condition for fulfilling the inequality (26) is

$$|g(s,x(s)) - g(s,y(s))| \le \frac{1}{(b-a)^{\alpha+\gamma}M} \frac{|x(t) - y(t)|}{\tau \rho(x,y) + 1},$$
(31)

for all $t, s \in [a, b], s \leq t, x, y \in C([a, b], \mathbb{R})$, where $M := \sup_{x \in [0, (b-a)^{\beta}]} |A(x)|$.

Example 1. For $\beta = 0$, $A(x) = \frac{1}{\Gamma(\alpha)}$, the fractional integral operator ${}^{A}I_{a+}^{\alpha,\beta}$ becomes the original *Riemann–Liouville fractional integral operator* (*RL integral operator*)

$${}^{RL}I^{\alpha}_{a+}g(t) := \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1}g(s)ds = {}^{A}I^{\alpha,0}_{a+}g(t), \ t \in [a,b].$$
(32)

If the functions $g : [a, b] \times \mathbb{R} \to \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ are continuous and there is $\tau > 0$ such that

$$\frac{1}{\Gamma(\alpha)}|(t-s)^{\alpha+\gamma-1}(g(s,x(s))-g(s,y(s)))| \le \frac{1}{b-a}\frac{|x(t)-y(t)|}{\tau\rho(x,y)+1},$$
(33)

for all $t, s \in [a, b], s \leq t, x, y \in C([a, b], \mathbb{R})$, then the fractional integral equation

$$x(t) = {}^{RL} I_{a+}^{\alpha+\gamma} g(t, x(t)) + f(t), \ t \in [a, b],$$
(34)

has a unique solution in $C([a, b], \mathbb{R})$. A sufficient condition for fulfilling the inequality (33) is

$$|g(s, x(s)) - g(s, y(s))| \le \frac{\Gamma(\alpha)}{(b-a)^{\alpha+\gamma}} \frac{|x(t) - y(t)|}{\tau \rho(x, y) + 1},$$
(35)

for all $t, s \in [a, b], s \leq t, x, y \in C([a, b], \mathbb{R})$.

Example 2. For $\beta = \delta \in (0, 1]$, $A(x) = \frac{1}{\Gamma(\alpha)\delta^{\alpha}} \exp(\frac{\delta - 1}{\delta}x)$, the fractional integral operator ${}^{A}I_{a+}^{\alpha,\beta}$ becomes the generalized proportional fractional integral operator (GPF integral operator)

$${}^{GPF}I^{\alpha,\delta}_{a+}g(t) :=$$

$$\frac{1}{\Gamma(\alpha)\delta^{\alpha}}\int_{a}^{t}(t-s)^{\alpha-1}\exp(\frac{\delta-1}{\delta}(t-s))g(s)ds = {}^{A}I^{\alpha,\delta}_{a+}g(t), t \in [a,b].$$
(36)

If the functions $g : [a, b] \times \mathbb{R} \to \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ *are continuous and there is* $\tau > 0$ *such that*

$$\frac{1}{\Gamma(\alpha)\delta^{\alpha}}|(t-s)^{\alpha+\gamma-1}\exp(\frac{\delta-1}{\delta}(t-s)^{\delta})(g(s,x(s))-g(s,y(s)))| \leq \frac{1}{b-a}\frac{|x(t)-y(t)|}{\tau\rho(x,y)+1},$$
(37)

for all $t, s \in [a, b], s \leq t, x, y \in C([a, b], \mathbb{R})$, then the fractional integral equation

$$x(t) = {}^{GPF} I_{a+}^{\alpha+\gamma,\delta} g(t, x(t)) + f(t), \ t \in [a, b],$$
(38)

has a unique solution in $C([a, b], \mathbb{R})$. A sufficient condition for fulfilling the inequality (37) is

$$|g(s,x(s)) - g(s,y(s))| \le \frac{\Gamma(\alpha)\delta^{\alpha}}{(b-a)^{\alpha+\gamma}} \frac{|x(t) - y(t)|}{\tau\rho(x,y) + 1},$$
(39)

for all $t, s \in [a, b], s \leq t, x, y \in C([a, b], \mathbb{R})$.

4. Conclusions

In this paper, we define the notion of *F*-contraction for cone metric spaces over topological left modules and we establish some fixed point results for self-mappings satisfying a contractive condition of this type. Applications of the main result to the study of the existence and uniqueness of the solutions for certain types of integral equations were presented in the last part of the article, one of them being an integral equation of fractional type.

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