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Analysis of Impulsive Boundary Value Pantograph Problems via Caputo Proportional Fractional Derivative under Mittag–Leffler Functions

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Abstract: This manuscript investigates an extended boundary value problem for a fractional pantograph differential equation with instantaneous impulses under the Caputo proportional fractional derivative with respect to another function. The solution of the proposed problem is obtained using Mittag–Leffler functions. The existence and uniqueness results of the proposed problem are established by combining the well-known fixed point theorems of Banach and Krasnoselskii with nonlinear functional techniques. In addition, numerical examples are presented to demonstrate our theoretical analysis.

Keywords: fixed point theorems; impulsive condition; boundary value problems; existence theory; fractional differential equation

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1. Introduction

Fractional differential equations (FDEs) have recently gained prominence and attention as a way to describe applications in a variety of domains, including chemistry, mechanics, fluid systems, electronics, electromagnetics, and other domains. The study of FDEs encompasses everything from the theoretical aspects of solution existence to the methodologies for discovering analytic and numerical solutions (see [1–5]). In both the physical and social sciences, impulsive differential equations have become essential mathematical models of phenomena. These equations are applied to describe the evolutionary processes that change their state abruptly at a certain moment. This problem has piqued the interest of researchers due to its rich theory and relevance in a wide range of scientific and technological disciplines, including mechanics, ecology, medicine, biology, and electrical engineering, (see [6–9]).

One form of well-known nonlinear delay differential equation has recently been investigated:

$$\begin{cases} x'(t) = ax(t) + bx(\mu t), \quad t \in [0,T], \quad T > 0, \\ x(0) = x_0, \quad \mu \in (0,1), \quad a,b, \in \mathbb{R}. \end{cases} \tag{1}$$

Equation (1) is called the pantograph equation. Ockendon and Tayler [10] have researched that there have been a wide range of applications in numerous disciplines of applied sciences and engineering—this is used to model various current processes and phenomena that are dependent on earlier ones (see [11–17] and the references cited therein for several works). The existence, uniqueness, and different kinds of Hyers–Ulam stability of solutions for nonlinear FDEs via impulse terms or non-impulse terms drew a lot of attention from researchers. For example, in 2013, Balachandran and co-workers [18] used fractional calculus and fixed point theorems to investigate the existence of solutions for nonlinear pantograph equations:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}x(t) = f(t, x(t), x(\mu t)), \quad \alpha \in (0,1], \quad t \in [0,1], \\ x(0) = x_0, \quad x_0 \in \mathbb{R}, \quad \mu \in (0,1), \end{cases} \tag{2}$$

where ${}^{C}\mathfrak{D}^{\alpha}$ denotes the Caputo fractional derivative of order α and $f \in \mathcal{C}([0,1] \times \mathbb{R}^2, \mathbb{R})$.

The investigated impulsive FDEs are not reliant on constant coefficients after reading the previous publications listed. The impulsive fractional boundary value problems (BVPs) with constant coefficients have received little attention. In physics, however, impulsive FDEs with constant coefficients have a stronger foundation and play an important role. Hooke and Newton laws are employed in mechanics to explain the behavior of particular materials under the influence of external forces. Certain researchers propose revising the classical Newton's law, which is considered a generalized Nutting's law, in order to change some possible modification qualities. On the other hand, a mass-spring-damper system is frequently exposed to short-term perturbations (an external force) that are sudden and manifest as instantaneous impulses involving the associated differential equations. In 2014, the author [19] established certain necessary conditions for the existence of a solution to an impulsive fractional anti-periodic BVP with constant coefficients of the form:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}_{t_k^+}x(t) + \lambda x(t) = f(t, x(t)), \quad t \in \mathcal{J}' = \mathcal{J}\backslash\{t_1, \ldots, t_m\}, \quad \mathcal{J} := [0,1], \\ \Delta x(t_k) = y_k, \quad k = 1, 2, \ldots, m, \\ x(0) + x(1) = 0, \quad y_k \in \mathbb{R}, \end{cases} \tag{3}$$

where $\lambda > 0$, ${}^{C}\mathfrak{D}^{\alpha}_{t_k^+}$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, $f \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$, the fixed impulsive time t_k satisfy $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1$, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump of $x(t)$ at $t = t_k$. The existence results of solutions were investigated by helping Lipschitz and nonlinear growth conditions. In addition, Mittag–Leffler functions attributes and computational formula are employed to construct examples. Based on the Banach contraction principle and Krasnoselskii's fixed point theorem, Zuo and co-workers [20] developed existence theorems for impulsive fractional integro-differential equations of mixed type with constant coefficient and anti-periodic boundary conditions in 2017:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}_{t_k^+}x(t) + \lambda x(t) = f(t, x(t), Kx(t), Sx(t)), \quad t \in \mathcal{J}' = \mathcal{J}\backslash\{t_1, \ldots, t_m\}, \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ x(0) + x(1) = 0, \quad \lambda > 0, \end{cases} \tag{4}$$

where $I_k \in \mathbb{R}$, $f \in \mathcal{C}(\mathcal{J} \times \mathbb{R}^3, \mathbb{R})$, $\mathcal{J} := [0,1]$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$, $x(t_k^-) = x(t_k)$ represent the right and left hand limits of $x(t)$ at $t = t_k$, respectively, K and S are linear operators. In 2020, Ahmed and co-workers [21] estbblished the existence and uniqueness of the solution for the impulsive fractional pantograph differential equation with a more broader anti-periodic boundary condition of the form:

$$\begin{cases} {}^{C}\mathfrak{D}^{\alpha}_{0^+}x(t) + \lambda x(t) = f(t, x(t), x(\mu t)), \quad t \in \mathcal{J}' = \mathcal{J}\backslash\{t_1, \ldots, t_k\}, \\ \Delta x|_{t=t_k}(0) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ ax(0) + bx(1) = 0, \quad a \geq b > 0, \quad \mu \in (0, 1), \end{cases} \tag{5}$$

where ${}^{C}\mathfrak{D}^{\alpha}_{0^+}$ denotes the Caputo fractional derivative of order α, $f \in \mathcal{C}(\mathcal{J} \times \mathbb{R}^2, \mathbb{R})$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, with $x(t_k^+)$ and $x(t_k^-)$ representing the right and left limits of $x(t)$ at $t = t_k$. Using Banach's and Krasnoselskii's fixed point theorems, they established the existence and uniqueness of the solution for impulsive problem (5). We recommend manuscripts [22–28] and the references given therein for contemporary papers on impulsive FDEs on existence, uniqueness, and stability. The qualitative feature of non-impulsive/impulsive FDEs is increasingly being studied in research.

Recently, Jarad and co-workers [29] constructed a novel brand of fractional operators builded from the modified conformable derivatives. After that, Jarad and co-workers formulated the proportional fractional calculus and shown certain features of the proportional fractional derivatives and fractional integrals of a function concerning another function. The kernel achieved in their consideration contains an exponential function and is function dependent (as specified in Section 2) in [30,31]. The proportional fractional operators have been applied to FDEs with and without impulsive conditions (see [32–38]). For more interesting work on FDEs, we refer to read [39–46] and references cited therein. Few works have been published on impulsive Caputo proportional fractional BVPs using function via proportional delay term, to the author's knowledge.

The existence and uniqueness results of the solutions for the following nonlinear impulsive pantograph fractional BVP under Caputo proportional fractional derivative concerns a particular function are considered in this manuscript:

$$\begin{cases} {}^{C\rho_k}\mathfrak{D}^{\alpha_k,\psi_k}_{t_k^+}x(t) + \rho_k^{\alpha_k}\lambda x(t) = f(t, x(t), x(\mu t)), \quad t \in \mathcal{J}', \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = \varphi_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ \beta x(0) + \eta x(T) = \gamma, \end{cases} \tag{6}$$

where ${}^{C\rho_k}\mathfrak{D}^{\alpha_k,\psi_k}_{t_k^+}$ denotes the Caputo proportional fractional derivative operator with respect to another increasing differentiable function ψ_k of order $0 < \alpha_k < 1$ with $0 < \rho_k \leq 1$, $t \in \mathcal{J}_k := (t_k, t_{k+1}] \subseteq \mathcal{J} := [0, T]$, $k = 0, 1, \ldots, m$, $\mathcal{J}' := \mathcal{J} \setminus \{t_1, t_2, \ldots, t_m\}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $0 < \mu < 1$, $\lambda > 0$, $f \in \mathcal{C}(\mathcal{J} \times \mathbb{R}^2, \mathbb{R})$, $\varphi_k \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \ldots, m$, $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$, $x(t_k^-) = x(t_k)$, $\beta, \eta, \gamma \in \mathbb{R}$, and

$$\beta + \eta \prod_{i=1}^{m+1} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i) - \psi_{i-1}(t_{i-1})\right)} \mathbb{E}_{\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i) - \psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\right) \neq 0.$$

The goal of this manuscript is to use the fixed point theorems of Banach and Krasnoselskii to investigate the existence and uniqueness of solutions to the impulsive problem (6). The following are the main points of this manuscript:

(i) We consider a new impulsive pantograph differential equations with Caputo proportional fractional derivative concerning a certain function.
(ii) Under the Caputo proportional fractional derivative, we explore more broader proportional BVPs with constant coefficients.

The development of qualitative analysis of impulsive fractional BVPs is encouraged in this manuscript. Notice that the significance of this discussion on the manuscript is that the problem (6) generates many types, including mixed types of impulsive FDEs with boundary conditions. For instance, if we set $\rho_k = 1$ in (6), then we have the Riemann–Liouville fractional operators [2] with $\psi_k(t) = t$, the Hadamard fractional operators [2] with

$\psi_k(t) = \log t$, the Katugampola fractional operators [47] with $\psi_k(t) = t^{\mu}/\mu$, $\mu > 0$, the conformable fractional operators [48] with $\psi_k(t) = (t-a)^{\mu}/\mu$, $\mu > 0$, and the generalized conformable fractional operators [49] with $\psi_k(t) = t^{\mu+\phi}/(\mu+\phi)$, respectively. In addition, several other special cases can be derived as well. To the best of the author's knowledge, there are some papers that have established either impulsive fractional BVPs [33–35] and few papers focused on impulsive Caputo proportional fractional BVPs with respect to another function via proportional delay term.

The remainder of the manuscript is organized in the following manner. Section 2 introduces some key concepts and lemmas linked to the major findings. We also present certain definitions of well-known fixed point theorems and construct the formulas for the solution involving Mittag–Leffler functions with the linear impulsive problem. We use Banach's and Krasnoselskii's fixed point theorems to analyze the existence and uniqueness of solutions for the impulsive problem (6) in Section 3. Finally, examples are provided to demonstrate the validity of our primary findings in Sections 4 and 5 contains the conclusion of our findings.

2. Preliminaries

This part introduces the generalized proportional fractional derivatives and fractional integral notations, definitions, and preliminary facts that will be utilized throughout the manuscript. For more details, (see [30,31,50,51]).

Let $\mathcal{J}_0 := [t_0, t_1]$, $\mathcal{J}_1 := (t_1, t_2]$, ..., $\mathcal{J}_{m-1} := (t_{m-1}, t_m]$, $\mathcal{J}_m := (t_m, T]$, and let us denote by $\mathcal{PC}(\mathcal{J}, \mathbb{R}) = \{x : \mathcal{J} \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-) = x(t_k)$ exist, $k = 1, 2, \ldots, m\}$ the space of piecewise continuous functions on the interval $\mathcal{J}$. It is clear that $\mathcal{PC}(\mathcal{J}, \mathbb{R})$ is a Banach space equipped with the norm $\|x\|_{\mathcal{PC}} = \sup_{t\in\mathcal{J}}\{|x(t)|\}$. Let the norm of a measurable function $\sigma : \mathcal{J} \to \mathbb{R}$ be defined by

$$\|\sigma\|_{L^q(\mathcal{J})} = \begin{cases} \left(\int_{\mathcal{J}} |\sigma(s)|^q ds\right)^{1/q}, & 1 \le q < \infty, \\ \inf\limits_{mes(\overline{\mathcal{J}})=0} \left\{ \sup\limits_{t\in\mathcal{J}\setminus\overline{\mathcal{J}}} |x(t)| \right\}, & q = \infty. \end{cases}$$

Then $L^q(\mathcal{J}, \mathbb{R})$ is a Banach space of Lebesque-measurable functions $\sigma : \mathcal{J} \to \mathbb{R}$ with $\|\sigma\|_{L^q(\mathcal{J})} < \infty$.

Definition 1 ([30,31]). *Take $\rho \in [0,1]$ and let the functions κ_0, $\kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous such that for all $t \in \mathbb{R}$ we have*

$$\lim_{\rho\to 0^+} \kappa_1(\rho, t) = 1, \quad \lim_{\rho\to 0^+} \kappa_0(\rho, t) = 0, \quad \lim_{\rho\to 1^-} \kappa_1(\rho, t) = 0, \quad \lim_{\rho\to 1^-} \kappa_0(\rho, t) = 1,$$

and $\kappa_1(\rho, t) \neq 0$, $\rho \in [0,1)$, $\kappa_0(\rho, t) \neq 0$, $\rho \in (0,1]$. Let $\psi(t)$ be a continuously differentiable and increasing function. Then, the proportional differential operator of order ρ of f with respect to ψ is defined by

$${}^{\rho}\mathfrak{D}^{\psi} f(t) = \kappa_1(\rho, t) f(t) + \kappa_0(\rho, t) \frac{f'(t)}{\psi'(t)}.$$

In particular, if $\kappa_1(\rho, t) = 1 - \rho$ and $\kappa_0(\rho, t) = \rho$, we obtain

$${}^{\rho}\mathfrak{D}^{\psi} f(t) = (1-\rho) f(t) + \rho \frac{f'(t)}{\psi'(t)}.$$

Definition 2 ([30,31]). *Take $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $\rho \in (0,1]$, $\psi \in \mathcal{C}^1([a,b])$, $\psi' > 0$. The proportional fractional integral of order α of the function $f \in L^1([a,b])$ with respect to another function ψ is defined by*

$$ {}^{\rho}\mathcal{I}_a^{\alpha,\psi} f(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)} \left(\psi(t)-\psi(s)\right)^{\alpha-1} f(s)\psi'(s)ds, $$

where $\Gamma(\cdot)$ *is the (Euler's) Gamma function defined by* $\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1}e^{-s}ds$, $s > 0$.

Definition 3 ([30,31]). *Take* $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $\rho \in (0,1]$, $\psi \in \mathcal{C}([a,b])$, $\psi'(t) > 0$. *The Riemann–Liouville proportional fractional derivative of order* α *of the function* $f \in \mathcal{C}^n([a,b])$ *with respect to another function* ψ *is defined by* ${}^{\rho}\mathfrak{D}_a^{\alpha,\psi} f(t) = {}^{\rho}\mathfrak{D}^{n,\psi}\,{}_a^{\rho}\mathcal{I}^{n-\alpha,\psi} f(t)$ *or*

$$ {}^{\rho}\mathfrak{D}_a^{\alpha,\psi} f(t) = \frac{{}^{\rho}\mathfrak{D}_t^{n,\psi}}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)} \left(\psi(t)-\psi(s)\right)^{n-\alpha-1} f(s)\psi'(s)ds, $$

where $n = [Re(\alpha)] + 1$, $[Re(\alpha)]$ *represents the integer part of the real number* α *and*

$$ {}^{\rho}\mathfrak{D}^{n,\psi} = \underbrace{{}^{\rho}\mathfrak{D}^{\psi} \,.\, {}^{\rho}\mathfrak{D}^{\psi} \ldots {}^{\rho}\mathfrak{D}^{\psi}}_{n \text{ times}}. $$

Definition 4 ([30,31]). *Take* $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $\rho \in (0,1]$, $\psi \in \mathcal{C}([a,b])$, $\psi'(t) > 0$. *The Caputo proportional fractional derivative of order* α *of the function* f *with respect to another function* ψ *is defined by* ${}^{C\rho}\mathfrak{D}_a^{\alpha,\psi} f(t) = {}^{\rho}\mathcal{I}_a^{n-\alpha,\psi}\,{}^{\rho}\mathfrak{D}^{n,\psi} f(t)$ *or*

$$ {}^{C\rho}\mathfrak{D}_a^{\alpha,\psi} f(t) = \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)} \left(\psi(t)-\psi(s)\right)^{n-\alpha-1} {}^{\rho}\mathfrak{D}^{n,\psi} f(s)\psi'(s)ds. $$

Next, we provide some properties of the classical and generalized Mittag–Leffler functions $\mathbb{E}_{\alpha}(\cdot)$ and $\mathbb{E}_{\alpha,\beta}(\cdot)$, which is used throughout in this paper.

Lemma 1 ([50,51]). *Let* $\alpha \in (0,1)$, $\beta > 0$ *be arbitrary constants. Then the functions* $\mathbb{E}_{\alpha}$ *and* $\mathbb{E}_{\alpha,\beta}$ *are nonnegative functions, and for any* $z < 0$, $\mathbb{E}_{\alpha}(z) \le 1$, $\mathbb{E}_{\alpha,\beta}(z) \le 1/\Gamma(\beta)$, *where the classical and generalized Mittag–Leffler functions* $\mathbb{E}_{\alpha}$ *and* $\mathbb{E}_{\alpha,\beta}$ *are defined by*

$$ \mathbb{E}_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \quad \text{and} \quad \mathbb{E}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad z \in \mathbb{R}. $$

Moreover, for any $\lambda < 0$ *and* $\tau_1, \tau_2 \in \mathcal{J}$, *we have the following property:*

$$ \mathbb{E}_{\alpha,\alpha+\beta}\left(\lambda\left(\psi(\tau_2)-\psi(a)\right)^{\alpha}\right) \to \mathbb{E}_{\alpha,\alpha+\beta}\left(\lambda\left(\psi(\tau_1)-\psi(a)\right)^{\alpha}\right) \quad as \quad \tau_2 \to \tau_1, $$

where $\mathbb{E}_{\alpha}(0) = 1$ *and* $\mathbb{E}_{\alpha,\beta}(0) = 1/\Gamma(\beta)$. *In Addition, Wang and co-workers [50] provide a possible calculational formula of* $\mathbb{E}_{\frac{1}{2}}(-z)$ *as follows:*

$$ \mathbb{E}_{\frac{1}{2}}(-z) = \frac{1 + \frac{\pi-2}{\sqrt{\pi}}z}{1 + \sqrt{\pi}z + (\pi-2)z^2}, \quad z \ge 0. \tag{7} $$

Definition 5. *A function* $x \in \mathcal{PC}(J,\mathbb{R})$ *is said to be a solution of problem* (6) *if it satisfies the equation* ${}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k} x(t) + \rho_k^{\alpha_k}\lambda x(t) = f(t,x(t),x(\mu t))$ *a.e on* $\mathcal{J}'$ *and the condition* $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = \varphi_k(x(t_k))$, $k = 1,2,\ldots,m$, *and* $\beta x(0) + \eta x(T) = \gamma$.

Lemma 2. *Let* $g : \mathcal{J} \to \mathbb{R}$ *be a continuous function. The function* x *is given by*

$$ \begin{aligned} x(t) &= x_a\, e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(a)\right)} \mathbb{E}_{\alpha}\left(-\lambda\left(\psi(t)-\psi(a)\right)^{\alpha}\right) \\ &\quad + \frac{1}{\rho^{\alpha}} \int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)} \left(\psi(t)-\psi(s)\right)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\left(-\lambda\left(\psi(t)-\psi(s)\right)^{\alpha}\right) g(s)\psi'(s)ds. \end{aligned} $$

is a solution of the linear Caputo proportional fractional proportional differential equation with constant coefficients of the form:

$$\begin{cases} {}^{C\rho}\mathfrak{D}_{a^+}^{\alpha,\psi} x(t) + \rho^{\alpha}\lambda x(t) = g(t), \quad t \in (a,T], \quad 0 < \alpha < 1, \\ x(a) = x_a, \quad x_a \in \mathbb{R}. \end{cases}$$

Proof. It is easy to derive by direct calculation. Please see Example 3.2 in [31]. □

The following lemma is used to create an equivalent integral equation for the impulsive problem (6). For the sake of calculation in this manuscript, we set the notation:

$$\Phi_n^c(t_b,t_a) := e^{\frac{\rho_n-1}{\rho_n}\left(\psi_n(t_b)-\psi_n(t_a)\right)}\left(\psi_n(t_b)-\psi_n(t_a)\right)^{c-1}, \tag{8}$$

where $t_a, t_b \in \{t_0, t_1, \ldots, t_m, T\}$ and $c \in \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$.

From (8) with $0 < e^{\frac{\rho_n-1}{\rho_n}\left(\psi_n(t_b)-\psi_n(t_a)\right)} \leq 1$, for $0 \leq t_a < t_b \leq T$, we obtain

$$|\Phi_n^c(t_b,t_a)| \leq \left|e^{\frac{\rho_n-1}{\rho_n}\left(\psi_n(t_b)-\psi_n(t_a)\right)}\right|\left(\psi_n(t_b)-\psi_n(t_a)\right)^{c-1} \leq \left(\psi_n(t_b)-\psi_n(t_a)\right)^{c-1}. \tag{9}$$

Lemma 3. *Suppose that $\alpha_k \in (0,1)$, $\rho_k \in (0,1]$, $\psi_k \in \mathcal{C}(\mathcal{J},\mathbb{R})$ with $\psi_k' > 0$ for $t \in \mathcal{J}$, $k = 0,1,\ldots,m$, $h \in \mathcal{C}(\mathcal{J},\mathbb{R})$, and $\Omega \neq 0$. The function $x \in \mathcal{PC}(\mathcal{J},\mathbb{R})$ is given by*

$$\begin{aligned} x(t) \quad = \quad & \left\{e^{\frac{\rho_k-1}{\rho_k}\left(\psi_k(t)-\psi_k(t_k)\right)}\,\mathbb{E}_{\alpha_k}\left(-\lambda\left(\psi_k(t)-\psi_k(t_k)\right)^{\alpha_k}\right)\right\}\left\{\left[\gamma\right.\right. \\ & -\eta\sum_{i=1}^{m}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i}\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}}\right)h(s)\psi_{i-1}'(s)ds\right. \\ & \left.+\varphi_i(x(t_i))\right)\prod_{j=i+1}^{m+1}e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\,\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right) \\ & \left.-\frac{\eta}{\rho_m^{\alpha_m}}\int_{t_m}^{T}\Phi_m^{\alpha_m}(T,s)\mathbb{E}_{\alpha_m,\alpha_m}\left(-\lambda\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m}\right)h(s)\psi_m'(s)ds\right] \\ & \times\left[\frac{1}{\Omega}\prod_{i=1}^{k}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)}\,\mathbb{E}_{\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\right)\right] \\ & +\sum_{i=1}^{k}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i}\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}}\right)h(s)\psi_{i-1}'(s)ds\right. \\ & \left.+\varphi_i(x(t_i))\right)\prod_{j=i+1}^{k}e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\,\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\Bigg\} \\ & +\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{t}\Phi_k^{\alpha_k}(t,s)\mathbb{E}_{\alpha_k,\alpha_k}\left(-\lambda\left(\psi_k(t)-\psi_k(s)\right)^{\alpha_k}\right)h(s)\psi_k'(s)ds, \end{aligned} \tag{10}$$

where

$$\Omega := \beta + \eta\prod_{i=1}^{m+1}e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)}\mathbb{E}_{\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\right), \tag{11}$$

is a solution of the impulsive problem:

$$\begin{cases} {}^{C\rho_k}\mathfrak{D}^{\alpha_k,\psi_k}_{t_k^+}x(t)+\rho_k^{\alpha_k}\lambda x(t)=h(t), \quad t\in(0,T]\setminus\{t_1,t_2,\ldots,t_m\},\\ \Delta x(t_k)=x(t_k^+)-x(t_k^-)=\varphi_k(x(t_k)), \quad k=1,2,\ldots,m,\\ \beta x(0)+\eta x(T)=\gamma. \end{cases} \tag{12}$$

Proof. Assume that x is a solution of (12). We consider the following several cases.

For $t\in[0,t_1)$, in view of Lemma 2, we have,

$$\begin{aligned} x(t) &= x_0 e^{\frac{\rho_0-1}{\rho_0}\left(\psi_0(t)-\psi_0(t_0)\right)}\,\mathbb{E}_{\alpha_0}\big(-\lambda\big(\psi_0(t)-\psi_0(0)\big)^{\alpha_0}\big)\\ &\quad+\frac{1}{\rho_0^{\alpha_0}}\int_0^t e^{\frac{\rho_0-1}{\rho_0}\left(\psi_0(t)-\psi_0(s)\right)}\big(\psi_0(t)-\psi_0(s)\big)^{\alpha_0-1}\\ &\qquad\times\mathbb{E}_{\alpha_0,\alpha_0}\big(-\lambda\big(\psi_0(t)-\psi_0(s)\big)^{\alpha_0}\big)h(s)\psi_0'(s)ds. \end{aligned}$$

In particular, for $t=t_1$, we obtain,

$$\begin{aligned} x(t_1) &= x_0\, e^{\frac{\rho_0-1}{\rho_0}\left(\psi_0(t_1)-\psi_0(t_0)\right)}\,\mathbb{E}_{\alpha_0}\big(-\lambda\big(\psi_0(t_1)-\psi_0(0)\big)^{\alpha_0}\big)\\ &\quad+\frac{1}{\rho_0^{\alpha_0}}\int_0^{t_1} e^{\frac{\rho_0-1}{\rho_0}\left(\psi_0(t_1)-\psi_0(s)\right)}\big(\psi_0(t_1)-\psi_0(s)\big)^{\alpha_0-1}\\ &\qquad\times\mathbb{E}_{\alpha_0,\alpha_0}\big(-\lambda\big(\psi_0(t_1)-\psi_0(s)\big)^{\alpha_0}\big)h(s)\psi_0'(s)ds. \end{aligned}$$

For $t\in[t_1,t_2)$, we have

$$\begin{aligned} x(t) &= x(t_1^+)\, e^{\frac{\rho_1-1}{\rho_1}\left(\psi_1(t)-\psi_1(t_1)\right)}\,\mathbb{E}_{\alpha_1}\big(-\lambda\big(\psi_1(t)-\psi_1(t_1)\big)^{\alpha_1}\big)\\ &\quad+\frac{1}{\rho_1^{\alpha_1}}\int_{t_1}^t e^{\frac{\rho_1-1}{\rho_1}\left(\psi_1(t)-\psi_1(s)\right)}\big(\psi_1(t)-\psi_1(s)\big)^{\alpha_1-1}\\ &\qquad\times\mathbb{E}_{\alpha_1,\alpha_1}\big(-\lambda\big(\psi_1(t)-\psi_1(s)\big)^{\alpha_1}\big)h(s)\psi_1'(s)ds. \end{aligned}$$

Using the impulsive condition in (12), $x(t_1^+)=x(t_1^-)+\varphi_1(x(t_1))$, we obtain

$$\begin{aligned} x(t) &= \left[x_0\, e^{\frac{\rho_0-1}{\rho_0}\left(\psi_0(t_1)-\psi_0(t_0)\right)}\,\mathbb{E}_{\alpha_0}\big(-\lambda\big(\psi_0(t_1)-\psi_0(0)\big)^{\alpha_0}\big)\right]\\ &\qquad\times e^{\frac{\rho_1-1}{\rho_1}\left(\psi_1(t)-\psi_1(t_1)\right)}\mathbb{E}_{\alpha_1}\big(-\lambda\big(\psi_1(t)-\psi_1(t_1)\big)^{\alpha_1}\big)\\ &\quad+\left[\frac{1}{\rho_0^{\alpha_0}}\int_0^{t_1} e^{\frac{\rho_0-1}{\rho_0}\left(\psi_0(t_1)-\psi_0(s)\right)}\big(\psi_0(t_1)-\psi_0(s)\big)^{\alpha_0-1}\right.\\ &\qquad\left.\times\mathbb{E}_{\alpha_0,\alpha_0}\big(-\lambda\big(\psi_0(t_1)-\psi_0(s)\big)^{\alpha_0}\big)h(s)\psi_0'(s)ds+\varphi_1(x(t_1))\right]\\ &\qquad\times e^{\frac{\rho_1-1}{\rho_1}\left(\psi_1(t)-\psi_1(t_1)\right)}\mathbb{E}_{\alpha_1}\big(-\lambda\big(\psi_1(t)-\psi_1(t_1)\big)^{\alpha_1}\big)\\ &\quad+\frac{1}{\rho_1^{\alpha_1}}\int_{t_1}^t e^{\frac{\rho_1-1}{\rho_1}\left(\psi_1(t)-\psi_1(s)\right)}\big(\psi_1(t)-\psi_1(s)\big)^{\alpha_1-1}\\ &\qquad\times\mathbb{E}_{\alpha_1,\alpha_1}\big(-\lambda\big(\psi_1(t)-\psi_1(s)\big)^{\alpha_1}\big)h(s)\psi_1'(s)ds. \end{aligned}$$

For $t\in[t_2,t_3)$, in the same previous process, we obtain

$$
\begin{aligned}
x(t) &= x(t_2^+)\, e^{\frac{\rho_2-1}{\rho_2}\left(\psi_2(t)-\psi_2(t_2)\right)}\, \mathbb{E}_{\alpha_2}\big(-\lambda\big(\psi_2(t)-\psi_2(t_2)\big)^{\alpha_2}\big) + \frac{1}{\rho_2^{\alpha_2}}\int_{t_2}^{t} e^{\frac{\rho_2-1}{\rho_2}\left(\psi_2(t)-\psi_2(s)\right)} \\
&\quad \big(\psi_2(t)-\psi_2(s)\big)^{\alpha_2-1}\mathbb{E}_{\alpha_2,\alpha_2}\big(-\lambda\big(\psi_2(t)-\psi_2(s)\big)^{\alpha_2}\big)h(s)\psi_2'(s)ds \\
&= \Big[x(t_2)+\varphi_2(x(t_2))\Big]\, e^{\frac{\rho_2-1}{\rho_2}\left(\psi_2(t)-\psi_2(t_2)\right)}\, \mathbb{E}_{\alpha_2}\big(-\lambda\big(\psi_2(t)-\psi_2(t_2)\big)^{\alpha_2}\big) \\
&\quad +\frac{1}{\rho_2^{\alpha_2}}\int_{t_2}^{t} e^{\frac{\rho_2-1}{\rho_2}\left(\psi_2(t)-\psi_2(s)\right)}\big(\psi_2(t)-\psi_2(s)\big)^{\alpha_2-1}\mathbb{E}_{\alpha_2,\alpha_2}\big(-\lambda\big(\psi_2(t)-\psi_2(s)\big)^{\alpha_2}\big)h(s)\psi_2'(s)ds \\
&= x_0\Bigg\{e^{\frac{\rho_0-1}{\rho_0}\left(\psi_0(t_1)-\psi_0(t_0)\right)}\, e^{\frac{\rho_1-1}{\rho_1}\left(\psi_1(t_2)-\psi_1(t_1)\right)}\mathbb{E}_{\alpha_0}\big(-\lambda\big(\psi_0(t_1)-\psi_0(0)\big)^{\alpha_0}\big) \\
&\quad \times\mathbb{E}_{\alpha_1}\big(-\lambda\big(\psi_1(t_2)-\psi_1(t_1)\big)^{\alpha_1}\big)\Bigg\}\Bigg\{e^{\frac{\rho_2-1}{\rho_2}\left(\psi_2(t)-\psi_2(t_2)\right)}\mathbb{E}_{\alpha_2}\big(-\lambda\big(\psi_2(t)-\psi_2(t_2)\big)^{\alpha_2}\big)\Bigg\} \\
&\quad +\Bigg\{\Bigg[\frac{1}{\rho_0^{\alpha_0}}\int_0^{t_1} e^{\frac{\rho_0-1}{\rho_0}\left(\psi_0(t_1)-\psi_0(s)\right)}\big(\psi_0(t_1)-\psi_0(s)\big)^{\alpha_0-1}\mathbb{E}_{\alpha_0,\alpha_0}\big(-\lambda\big(\psi_0(t_1)-\psi_0(s)\big)^{\alpha_0}\big)h(s)\psi_0'(s)ds \\
&\quad +\varphi_1(x(t_1))\Bigg]e^{\frac{\rho_1-1}{\rho_1}\left(\psi_1(t_2)-\psi_1(t_1)\right)}\, \mathbb{E}_{\alpha_1}\big(-\lambda\big(\psi_1(t_2)-\psi_1(t_1)\big)^{\alpha_1}\big) \\
&\quad +\Bigg[\frac{1}{\rho_1^{\alpha_1}}\int_{t_1}^{t_2} e^{\frac{\rho_1-1}{\rho_1}\left(\psi_1(t_2)-\psi_1(s)\right)}\big(\psi_1(t_2)-\psi_1(s)\big)^{\alpha_1-1}\mathbb{E}_{\alpha_1,\alpha_1}\big(-\lambda\big(\psi_1(t_2)-\psi_1(s)\big)^{\alpha_1}\big) \\
&\quad +\varphi_2(x(t_2))\Bigg]\Bigg\}\Bigg\{e^{\frac{\rho_2-1}{\rho_2}\left(\psi_2(t)-\psi_2(t_2)\right)}\, \mathbb{E}_{\alpha_2}\big(-\lambda\big(\psi_2(t)-\psi_2(t_2)\big)^{\alpha_2}\big)\Bigg\} \\
&\quad +\frac{1}{\rho_2^{\alpha_2}}\int_{t_2}^{t} e^{\frac{\rho_2-1}{\rho_2}\left(\psi_2(t)-\psi_2(s)\right)}\big(\psi_2(t)-\psi_2(s)\big)^{\alpha_2-1}\mathbb{E}_{\alpha_2,\alpha_2}\big(-\lambda\big(\psi_2(t)-\psi_2(s)\big)^{\alpha_2}\big)h(s)\psi_2'(s)ds.
\end{aligned}
$$

Repeating the previous process, for $t \in [t_k, t_{k+1})$, $k = 0, 1, 2, \ldots, m$, we have

$$
\begin{aligned}
x(t) &= \Bigg\{x_0\prod_{i=1}^{k} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)}\, \mathbb{E}_{\alpha_{i-1}}\big(-\lambda\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}\big) \\
&\quad +\sum_{i=1}^{k}\Bigg(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\alpha_{i-1}-1} \\
&\quad \times\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\big(-\lambda\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\alpha_{i-1}}\big)h(s)\psi_{i-1}'(s)ds+\varphi_i(x(t_i))\Bigg) \\
&\quad \times\prod_{j=i+1}^{k} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\, \mathbb{E}_{\alpha_{j-1}}\big(-\lambda\big(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\big)^{\alpha_{j-1}}\big)\Bigg\} \\
&\quad \times\Bigg\{e^{\frac{\rho_k-1}{\rho_k}\left(\psi_k(t)-\psi_k(t_k)\right)}\, \mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t)-\psi_k(t_k)\big)^{\alpha_k}\big)\Bigg\}+\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{t} e^{\frac{\rho_k-1}{\rho_k}\left(\psi_k(t)-\psi_k(s)\right)} \\
&\quad \times\big(\psi_k(t)-\psi_k(s)\big)^{\alpha_k-1}\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(t)-\psi_k(s)\big)^{\alpha_k}\big)h(s)\psi_k'(s)ds.
\end{aligned}
\tag{13}
$$

From the boundary condition, $\beta x(0) + \eta x(T) = \gamma$, it follows that

$$
\begin{aligned}
x_0 \;=\; & \frac{1}{\Omega}\Bigg\{\gamma-\eta\sum_{i=1}^{m}\Bigg(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\\
& \times\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\big(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}}\big)h(s)\psi_{i-1}'(s)ds+\varphi_i(x(t_i))\Bigg)\\
& \times\prod_{j=i+1}^{m+1} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\,\mathbb{E}_{\alpha_{j-1}}\big(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\big)\\
& -\frac{\eta}{\rho_m^{\alpha_m}}\int_{t_m}^{T} e^{\frac{\rho_m-1}{\rho_m}\left(\psi_m(T)-\psi_m(s)\right)}\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\\
& \times\mathbb{E}_{\alpha_m,\alpha_m}\big(-\lambda\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m}\big)h(s)\psi_m'(s)ds\Bigg\},
\end{aligned}
$$

where Ω is defined by (11). In the last step, we insert the value x_0 into (13) to obtain (10).

Conversely, it is easy to show by direct computation that the solution $x(t)$ is given by (10) fulfills the impulsive problem (12) with the boundary conditions. □

3. Existence Analysis

This section investigates some sufficient conditions for the existence and uniqueness of a solutions to the impulsive problem (6) using Banach's and Krasnoselskii's fixed point theorems.

In view of Lemma 3, we define an operator $\mathcal{Q}:\mathcal{PC}(\mathcal{J},\mathbb{R})\to\mathcal{PC}(\mathcal{J},\mathbb{R})$ as

$$
\begin{aligned}
(\mathcal{Q}x)(t) \;=\; & \Big\{e^{\frac{\rho_k-1}{\rho_k}\left(\psi_k(t)-\psi_k(t_k)\right)}\,\mathbb{E}_{\alpha_k}\big(-\lambda\left(\psi_k(t)-\psi_k(t_k)\right)^{\alpha_k}\big)\Big\}\Bigg\{\Bigg[\gamma\\
& -\eta\sum_{i=1}^{m}\Bigg(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i}\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\big(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}}\big)F_x(s)\psi_{i-1}'(s)ds\\
& +\varphi_i(x(t_i))\Bigg)\prod_{j=i+1}^{m+1} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\mathbb{E}_{\alpha_{j-1}}\big(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\big)\\
& -\frac{\eta}{\rho_m^{\alpha_m}}\int_{t_m}^{T}\Phi_m^{\alpha_m}(T,s)\mathbb{E}_{\alpha_m,\alpha_m}\big(-\lambda\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m}\big)F_x(s)\psi_m'(s)ds\Bigg]\\
& \times\Bigg[\frac{1}{\Omega}\prod_{i=1}^{k} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)}\,\mathbb{E}_{\alpha_{i-1}}\big(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\big)\Bigg]\\
& +\sum_{i=1}^{k}\Bigg(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i}\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\big(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}}\big)F_x(s)\psi_{i-1}'(s)ds\\
& +\varphi_i(x(t_i))\Bigg)\prod_{j=i+1}^{k} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\,\mathbb{E}_{\alpha_{j-1}}\big(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\big)\Bigg\}\\
& +\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{t}\Phi_k^{\alpha_k}(t,s)\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\left(\psi_k(t)-\psi_k(s)\right)^{\alpha_k}\big)F_x(s)\psi_k'(s)ds,
\end{aligned}
\tag{14}
$$

where $F_x(t)=f(t,x(t),x(\mu t))$. Notice that $\mathcal{Q}$ has fixed points if and only if the impulsive problem (6) has solutions.

3.1. Uniqueness Property

Theorem 1. *Let $f\in\mathcal{C}(\mathcal{J}\times\mathbb{R}^2,\mathbb{R})$ and $\varphi_k\in\mathcal{C}(\mathbb{R},\mathbb{R})$ for $k=1,2,\ldots,m$. Assume that*

Hypothesis 1 (H1). *there exists a constant $\mathbb{L}_1>0$ such that $|f(t,x_1,y_1)-f(t,x_2,y_2)|\le\mathbb{L}_1(|x_1-x_2|+|y_1-y_2|)$, for all $t\in\mathcal{J}$, $x_i,y_i\in\mathbb{R}$, $i=1,2$.*

Hypothesis 2 (H2). *there exists a constant* $\mathbb{M}_1 > 0$ *such that* $|\varphi_k(x) - \varphi_k(y)| \leq \mathbb{M}_1|x-y|$, *for each* $x, y \in \mathbb{R}$, $k = 1, 2, \ldots, m$.

Then, the impulsive problem (6) *has a unique solution on* $\mathcal{J}$ *if*

$$\left(1 + \frac{|\eta|}{|\Omega|}\right)\left(2\mathbb{L}_1 \sum_{i=1}^{m+1} \frac{\left(\psi_{i-1}(t_i) - \psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)} + m\mathbb{M}_1\right) < 1. \tag{15}$$

Proof. Before proving this theorem, we convert the impulsive problem (6) into $x = \mathcal{Q}x$ (a fixed point problem), where the operator $\mathcal{Q}$ is defined by (14). It is clear that the fixed points of the operator $\mathcal{Q}$ are solutions of the impulsive problem (6).

Let $\sup_{t\in\mathcal{J}} |f(t,0,0)| := \mathbb{N}_1 < \infty$ and $\mathbb{N}_2 = \max\{|\varphi_k(0)|, k = 1, 2, \ldots, m\}$. Next, we set $B_{r_1} := \{x \in \mathcal{PC}(\mathcal{J}, \mathbb{R}) : \|x\| \leq r_1\}$ with

$$r_1 \geq \frac{\left(1 + \frac{|\eta|}{|\Omega|}\right)\left(\mathbb{N}_1 \sum_{i=1}^{m+1} \frac{\left(\psi_{i-1}(t_i) - \psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)} + m\mathbb{N}_2\right) + \frac{|\gamma|}{|\Omega|}}{1 - \left(1 + \frac{|\eta|}{|\Omega|}\right)\left(2\mathbb{L}_1 \sum_{i=1}^{m+1} \frac{\left(\psi_{i-1}(t_i) - \psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)} + m\mathbb{M}_1\right)}. \tag{16}$$

Clearly, B_{r_1} is a bounded, closed, and convex subset of $\mathcal{PC}(\mathcal{J}, \mathbb{R})$. The argument of the proof is separated into two steps:

Step 1. We show that $\mathcal{Q}B_{r_1} \subset B_{r_1}$.

For any $x \in B_{r_1}$, we have

$$\begin{aligned}
|(\mathcal{Q}x)(t)| \leq\ & \left\{\left|e^{\frac{\rho_k - 1}{\rho_k}\left(\psi_k(t) - \psi_k(t_k)\right)}\right| \left|\mathbb{E}_{\alpha_k}\left(-\lambda\left(\psi_k(t) - \psi_k(t_k)\right)^{\alpha_k}\right)\right|\right\}\left\{\left[|\gamma|\right.\right. \\
& + |\eta| \sum_{i=1}^{m}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i} |\Phi_{i-1}^{\alpha_{i-1}}(t_i, s)| \left|\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i) - \psi_{i-1}(s)\right)^{\alpha_{i-1}}\right)\right| |F_x(s)|\psi'_{i-1}(s)ds\right. \\
& \left.\left. + |\varphi_i(x(t_i))|\right) \prod_{j=i+1}^{m+1} \left|e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j) - \psi_{j-1}(t_{j-1})\right)}\right| \left|\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j) - \psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\right| \right. \\
& \left. + \frac{|\eta|}{\rho_m^{\alpha_m}}\int_{t_m}^{T} |\Phi_m^{\alpha_m}(T, s)| \left|\mathbb{E}_{\alpha_m,\alpha_m}\left(-\lambda\left(\psi_m(T) - \psi_m(s)\right)^{\alpha_m}\right)\right| |F_x(s)|\psi'_m(s)ds\right] \\
& \times \left[\frac{1}{|\Omega|}\prod_{i=1}^{m}\left|e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i) - \psi_{i-1}(t_{i-1})\right)}\right| \left|\mathbb{E}_{\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i) - \psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\right)\right|\right] \\
& + \sum_{i=1}^{k}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i} \left|\Phi_{i-1}^{\alpha_{i-1}}(t_i, s)\right| \left|\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i) - \psi_{i-1}(s)\right)^{\alpha_{i-1}}\right)\right| |F_x(s)|\psi'_{i-1}(s)ds\right. \\
& \left. + |\varphi_i(x(t_i))|\right) \prod_{j=i+1}^{k} \left|e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j) - \psi_{j-1}(t_{j-1})\right)}\right| \left|\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j) - \psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\right| \Bigg\} \\
& + \frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{t} \left|\Phi_k^{\alpha_k}(t, s)\right| \left|\mathbb{E}_{\alpha_k,\alpha_k}\left(-\lambda\left(\psi_k(t) - \psi_k(s)\right)^{\alpha_k}\right)\right| |F_x(s)|\psi'_k(s)ds.
\end{aligned} \tag{17}$$

Applying Lemma 1 and (9) with $0 < e^{\frac{\rho_i - 1}{\rho_i}\left(\psi_i(u) - \psi_i(v)\right)} \leq 1$, for any $0 < v < u < T$, $0 < \rho_i \leq 1$, $i = 0, 1, \ldots, m$, we have

$$
\begin{aligned}
|(\mathcal{Q}x)(t)| \leq\ & \frac{1}{|\Omega|}\Bigg[|\gamma|+|\eta|\sum_{i=1}^{m}\Bigg(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}|F_x(s)|\psi_{i-1}'(s)ds+|\varphi_i(x(t_i))|\Bigg)\\
&+\frac{|\eta|}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}|F_x(s)|\psi_m'(s)ds\Bigg] \qquad (18)\\
&+\sum_{i=1}^{m}\Bigg(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}|F_x(s)|\psi_{i-1}'(s)ds+|\varphi_i(x(t_i))|\Bigg)\\
&+\frac{1}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}|F_x(s)|\psi_m'(s)ds.
\end{aligned}
$$

From (H_1) and (H_2), we can compute that

$$|F_x(t)| \leq |f(t,x(t),x(\mu t))-f(t,0,0)|+|f(t,0,0)| \leq 2\mathbb{L}_1\|x\|+\mathbb{N}_1, \qquad (19)$$

and

$$|\varphi_k(x(t_k))| \leq |\varphi_k(x(t_k))-\varphi_k(0)|+|\varphi_k(0)| \leq \mathbb{M}_1\|x\|+\mathbb{N}_2. \qquad (20)$$

Inserting (19) and (20) into (18), we have

$$
\begin{aligned}
|(\mathcal{Q}x)(t)| \leq\ & \frac{1}{|\Omega|}\Bigg[|\gamma|+|\eta|\sum_{i=1}^{m}\Bigg(\frac{2\mathbb{L}_1\|x\|+\mathbb{N}_1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\psi_{i-1}'(s)ds+\mathbb{M}_1\|x\|+\mathbb{N}_2\Bigg)\\
&+\frac{|\eta|(2\mathbb{L}_1\|x\|+\mathbb{N}_1)}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\psi_m'(s)ds\Bigg]\\
&+\sum_{i=1}^{m}\Bigg(\frac{2\mathbb{L}_1\|x\|+\mathbb{N}_1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\psi_{i-1}'(s)ds+\mathbb{M}_1\|x\|+\mathbb{N}_2\Bigg)\\
&+\frac{2\mathbb{L}_1\|x\|+\mathbb{N}_1}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\psi_m'(s)ds\\
=\ & \frac{1}{|\Omega|}\Bigg[|\gamma|+|\eta|\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}(2\mathbb{L}_1\|x\|+\mathbb{N}_1)+|\eta|m(\mathbb{M}_1\|x\|+\mathbb{N}_2)\Bigg]\\
&+\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}-1}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}(2\mathbb{L}_1\|x\|+\mathbb{N}_1)+m(\mathbb{M}_1\|x\|+\mathbb{N}_2)\\
=\ & \left(1+\frac{|\eta|}{|\Omega|}\right)\left(2\mathbb{L}_1\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+m\mathbb{M}_1\right)\|x\|\\
&+\left(1+\frac{|\eta|}{|\Omega|}\right)\left(\mathbb{N}_1\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+m\mathbb{N}_2\right)+\frac{|\gamma|}{|\Omega|}\leq r_1.
\end{aligned}
$$

Then, $\|\mathcal{Q}x\| \leq r_1$, implies that, $\mathcal{Q}B_{r_1} \subset B_{r_1}$.

Step 2. We show that $\mathcal{Q}$ is a contraction.

For any $x, y \in B_{r_1}$, and for each $t \in \mathcal{J}$, we obtain

$$
\begin{aligned}
&|(\mathcal{Q}x)(t)-(\mathcal{Q}y)(t)|\\
\leq\ & \left\{\left|e^{\frac{\rho_k-1}{\rho_k}\left(\psi_k(t)-\psi_k(t_k)\right)}\right|\left|\mathbb{E}_{\alpha_k}\left(-\lambda\left(\psi_k(t)-\psi_k(t_k)\right)^{\alpha_k}\right)\right|\right\}\\
&\times\left\{\left[|\eta|\sum_{i=1}^{m}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i}\left|\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\right|\left|\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}}\right)\right|\right.\right.\right.\\
&\left.\left.\left.\times\left|F_x(s)-F_y(s)\right|\psi_{i-1}'(s)ds+|\varphi_i(x(t_i))-\varphi_i(y(t_i))|\right)\right.\right.\\
&\left.\left.\times\prod_{j=i+1}^{m+1}\left|e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\right|\left|\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\right|\right.\\
&\left.+\frac{|\eta|}{\rho_m^{\alpha_m}}\int_{t_m}^{T}|\Phi_m^{\alpha_m}(T,s)|\left|\mathbb{E}_{\alpha_m,\alpha_m}\left(-\lambda\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m}\right)\right|\left|F_x(s)-F_y(s)\right|\psi_m'(s)ds\right]\\
&\times\left[\frac{1}{|\Omega|}\prod_{i=1}^{m}\left|e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)}\right|\left|\mathbb{E}_{\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\right)\right|\right]\\
&+\sum_{i=1}^{k}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i}\left|\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\right|\left|\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}}\right)\right|\right.\\
&\left.\times\left|F_x(s)-F_y(s)\right|\psi_{i-1}'(s)ds+|\varphi_i(x(t_i))-\varphi_i(y(t_i))|\right)\\
&\times\prod_{j=i+1}^{k}\left|e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\right|\left|\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\right|\Bigg\}\\
&+\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{t}\left|\Phi_k^{\alpha_k}(t,s)\right|\left|\mathbb{E}_{\alpha_k,\alpha_k}\left(-\lambda\left(\psi_k(t)-\psi_k(s)\right)^{\alpha_k}\right)\right|\left|F_x(s)-F_y(s)\right|\psi_k'(s)ds.
\end{aligned}
$$

Applying Lemma 1 and (9) with $0<e^{\frac{\rho_i-1}{\rho_i}\left(\psi_i(u)-\psi_i(v)\right)}\leq 1$, for any $0<v<u<T$, $0<\rho_i\leq 1$, $i=0,1,\ldots,m$, we have

$$
\begin{aligned}
&|(\mathcal{Q}x)(t)-(\mathcal{Q}y)(t)|\\
\leq\ & \frac{1}{|\Omega|}\left[|\eta|\sum_{i=1}^{m}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\left|F_x(s)-F_y(s)\right|\psi_{i-1}'(s)ds\right.\right.\\
&\left.\left.+|\varphi_i(x(t_i))-\varphi_i(y(t_i))|\right)+\frac{|\eta|}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\left|F_x(s)-F_y(s)\right|\psi_m'(s)ds\right]\\
&+\sum_{i=1}^{m}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\left|F_x(s)-F_y(s)\right|\psi_{i-1}'(s)ds\right.\\
&\left.+|\varphi_i(x(t_i))-\varphi_i(y(t_i))|\right)+\frac{1}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left(\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\left|F_x(s)-F_y(s)\right|\psi_m'(s)ds.\right.
\end{aligned}
$$

From (H_1) and (H_2), one has

$$
\begin{aligned}
& |(\mathcal{Q}x)(t) - (\mathcal{Q}y)(t)| \\
\leq\ & \frac{1}{|\Omega|}\Bigg[|\eta|\sum_{i=1}^{m}\Bigg(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\psi_{i-1}'(s)ds\, 2\mathbb{L}_1\|x-y\| \\
& +\mathbb{M}_1\|x-y\|\Bigg) + \frac{|\eta|}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\psi_m'(s)ds\, 2\mathbb{L}_1\|x-y\|\Bigg] \\
& +\sum_{i=1}^{m}\Bigg(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\psi_{i-1}'(s)ds\, 2\mathbb{L}_1\|x-y\| \\
& +\mathbb{M}_1\|x-y\|\Bigg) + \frac{1}{\rho_m^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\psi_m'(s)ds\, 2\mathbb{L}_1\|x-y\| \\
=\ & \Bigg[\frac{|\eta|}{|\Omega|}\Bigg(\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}2\mathbb{L}_1+m\mathbb{M}_1\Bigg) \\
& +\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}2\mathbb{L}_1+m\mathbb{M}_1\Bigg]\|x-y\| \\
=\ & \left(1+\frac{|\eta|}{|\Omega|}\right)\left(2\mathbb{L}_1\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+m\mathbb{M}_1\right)\|x-y\|.
\end{aligned}
$$

By (15), the operator $\mathcal{Q}$ is a contraction map. According to Banach's fixed point theorem, we can conclude that the impulsive problem (6) has a unique solution. □

3.2. Existence Property

Lemma 4 (Generalized Arzelá–Ascoli theorem, Theorem 2.1, [52]). *Let $\mathbb{E}$ be a Banach space, $\mathcal{J} := [0,T]$ and $\mathcal{M} \subset \mathcal{PC}(\mathcal{J},\mathbb{R})$. If the following assumptions are satisfied: (i) $\mathcal{M}$ is a uniformly bounded subset of $\mathcal{PC}(\mathcal{J},\mathbb{R})$; (ii) $\mathcal{M}$ is equicontinuous in (t_k,t_{k+1}), $k = 0,1,\ldots,m$, where $t_0 = 0$, $t_{m+1} = T$; (iii) Its t-sections $\mathcal{M}(t) = \{x(t) : x \in \mathcal{M}, t \in \mathcal{J}\setminus\{t_1,t_2,\ldots,t_m\}, \mathcal{M}(t_k^+) = \{x(t_k^+) : x \in \mathcal{M} \text{ and } \mathcal{M}(t_k^-) = \{x(t_k^-) : x \in \mathcal{M}\}\}$ are relatively compact subsets of $\mathbb{E}$, then $\mathcal{M}$ is a relatively compact subset of $\mathcal{PC}(\mathcal{J},\mathbb{R})$.*

Lemma 5 (Krasnoselskii's fixed point theorem [53]). *Let $\mathbb{D}$ be a closed, convex, and nonempty subset of a Banach space $\mathbb{E}$, and let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be operators such that: (i) $\mathcal{Q}_1x + \mathcal{Q}_2y \in \mathbb{D}$ whenever $x, y \in \mathbb{D}$; (ii) $\mathcal{Q}_1$ is compact and continuous; (iii) $\mathcal{Q}_2$ is a contraction mapping. Then there exists $z \in \mathbb{D}$ such that $z = \mathcal{Q}_1z + \mathcal{Q}_2z$.*

The existence result is based on Krasnoselskii's fixed point theorem

Theorem 2. *Let $f : \mathcal{J}\times\mathbb{R}^2 \to \mathbb{R}$ and $\varphi_k : \mathbb{R}\to\mathbb{R}$, $k = 1,2,\ldots,m$, be continuous functions. Assume the assumption (H_2) holds and the following assumptions are satisfied:*

Hypothesis 3 (H3). *there exists a function $\xi \in L^{\frac{1}{q}}(\mathcal{J},\mathbb{R}^+)$, $(0 < q < \alpha_k < 1)$, $k = 0,1,\ldots,m+1$, exists, and $\omega \in \mathcal{C}(\mathcal{J},[0,\infty))$ is a nondecreasing function satisfying the following inequality $|f(t,x(s),x(\mu s))| \leq \xi(t)\omega(\|x\|)$, for all $t \in \mathcal{J}$, $x \in \mathcal{PC}(\mathcal{J},\mathbb{R})$.*

Then, the impulsive problem (6) has at least one solution on $\mathcal{J}$ if

$$
\left(1+\frac{|\eta|}{|\Omega|}\right)\left(\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}-q}}{\left(\frac{\alpha_{i-1}-q}{1-q}\right)^{1-q}\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\|\xi\|_{L^{\frac{1}{q}}(\mathcal{J})}\liminf_{r_2\to+\infty}\frac{\omega(r_2)}{r_2}+m\mathbb{M}_1\right) < 1. \tag{21}
$$

Proof. Let us define a suitable $B_{r_2} = \{x \in \mathcal{PC}(\mathcal{J},\mathbb{R}) : \|x\| \leq r_2\}$. Obviously, B_{r_2} is a bounded, closed, and convex subset of $\mathcal{PC}(\mathcal{J},\mathbb{R})$, for each $r_2 > 0$. Next, we define the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on B_{r_2} for $t \in \mathcal{J}$ as

$$\begin{aligned}
&(\mathcal{Q}_1 x)(t)\\
=\ & \left\{ e^{\frac{\rho_k-1}{\rho_k}\left(\psi_k(t)-\psi_k(t_k)\right)}\ \mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t)-\psi_k(t_k)\big)^{\alpha_k}\big)\right\}\Bigg\{\Bigg[\gamma\\
&-\eta\sum_{i=1}^{m}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i}\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\big(-\lambda\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\alpha_{i-1}}\big)F_x(s)\psi_{i-1}'(s)ds\right)\\
&\times\prod_{j=i+1}^{m+1} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\ \mathbb{E}_{\alpha_{j-1}}\big(-\lambda\big(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\big)^{\alpha_{j-1}}\big)\\
&-\frac{\eta}{\rho_m^{\alpha_m}}\int_{t_m}^{T}\Phi_m^{\alpha_m}(T,s)\mathbb{E}_{\alpha_m,\alpha_m}\big(-\lambda\big(\psi_m(T)-\psi_m(s)\big)^{\alpha_m}\big)F_x(s)\psi_m'(s)ds\Bigg]\\
&\times\left[\frac{1}{\Omega}\prod_{i=1}^{k} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)}\ \mathbb{E}_{\alpha_{i-1}}\big(-\lambda\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}\big)\right]\\
&+\sum_{i=1}^{k}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i}\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\big(-\lambda\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\alpha_{i-1}}\big)F_x(s)\psi_{i-1}'(s)ds\right)\\
&\times\prod_{j=i+1}^{k} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\ \mathbb{E}_{\alpha_{j-1}}\big(-\lambda\big(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\big)^{\alpha_{j-1}}\big)\Bigg\}\\
&+\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{t}\Phi_k^{\alpha_k}(t,s)\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(t)-\psi_k(s)\big)^{\alpha_k}\big)F_x(s)\psi_k'(s)ds,
\end{aligned}$$

and

$$\begin{aligned}
&(\mathcal{Q}_2 x)(t)\\
=\ & \left\{ e^{\frac{\rho_k-1}{\rho_k}\left(\psi_k(t)-\psi_k(t_k)\right)}\ \mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t)-\psi_k(t_k)\big)^{\alpha_k}\big)\right\}\Bigg\{\Bigg[-\eta\sum_{i=1}^{m}\varphi_i(x(t_i))\\
&\times\prod_{j=i+1}^{m+1} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\ \mathbb{E}_{\alpha_{j-1}}\big(-\lambda\big(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\big)^{\alpha_{j-1}}\big)\Bigg]\\
&\times\left[\frac{1}{\Omega}\prod_{i=1}^{k} e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)}\ \mathbb{E}_{\alpha_{i-1}}\big(-\lambda\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}\big)\right]\\
&+\sum_{i=1}^{k}\varphi_i(x(t_i))\prod_{j=i+1}^{k} e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\ \mathbb{E}_{\alpha_{j-1}}\big(-\lambda\big(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\big)^{\alpha_{j-1}}\big)\Bigg\}.
\end{aligned}$$

Note that $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$.

Step 1. We show that $r_2^* > 0$ with $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_{r_2^*}$ for each $x, y \in B_{r_2^*}$.

Suppose by contradiction that for any $r_2 > 0$ there exist x_{r_2}, $y_{r_2} \in B_{r_2^*}$ and $t_{r_2} \in \mathcal{J}$ such that $|(\mathcal{Q}_2 x_{r_2})(t_{r_2}) + (\mathcal{Q}_1 x_{r_2})(t_{r_2})| > r_2$.

By using Lemma 1, (9), and (H_3) with the Hölder inequality, for any $x \in B_{r_2}$, we have

$$
\begin{aligned}
&\int_{t_{i-1}}^{t_i}\left|\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\right|\left|\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}}\right)\right||F_x(s)|\psi'_{i-1}(s)ds\\
\leq\ & \frac{1}{\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\xi(s)\omega(r_2)\psi'_{i-1}(s)ds\\
\leq\ & \frac{1}{\Gamma(\alpha_{i-1})}\left(\int_{t_{i-1}}^{t_i}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\frac{\alpha_{i-1}-1}{1-q}}\psi'_{i-1}(s)ds\right)^{1-q}\left(\int_{t_{i-1}}^{t_i}\left(\xi(s)\omega(r_2)\right)^{\frac{1}{q}}ds\right)^{q}\\
\leq\ & \frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}-q}}{\left(\frac{\alpha_{i-1}-q}{1-q}\right)^{1-q}\Gamma(\alpha_{i-1})}\left\|\xi\right\|_{L^{\frac{1}{q}}(\mathcal{J})}\omega(r_2). \qquad (22)
\end{aligned}
$$

By direct calculation with Lemma 1, (9) and (20), we have

$$
\begin{aligned}
r_2 <\ & |(\mathcal{Q}_1x_{r_2})(t_{r_2})+(\mathcal{Q}_2y_{r_2})(t_{r_2})|\\
\leq\ & \left\{\left|e^{\frac{\rho_m-1}{\rho_m}\left(\psi_m(t_{r_2})-\psi_m(t_m)\right)}\right|\left|\mathbb{E}_{\alpha_m}\left(-\lambda\left(\psi_m(t_{r_2})-\psi_m(t_m)\right)^{\alpha_m}\right)\right|\right\}\left\{\left[|\gamma|\right.\right.\\
&+|\eta|\sum_{i=1}^{m}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i}\left|\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\right|\left|\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}}\right)\right|\left|F_{x_{r_2}}(s)\right|\psi'_{i-1}(s)ds\right)\\
&\times\prod_{j=i+1}^{m+1}\left|e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\right|\left|\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\right|\\
&\left.+\frac{|\eta|}{\rho_m^{\alpha_m}}\int_{t_m}^{T}|\Phi_m^{\alpha_m}(T,s)|\left|\mathbb{E}_{\alpha_m,\alpha_m}\left(-\lambda\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m}\right)\right|\left|F_{x_{r_2}}(s)\right|\psi'_m(s)ds\right]\\
&\times\left[\frac{1}{|\Omega|}\prod_{i=1}^{m}\left|e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)}\right|\left|\mathbb{E}_{\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\right)\right|\right]\\
&+\sum_{i=1}^{m}\left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}}\int_{t_{i-1}}^{t_i}\left|\Phi_{i-1}^{\alpha_{i-1}}(t_i,s)\right|\left|\mathbb{E}_{\alpha_{i-1},\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}}\right)\right|\left|F_{x_{r_2}}(s)\right|\psi'_{i-1}(s)ds\right)\\
&\times\left.\prod_{j=i+1}^{m}\left|e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\right|\left|\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\right|\right\}\\
&+\frac{1}{\rho_m^{\alpha_m}}\int_{t_m}^{t_{r_2}}|\Phi_m^{\alpha_m}(t_{r_2},s)|\left|\mathbb{E}_{\alpha_m,\alpha_m}\left(-\lambda\left(\psi_m(t_{r_2})-\psi_m(s)\right)^{\alpha_m}\right)\right|\left|F_{x_{r_2}}(s)\right|\psi'_m(s)ds\\
&+\left\{\left|e^{\frac{\rho_m-1}{\rho_m}\left(\psi_m(t_{r_2})-\psi_m(t_m)\right)}\right|\left|\mathbb{E}_{\alpha_m}\left(-\lambda\left(\psi_m(t_{r_2})-\psi_m(t_m)\right)^{\alpha_m}\right)\right|\right\}\left\{\left[|\eta|\sum_{i=1}^{m}|\varphi_i(y_{r_2}(t_i))|\right.\right.\\
&\times\prod_{j=i+1}^{m+1}\left|e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\right|\left|\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\right|\Bigg]\\
&\times\left[\frac{1}{|\Omega|}\prod_{i=1}^{m}\left|e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)}\right|\left|\mathbb{E}_{\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\right)\right|\right]\\
&\left.+\sum_{i=1}^{m}|\varphi_i(y_{r_2}(t_i))|\prod_{j=i+1}^{m}\left|e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\right|\left|\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\right|\right\}\\
\leq\ & \frac{|\gamma|}{|\Omega|}+\left(1+\frac{|\eta|}{|\Omega|}\right)\left(\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}-q}}{\left(\frac{\alpha_{i-1}-q}{1-q}\right)^{1-q}\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\left\|\xi\right\|_{L^{\frac{1}{q}}(\mathcal{J})}\omega(r_2)+m(\mathbb{M}_1r_2+\mathbb{N}_2)\right).
\end{aligned}
$$

Dividing both sides in the above inequality by r_2 and taking the lower limit as $r_2 \to +\infty$, we obtain

$$1 \leq \left(1+\frac{|\eta|}{|\Omega|}\right)\left(\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}-q}}{\left(\frac{\alpha_{i-1}-q}{1-q}\right)^{1-q}\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\|\xi\|_{L^{\frac{1}{q}}(\mathcal{J})}\liminf_{r_2\to+\infty}\frac{\omega(r_2)}{r_2}+m\mathbb{M}_1\right),$$

which contradicts (21). Then, there exists $r_2 > 0$ so that $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_{r_2}$, for $x, y \in B_{r_2}$.

Step 2. We show that $\mathcal{Q}_2$ is a contraction mapping on B_{r_2}.

For each $t \in \mathcal{J}$ and $x, y \in B_{r_2}$, we have

$$\begin{aligned}
&|(\mathcal{Q}_2 x)(t) - (\mathcal{Q}_2 y)(t)| \\
\leq\ & \left\{\left|e^{\frac{\rho_m-1}{\rho_m}\left(\psi_m(T)-\psi_m(t_m)\right)}\right|\left|\mathbb{E}_{\alpha_m}\left(-\lambda\left(\psi_m(T)-\psi_m(t_m)\right)^{\alpha_m}\right)\right|\right\} \\
&\times\left\{\left[|\eta|\sum_{i=1}^{m}|\varphi_i(x(t_i))-\varphi_i(y(t_i))|\prod_{j=i+1}^{m+1}\left|e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\right|\right.\right. \\
&\left.\left.\times\left|\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\right|\right] \\
&\times\left[\frac{1}{|\Omega|}\prod_{i=1}^{m}\left|e^{\frac{\rho_{i-1}-1}{\rho_{i-1}}\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)}\right|\left|\mathbb{E}_{\alpha_{i-1}}\left(-\lambda\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\right)\right|\right] \\
&+\sum_{i=1}^{m}|\varphi_i(x(t_i))-\varphi_i(y(t_i))|\prod_{j=i+1}^{m}\left|e^{\frac{\rho_{j-1}-1}{\rho_{j-1}}\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)}\right| \\
&\times\left|\mathbb{E}_{\alpha_{j-1}}\left(-\lambda\left(\psi_{j-1}(t_j)-\psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right)\right|\Bigg\} \\
\leq\ & \left(1+\frac{|\eta|}{|\Omega|}\right)m\mathbb{M}_1\|x-y\|.
\end{aligned}$$

By setting $\epsilon^* = (1 + (|\eta|/|\Omega|))m\mathbb{M}_1$ with (21), we obtain $0 < \epsilon^* < 1$ and $\|\mathcal{Q}_2 x - \mathcal{Q}_2 y\|_{\mathcal{PC}} \leq \epsilon^* \|x - y\|$. Then, $\mathcal{Q}_2$ is a contraction mapping.

Step 3. We show that $\mathcal{Q}_1$ is compact and continuous on B_{r_2}.

From the property of continuity of f implies that $\mathcal{Q}_1$ is also continuous. Next, we show that $\mathcal{Q}_1$ is compact. By the same process as in the first part of Theorem 1, which implies that $\mathcal{Q}_1(B_{r_2})$ is uniformly bounded on $\mathcal{PC}(\mathcal{J}, \mathbb{R})$. We will show that $\mathcal{Q}_1(B_{r_2})$ is an equicontinuous on $\mathcal{J}_k$, for $k = 1, 2, \ldots, m$.

Let $\mathbb{S} = \mathcal{J} \times B_{r_2}^2$ and $f^* = \sup_{t\in\mathcal{J}}|F_x(t)| = \sup_{(t,x(t),x(\mu t))\in\mathbb{S}}|f(t, x(t), x(\mu t))|$. Then, for any $t_k < \tau_1 < \tau_2 \leq t_{k+1}$, we have

$$
\begin{aligned}
&|(\mathcal{Q}_1 x)(\tau_2) - (\mathcal{Q}_1 x)(\tau_1)| \\
\leq\ & \left\{ \left| e^{\frac{\rho_k - 1}{\rho_k}\left(\psi_k(\tau_2) - \psi_k(t_k)\right)} \mathbb{E}_{\alpha_k}\left(-\lambda \left(\psi_k(t_2) - \psi_k(t_k)\right)^{\alpha_k}\right) - e^{\frac{\rho_k - 1}{\rho_k}\left(\psi_k(\tau_1) - \psi_k(t_k)\right)} \mathbb{E}_{\alpha_k}\left(-\lambda \left(\psi_k(t_1) - \psi_k(t_k)\right)^{\alpha_k}\right) \right| \right\} \\
&\times \left\{ \left[|\gamma| + |\eta| \sum_{i=1}^{m} \left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}} \int_{t_{i-1}}^{t_i} \left| \Phi_{i-1}^{\alpha_{i-1}}(t_i, s) \right| \left| \mathbb{E}_{\alpha_{i-1}, \alpha_{i-1}}\left(-\lambda \left(\psi_{i-1}(t_i) - \psi_{i-1}(s)\right)^{\alpha_{i-1}}\right) \right| |F_x(s)| \psi_{i-1}'(s) ds \right) \right. \right. \\
&\times \prod_{j=i+1}^{m+1} \left| e^{\frac{\rho_{j-1} - 1}{\rho_{j-1}}\left(\psi_{j-1}(t_j) - \psi_{j-1}(t_{j-1})\right)} \right| \left| \mathbb{E}_{\alpha_{j-1}}\left(-\lambda \left(\psi_{j-1}(t_j) - \psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right) \right| \\
&\left. + \frac{|\eta|}{\rho_m^{\alpha_m}} \int_{t_m}^{T} |\Phi_m^{\alpha_m}(T, s)| \left| \mathbb{E}_{\alpha_m, \alpha_m}\left(-\lambda \left(\psi_m(T) - \psi_m(s)\right)^{\alpha_m}\right) \right| |F_x(s)| \psi_m'(s) ds \right] \\
&\times \left[\frac{1}{|\Omega|} \prod_{i=1}^{k} \left| e^{\frac{\rho_{i-1} - 1}{\rho_{i-1}}\left(\psi_{i-1}(t_i) - \psi_{i-1}(t_{i-1})\right)} \right| \left| \mathbb{E}_{\alpha_{i-1}}\left(-\lambda \left(\psi_{i-1}(t_i) - \psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\right) \right| \right] \\
&+ \sum_{i=1}^{k} \left(\frac{1}{\rho_{i-1}^{\alpha_{i-1}}} \int_{t_{i-1}}^{t_i} \left| \Phi_{i-1}^{\alpha_{i-1}}(t_i, s) \right| \left| \mathbb{E}_{\alpha_{i-1}, \alpha_{i-1}}\left(-\lambda \left(\psi_{i-1}(t_i) - \psi_{i-1}(s)\right)^{\alpha_{i-1}}\right) \right| |F_x(s)| \psi_{i-1}'(s) ds \right) \\
&\times \prod_{j=i+1}^{k} \left| e^{\frac{\rho_{j-1} - 1}{\rho_{j-1}}\left(\psi_{j-1}(t_j) - \psi_{j-1}(t_{j-1})\right)} \right| \left| \mathbb{E}_{\alpha_{j-1}}\left(-\lambda \left(\psi_{j-1}(t_j) - \psi_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}\right) \right| \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{\tau_2}\Phi_k^{\alpha_k}(\tau_2,s)\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k}\big)F_x(s)\psi_k'(s)ds\right.\\
&\left.-\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{\tau_1}\Phi_k^{\alpha_k}(\tau_1,s)\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_1)-\psi_k(s)\big)^{\alpha_k}\big)F_x(s)\psi_k'(s)ds\right|\\
\leq\ &\left\{\left|\mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t_2)-\psi_k(t_k)\big)^{\alpha_k}\big)-\mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t_1)-\psi_k(t_k)\big)^{\alpha_k}\big)\right|\right\}\left\{\frac{|\gamma|}{|\Omega|}\right.\\
&\left.+f^*\left(\frac{|\eta|}{|\Omega|}\sum_{i=1}^{m+1}\frac{\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\sum_{i=1}^{k}\frac{\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}\right)\right\}\\
&+\left|\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{\tau_1}\Phi_k^{\alpha_k}(\tau_2,s)\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k}\big)F_x(s)\psi_k'(s)ds\right.\\
&-\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{\tau_1}\Phi_k^{\alpha_k}(\tau_2,s)\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_1)-\psi_k(s)\big)^{\alpha_k}\big)F_x(s)\psi_k'(s)ds\\
&+\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{\tau_1}\Phi_k^{\alpha_k}(\tau_2,s)\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_1)-\psi_k(s)\big)^{\alpha_k}\big)F_x(s)\psi_k'(s)ds\\
&-\frac{1}{\rho_k^{\alpha_k}}\int_{t_k}^{\tau_1}\Phi_k^{\alpha_k}(\tau_1,s)\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_1)-\psi_k(s)\big)^{\alpha_k}\big)F_x(s)\psi_k'(s)ds\\
&\left.+\frac{1}{\rho_k^{\alpha_k}}\int_{\tau_1}^{\tau_2}\Phi_k^{\alpha_k}(\tau_2,s)\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k}\big)F_x(s)\psi_k'(s)ds\right|\\
\leq\ &\left\{\left|\mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t_2)-\psi_k(t_k)\big)^{\alpha_k}\big)-\mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t_1)-\psi_k(t_k)\big)^{\alpha_k}\big)\right|\right\}\left\{\frac{|\gamma|}{|\Omega|}\right.\\
&\left.+f^*\left(\frac{|\eta|}{|\Omega|}\sum_{i=1}^{m+1}\frac{\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\sum_{i=1}^{k}\frac{\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}\right)\right\}\\
&+\frac{f^*}{\rho_k^{\alpha_k}\Gamma(\alpha_k)}\int_{t_k}^{\tau_1}\big|\Phi_k^{\alpha_k}(\tau_2,s)-\Phi_k^{\alpha_k}(\tau_1,s)\big|\psi_k'(s)ds+\frac{f^*}{\rho_k^{\alpha_k}\Gamma(\alpha_k)}\int_{\tau_1}^{\tau_2}\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k}\psi_k'(s)ds\\
&+\frac{f^*}{\rho_k^{\alpha_k}}\int_{t_k}^{\tau_1}\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k-1}\Big|\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k}\big)-\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_1)-\psi_k(s)\big)^{\alpha_k}\big)\Big|\psi_k'(s)ds\\
\leq\ &\left\{\left|\mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t_2)-\psi_k(t_k)\big)^{\alpha_k}\big)-\mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t_1)-\psi_k(t_k)\big)^{\alpha_k}\big)\right|\right\}\left\{\frac{|\gamma|}{|\Omega|}\right.\\
&\left.+f^*\left(\frac{|\eta|}{|\Omega|}\sum_{i=1}^{m+1}\frac{\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\sum_{i=1}^{k}\frac{\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}\right)\right\}\\
&+\frac{f^*}{\rho_k^{\alpha_k}\Gamma(\alpha_k+1)}\Big|2\big(\psi_k(\tau_2)-\psi_k(\tau_1)\big)^{\alpha_k}+\big(\psi_k(\tau_2)-\psi_k(t_k)\big)^{\alpha_k}-\big(\psi_k(\tau_1)-\psi_k(t_k)\big)^{\alpha_k}\Big|\\
&+\frac{f^*}{\rho_k^{\alpha_k}}\int_{t_k}^{\tau_1}\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k-1}\Big|\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k}\big)-\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_1)-\psi_k(s)\big)^{\alpha_k}\big)\Big|\psi_k'(s)ds.
\end{aligned}
$$

By (ii) as in Lemma 1, which implies that $\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(t)-\psi_k(s)\big)^{\alpha_k}\big)$ is continuous on $t\in\mathcal{J}$, and then $\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(t)-\psi_k(s)\big)^{\alpha_k}\big)$ is uniformly continuous on $t\in\mathcal{J}$; therefore, for any $\epsilon>0$, there is a sufficiently small $\delta>0$ such that, for τ_1, $\tau_2\in\mathcal{J}$ with $|\tau_2-\tau_1|\leq\delta$, we obtain

$$
\Big|\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_2)-\psi_k(t_k)\big)^{\alpha_k}\big)-\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_1)-\psi_k(t_k)\big)^{\alpha_k}\big)\Big|<\frac{\epsilon}{\big(\psi_k(\tau_1)-\psi_k(t_k)\big)^{\frac{\alpha_k}{2-\alpha_k}}}.
$$

Let $\xi_1(2-\alpha_k)/(2(1-\alpha_k))$ and $\xi_2 = (2-\alpha_k)/\alpha_k$. Thus, $\xi_1 > 1$, $\xi_2 > 1$, and $1/\xi_1 + 1/\xi_2 = 1$. By using the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{t_k}^{\tau_1} \big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k-1}\Big|\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k}\big)-\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_1)-\psi_k(s)\big)^{\alpha_k}\big)\Big|\psi_k'(s)ds\\
\leq\ &\left[\int_{t_k}^{\tau_1}\big(\psi_k(\tau_2)-\psi_k(s)\big)^{(\alpha_k-1)\frac{2-\alpha_k}{2(1-\alpha_k)}}\psi_k'(s)ds\right]^{\frac{2(1-\alpha_k)}{2-\alpha_k}}\\
&\times\left[\int_{t_k}^{\tau_1}\left(\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\alpha_k}\big)-\mathbb{E}_{\alpha_k,\alpha_k}\big(-\lambda\big(\psi_k(\tau_1)-\psi_k(s)\big)^{\alpha_k}\big)\right)^{\frac{2-\alpha_k}{\alpha_k}}\psi_k'(s)ds\right]^{\frac{\alpha_k}{2-\alpha_k}}\\
\leq\ &\left[\int_{t_k}^{\tau_1}\big(\psi_k(\tau_2)-\psi_k(s)\big)^{\frac{\alpha_k-2}{2}}\psi_k'(s)ds\right]^{\frac{2(1-\alpha_k)}{2-\alpha_k}}\left[\int_{t_k}^{\tau_1}\left(\frac{\epsilon}{\big(\psi_k(\tau_1)-\psi_k(t_k)\big)^{\frac{\alpha_k}{2-\alpha_k}}}\right)^{\frac{2-\alpha_k}{\alpha_k}}\psi_k'(s)ds\right]^{\frac{\alpha_k}{2-\alpha_k}}\\
\leq\ &\left[\frac{2\big(\psi_k(\tau_2)-\psi_k(t_k)\big)^{\frac{\alpha_k}{2}}-2\big(\psi_k(\tau_2)-\psi_k(\tau_1)\big)^{\frac{\alpha_k}{2}}}{\alpha_k}\right]^{\frac{2(1-\alpha_k)}{2-\alpha_k}}\epsilon.
\end{aligned}$$

Then

$$\begin{aligned}
&|(\mathcal{Q}_1x)(\tau_2)-(\mathcal{Q}_1x)(\tau_1)|\\
\leq\ &\left\{\Big|\mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t_2)-\psi_k(t_k)\big)^{\alpha_k}\big)-\mathbb{E}_{\alpha_k}\big(-\lambda\big(\psi_k(t_1)-\psi_k(t_k)\big)^{\alpha_k}\big)\Big|\right\}\left\{\frac{|\gamma|}{|\Omega|}\right.\\
&\left.+f^*\left(\frac{|\eta|}{|\Omega|}\sum_{i=1}^{m+1}\frac{\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+\sum_{i=1}^{k}\frac{\big(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\big)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}\right)\right\}\\
&+\frac{f^*}{\rho_k^{\alpha_k}\Gamma(\alpha_k+1)}\Big|2\big(\psi_k(\tau_2)-\psi_k(\tau_1)\big)^{\alpha_k}+\big(\psi_k(\tau_2)-\psi_k(t_k)\big)^{\alpha_k}-\big(\psi_k(\tau_1)-\psi_k(t_k)\big)^{\alpha_k}\Big|\\
&+\frac{f^*}{\rho_k^{\alpha_k}}\left[\frac{2\big(\psi_k(\tau_2)-\psi_k(t_k)\big)^{\frac{\alpha_k}{2}}-2\big(\psi_k(\tau_2)-\psi_k(\tau_1)\big)^{\frac{\alpha_k}{2}}}{\alpha_k}\right]^{\frac{2(1-\alpha_k)}{2-\alpha_k}}\quad \epsilon\to 0,\ \text{as } \tau_2\to\tau_1.
\end{aligned}$$

Hence, $\mathcal{Q}_1$ is equicontinuous on $\mathcal{J}_k$. Combining the above processes, and the $\mathcal{PC}(\mathcal{J},\mathbb{R})$-type Arzelá-Ascoli theorem (Lemma 4 in the case $\mathbb{E}=\mathbb{R}$), we conclude that $\mathcal{Q}_1 : B_{r_2}\to B_{r_2}$ is compact and completely continuous; therefore, it now follows by Lemma 5 that the problem (6) has at least one solution. □

4. Numerical Examples

This section presents three examples to illustrate our results.

Example 1. *Consider the following nonlinear impulsive Caputo proportional fractional BVPs.*

$$\begin{cases}
{}^{\frac{k+1}{k+2}}\mathfrak{D}_{t_k^+}^{\frac{k+2}{k+3},t^{\frac{1}{k+2}}}x(t)+\dfrac{2}{5}\left(\dfrac{k+1}{k+2}\right)^{\frac{k+2}{k+3}}x(t)=\dfrac{\cos(2t)}{16}\left(\dfrac{|x(t)|}{1+|x(t)|}+x\left(\dfrac{t}{4}\right)\right)+\dfrac{1}{2},\ t\neq\dfrac{1}{2},\\
\Delta x(\dfrac{1}{2})=\dfrac{1}{8}\tan^{-1}(x(\dfrac{1}{2})),\\
2x(0)+3x(1)=1.
\end{cases}\tag{23}$$

Here, $a=0$, $T=1$, $\alpha_k=(k+2)/(k+3)$, $\rho_k=(k+1)(k+2)$, $\psi_k=t^{1/(k+2)}$, $k=0,1$, $\lambda=2/5$, $\mu=1/4$,$\beta=2$, $\eta=3$, $\gamma=1$. *From the given data, we obtain the constants*

$\Omega \approx 2.84184$ *and* $\sum_{i=1}^{m+1}\left(\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}\right)/(\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)) \approx 1.84709$. *For* $u_1,u_2,v_1,v_2\in\mathbb{R}$ *and* $t\in[0,1]$, *we have*

$$\begin{aligned}|f(t,u_1,v_1)-f(t,u_2,v_2)| &\leq \frac{\cos(2t)}{16}\Big(|u_1-u_2|+|v_1-v_2|\Big),\\ |\varphi_k(x_1)-\varphi_k(x_2)| &\leq \frac{1}{8}|x_1-x_2|.\end{aligned}$$

The assumptions (H_1)-(H_2) *are satisfied with* $\mathbb{L}_1=1/16$ *and* $\mathbb{M}_1=1/8$. *Hence*

$$\left(1+\frac{|\eta|}{|\Omega|}\right)\left(2\mathbb{L}_1\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}}}{\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+m\mathbb{M}_1\right)\approx 0.731579<1.$$

Since, all assumptions of Theorem 1 are fulfilled, then (23) has a unique solution on $[0,1]$.

Example 2. *Consider the following nonlinear impulsive Caputo proportional fractional BVPs:*

$$\begin{cases} {}^{\frac{k+1}{7}}\mathfrak{D}_{t_k^+}^{\frac{2}{k+3},t^{\frac{2}{k+1}}}x(t)+\frac{1}{5}\left(\frac{k+1}{7}\right)^{\frac{2}{k+3}}x(t)=\frac{\sqrt[3]{t+1}}{16}\left(\frac{|x(t)|}{1+|x(t)|}+x\left(\frac{t}{2}\right)\right),\ t\neq\left\{\frac{1}{3},\frac{2}{3}\right\},\\ \Delta x(t_k)=\frac{x(t_k)}{8+x(t_k)},\quad k=1,2,\\ 3x(0)+4x(1)=2.\end{cases}\tag{24}$$

Here, $a=0$, $T=1$, $\alpha_k=2/(k+k)$, $\rho_k=(k+2)/4$, $\psi_k=t^{2/(k+1)}$, $k=0,1,2$, $\lambda=1/5$, $\mu=1/2$, $\beta=3$, $\eta=4$, $\gamma=2$. *From the given data, we have the constant* $\Omega\approx 3.48341$. *By setting* $q=1/6<\alpha_k$ *for* $k=0,1,2$, *then* $\sum_{i=1}^{m+1}\left(\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}-q}\right)/(((\alpha_{i-1}-q/(1-q))^{1-q}\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1}))\approx 4.25528$. *The assumption* (H_3) *is fulfilled, this implies that*

$$\left|f\left(t,x(t),x\left(\frac{t}{2}\right)\right)\right|\leq\frac{\sqrt[3]{t+1}}{16}\Big(\|x\|+1\Big),\quad \zeta(t)=\frac{\sqrt[3]{t+1}}{16},\quad \omega(r)=r+1.$$

For any $x_1,x_2\in\mathbb{R}$ *and* $t\in[0,1]$, *we have*

$$|\varphi_k(x_1)-\varphi_k(x_2)|\leq\frac{1}{8}|x_1-x_2|.$$

So, $\mathbb{M}_1=1/8$, $\|\xi\|_{L^{\frac{1}{q}}(\mathcal{J})}=\sqrt[3]{2}/16$ *and* $\liminf_{r_2\to+\infty}\omega(r_2)/r_2=1$. *Then, by (21), we obtain*

$$\mathfrak{C}:=\left(1+\frac{|\eta|}{|\Omega|}\right)\left(\sum_{i=1}^{m+1}\frac{\left(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1})\right)^{\alpha_{i-1}-q}}{\left(\frac{\alpha_{i-1}-q}{1-q}\right)^{1-q}\rho_{i-1}^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\|\xi\|_{L^{\frac{1}{q}}(\mathcal{J})}\liminf_{r_2\to+\infty}\frac{\omega(r_2)}{r_2}+m\mathbb{M}_1\right).$$

Then, $\mathfrak{C}\approx 0.988395<1$. *Since, all the assumptions of Theorem 2 are fulfilled, hence (24) has at least one solution on* $[0,1]$.

Example 3. *Consider the following impulsive fractional differential equation with boundary conditions.*

$$\begin{cases} {}^{C\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\psi_k}x(t)+\frac{\rho_k^{\alpha_k}}{4}x(t)=0,\quad t\in[0,1]\setminus\left\{\frac{1}{2}\right\},\\ \Delta x(\frac{1}{2})=1,\\ x(0)+x(1)=1.\end{cases}\tag{25}$$

Here $a=0$, $T=1$, $\lambda=1/4$, $\beta=\eta=\gamma=1$, *and* $f(t,x(t),x(\mu t))=0$. *Clearly, all assumptions of Theorem 1 are achieved. Then, (25) has a unique solution on* $[0,1]$. *It follows Lemma 3 we obtain*

$$x(t)=\begin{cases}\left(\dfrac{1-e^{\frac{\rho_1-1}{2\rho_1}}\mathbb{E}_{\alpha_1}\left(-\left(\frac{1}{2}\right)^{\alpha_1+2}\right)}{1+e^{\frac{\rho_0-1}{4\rho_0}+\frac{\rho_1-1}{2\rho_1}}\mathbb{E}_{\alpha_0}\left(-\left(\frac{1}{4}\right)^{\alpha_0+1}\right)\mathbb{E}_{\alpha_1}\left(-\left(\frac{1}{2}\right)^{\alpha_1+2}\right)}\right)\left(e^{\frac{\rho_0-1}{\rho_0}t^2}\mathbb{E}_{\alpha_0}\left(-\frac{1}{4}t^{2\alpha_0}\right)\right), & t\in[0,\frac{1}{2}),\\ \left(\dfrac{\left(1-e^{\frac{\rho_1-1}{2\rho_1}}\mathbb{E}_{\alpha_1}\left(-\left(\frac{1}{2}\right)^{\alpha_1+2}\right)\right)e^{\frac{\rho_0-1}{4\rho_0}}\mathbb{E}_{\alpha_0}\left(-\left(\frac{1}{4}\right)^{\alpha_0+1}\right)}{1+e^{\frac{\rho_0-1}{4\rho_0}+\frac{\rho_1-1}{2\rho_1}}\mathbb{E}_{\alpha_0}\left(-\left(\frac{1}{4}\right)^{\alpha_0+1}\right)\mathbb{E}_{\alpha_1}\left(-\left(\frac{1}{2}\right)^{\alpha_1+2}\right)}+1\right)\left(e^{\frac{\rho_1-1}{\rho_1}(t-\frac{1}{2})}\mathbb{E}_{\alpha_1}\left(-\frac{1}{4}(t-\frac{1}{2})^{\alpha_1}\right)\right), & t\in[\frac{1}{2},1].\end{cases}$$

Thanks (7) again, we can derive the numerical solution of (25) with the different of α_k, ρ_k and $\psi_k(t)$ as shown in Figures 1–3. In addition, the different values for Ω can be obtained corresponding to the different values of α_k, ρ_k, and $\psi_k(t)$ as shown in Table 1.

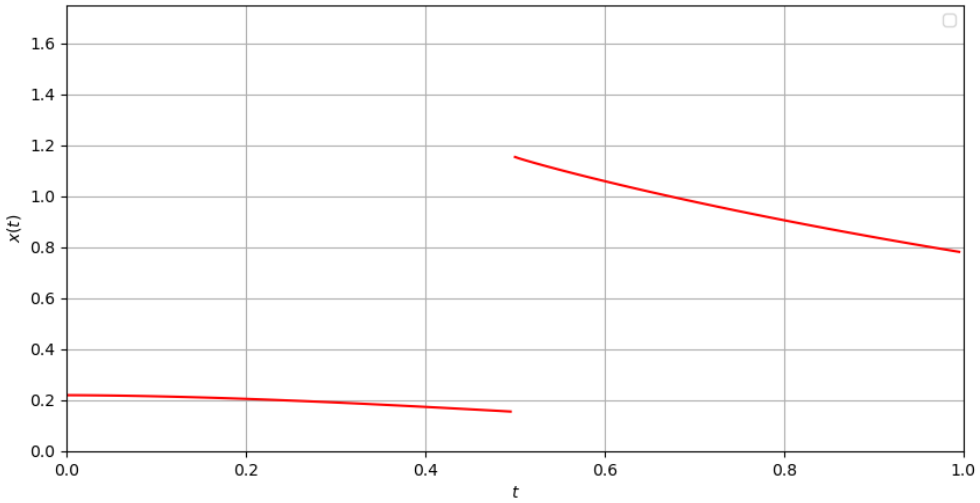

Figure 1. The solution of Example (25) via $\alpha_k=\sin(\frac{\pi}{4-k})$, $\rho_k=\frac{k+1}{k+2}$, and $\psi_k(t)=t^{\frac{2}{k+1}}$.

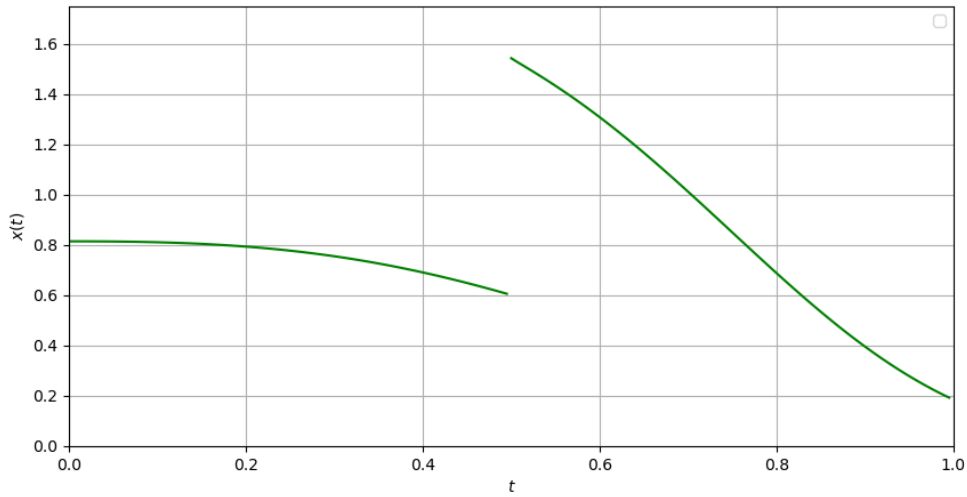

Figure 2. The solution of Example (25) via $\alpha_k=\frac{2k+2}{2k+3}$, $\rho_k=\frac{\sqrt{k^2+3}}{k^2+5}$, and $\psi_k(t)=t^{k+3}$.

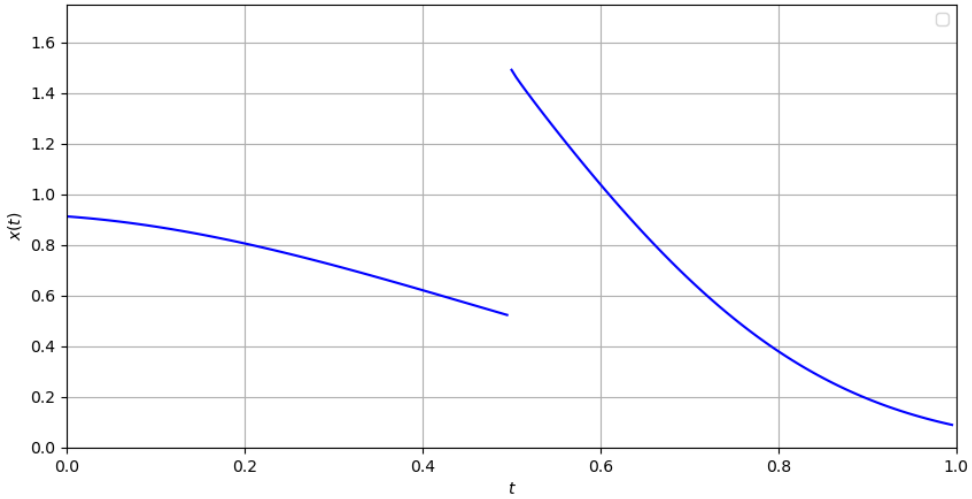

Figure 3. The solution of Example (25) via $\alpha_k = \frac{\sqrt{k+1}}{2}$, $\rho_k = e^{-\sqrt{k+1}}$, and $\psi_k(t) = t^{\sqrt{(k+1)^2+3}}$.

Table 1. The values of Ω for different values α_k, ρ_k, and $\psi_k(t)$ $(k = 0, 1)$.

	$\boldsymbol{\alpha_k}$	$\boldsymbol{\rho_k}$	$\boldsymbol{\psi_k(t)}$	$\boldsymbol{\Omega}$
I	$\sin(\frac{\pi}{4-k})$	$\frac{k+1}{k+2}$	$t^{\frac{2}{k+1}}$	1.44637
II	$\frac{2k+2}{2k+3}$	$\frac{\sqrt{k^2+3}}{k^2+5}$	t^{k+3}	1.30799
III	$\frac{\sqrt{k+1}}{2}$	$e^{-\sqrt{k+1}}$	$t^{\sqrt{(k+1)^2+3}}$	1.032894

5. Conclusions

A variety of novel forms of fractional derivatives have recently been constructed and employed to better describe real-world phenomena. The so-called generalized proportional fractional derivatives are one of the most recently introduced fractional derivatives, which is an extension of the classical Riemann–Liouville and Caputo fractional derivatives. In this manuscript, the impulsive proportional fractional pantograph differential equations with a constant coefficient and generalized boundary conditions were examined in this manuscript. The Mittag–Leffler functions were utilized to present the solutions for the proposed problem. The existence and uniqueness results are based on the well-known fixed point theorems of Banach and Krasnoselskii. Finally, to guarantee the accuracy of the results, three numerical examples illustrating the implementation of our important conclusions have been provided. By the way, we have accomplished in showing certain particular cases connected to the results as a result of our discussion of this study [18–21]. This research has enriched the qualitative theory literature on nonlinear impulsive fractional initial/boundary value problems involving a specific function in future research such as the linear Cauchy problem with variable coefficient or convergence analysis.

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