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Existence Results for Hilfer Fractional Differential Equations with Variable Coefficient

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Abstract: The aim of this paper is to establish the existence and uniqueness results for differential equations of Hilfer-type fractional order with variable coefficient. Firstly, we establish the equivalent Volterra integral equation to an initial value problem for a class of nonlinear fractional differential equations involving Hilfer fractional derivative. Secondly, we obtain the existence and uniqueness results for a class of Hilfer fractional differential equations with variable coefficient. We verify our results by providing two examples.

Keywords: fractional differential equation; Hilfer fractional derivative; variable coefficient

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1. Introduction

Fractional differential equation is an interesting research field. The reason is that it can be used to solve practical problems from the fields of science and engineering, such as physics, chemistry, electrodynamics of complex media, polymer rheology and so on ([1–6]). Recently the research of fractional differential equations has made great progress. In the literature, there are several definitions of fractional integrals and derivatives, the most popular definitions are in the sense of the Riemann–Liouville and Caputo derivatives. In 2000, Hilfer introduced a generalized Riemann–Liouville fractional derivative, the socalled Hilfer fractional derivative. Many authors studied the existence of solutions for fractional differential equations involving Hilfer fractional derivative (see [7–11]).

However, there are few studies on fractional differential equations involving Hilfer fractional derivative with variable coefficient. Due to the variable coefficient function, it is difficult to obtain the analytical solution of such equations ([12–18]). The representation of explicit solutions has become a problem, even for the case involving Riemann–Liouville or Caputo derivative, the problem is not completely solved. In [15], the authors considered the following linear Caputo fractional differential equation with constant coefficients and obtained solutions by Adomian decomposition method:

$$\sum_{i=0}^{n} \alpha_i \, {}^{L}D^{\beta_i} y_i(t) = f(t), \quad t \in [a,b].$$

In [16], using Neumann series for the corresponding Volterra integral equations and the generalized Mittag–Leffler functions, the authors obtained solutions of initial value problems of fractional differential equations with constant coefficients:

$${}^{L}D^{\beta_n} y_n(t) + \sum_{i=0}^{n-1} \alpha_i \, {}^{L}D^{\beta_i} y_i(t) = f(t), \quad t \in [a,b].$$

In [17], the authors investigated solutions around an ordinary point for linear homogeneous sequential Caputo fractional differential equations with variable coefficients:

$$ {}^{L}D^{\beta_n} y_n(t) + \sum_{i=0}^{n-1} \alpha_i(t)\, {}^{L}D^{\beta_i} y_i(t) = f(t), \quad t \in [a,b]. $$

In [18], the authors study the existence, uniqueness and stability of solutions of the implicit fractional differential equations as follows:

$$ \begin{cases} {}^{H}D_{0^+}^{\alpha,\beta} x(t) = f(t, {}^{H}D_{0^+}^{\alpha,\beta} x(t)), \quad t \in J := (0,T], \\ (I_{0^+}^{n-\gamma} x)^{(n-j)}(0+) = \sum\limits_{i=0}^{m} c_i x(\tau_i), \quad \alpha < \gamma = \alpha + \beta - \alpha\beta < 1, \quad j = 1, 2, \cdots, n, \end{cases} $$

where ${}^{H}D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative, $\alpha, \beta \in (0,1)$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

In [19], the authors study the following fractional differential equation with continuous variable coefficients and Hilfer fractional derivatives with respect to another function:

$$ \begin{cases} {}^{H}D_{0^+}^{\beta_0,\mu_0,\varphi} y(t) + \sum\limits_{i=1}^{m} \sigma_i(t)\, {}^{H}D_{0^+}^{\beta_i,\mu_i,\varphi} y(t) = g(t), \quad t \in [0, t_0],\ m \in \mathbb{N}, \\ \left(\dfrac{1}{\varphi'(t)} \dfrac{d}{dt}\right)^k (I_{0^+}^{(1-\mu_0)(n-\beta_0);\varphi} y)(0+) = a_k \in \mathbb{R}, \quad k = 0, 1, \cdots, n_0 - 1, \end{cases} $$

where $\beta_0 > \beta_1 > \cdots > \beta_m \geq 0$, $0 \leq \mu_i \leq 1$, σ_i, $g \in C[0, t_0]$ and n_i are non-negative integers satisfying $n_i - 1 < \beta_i < n_i$, $i = 0, 1, \cdots, m$.

In this paper, we consider the following initial value problem for Hilfer fractional differential equations with variable coefficient:

$$ \begin{cases} {}^{H}D_{0^+}^{\alpha,\beta} x(t) - \lambda(t) x(t) = f(t, x(t)), \quad t \in J := (0,T], \\ (I_{0^+}^{n-\gamma} x)^{(n-j)}(0+) = c_j, \quad \alpha < \gamma = \alpha + n\beta - \alpha\beta < n, \quad j = 1, 2, \cdots, n, \end{cases} \tag{1} $$

where $\alpha \in (n-1, n)$, $\beta \in (0,1)$, ${}^{H}D_{0^+}^{\alpha,\beta}$ is the Hilfer fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a function to be specified later. $I_{0^+}^{n-\gamma}$ is the left-sided Riemann–Liouville fractional integral of order $n - \gamma$.

Using new techniques, we prove the existence result under the weak assumptions for the variable coefficient $\lambda \in L^{\frac{1}{p}}$ and nonlinearity $f(t, x(t))$ (Theorem 6), the main advantage of our techniques is that we are able to consider a wide range of function spaces, and in which the Hilfer fractional derivative of f exists.

In many cases, a representation of the solution of variable coefficient fractional differential equations is still an open question. In this paper, we give the structure of solutions for variable coefficient fractional differential equations with the Hilfer-type fractional derivative and provide a better understanding of the structure of solutions of Hilfer-type fractional differential equations with variable coefficients. The results are new even for the special case: $\beta = 0$ or $\beta = 1$. Moreover, we find the explicit solutions for the linear Hilfer fractional differential equations with variable coefficient as follows:

$$ \begin{cases} {}^{H}D_{0^+}^{\alpha,\beta} x(t) - \lambda(t) x(t) = f(t), \quad t \in J := (0,T], \\ (I_{0^+}^{n-\gamma} x)^{(n-j)}(0+) = c_j, \quad \alpha < \gamma = \alpha + n\beta - \alpha\beta < n, \quad j = 1, 2, \cdots, n. \end{cases} $$

Under appropriate assumptions, we can prove the solution x is given by

$$ x(t) = \sum_{j=1}^{n} \sum_{k=0}^{\infty} \frac{c_j (I_{0^+}^{\alpha} \lambda(\cdot))^k t^{\gamma - j}}{\Gamma(\gamma - j + 1)} + \sum_{k=1}^{\infty} (I_{0^+}^{\alpha} \lambda(\cdot))^{k-1} (I_{0^+}^{\alpha} f)(t). $$

Furthermore, we obtain the existence of solutions for the problem (1) under the new assumptions for f, and we can find that the linear nonhomogeneous Hilfer fractional differential equations and the Hilfer fractional differential equations with constant coefficient are the special cases of our conclusion.

The article is partitioned as follows: In Section 2, the basic definitions and conclusions are presented. In Section 3, we present the equivalent Volterra integral equation. The main results are obtained in Section 4. The applications are shown in Section 5.

2. Preliminaries

In this paper, let $\Omega = [a,b]$ be a finite interval on $\mathbb{R}$. We denote by $L^p(\Omega,\mathbb{R})$ the Banach space of all Lebesgue measurable functions $l : \Omega \to \mathbb{R}$ with the norm $\|l\|_{L^p} = \left(\int_\Omega |l(t)|^p dt\right)^{\frac{1}{p}} < \infty$ and by $AC([a,b],\mathbb{R})$ the space of all the absolutely continuous functions defined on $[a,b]$. Moreover, we use the following notation:

$$AC^n([a,b],\mathbb{R}) = \{f; f \in C^{n-1}([a,b],\mathbb{R}) \text{ and } f^{(n-1)} \in AC([a,b],\mathbb{R})\}.$$

In particular, $AC^1([a,b],\mathbb{R}) = AC([a,b],\mathbb{R})$.

For $0 \le \nu < 1$, we denote the weighted spaces

$$C_\nu[a,b] := \{x : (a,b] \to \mathbb{R};\ (t-a)^\nu x(t) \in C([a,b],\mathbb{R})\} \quad \text{with the norm} \quad \|x\|_{C_\nu} = \max_{t\in[a,b]} (t-a)^\nu |x(t)|.$$

First, we recall some basic concepts and results which will be used in the sequel.

Definition 1 ([3,4])**.** *The left-sided fractional integral of order q for a function $x(t) \in L^1$ is defined by*

$$(I^q_{a^+} x)(t) \;=\; \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} x(s) ds, \quad t > a, \quad q > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 ([3,4])**.** *If $k(t) \in AC^n([a,b],\mathbb{R})$, then the left-sided Riemann–Liouville fractional derivative $({}^L D^q_{a^+} k)(t)$ of order q exists almost everywhere on $[a, b]$ and can be written as*

$$({}^L D^q_{a^+} k)(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-q-1} k(s) ds, \quad t > a,\ n-1 < q < n.$$

Lemma 1 ([4])**.** *If $\mu > 0$ and $\theta > 0$, then*

$$\begin{aligned}
[I^\mu_{a^+} (s-a)^{\theta-1}](t) &= \frac{\Gamma(\theta)}{\Gamma(\theta+\mu)} (t-a)^{\theta+\mu-1}, \\
\left({}^L D^\mu_{a^+} (s-a)^{\theta-1}\right)(t) &= \frac{\Gamma(\theta)}{\Gamma(\theta-\mu)} (t-a)^{\theta-\mu-1}, \\
\left({}^L D^\mu_{a^+} (s-a)^{\mu-j}\right)(t) &= 0, \quad j = 1, 2, \cdots, [\mu]+1. \qquad (2)
\end{aligned}$$

Lemma 2 ([4])**.** *If $k(t) \in L^p(a,b)(1 \le p \le \infty)$ and $\theta_1, \theta_2 > 0$, then the following relations hold:*

(i) $({}^L D^{\theta_1}_{a^+} I^{\theta_1}_{a^+} k)(t) = k(t)$ *a.e.* $t \in [a,b]$;

(ii) *For* $\theta_1 > n$, $(\frac{d^n}{dt^n} I^{\theta_1}_{a^+})k(t) = I^{\theta_1 - n}_{a^+} k(t)$.

(iii) *For almost every point* $t \in [a,b]$, $(I^{\theta_1}_{a^+} I^{\theta_2}_{a^+} k)(t) = (I^{\theta_1+\theta_2}_{a^+} k)(t)$.

Lemma 3 ([4])**.** *Let $\nu \in (n-1,n)(n \in \mathbb{N})$. If $y \in L^1(a,b)$ and $I_{a^+}^{n-\nu}y \in AC^n[a,b]$, then*

$$\left(I_{a^+}^{\nu}\, {}^{L}D_{a^+}^{\nu}y\right)(t) = y(t) - \sum_{j=1}^{n} \frac{(I_{a^+}^{n-\nu}y)^{(n-j)}(a+)}{\Gamma(\nu-j+1)}(t-a)^{\nu-j}$$

holds almost everywhere on $[a,b]$.

In [7], R. Hilfer studied a generalized fractional operator having the Riemann–Liouville and Caputo derivatives as specific cases.

Definition 3 ([4])**.** *The left-sided Hilfer fractional derivative of order $\alpha \in (n-1,n), \beta \in [0,1]$ of $y(t)$ is defined by:*

$${}^{H}D_{a^+}^{\alpha,\beta}y(t) = (I_{a^+}^{\beta(n-\alpha)}D^n I_{a^+}^{(1-\beta)(n-\alpha)}y)(t) = (I_{a^+}^{\gamma-\alpha}D^n I_{a^+}^{n-\gamma}y)(t), \quad t \in [a,b],$$

where $D := \frac{d}{dt}$, $\gamma = \alpha + n\beta - \alpha\beta$.

Remark 1. *(i) The operator ${}^{H}D_{a^+}^{\alpha,\beta}$ can be written as:*

$${}^{H}D_{a^+}^{\alpha,\beta} = I_{a^+}^{\beta(n-\alpha)}D^n(I_{a^+}^{(1-\beta)(n-\alpha)}) = I_{a^+}^{\beta(n-\alpha)}\, {}^{L}D_{a^+}^{\gamma} = I_{a^+}^{\gamma-\alpha}\, {}^{L}D_{a^+}^{\gamma}.$$

(ii) when $\beta = 0$, the left-sided Riemann–Liouville fractional derivative can be presented as ${}^{L}D_{a^+}^{\alpha} := {}^{H}D_{a^+}^{\alpha,0}$.

(iii) when $\beta = 1$, the left-sided Caputo fractional derivative can be presented as ${}^{C}D_{a^+}^{\alpha} := {}^{H}D_{a^+}^{\alpha,1}$.

From Lemma 2 (iii), we can obtain the following result.

Lemma 4. *Let $y \in L^1(a,b)$ and ${}^{L}D_{a^+}^{\gamma-\alpha}y \in L^1(a,b)$ exists, then*

$$\left({}^{H}D_{a^+}^{\alpha,\beta}I_{a^+}^{\alpha}y\right)(t) = I_{a^+}^{\gamma-\alpha}\, {}^{L}D_{a^+}^{\gamma-\alpha}y(t).$$

Proof.

$$\left({}^{H}D_{a^+}^{\alpha,\beta}I_{a^+}^{\alpha}y\right)(t) = I_{a^+}^{\gamma-\alpha}D^n I_{a^+}^{n-\gamma}I_{a^+}^{\alpha}y(t) = I_{a^+}^{\gamma-\alpha}D^n I_{a^+}^{n+\alpha-\gamma}y(t) = I_{a^+}^{\gamma-\alpha}\, {}^{L}D_{a^+}^{\gamma-\alpha}y(t).$$

□

From (2), we can derive the following result.

Lemma 5. *For $\gamma = \alpha + n\beta - \alpha\beta \in (n-1,n)$, a general solution of the fractional differential equation ${}^{H}D_{a^+}^{\alpha,\beta}x(t) = 0$ is given by*

$$x(t) = \sum_{i=1}^{n} c_i(t-a)^{\gamma-i}, \quad t > a,$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$ are arbitrary constants.

Next, we present the following lemmas.

Lemma 6. *If $\omega > 0$, $-1 < \tau \leq 0$, then for $\psi \in L^{\frac{1}{p}}(0 < p < 1)$,*

(i) $$\int_a^t (t-s)^{\omega-1}\psi(s)ds \leq \left(\frac{1-p}{\omega-p}\right)^{1-p}(t-a)^{\omega-p}\|\psi\|_{L^{\frac{1}{p}}}, \quad \text{for} \quad \omega > p, \quad t > a,$$

(ii) $\int_0^t (t-s)^{\omega-1} s^{\tau}\psi(s)ds \leq \left(B\left(\frac{\omega-p}{1-p}, \frac{\tau+1-p}{1-p}\right)\right)^{1-p} t^{\omega+\tau-p}\|\psi\|_{L^{\frac{1}{p}}}$, *for* $p < \min\{\omega, 1+\tau\}$, $t > 0$,

where $B(\cdot,\cdot)$ *is the Beta function.*

Proof. With the help of Hölder's inequality, we get

$$\int_a^t (t-s)^{\omega-1}\psi(s)ds \leq \left(\int_a^t (t-s)^{\frac{\omega-1}{1-p}}ds\right)^{1-p}\|\psi\|_{L^{\frac{1}{p}}} = \left(\frac{1-p}{\omega-p}\right)^{1-p}(t-a)^{\omega-p}\|\psi\|_{L^{\frac{1}{p}}}, \quad \text{for} \quad \omega > p,\ t > a.$$

$$\begin{aligned}\int_0^t (t-s)^{\omega-1}s^{\tau}\psi(s)ds &\leq \left(\int_0^t (t-s)^{\frac{\omega-1}{1-p}}s^{\frac{\tau}{1-p}}ds\right)^{1-p}\|\psi\|_{L^{\frac{1}{p}}} \\ &= \left(B\left(\frac{\omega-p}{1-p}, \frac{1+\tau-p}{1-p}\right)\right)^{1-p} t^{\omega+\tau-p}\|\psi\|_{L^{\frac{1}{p}}}, \quad \text{for} \quad p < \min\{\omega, 1+\tau\},\ t > 0.\end{aligned}$$

□

Lemma 7. *If* $\omega \in (0,1)$, $\tau \in (-1,0]$ *and* $\omega+\tau > 0$, *then for* $0 < p < \frac{\omega+\tau}{2}$ *and* $y \in L^{\frac{1}{p}}$, $(I_{0^+}^{\omega}s^{\tau}y(s))(t) \in AC([0,T],\mathbb{R})$.

Proof. For every finite collection $\{(a_i, b_i)\}_{1\leq i\leq n}$ on J with $\sum\limits_{i=1}^{n}(b_i - a_i) \to 0$, it follows from Hölder's inequality that

$$\begin{aligned}
&\sum_{i=1}^{n}\left|(I_{0^+}^{\omega}s^{\tau}y(s))(b_i) - (I_{0^+}^{\omega}s^{\tau}y(s))(a_i)\right| \\
\leq\ & \frac{1}{\Gamma(\omega)}\left\{\sum_{i=1}^{n}\left|\int_0^{a_i}[(b_i-s)^{\omega-1}-(a_i-s)^{\omega-1}]s^{\tau}y(s)ds\right| + \sum_{i=1}^{n}\int_{a_i}^{b_i}(b_i-s)^{\omega-1}s^{\tau}|y(s)|ds\right\} \\
\leq\ & \frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\omega)}\left\{\sum_{i=1}^{n}\left[\int_0^{a_i}|(b_i-s)^{\omega-1}-(a_i-s)^{\omega-1}|^{\frac{1}{1-p}}s^{\frac{\tau}{1-p}}ds\right]^{1-p} + \sum_{i=1}^{n}\left[\int_{a_i}^{b_i}(b_i-s)^{\frac{\omega-1}{1-p}}s^{\frac{\tau}{1-p}}ds\right]^{1-p}\right\} \\
\leq\ & \frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\omega)}\left\{(1-\omega)\sum_{i=1}^{n}\left(\int_0^{a_i}\left|\int_{a_i}^{b_i}(\zeta-s)^{\omega-2}d\zeta\right|^{\frac{1}{1-p}}s^{\frac{\tau}{1-p}}ds\right)^{1-p}\right. \\
&\left. + \left(B\left(\frac{\omega-p}{1-p}, \frac{\tau+1-p}{1-p}\right)\right)^{1-p}\sum_{i=1}^{n}(b_i-a_i)^{\omega+\tau-p}\right\} \\
\leq\ & \overline{M}\sum_{i=1}^{n}\left[\int_0^{a_i}((a_i-s)^{\theta}-(b_i-s)^{\theta})s^{\frac{\tau}{1-p}}ds\right]^{1-p} \\
&+\frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\omega)}\left(B\left(\frac{\omega-p}{1-p}, \frac{\tau+1-p}{1-p}\right)\right)^{1-p}\sum_{i=1}^{n}(b_i-a_i)^{\omega+\tau-p} \\
=\ & \overline{M}\sum_{i=1}^{n}\left\{(a_i^{\theta+\frac{\tau+1-p}{1-p}} - b_i^{\theta+\frac{\tau+1-p}{1-p}})B\left(\theta+1, \frac{\tau+1-p}{1-p}\right)\right. \\
&\left.+b_i^{\theta+\frac{\tau+1-p}{1-p}}\left[B\left(\theta+1, \frac{\tau+1-p}{1-p}\right) - \int_0^{\frac{a_i}{b_i}}(1-\mu)^{\theta}\mu^{\frac{\tau}{1-p}}d\mu\right]\right\}^{1-p} \\
&+\frac{\|y\|_{L^{\frac{1}{p}}}}{\Gamma(\omega)}\left(B\left(\frac{\omega-p}{1-p}, \frac{\tau+1-p}{1-p}\right)\right)^{1-p}\sum_{i=1}^{n}(b_i-a_i)^{\omega+\tau-p} \\
\to\ & 0,
\end{aligned}$$

where $\overline{M} > 0$ is a constant and $\theta = \dfrac{\omega - 1 - p}{1 - p} \in (-1, 0)$. Now, $(I_{0^+}^{\omega} s^{\tau} y(s))(t) \in AC([0, T], \mathbb{R})$. □

The following theorem will be used to prove the main result.

Theorem 1 ([20] Schauder Fixed Point Theorem)**.** *If U is a nonempty, closed, bounded convex subset of a Banach space X and $T : U \to U$ is completely continuous, then T has a fixed point in U.*

3. Equivalent Volterra Integral Equation

We consider the following linear Hilfer fractional differential equation with initial value conditions

$$\begin{cases} {}^{H}D_{0^+}^{\alpha,\beta} x(t) = f(t), \quad t \in J := (0, T], \\ (I_{0^+}^{n-\gamma} x)^{(n-j)}(0+) = c_j, \quad \alpha < \gamma = \alpha + n\beta - \alpha\beta < n, \quad j = 1, 2, \cdots, n, \end{cases} \tag{3}$$

where $f \in L^{\frac{1}{p}}(J)(0 < p < \frac{1-\gamma+\alpha}{2})$.

Definition 4. *A function $x \in C_{n-\gamma}$ satisfying (3) is called a solution of (3).*

Theorem 2. *A function $x \in C_{n-\gamma}$ satisfies a.e. (3) if and only if x satisfies a.e. the following integral equation*

$$x(t) = \sum_{j=1}^{n} \frac{c_j t^{\gamma-j}}{\Gamma(\gamma - j + 1)} + (I_{0^+}^{\alpha} f)(t). \tag{4}$$

Proof. (Necessity) For $t \in J$, $\gamma - \alpha \in (0, 1)$, it follows from Lemma 7 ($\tau = 0$) that $I_{0^+}^{1-(\gamma-\alpha)} f \in AC[0, T]$ and ${}^{L}D_{0^+}^{\gamma-\alpha} f \in L^1(J)$. From Lemmas 3 and 4, we arrive at

$$({}^{H}D_{0^+}^{\alpha,\beta} I_{0^+}^{\alpha} f)(t) = (I_{0^+}^{\gamma-\alpha} \, {}^{L}D_{0^+}^{\gamma-\alpha} f)(t) = f(t) - \frac{(I_{0^+}^{1-\gamma+\alpha} f)(0+)}{\Gamma(\gamma - \alpha)} t^{\gamma-\alpha-1}, \quad a.e. \quad t \in [0, T],$$

then

$${}^{H}D_{0^+}^{\alpha,\beta} \Big(x(t) - (I_{0^+}^{\alpha} f)(t) \Big) - \frac{(I_{0^+}^{1-\gamma+\alpha} f)(0+)}{\Gamma(\gamma - \alpha)} t^{\gamma-\alpha-1} = 0.$$

Since $(I_{0^+}^{1-\gamma+\alpha} f)(0+) = 0$, then

$${}^{H}D_{0^+}^{\alpha,\beta} \Big(x(t) - (I_{0^+}^{\alpha} f)(t) \Big) = 0.$$

Using Lemma 5, we find

$$x(t) \;=\; \sum_{j=1}^{n} d_j t^{\gamma-j} + (I_{0^+}^{\alpha} f)(t).$$

Noting that the initial value conditions $(I_{0^+}^{n-\gamma} x)^{(n-j)}(0+) = c_j (j = 1, 2, \cdots, n)$, we can obtain $d_j = \Gamma(\gamma - j + 1)(I_{0^+}^{n-\gamma} x)^{(n-j)}(0+) = c_j (j = 1, 2, \cdots, n)$.

(Sufficiency) Let $x(t)$ satisfy (4). It follows from Lemmas 1, 3 and 7 that ${}^{H}D_{0^+}^{\alpha,\beta} x(t)$ exists and $x(t)$ satisfies (3) by direct computation. This completes the proof. □

Next, we consider the following nonlinear Hilfer fractional differential equation with initial value conditions

$$\begin{cases} {}^{H}D_{0^+}^{\alpha,\beta} x(t) = f(t, x(t)), \quad t \in J := (0, T], \\ (I_{0^+}^{n-\gamma} x)^{(n-j)}(0+) = c_j, \quad \alpha < \gamma = \alpha + n\beta - \alpha\beta < n, \quad j = 1, 2, \cdots, n. \end{cases} \tag{5}$$

Definition 5. *A function $x \in C_{n-\gamma}$ satisfying (5) is called a solution of (5).*

Theorem 3. *Let $f : (0,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, x(\cdot)) \in L^{\frac{1}{p}}(J, \mathbb{R})(0 < p < \frac{1-\gamma+\alpha}{2})$ for any $x \in C_{n-\gamma}([0, T], \mathbb{R})$. If $x(t) \in C_{n-\gamma}([0, T], \mathbb{R})$, then x satisfies a.e. (5) if and only if x satisfies a.e. the following equation*

$$x(t) = \sum_{j=1}^{n} \frac{c_j t^{\gamma-j}}{\Gamma(\gamma-j+1)} + \int_0^t \frac{(t-s)^{\alpha-1} f(s, x(s))}{\Gamma(\alpha)} ds.$$

The proof of Theorem 3 is nearly same as Theorem 2, we omit it here.

4. Existence and Uniqueness Result

Theorem 4. *Let $f : (0,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, x(\cdot)) \in L^{\frac{1}{p}}(J, \mathbb{R})(n-\gamma < p < \frac{1+\alpha-n}{2})$ for any $x \in C_{n-\gamma}([0, T], \mathbb{R})$ and there exists a function $\mu \in L^{\frac{1}{p}}(J, \mathbb{R}^+)(n-\gamma < p < \frac{1+\alpha-n}{2})$ such that*

$$|f(t, u(t)) - f(t, v(t))| \leq \mu(t)|u(t) - v(t)| \tag{6}$$

and $M_f := \sup_{t \in J} |f(t, 0)| < \infty$. Then the problem (5) has a unique solution $x \in C_{n-\gamma}$.

Proof. Clearly, if $x \in C_{n-\gamma}$, then

$$\begin{aligned} |f(t, x(t))| &\leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \leq \mu(t)|x(t)| + M_f, \\ t^{n-\gamma}|f(t, x(t))| &\leq \mu(t) t^{\gamma-n} \|x\|_{C_{n-\gamma}} + t^{n-\gamma} M_f. \end{aligned}$$

Applying with Lemma 7 (with ω and τ replaced by $1-\gamma+\alpha$ and $\gamma-n$, respectively), we can verify that $(I_{0^+}^{1-(\gamma-\alpha)} f)(t) \in AC[0, T]$ and $({}^L D_{0^+}^{\gamma-\alpha} f)(t) \in L^1(J)$ for any $x \in C_{n-\gamma}$.

Define an operator $\mathcal{F} : C_{n-\gamma} \to C_{n-\gamma}$ as

$$(\mathcal{F}x)(t) = \sum_{j=1}^{n} \frac{c_j t^{\gamma-j}}{\Gamma(\gamma-j+1)} + (I_{0^+}^{\alpha} F_x)(t),$$

where $F_x(t) = f(t, x(t))$. Clearly, $\mathcal{F}$ is well defined and the fixed point of $\mathcal{F}$ is the solution of the problem (5).

It is easy to check that

$$\begin{aligned} t^{n-\gamma}|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| &\leq \frac{t^{n-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\mu(s)||x(s) - y(s)| ds \\ &\leq \frac{t^{n-\gamma}}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-p}} s^{\frac{\gamma-n}{1-p}} ds \right)^{1-p} \cdot \|\mu\|_{L_{\frac{1}{p}}} \|x - y\|_{C_{n-\gamma}} \\ &\leq \frac{t^{\alpha-p}}{\Gamma(\alpha)} \left[B(\frac{\alpha-p}{1-p}, \frac{\gamma-n+1-p}{1-p}) \right]^{1-p} \cdot \|\mu\|_{L_{\frac{1}{p}}} \|x - y\|_{C_{n-\gamma}} \end{aligned}$$

and

$$
\begin{aligned}
& t^{n-\gamma}|(\mathcal{F}^2x)(t)-(\mathcal{F}^2y)(t)| \\
\leq\ & t^{n-\gamma}I_{0+}^{\alpha}|\mu(s)|s^{\gamma-n}\Big[s^{n-\gamma}|(\mathcal{F}x)(s)-(\mathcal{F}y)(s)|\Big](t) \\
\leq\ & \frac{t^{n-\gamma}}{\Gamma^2(\alpha)}\int_0^t(t-s)^{\alpha-1}|\mu(s)|s^{\gamma-n}s^{\alpha-p}ds\Big[B(\frac{\alpha-p}{1-p},\frac{\gamma-n+1-p}{1-p})\Big]^{1-p}\cdot\|\mu\|_{L_{\frac{1}{p}}}\|x-y\|_{C_{n-\gamma}} \\
\leq\ & \frac{t^{2(\alpha-p)}}{\Gamma^2(\alpha)}\Big[B(\frac{\alpha-p}{1-p},\frac{\gamma-n+1-p+\alpha-p}{1-p})\Big]^{1-p}\Big[B(\frac{\alpha-p}{1-p},\frac{\gamma-n+1-p}{1-p})\Big]^{1-p}\cdot\|\mu\|^2_{L_{\frac{1}{p}}}\|x-y\|_{C_{n-\gamma}} \\
=\ & \frac{t^{2(\alpha-p)}}{\Gamma^2(\alpha)}\Big[\frac{\Gamma^2(\frac{\alpha-p}{1-p})\Gamma(\frac{\gamma-n}{1-p}+1)}{\Gamma(\frac{\gamma-n+2(\alpha-p)}{1-p}+1)}\Big]^{1-p}\cdot\|\mu\|^2_{L_{\frac{1}{p}}}\|x-y\|_{C_{n-\gamma}}.
\end{aligned}
$$

By induction we deduce that

$$
t^{n-\gamma}|(\mathcal{F}^kx)(t)-(\mathcal{F}^ky)(t)| \leq \frac{T^{k(\alpha-p)}}{\Gamma^k(\alpha)}\Big[\frac{\Gamma^k(\frac{\alpha-p}{1-p})\Gamma(\frac{\gamma-n}{1-p}+1)}{\Gamma(\frac{\gamma-n+k(\alpha-p)}{1-p}+1)}\Big]^{1-p}\cdot\|\mu\|^k_{L_{\frac{1}{p}}}\|x-y\|_{C_{n-\gamma}},
$$

for k large enough one has $\frac{T^{k(\alpha-p)}}{\Gamma^k(\alpha)}\Big[\frac{\Gamma^k(\frac{\alpha-p}{1-p})\Gamma(\frac{\gamma-n}{1-p}+1)}{\Gamma(\frac{\gamma-n+k(\alpha-p)}{1-p}+1)}\Big]^{1-p}\cdot\|\mu\|^k_{L_{\frac{1}{p}}}<1$ and by Banach contraction principle, $\mathcal{F}$ has a unique fixed point $x\in C_{n-\gamma}$. □

Remark 2. *Obviously, if* $n-\gamma<p<\frac{1-\alpha+n}{2}$*, then the relation* $C_{n-\gamma}\subset L_{\frac{1}{p}}$ *holds, but the converse may not be true, for example,* $n=1$*,* $\alpha=0.9$*,* $\beta=0.1$*,* $\gamma=\alpha+\beta-\alpha\beta=0.91$*,* $t^{-\frac{1}{10}}\in L^9(0,T)$ *but* $t^{-\frac{1}{10}}\notin C_{0.09}[0.T]$*.*

Special Case I. If we take $f(t,x(t))=\lambda(t)x(t)+f(t)$, obviously, $|f(t,x(t))-f(t,y(t))|\leq|\lambda(t)||x(t)-y(t)|$, then the following result holds.

Corollary 1. *Let* $\lambda(t)\in C([0,T])$ *and* $f(t)\in L^{\frac{1}{p}}(J)(n-\gamma<p<\frac{1+\alpha-n}{2})$ *and* $\sup\limits_{t\in J}|f(t)|<\infty$*, then the linear Hilfer fractional differential equations with variable coefficient*

$$
\begin{cases} {}^HD_{0+}^{\alpha,\beta}x(t)-\lambda(t)x(t)=f(t), \quad t\in J:=(0,T], \\ (I_{0+}^{n-\gamma}x)^{(n-j)}(0+)=c_j, \quad \alpha<\gamma=\alpha+n\beta-\alpha\beta<n, \quad j=1,2,\cdots,n \end{cases} \tag{7}
$$

has a unique solution $x\in C_{n-\gamma}$ *and* x *is given by*

$$
x(t)=\sum_{j=1}^n\sum_{k=0}^\infty\frac{c_j(I_{0+}^\alpha\lambda(\cdot))^kt^{\gamma-j}}{\Gamma(\gamma-j+1)}+\sum_{k=1}^\infty(I_{0+}^\alpha\lambda(\cdot))^{k-1}(I_{0+}^\alpha f)(t).
$$

Proof. By Theorem 4, we can see that the solution of the problem (7) is given by

$$
x(t)=\sum_{j=1}^n\frac{c_jt^{\gamma-j}}{\Gamma(\gamma-j+1)}+(I_{0+}^\alpha\lambda(\cdot)x)(t)+(I_{0+}^\alpha f)(t).
$$

It follows from Theorem 4 that the operator $\mathcal{F}:C_{n-\gamma}\to C_{n-\gamma}$ defined as

$$
(\mathcal{F}x)(t)=\sum_{j=1}^n\frac{c_jt^{\gamma-j}}{\Gamma(\gamma-j+1)}+(I_{0+}^\alpha\lambda(\cdot)x)(t)+(I_{0+}^\alpha f)(t)
$$

has a unique fixed point which is the solution of the problem (7).

Let

$$\begin{cases} x_0(t) = \sum_{j=1}^{n} \frac{c_j t^{\gamma-j}}{\Gamma(\gamma-j+1)}, \\ x_m(t) = x_0(t) + (I_{0^+}^{\alpha}\lambda(\cdot)x_{m-1})(t) + (I_{0^+}^{\alpha}f)(t), \end{cases}$$

iterating, for $m \in \mathbb{N}$, we can write

$$x_m(t) \quad = \quad \sum_{k=0}^{m} (I_{0^+}^{\alpha}\lambda(\cdot))^k x_0(t) + \sum_{k=1}^{m} (I_{0^+}^{\alpha}\lambda(\cdot))^{k-1}(I_{0^+}^{\alpha}f)(t).$$

Taking the limit as $m \to \infty$, we can get

$$x(t) \quad = \quad \sum_{j=1}^{n}\sum_{k=0}^{\infty} \frac{c_j(I_{0^+}^{\alpha}\lambda(\cdot))^k t^{\gamma-j}}{\Gamma(\gamma-j+1)} + \sum_{k=1}^{\infty} (I_{0^+}^{\alpha}\lambda(\cdot))^{k-1}(I_{0^+}^{\alpha}f)(t).$$

□

Special Case II. In particular, when $\lambda(t) \equiv \lambda \in \mathbb{R}$ is a constant, we derive the following result.

Corollary 2. *Let $f(t) \in L^{\frac{1}{p}}(J)(n-\gamma < p < \frac{1+\alpha-n}{2})$ and $\sup_{t\in J}|f(t)| < \infty$. Then the initial value problem*

$$\begin{cases} {}^{H}D_{0^+}^{\alpha,\beta}x(t) - \lambda x(t) = f(t), \quad t \in J := (0,T], \quad \lambda \in \mathbb{R} \\ (I_{0^+}^{n-\gamma}x)^{(n-j)}(0+) = c_j, \quad \alpha < \gamma = \alpha + n\beta - \alpha\beta < n, \quad j = 1,2,\cdots,n \end{cases} \tag{8}$$

has a unique solution $x \in C_{n-\gamma}$ and x is given by

$$x(t) \quad = \quad \sum_{j=1}^{n} c_j t^{\gamma-j} E_{\alpha,\gamma-j+1}(\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) f(s) ds,$$

where

$$E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k+\nu)}, \quad \mu, \nu, z \in \mathbb{C}, Re(\mu) > 0$$

is a Mittag–Leffler function ([4]).

Proof. It follows from Theorem 4 that the problem (8) has a unique solution $x \in C_{n-\gamma}$ and the solution x is given by the limit $x(t) = \lim_{m\to+\infty} x_m(t)$ of the sequence

$$\begin{cases} x_0(t) = \sum_{j=1}^{n} \frac{c_j t^{\gamma-j}}{\Gamma(\gamma-j+1)}, \\ x_m(t) = x_0(t) + \lambda(I_{0^+}^{\alpha}x_{m-1})(t) + (I_{0^+}^{\alpha}f)(t). \end{cases}$$

Iterating, for $m \in \mathbb{N}$, we arrive at

$$\begin{aligned} x_m(t) \quad &= \quad \sum_{k=0}^{m} \lambda^k I_{0^+}^{\alpha k} x_0(t) + \sum_{k=1}^{m} \lambda^{k-1} I_{0^+}^{\alpha k} f(t) \\ &= \quad \sum_{j=1}^{n}\sum_{k=0}^{m} \frac{\lambda^k c_j}{\Gamma(\gamma-j+1)} I_{0^+}^{\alpha k} t^{\gamma-j} + \int_0^t \Big[\sum_{k=1}^{m} \frac{\lambda^{k-1}(t-s)^{\alpha k-1}}{\Gamma(\alpha k)}\Big] f(s) ds \\ &= \quad \sum_{j=1}^{n}\sum_{k=0}^{m} \frac{\lambda^k c_j t^{\gamma-j+\alpha k+1}}{\Gamma(\gamma-j+\alpha k)} + \int_0^t \Big[\sum_{k=1}^{m} \frac{\lambda^{k-1}(t-s)^{\alpha k-1}}{\Gamma(\alpha k)}\Big] f(s) ds. \end{aligned}$$

Taking the limit as $m \to \infty$, we have

$$\begin{aligned} x(t) &= \lim_{m\to\infty} x_m(t) = \sum_{j=1}^{n}\sum_{k=0}^{\infty} \frac{\lambda^k c_j t^{\gamma-j+\alpha k}}{\Gamma(\gamma-j+\alpha k+1)} + \int_0^t \Big[\sum_{k=1}^{\infty} \frac{\lambda^{k-1}(t-s)^{\alpha k-1}}{\Gamma(\alpha k)}\Big] f(s)ds \\ &= \sum_{j=1}^{n} c_j t^{\gamma-j} E_{\alpha,\gamma-j+1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) f(s)ds. \end{aligned}$$

This yields the explicit solution to the problem (8). □

Next, we deal with the existence of solutions to the problem (1). We consider the following assumptions.

(H1) $f : (0, T] \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot)) \in L^{\frac{1}{p}}(J, \mathbb{R})(n - \gamma < p < \frac{1+\alpha-n}{2})$ for any $x \in C_{n-\gamma}([0, T], \mathbb{R})$ and $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in J$;

(H2) there exists a function $\mu \in L^{\frac{1}{p_1}}(J, \mathbb{R}^+)(n - \gamma < p_1 < \frac{1+\alpha-n}{2})$ such that

$$|f(t, u)| \le \mu(t)|u(t)|^\sigma, \quad \sigma \in (0, 1).$$

Theorem 5. *Assume that $\lambda \in L^{\frac{1}{p_2}}(n - \gamma < p_2 < \frac{1+\alpha-n}{2})$ and (H1) and (H2) are satisfied, a function $x \in C_{n-\gamma}$ is a solution of (1) if and only if x is a solution of the following equation*

$$x(t) = \sum_{j=1}^{n} \frac{c_j t^{\gamma-j}}{\Gamma(\gamma-j+1)} + (I^\alpha_{0^+}\lambda(\cdot)x)(t) + (I^\alpha_{0^+}F_x)(t).$$

Proof. Clearly, if $x \in C_{n-\gamma}$, then

$$t^{n-\gamma}|f(t, x(t))| \;\; \le \;\; \mu(t)t^{\sigma(\gamma-n)}\|x\|^\sigma_{C_{n-\gamma}}. \tag{9}$$

According to Lemma 7 (with ω and τ replaced by $1 - \gamma + \alpha$ and $\sigma(\gamma - n)$, respectively), we can see that $I^{1-(\gamma-\alpha)}_{0^+}(\lambda(\cdot)x + F_x)(t) \in AC[0, T]$ and ${}^L D^{\gamma-\alpha}_{a^+}(\lambda(\cdot)x + F_x)(t) \in L^1(J)$ for any $x \in C_{n-\gamma}$. Then the above result can be proved in a similar way as in the proof of Theorem 2. □

For the sake of convenience, we set two constants

$$\begin{aligned} \Lambda_1 &:= \left(B\left(\frac{\alpha-p_2}{1-p_2}, \frac{\gamma-n+1-p_2}{1-p_2}\right)\right)^{1-p_2} \frac{\|\lambda\|_{L^{\frac{1}{p_2}}}}{\Gamma(\alpha)}, \\ \Lambda_2 &:= \left(B\left(\frac{\alpha-p_1}{1-p_1}, \frac{\sigma(\gamma-n)+1-p_1}{1-p_1}\right)\right)^{1-p_1} \frac{\|\mu\|_{L^{\frac{1}{p_1}}}}{\Gamma(\alpha)}. \end{aligned}$$

Theorem 6. *Assume that $\lambda \in L^{\frac{1}{p_2}}(n - \gamma < p_2 < \frac{1+\alpha-n}{2})$ and (H1) and (H2) are satisfied, then the problem (1) has at least a solution $x \in C_{n-\gamma}$ if $\Lambda_1 T^{\alpha-p_2} < 1$.*

Proof. We define an operator $\mathcal{F} : C_{n-\gamma} \to C_{n-\gamma}$ as

$$(\mathcal{F}x)(t) = \sum_{j=1}^{n} \frac{c_j t^{\gamma-j}}{\Gamma(\gamma-j+1)} + (I^\alpha_{0^+}\lambda(\cdot)x)(t) + (I^\alpha_{0^+}F_x)(t).$$

Clearly, by Lemma 5, $\mathcal{F}$ is well defined, and the fixed point of $\mathcal{F}$ is the solution of the problem (1).

For $x \in C_{n-\gamma}$, from Lemma 6 (ii) and (9), we have

$$\begin{aligned}
\left|\left(I_{0^+}^{\alpha}\lambda(s)x(s)\right)(t)\right| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\gamma-n}|\lambda(s)|ds\cdot\|x\|_{C_{n-\gamma}} \leq \Lambda_1 t^{\alpha-n+\gamma-p_2}\cdot\|x\|_{C_{n-\gamma}},\\
|(I_{0^+}^{\alpha}F_x)(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\sigma(\gamma-n)}\mu(s)ds\cdot\|x\|^{\sigma}_{C_{n-\gamma}} \leq \Lambda_2 t^{\alpha+\sigma(\gamma-n)-p_1}\cdot\|x\|^{\sigma}_{C_{n-\gamma}},\\
t^{n-\gamma}|(\mathcal{F}x)(t)| &\leq t^{n-\gamma}\left(\sum_{j=1}^{n}\frac{c_j t^{\gamma-j}}{\Gamma(\gamma-j+1)} + |(I_{0^+}^{\alpha}\lambda(\cdot)x)(t)| + |(I_{0^+}^{\alpha}F_x)(t)|\right)\\
&\leq \sum_{j=1}^{n}\frac{c_j t^{n-j}}{\Gamma(\gamma-j+1)} + \Lambda_1 t^{\alpha-p_2}\cdot\|x\|_{C_{n-\gamma}} + \Lambda_2 t^{\alpha-p_1+(1-\sigma)(n-\gamma)}\cdot\|x\|^{\sigma}_{C_{n-\gamma}}.
\end{aligned} \tag{10}$$

Step I. For an $r>0$, we set the ball $B_r\subset C_{n-\gamma}$ as $B_r=\{x\in C_{n-\gamma};\|x\|_{C_{n-\gamma}}\leq r\}$. We claim that there exists an $r_0>0$ such that $\mathcal{F}B_{r_0}\subseteq B_{r_0}$. If this is not true, then for each positive number r, there exists a function $\widetilde{x}(\cdot)\in B_r$, for some $t_0\in J$,

$$\|(\mathcal{F}\widetilde{x})\|_{C_{n-\gamma}} := t_0^{n-\gamma}|(\mathcal{F}\widetilde{x})(t_0)|>r.$$

It follows from (10) that

$$r \;<\; t_0^{n-\gamma}|(\mathcal{F}\widetilde{x})(t_0)| \leq \sum_{j=1}^{n}\frac{c_j t_0^{n-j}}{\Gamma(\gamma-j+1)} + \Lambda_1 T^{\alpha-p_2}r + \Lambda_2 T^{\alpha-p_1+(1-\sigma)(n-\gamma)}r^{\sigma}.$$

Dividing both sides of the above inequality by r and taking the limit as $r\to\infty$, we deduce that $\Lambda_1 T^{\alpha-p_2}\geq 1$. This is a contradiction. Hence, $\mathcal{F}B_r\subseteq B_r$.

Step II. We show that $\mathcal{F}$ is continuous. Let $\{x_m\}$ be a sequence such that $x_m\to x$ in B_r, then there exists $\varepsilon>0$ such that $\|x_m-x\|_{C_{n-\gamma}}<\varepsilon$ for n sufficiently large. By (9), we have

$$|f(t,x_m(t))-f(t,x(t))| \;\leq\; t^{\sigma(\gamma-n)}\mu(t)(\varepsilon^{\sigma}+2r^{\sigma}).$$

Moreover, by (H1), for almost every $t\in J$, we see that $|f(t,x_m(t))-f(t,x(t))|\to 0$. From the Lebesgue's dominated convergence theorem, it follows that $|(\mathcal{F}x_m)(t)-(\mathcal{F}x)(t)|\to 0$ as $m\to\infty$. Now we see that $\mathcal{F}$ is continuous.

Step III. We prove that $\mathcal{F}$ maps bounded sets into equicontinuous sets of B_r. Let $t_1,t_2\in J$, $t_1<t_2$ and $x\in B_r$. Similar to Lemma 7, we derive

$$\begin{aligned}
& t_1^{n-\gamma}\Big[|(I_{0^+}^{\alpha}\lambda(\cdot)x)(t_2)-(I_{0^+}^{\alpha}\lambda(\cdot)x)(t_1)| + |(I_{0^+}^{\alpha}F_x)(t_2)-(I_{0^+}^{\alpha}F_x)(t_1)|\Big]\\
\leq\; & \frac{t_1^{n-\gamma}}{\Gamma(\alpha)}\Big[\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|\left(s^{\gamma-n}|\lambda(s)|r+s^{-\sigma(n-\gamma)}|\mu(s)|r^{\sigma}\right)ds\\
& + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left(s^{\gamma-n}|\lambda(s)|r+s^{-\sigma(n-\gamma)}|\mu(s)|r^{\sigma}\right)ds\Big]\\
\to\; & 0, \qquad \text{as} \quad t_1\to t_2.
\end{aligned} \tag{11}$$

Then by (10) and (11),

$$
\begin{aligned}
& |t_2^{n-\gamma}(\mathcal{F}x)(t_2) - t_1^{n-\gamma}(\mathcal{F}x)(t_1)| \\
\leq\ & |t_2^{n-\gamma} - t_1^{n-\gamma}||(\mathcal{F}x)(t_2)| + t_1^{n-\gamma}|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \\
\leq\ & |t_2^{n-\gamma} - t_1^{n-\gamma}||(\mathcal{F}x)(t_2)| + t_1^{n-\gamma}\sum_{j=1}^{n}\frac{c_j|t_2^{\gamma-j} - t_1^{\gamma-j}|}{\Gamma(\gamma-j+1)} + t_1^{n-\gamma}\Big[|(I_{0+}^{\alpha}\lambda(\cdot)x)(t_2) - (I_{0+}^{\alpha}\lambda(\cdot)x)(t_1)| \\
& +|(I_{0+}^{\alpha}F_x)(t_2) - (I_{0+}^{\alpha}F_x)(t_1)|\Big] \\
\to\ & 0, \qquad \text{as} \quad t_1 \to t_2.
\end{aligned}
$$

From the above steps we deduce that $\mathcal{F}$ is completely continuous. It follows from Schauder's fixed point theorem that $\mathcal{F}$ has a fixed point $x \in B_r$. $\square$

5. Applications

In this section, we present two examples to illustrate our results.

Example 1. *We consider the following initial value problem for the variable parameter fractional differential equation:*

$$
\begin{cases}
{}^{H}D_{0^+}^{\frac{9}{10},\frac{1}{5}}x(t) - \dfrac{\sin t}{2\sqrt[5]{t}}x(t) = \dfrac{\cos t}{\sqrt[5]{t}}|x(t)|^{\frac{1}{2}}, \quad t \in J := (0,1], \\
(I_{0^+}^{\frac{2}{25}}x)(0) = c_2,\ (I_{0^+}^{\frac{2}{25}}x)'(0) = c_1.
\end{cases} \tag{12}
$$

Set $\alpha = \frac{9}{10}$, $\beta = \frac{1}{5}$, $n = 1$, $\gamma = \alpha + \beta - \alpha\beta = \frac{23}{25}$, $n - \gamma = \frac{2}{25}$, $\lambda(t) = \frac{\sin t}{2\sqrt[5]{t}}$, $f(t, x(t)) = \frac{\cos t}{\sqrt[5]{t}}|x(t)|^{\frac{1}{2}}$. *Notice that this problem is a particular case of (1).*

We choose $p = \frac{1}{4} \in (\frac{2}{25}, \frac{9}{20})$, *then for* $x(t) \in C_{\frac{2}{25}}$,

$$
\left(\int_0^1 \left|\frac{\cos t}{\sqrt[5]{t}}|x(t)|^{\frac{1}{2}}\right|^4 dt\right)^{\frac{1}{4}} \leq \left(\int_0^1 \frac{1}{t^{\frac{24}{25}}}dt\right)^{\frac{1}{4}} \cdot \|x\|_{C_{\frac{2}{25}}}^{\frac{1}{2}} < +\infty,
$$

i.e., f *satisfies (H1). Moreover,* $|f(t, x(t))| \leq \mu(t)|x(t)|^{\frac{1}{2}}$ *where* $\mu(t) = \frac{1}{\sqrt[5]{t}} \in L^{\frac{1}{p_1}}(p_1 = \frac{1}{3})$, $\|\mu\|_{L^3} = (\frac{5}{2})^{\frac{1}{3}}$ *and* $\lambda \in L^{\frac{1}{p_2}}(p_2 = \frac{1}{4})$, $\|\lambda\|_{L^4} \leq \frac{5^{\frac{1}{4}}}{2}$. *By simple calculation one can obtain*

$$
\Lambda_1 := \left(B\left(\frac{\alpha - p_2}{1 - p_2}, \frac{\gamma - n + 1 - p_2}{1 - p_2}\right)\right)^{1-p_2}\frac{\|\lambda\|_{L^{\frac{1}{p_2}}}}{\Gamma(\alpha)} \leq \left(B\left(\frac{13}{15}, \frac{67}{75}\right)\right)^{\frac{3}{4}}\frac{5^{\frac{1}{4}}}{2\Gamma(\frac{9}{10})} \approx 0.84 < 1.
$$

By Theorem 6, the problem (12) has at least one solution $x \in C_{\frac{2}{25}}$.

Example 2. *We consider the following initial value problem for the variable parameter fractional differential equation:*

$$
\begin{cases}
{}^{H}D_{0^+}^{\frac{19}{10},\frac{1}{10}}x(t) - \lambda t^{\nu}x(t) = bt^{\omega}, \quad t \in J := (0,1], \\
(I_{0^+}^{\frac{9}{100}}x)(0+) = c_2,\ (I_{0^+}^{\frac{9}{100}}x)'(0+) = c_1,
\end{cases} \tag{13}
$$

where $\lambda, b \in \mathbb{R}$ *and* $\nu > -\frac{1}{100}, \omega > 0$.

Set $\alpha = \frac{19}{10}$, $\beta = \frac{1}{10}$, $n = 2$, $\gamma = \alpha + 2\beta - \alpha\beta = \frac{191}{100}$, $n - \gamma = \frac{9}{100}$, $\lambda(t) = \lambda t^{\nu}$, $f(t, x(t)) = \lambda t^{\nu}x(t) + bt^{\omega}$. *Notice that this problem is a particular case of (5).*

We choose $p = \frac{1}{10} \in (\frac{9}{100}, \frac{9}{20})$, *then for* $x(t) \in C_{\frac{9}{100}}$,

$$\Big(\int_0^1 \big|\lambda t^\nu x(t)\big|^{10} dt\Big)^{\frac{1}{10}} \le |\lambda| \Big(\int_0^1 t^{10\nu - \frac{9}{10}} dt\Big)^{\frac{1}{10}} \|x\|_{C_{\frac{9}{100}}} = |\lambda| \Big(10\nu + \frac{1}{10}\Big)^{-\frac{1}{10}} \|x\|_{C_{\frac{9}{100}}} < +\infty,$$

i.e., $f(\cdot, x(\cdot)) \in L^{10}$. *Moreover,*

$$|f(t, x(t)) - f(t, y(t))| \le |\lambda| t^\nu |x(t) - y(t)| := \mu(t)|x(t) - y(t)|, \quad \mu(t) \in L^{10}$$

and $M_f := \sup_{t\in J} |f(t,0)| = |b|$. *Thus, by Theorem 4, the problem (13) has a unique solution* $x \in C_{\frac{9}{100}}$.

Let

$$\begin{cases} x_0(t) = \sum_{j=1}^{2} \frac{c_j t^{\gamma-j}}{\Gamma(\gamma - j + 1)}, \\ x_m(t) = x_0(t) + (I_{0^+}^\alpha \lambda(\cdot) x_{m-1})(t) + b I_{0^+}^\alpha t^\omega, \end{cases}$$

iterating, for $m \in \mathbb{N}$, *we can write*

$$x(t) = \sum_{j=1}^{2} \sum_{k=0}^{\infty} \frac{c_j (I_{0^+}^\alpha \lambda(\cdot))^k t^{\gamma-j}}{\Gamma(\gamma - j + 1)} + \sum_{k=1}^{\infty} (I_{0^+}^\alpha \lambda(\cdot))^{k-1} (b I_{0^+}^\alpha t^\omega).$$

Noting that

$$(I_{0^+}^\alpha \lambda(\cdot)) t^\theta = \lambda I_{0^+}^\alpha t^{\nu+\theta} = \frac{\lambda \Gamma(\nu + \theta + 1)}{\Gamma(\alpha + \nu + \theta + 1)} t^{\alpha+\nu+\theta},$$

$$(I_{0^+}^\alpha \lambda(\cdot))^2 t^\theta = \frac{\lambda^2 \Gamma(\nu + \theta + 1)}{\Gamma(\alpha + \nu + \theta + 1)} I_{0^+}^\alpha t^{\alpha+2\nu+\theta} = \frac{\lambda^2 \Gamma(\nu + \theta + 1) \Gamma(\alpha + 2\nu + \theta + 1)}{\Gamma(\alpha + \nu + \theta + 1) \Gamma(2(\alpha + \nu) + \theta + 1)} t^{2(\alpha+\nu)+\theta},$$

we obtain

$$\begin{aligned} (I_{0^+}^\alpha \lambda(\cdot))^k t^\theta &= \prod_{i=0}^{k-1} \frac{\Gamma[i\alpha + (i+1)\nu + \theta + 1]}{\Gamma[(i+1)(\alpha + \nu) + \theta + 1]} (\lambda t^{\alpha+\nu})^k \cdot t^\theta \\ &= \prod_{i=0}^{k-1} \frac{\Gamma\Big[\alpha\Big(i(1 + \frac{\nu}{\alpha}) + \frac{\nu+\theta}{\alpha}\Big) + 1\Big]}{\Gamma\Big[\alpha\Big(i(1 + \frac{\nu}{\alpha}) + \frac{\nu+\theta}{\alpha} + 1\Big) + 1\Big]} (\lambda t^{\alpha+\nu})^k \cdot t^\theta, \quad k \in \mathbb{N}. \end{aligned}$$

Now we have

$$\sum_{j=1}^{2} \sum_{k=0}^{\infty} \frac{c_j (I_{0^+}^\alpha \lambda(\cdot))^k t^{\gamma-j}}{\Gamma(\gamma - j + 1)} = \sum_{j=1}^{2} \frac{c_j t^{\gamma-j}}{\Gamma(\gamma - j + 1)} E_{\alpha, 1+\frac{\nu}{\alpha}, \frac{\nu+\gamma-j}{\alpha}}(\lambda t^{\alpha+\nu}),$$

where

$$E_{\alpha,\zeta,l}(z) = \sum_{k=0}^{\infty} d_k z^k, \quad z \in \mathbb{R}, \ \alpha > 0, \ \zeta > 0, \ \alpha(i\zeta + l) \neq -1, -2, \cdots$$

is a generalized Mittag–Leffler function ([4]) *and* $d_0 = 1$,

$$d_k = \prod_{i=0}^{k-1} \frac{\Gamma[\alpha(i\zeta + l) + 1]}{\Gamma[\alpha(i\zeta + l + 1) + 1]}.$$

Similarly, we have

$$\sum_{k=1}^{\infty}(I_{0^+}^{\alpha}\lambda(\cdot))^{k-1}t^{\alpha+\omega} \quad = \quad t^{\alpha+\omega}E_{\alpha,1+\frac{\nu}{\alpha},\frac{\nu+\alpha+\omega}{\alpha}}(\lambda t^{\alpha+\nu}).$$

Finally, we get the following explicit solution for (13)

$$x(t) = \sum_{j=1}^{2}\frac{c_j t^{\gamma-j}}{\Gamma(\gamma-j+1)}E_{\alpha,1+\frac{\nu}{\alpha},\frac{\nu+\gamma-j}{\alpha}}(\lambda t^{\alpha+\nu}) + \frac{bt^{\alpha+\omega}\Gamma(\omega+1)}{\Gamma(\alpha+\omega+1)}E_{\alpha,1+\frac{\nu}{\alpha},\frac{\nu+\alpha+\omega}{\alpha}}(\lambda t^{\alpha+\nu}).$$

6. Conclusions

In this paper, using new techniques, the existence and uniqueness results for differential equations of Hilfer-type fractional order with variable coefficient are successfully obtained. The results are new even for the special case: $\beta = 0$ or $\beta = 1$. The results hold for all $\alpha \in (n-1, n)(n \in \mathbb{N})$. The proposed techniques can be extended to other hybrid fractional differential equations.

7. Future Research

In this paper, the variable coefficient $\lambda(\cdot) \in L^{\frac{1}{p}}$ and is not necessarily continuous, therefore, the technique can be used to solve other types of equations. For example, inspired by the reference [19], as future work, we will study the fractional differential equation with variable coefficients with respect to another function under the weak assumptions for σ_i, g:

$${}^{H}D_{0^+}^{\beta_0,\mu_0,\varphi}y(t) + \sum_{i=1}^{m}\sigma_i(t)\,{}^{H}D_{0^+}^{\beta_i,\mu_i,\varphi}y(t) = g(t), \quad t \in [0, t_0],\, m \in \mathbb{N},$$

where $\beta_0 > \beta_1 > \cdots > \beta_m \geq 0$, $0 \leq \mu_i \leq 1$ and n_i are non-negative integers satisfying $n_i - 1 < \beta_i < n_i$, $i = 0, 1, \cdots, m$.

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