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Mawhin's Continuation Technique for a Nonlinear BVP of Variable Order at Resonance via Piecewise Constant Functions

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Abstract: In this paper, we establish the existence of solutions to a nonlinear boundary value problem (BVP) of variable order at resonance. The main theorem in this study is proved with the help of generalized intervals and piecewise constant functions, in which we convert the mentioned Caputo BVP of fractional variable order to an equivalent standard Caputo BVP at resonance of constant order. In fact, to use the Mawhin's continuation technique, we have to transform the variable order BVP into a constant order BVP. We prove the existence of solutions based on the existing notions in the coincidence degree theory and Mawhin's continuation theorem (MCTH). Finally, an example is provided according to the given variable order BVP to show the correctness of results.

Keywords: piecewise constant function; Mawhin's continuation technique; variable order; resonance; existence

1. Introduction

The initial idea of fractional calculus is taken from the powers of real or complex numbers in the order of differentiation and integration operators. In recent decades, fractional operators of variable order are appeared extensively in a vast domain of sciences including chaotic dynamical systems, fractal theory, rheology, signal processing, mathematical modeling, control theory, and biomedical applications. This range of applications is due to the fact that fractional derivatives provide a strong tool in the mathematics to describe the memory and hereditary properties of processes and various materials; see, for example [1–3].

Before the variable order systems, discussion of boundary value problems with fractional constant orders has attracted the attention of most researchers, and valuable findings have been established. Various researches have been conducted to study the behaviors of different fractional BVPs by means of some known methods such as fixed point theorems, numerical methods, monotone iterative methods, variational methods, and etc. [4–12].

Nevertheless, in addition to numerous published papers on fractional constant order problems, few studies on the existence theory have been done in relation to variable order problems [13–19]. Hence, investigation of this interesting and general topic makes all our findings worthy.

In 1970, Gaines and Mawhin [20] introduced the theory of coincidence degree for analysis of differential and functional equations. Mawhin has made important contributions since then, and the mentioned theory is also famous to the Mawhin's coincidence theory. Coincidence theory is considered as a powerful technique, especially with regard to questions about the existence of solutions for nonlinear differential equations. Mawhin's theory permits the use of a method based on the topological degree notion for some problems which can be written as an abstract operator equation of the form $\Theta x = Wx$, where Θ is a linear non-invertible operator and W is a nonlinear operator acting on a Banach space.

In 1972, Mawhin extended a technique to solve this operator equation in his famous paper [21]. He assumed that Θ is a Fredholm operator of index zero. Then, he developed a new theory of topological degree known as the degree of coincidence for (Θ, W) , that is also known as Mawhin's coincidence degree theory in honor of him.

A given boundary value problem is said to be at resonance if the corresponding linear homogeneous BVP has a non-trivial solution. Many authors studied ordinary BVPs at resonance using Mawhin's coincidence degree theory; we can cite some works done by Feng and Webb [22], Guezane-Lakoud and Frioui [23], Mawhin and Ward [24], Infante [25], and references therein.

Based on the aforementioned technique in relation to Mawhin's method, in this paper, we shall investigate a nonlinear boundary value problem of variable order at resonance which takes the form as follows

$$\begin{cases} {}^c D_{0+}^{u(t)} \phi(t) = g(t, \phi(t)), & t \in A, \\ \phi(0) = \phi(T), \end{cases} \quad (1)$$

where $A = [0, T]$, $T \in (0, \infty)$, the function $u(t) : A \rightarrow (0, 1]$ is the order of the existing derivative in the above boundary problem, ${}^c D_{0+}^{u(t)}$ is the variable order Caputo derivative, and also $g \in C(A \times \mathbb{R}, \mathbb{R})$.

The important aim of this research is to investigate some qualitative properties of solutions of the given Caputo boundary value problem of variable order (1). The main novelty of this paper is that we use the Mawhin's continuation technique for the first time for proving the existence of solutions of a Caputo boundary value problem at resonance equipped with variable order. Most papers apply this technique on the constant order systems, while we here try to derive the necessary conditions on a variable order system. In comparison to variable order partial systems, a linear analogue of this problem can be observed in the framework of partial differential equation [26] and this shows another version of such problems and specify our main contribution in this work. It is notable for young researchers that they can implement and investigate this methods and techniques on hidden-memory variable-order fractional problems introduced in [27,28] in the future.

The structure of the paper is organized as follows: Initially, some auxiliary definitions and remarks are collected for recalling the required notions in Section 2. Further, in Section 3, based on coincidence degree theory, a partition of the given interval A is applied, and by defining the relevant piecewise constant functions, the existence results are derived for an equivalent constant-order BVP at resonance and accordingly, for the given Caputo BVP of variable order (1). This proof is completed in some steps. In Section 4, we give an example to illustrate the theoretical existence theorems. The paper is completed with conclusions in Section 5.

2. Auxiliary Concepts

At first, some needed concepts about our study are collected from different sources. Here, the Banach space $C(A, \mathbb{R})$ consisting of continuous functions like $\phi : A \rightarrow \mathbb{R}$ is equipped with the sup-norm $\|\phi\| = \sup\{|\phi(t)| : t \in A\}$.

Definition 1 ([29,30]). The Riemann-Liouville fractional integral (RLFI) of variable order $u(t)$ for the function ϕ is defined by

$$I_{0+}^{u(t)}\phi(t) = \frac{1}{\Gamma(u(t))} \int_0^t (t-s)^{u(t)-1} \phi(s) ds, \quad t \in A, \quad (2)$$

where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$, and the left Caputo fractional derivative (CFD) of variable order $u(t)$ for $\phi(t)$ is defined by

$${}^c D_{0+}^{u(t)}\phi(t) = \frac{1}{1-\Gamma(u(t))} \int_0^t (t-s)^{-u(t)} \phi'(s) ds, \quad t \in A. \quad (3)$$

Remark 1. Notice that in (2), we have specified the variable order as the function $u : A \rightarrow (0, 1]$, while for defining RLFI, we can consider it as a function with extended values like $u : A \rightarrow (0, \infty)$.

Remark 2 ([30]). When we define the variable order u as a constant-valued function in both (2) and (3), then the variable order RLFI and CFD operators are the same as the usual RLFI and CFD operators, respectively.

Remark 3 ([29]). As we know, the semigroup property is satisfied for the standard RLFI operators equipped with constant order, while it is not valid for extended case of variable orders $\beta_1(t)$ and $\beta_2(t)$. In other words, $I_{0+}^{\beta_1(t)}(I_{0+}^{\beta_2(t)}\phi(t)) \neq I_{0+}^{\beta_1(t)+\beta_2(t)}\phi(t)$.

To see this problem, we give the following example.

Example 1. Let $A = [0, 3]$ and $\phi(t) \equiv 1, \forall t \in A$. The variable orders of RLFI operator can be taken as: $\beta_1(t) = \frac{t}{2}$ and $\beta_2(t) = \begin{cases} 1, & t \in [0, 1] \\ 2, & t \in [1, 3] \end{cases}$.

Then for all $t \in A$, and according to Definition (2), we compute

$$\begin{aligned} I_{0+}^{\beta_1(t)}(I_{0+}^{\beta_2(t)}\phi(t)) &= \int_0^t \frac{(t-s)^{\beta_1(t)-1}}{\Gamma(\beta_1(t))} \int_0^s \frac{(s-\tau)^{\beta_2(s)-1}}{\Gamma(\beta_2(s))} \phi(\tau) d\tau ds \\ &= \int_0^t \frac{(t-s)^{\beta_1(t)-1}}{\Gamma(u(t))} \left[\int_0^1 \frac{(s-\tau)^0}{\Gamma(1)} d\tau + \int_1^s \frac{(s-\tau)}{\Gamma(2)} d\tau \right] ds \\ &= \int_0^t \frac{(t-s)^{\beta_1(t)-1}}{\Gamma(\beta_1(t))} \left[\frac{s^2}{2} - s + \frac{3}{2} \right] ds, \end{aligned}$$

and

$$I_{0+}^{\beta_1(t)+\beta_2(t)}\phi(t) = \int_0^t \frac{(t-s)^{\beta_1(t)+\beta_2(t)-1}}{\Gamma(\beta_1(t)+\beta_2(t))} \phi(s) ds.$$

For $t = 2$, it becomes

$$\begin{aligned} I_{0+}^{\beta_1(t)}(I_{0+}^{\beta_2(t)}\phi(t))|_{t=2} &= \int_0^2 \frac{(2-s)^0}{\Gamma(1)} \left[\frac{s^2}{2} - s + \frac{3}{2} \right] ds \\ &= \int_0^2 \left(\frac{s^2}{2} - s + \frac{3}{2} \right) ds \\ &= \frac{7}{3}, \end{aligned}$$

and

$$\begin{aligned}
 I_{0+}^{\beta_1(t)+\beta_2(t)} \phi(t)|_{t=2} &= \int_0^2 \frac{(2-s)^{\beta_1(t)+\beta_2(t)-1}}{\Gamma(\beta_1(t)+\beta_2(t))} \phi(s) ds \\
 &= \int_0^1 \frac{(2-s)^1}{\Gamma(2)} ds + \int_1^2 \frac{(2-s)^2}{\Gamma(3)} ds \\
 &= \frac{3}{2} + \frac{1}{6} \\
 &= \frac{5}{3}.
 \end{aligned}$$

Hence, it is simply seen that the mentioned property is not correct for the generalized RLFI operators with respect to variable orders.

The following expansion is key for our argument.

Lemma 1 ([31]). Let $a_1, \alpha_1 > 0$ and $n = 1 + [\alpha_1]$. Then

$$I_{a_1+}^{\alpha_1} ({}^c D_{a_1+}^{\alpha_1} \phi(t)) = \phi(t) - \sum_{k=0}^{n-1} \frac{\phi^{(k)}(a_1)}{k!} t^k.$$

Lemma 2 ([32]). Let $\alpha_1, \alpha_2 > 0$, $\phi, {}^c D_{a_1+}^{\alpha_1} \phi \in L^1(a_1, a_2)$. Then, the differential equation

$${}^c D_{a_1+}^{\alpha_1} \phi(t) = 0,$$

has unique solution

$$\phi(t) = r_0 + r_1(t - a_1) + r_2(t - a_1)^2 + \dots + r_{n-1}(t - a_1)^{n-1},$$

and we have

$$I_{a_1+}^{\alpha_1} ({}^c D_{a_1+}^{\alpha_1} \phi(t)) = \phi(t) + r_0 + r_1(t - a_1) + r_2(t - a_1)^2 + \dots + r_{n-1}(t - a_1)^{n-1},$$

such that $n - 1 < \alpha_1 \leq n$, $r_j \in \mathbb{R}$, $j = 1, 2, \dots, n$.

Furthermore, we have

$${}^c D_{a_1+}^{\alpha_1} (I_{a_1+}^{\alpha_1} \phi(t)) = \phi(t),$$

and

$$I_{a_1+}^{\alpha_1} (I_{a_1+}^{\alpha_2} \phi(t)) = I_{a_1+}^{\alpha_2} (I_{a_1+}^{\alpha_1} \phi(t)) = I_{a_1+}^{\alpha_1+\alpha_2} \phi(t).$$

We recall some properties of variable order RLFI operator formulated by (2) which will be used in the sequel.

Lemma 3 ([33]). If $u : A \rightarrow (0, 1]$ has the continuity property, then for

$$h \in C_\delta(A, \mathbb{R}) = \{h(t) \in C(A, \mathbb{R}), t^\delta h(t) \in C(A, \mathbb{R})\}, 0 \leq \delta \leq 1,$$

the integral $I_{0+}^{u(t)} h(t)$ admits a finite value for all $t \in A$.

Lemma 4 ([33]). Assume that $u : A \rightarrow (0, 1]$ has the continuity property. Then

$$I_{0+}^{u(t)} h(t) \in C(A, \mathbb{R}) \text{ for } h \in C(A, \mathbb{R}).$$

Definition 2 ([34,35]). An interval $J \subseteq \mathbb{R}$ is termed as a generalized interval if I is either an interval, or $\{a_1\}$, or $\emptyset$. A finite set $\mathcal{F}$ is defined to be a partition of J if every $x \in J$ belongs to exactly one and one generalized interval $\mathbb{I}$ in $\mathcal{F}$. Finally, $w : J \rightarrow \mathbb{R}$ is piecewise constant w.r.t $\mathcal{F}$ as a partition of J , if for each $\mathbb{I} \in \mathcal{F}$, w is constant on $\mathbb{I}$.

The next definitions and basic lemmas from coincidence degree theory are fundamental in the proof of theorems which we will establish them later.

Definition 3 ([20,36]). Consider two normed spaces $\mathbb{S}_1$ and $\mathbb{S}_2$. A Fredholm operator of index zero is a linear operator like $\Theta : \text{Dom}(\Theta) \subset \mathbb{S}_1 \rightarrow \mathbb{S}_2$ satisfying:

- (a) $\text{IMG}(\Theta) \subseteq \mathbb{S}_2$ is closed;
- (b) $\dim \text{KER}(\Theta) = \text{codim IMG}(\Theta) < +\infty$.

In view of Definition 3, it is followed the existence of continuous projections $\Psi : \mathbb{S}_1 \rightarrow \mathbb{S}_1$ and $\Phi : \mathbb{S}_2 \rightarrow \mathbb{S}_2$ such that $\text{IMG}(\Psi) = \text{KER}(\Theta)$, $\text{KER}(\Phi) = \text{IMG}(\Theta)$, $\mathbb{S}_1 = \text{KER}(\Theta) \oplus \text{KER}(\Psi)$, and $\mathbb{S}_2 = \text{IMG}(\Theta) \oplus \text{IMG}(\Phi)$.

It is known that the restriction of Θ to $\text{Dom}(\Theta) \cap \text{KER}(\Psi)$, which we shall represent by Θ_Ψ , will be an isomorphism onto its image [20,36].

Definition 4 ([20,36]). Let Θ be a Fredholm operator of index zero and $\Omega \subseteq \mathbb{S}_1$ be bounded with $\text{Dom}(\Theta) \cap \Omega \neq \emptyset$. We say $W : \overline{\Omega} \rightarrow \mathbb{S}_2$ has the Θ -compactness property in $\overline{\Omega}$ whenever:

- (H1) $\Phi W : \overline{\Omega} \rightarrow \mathbb{S}_2$ is continuous, and $\Phi W(\overline{\Omega}) \subseteq \mathbb{S}_2$ is bounded,
- (H2) $(\Theta_\Psi)^{-1}(I - \Phi)W : \overline{\Omega} \rightarrow \mathbb{S}_1$ is completely continuous.

The next theorem entitled *Mawhin's Continuation Theorem* is our main criterion in the present study which proves the existence of solution.

Theorem 1 ([37]). Assume that $\mathbb{S}_1$ and $\mathbb{S}_2$ are two Banach spaces and $\Omega \subset \mathbb{S}_1$ is an open, bounded and symmetric set with $0 \in \Omega$. Also, assume that:

- (A1) the Fredholm operator $\Theta : \text{Dom}(\Theta) \subset \mathbb{S}_1 \rightarrow \mathbb{S}_2$ of index zero is such that

$$\text{Dom}(\Theta) \cap \overline{\Omega} \neq \emptyset,$$

- (A2) the operator $W : \mathbb{S}_1 \rightarrow \mathbb{S}_2$ is Θ -compact on $\overline{\Omega}$,
- (A3) $\forall x \in \text{Dom}(\Theta) \cap \partial\Omega$ and $\forall \lambda \in (0, 1]$,

$$\Theta x - Wx \neq -\lambda(\Theta x + W(-x)),$$

where $\partial\Omega$ denotes the boundary of Ω w.r.t. $\mathbb{S}_1$.

Then the operator equation $\Theta x = Wx$ has at least one solution on $\text{Dom}(\Theta) \cap \overline{\Omega}$.

3. Existence of Solutions

To begin the desired analysis, we consider the following assumptions:

- (AS1) Consider a sequence of finite many points $\{T_k\}_{k=0}^n$ so that $1 = T_0 < T_k < T_n = T$, $k \in \mathbb{N}_1^{n-1}$. For $k \in \mathbb{N}_1^n$, denote the subintervals A_k as $A_k := (T_{k-1}, T_k]$. Then $\mathcal{P} = \bigcup_{k=1}^n A_k$ is a partition of A .

- (AS2) Let $g \in C(A_j \times \mathbb{R}, \mathbb{R})$ and there exists $\delta \in (0, 1)$ such that $t^\delta g \in C(A_j \times \mathbb{R}, \mathbb{R})$ and there exists $K > 0$ with $K < \min \left\{ 1, \frac{\Gamma(u_j + 1)}{(T_j - T_{j-1})^{u_j}} \right\}$ so that $t^\delta |g(t, \phi_1) - g(t, \phi_2)| \leq K|\phi_1 - \phi_2|$, for any $\phi_1, \phi_2 \in \mathbb{R}$ and $t \in A_j$.

For each $j \in \mathbb{N}_1^n$, the notation $E_j = C(A_j, \mathbb{R})$ denotes the Banach space of continuous functions $\phi : A_j \rightarrow \mathbb{R}$ with the sup-norm $\|\phi\|_{E_j} = \sup_{t \in A_j} |\phi(t)|$.

On the other side, consider the piecewise constant mapping $u(t) : A \rightarrow (0, 1]$ w.r.t. $\mathcal{P}$, i.e.,

$$u(t) = \sum_{j=1}^n u_j I_j(t),$$

where $0 < u_j \leq 1$ are real numbers, and I_j denotes the indicator of A_j , $j \in \mathbb{N}_1^n$; that is, $I_j(t) = 1$ if $t \in A_j$ and $I_j(t) = 0$ otherwise. In this case, the left CFD of variable order $u(t)$ for $\phi(t) \in C(A, \mathbb{R})$, defined as (3), can be formulated as a sum of the left CFD operators of constant orders $u_k \in \mathbb{R}$ which takes the form

$${}^c D_{0^+}^{u(t)} \phi(t) = \sum_{k=1}^{j-1} \int_{T_{k-1}}^{T_k} \frac{(t-s)^{-u_k}}{\Gamma(1-u_k)} \phi'(s) ds + \int_{T_{j-1}}^t \frac{(t-s)^{-u_j}}{\Gamma(1-u_j)} \phi'(s) ds. \quad (4)$$

Thus, the given Caputo BVP of variable order (1) can be reformulated for each $t \in A_j$, $j \in \mathbb{N}_1^n$ in the following structure

$$\sum_{k=1}^{j-1} \int_{T_{k-1}}^{T_k} \frac{(t-s)^{-u_k}}{\Gamma(1-u_k)} \phi'(s) ds + \int_{T_{j-1}}^t \frac{(t-s)^{-u_j}}{\Gamma(1-u_j)} \phi'(s) ds = g(t, \phi(t)). \quad (5)$$

Let the function $\tilde{\phi} \in E_j$ be so that $\tilde{\phi}(t) \equiv 0$ on $t \in [0, T_{j-1}]$ and it satisfies the above integral Equation (5). In such a situation, (5) is converted to the standard constant order fractional differential equation (FDE) as

$${}^c D_{T_{j-1}^+}^{u_j} \tilde{\phi}(t) = g(t, \tilde{\phi}(t)), \quad t \in A_j.$$

In accordance with above equation, for each $j \in \mathbb{N}_1^n$, we have the auxiliary FBVP equipped with Caputo constant order CFD operator

$$\begin{cases} {}^c D_{T_{j-1}^+}^{u_j} \phi(t) = g(t, \phi(t)), & t \in A_j, \\ \phi(T_{j-1}) = \phi(T_j). \end{cases} \quad (6)$$

A resonance problem is a boundary problem in which the corresponding homogeneous BVP has a non-trivial solution. Hence, we consider the homogeneous version of the given equivalent constant order FBVP (6) by

$$\begin{cases} {}^c D_{T_{j-1}^+}^{u_j} \phi(t) = 0, & t \in A_j, \\ \phi(T_{j-1}) = \phi(T_j). \end{cases} \quad (7)$$

By Lemma 2, the homogeneous constant order FBVP (7) has nontrivial solution $\phi(t) = c$ which converts the equivalent constant order FBVP (6) to a resonance FBVP.

As well as, on the given subintervals, let $\mathbb{S}_1 = \{\phi \in E_j : \phi(t) = I_{T_{j-1}^+}^{u_j} v(t) : v \in E_j, t \in A_j\}$ with the norm

$$\|\phi\|_{\mathbb{S}_1} = \|\phi\|_{E_j}.$$

The linear operator $\Theta : \text{Dom}(\Theta) \subseteq \mathbb{S}_1 \rightarrow E_j$ along with the operator $W : \mathbb{S}_1 \rightarrow E_j$ are defined as

$$\Theta[\phi(t)] := {}^c D_{T_{j-1}^+}^{u_j} \phi(t), \quad (8)$$

and

$$W[\phi(t)] := g(t, \phi(t)), \quad t \in A_j, \quad (9)$$

where

$$\text{Dom}(\Theta) = \{\phi \in \mathbb{S}_1 : {}^c D_{T_{j-1}^+}^{u_j} \phi \in E_j \text{ and } \phi(T_{j-1}) = \phi(T_j)\}.$$

Then the equivalent constant order resonance FBVP (6) can be reformulated by the equation $\Theta\phi = W\phi$.

The first theorem on the existence of solutions for the equivalent constant order resonance FBVP (6) is established in this position.

Theorem 2. *If the condition (AS2) holds, then the equivalent constant order resonance FBVP (6) has at least one solution.*

Proof. The proof will be followed in a sequence of claims.

Claim 1. We show that

$$\text{KER}(\Theta) = \{c : c \in \mathbb{R}\},$$

and

$$\text{IMG}(\Theta) = \{\phi \in E_j : \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} \phi(s) ds = 0\}.$$

Let Θ (defined by (8)) be such that for $t \in A_j$ and by Lemma 2, the equation $\Theta[\phi(t)] = {}^c D_{T_{j-1}^+}^{u_j} \phi(t) = 0$ has the solution $\phi(t) = c$, $c \in \mathbb{R}$. Then

$$\text{KER}(\Theta) = \{\phi(t) = c : c \in \mathbb{R}\}.$$

On the other hand, for $v \in \text{IMG}(\Theta)$, there exists $\phi \in \text{Dom}(\Theta)$ such that $v = \Theta\phi \in E_j$. By Lemma 1, for any $t \in A_j$, we have

$$\phi(t) = \phi(T_{j-1}) + \frac{1}{\Gamma(u_j)} \int_{T_{j-1}}^t (t-s)^{u_j-1} v(s) ds.$$

Since $\phi \in \text{Dom}(\Theta)$, v satisfies

$$\frac{1}{\Gamma(u_j)} \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} v(s) ds = 0.$$

Also, assume that $v \in E_j$ satisfies

$$\int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} v(s) ds = 0.$$

Let $\phi(t) = I_{T_{j-1}^+}^{u_j} v(t)$. Then $v(t) = {}^c D_{T_{j-1}^+}^{u_j} \phi(t)$ and so $\phi \in \text{Dom}(\Theta)$. Hence, $v \in \text{IMG}(\Theta)$, so

$$\text{IMG}(\Theta) = \{\phi \in \mathbb{S}_2 : \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} \phi(s) ds = 0\}.$$

Claim 2. Θ is a Fredholm operator of index zero.

The linear continuous projector operators $\Psi : \mathbb{S}_1 \rightarrow \mathbb{S}_1$ and $\Phi : E_j \rightarrow E_j$ can be considered by the following forms

$$\Psi\phi = \phi(T_{j-1}), \quad \Phi v(t) = \frac{u_j}{(T_j - T_{j-1})^{u_j}} \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} v(s) ds.$$

Clearly, $\text{IMG}(\Psi) = \text{KER}(\Theta)$ and $\Psi^2 = \Psi$. It follows that for any $\phi \in \mathbb{S}_1$,

$$\phi = (\phi - \Psi\phi) + \Psi\phi,$$

i.e., $\mathbb{S}_1 = \text{KER}(\Psi) + \text{KER}(\Theta)$. A simple computation shows that $\text{KER}(\Psi) \cap \text{KER}(\Theta) = 0$. Therefore, $\mathbb{S}_1 = \text{KER}(\Psi) \oplus \text{KER}(\Theta)$. A similar argument shows that for every $v \in E_j$, $\Phi^2 v = \Phi v$ and $v = (v - \Phi(v)) + \Phi(v)$, where $(v - \Phi(v)) \in \text{KER}(\Phi) = \text{IMG}(\Theta)$.

From $\text{IMG}(\Theta) = \text{KER}(\Phi)$ and $\Phi^2 = \Phi$, we have

$$\text{IMG}(\Phi) \cap \text{IMG}(\Theta) = 0.$$

Then, $E_j = \text{IMG}(\Theta) \oplus \text{IMG}(\Phi)$.

In this case,

$$\dim(\text{KER}(\Theta)) = \dim \text{IMG}(\Phi) = \text{codim} \text{IMG}(\Theta).$$

The obtained result shows that Θ is a Fredholm operator of index zero.

Claim 3. $\Theta_{\Psi}^{-1} = (\Theta|_{\text{Dom}(\Theta) \cap \text{KER}(\Psi)})^{-1}$ (the inverse of $\Theta|_{\text{Dom}(\Theta) \cap \text{KER}(\Psi)}$).

Clearly, $\Theta_{\Psi}^{-1} : \text{IMG}(\Theta) \rightarrow \mathbb{S}_1 \cap \text{KER}(\Psi)$ satisfies

$$\Theta_{\Psi}^{-1}(v)(t) = I_{T_{j-1}^+}^{u_j} v(t).$$

Let $v \in \text{IMG}(\Theta)$. Then

$$\Theta \Theta_{\Psi}^{-1}(v) = {}^c D_{T_{j-1}^+}^{u_j} (I_{T_{j-1}^+}^{u_j} v) = v. \quad (10)$$

Furthermore, for $\phi \in \text{Dom}(\Theta) \cap \text{KER}(\Psi)$, we get

$$\Theta_{\Psi}^{-1}(\Theta(\phi(t))) = I_{T_{j-1}^+}^{u_j} ({}^c D_{T_{j-1}^+}^{u_j} \phi(t)) = \phi(t) - \phi(T_{j-1}).$$

Since $\phi \in \text{Dom}(\Theta) \cap \text{KER}(\Psi)$, we know that $\phi(T_{j-1}) = 0$. Thus

$$\Theta_{\Psi}^{-1}(\Theta(\phi(t))) = \phi(t). \quad (11)$$

Combining (10) and (11) shows that $\Theta_{\Psi^{-1}} = (\Theta|_{\text{Dom}(\Theta) \cap \text{KER}(\Psi)})^{-1}$.

Claim 4. On every bounded and open set $\Omega \subset \mathbb{S}_1$, W is Θ -compact.

Define $\Omega = \{\phi \in \mathbb{S}_1 : \|\phi\|_{\mathbb{S}_1} < M\}$ as a bounded and open set, where $M > 0$.

The proof of this claim will be done in three steps.

Step 1. ΦW is continuous.

This property for ΦW is derived due to the imposed conditions on the nonlinear function g and the Lebesgue dominated convergence criterion, immediately.

Step 2. $\Phi W(\overline{\Omega})$ is bounded.

Now, for each $\phi \in \overline{\Omega}$ and for all $t \in A_j$, we have

$$\begin{aligned} |\Phi W(\phi)(t)| &\leq \frac{u_j}{(T_j - T_{j-1})^{u_j}} \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} |g(s, \phi(s))| ds \\ &\leq \frac{u_j}{(T_j - T_{j-1})^{u_j}} \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} |g(s, \phi(s)) - g(s, 0)| ds \\ &\quad + \frac{u_j}{(T_j - T_{j-1})^{u_j}} \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} |g(s, 0)| ds \\ &\leq g^* + \frac{u_j}{(T_j - T_{j-1})^{u_j}} \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} s^{-\delta} (K|\phi(s)|) ds \\ &\leq g^* + MKT_{j-1}^{-\delta}, \end{aligned}$$

by assuming $g^* = \sup_{t \in A_j} |g(t, 0)|$. Thus,

$$\|\Phi W(\phi)\|_{E_j} \leq g^* + MKT_{j-1}^{-\delta} := R > 0.$$

This shows that $\Phi W(\overline{\Omega}) \subseteq E_j$ is bounded.

Step 3. $\Theta_{\Psi}^{-1}(I - \Phi)W : \overline{\Omega} \rightarrow \mathbb{S}_1$ is completely continuous.

By considering the existing hypotheses in relation to Ascoli-Arzelà theorem, it is necessary that we prove two properties of the boundedness and equi-continuity for $\Theta_{\Psi}^{-1}(I - \Phi)W(\overline{\Omega}) \subset \mathbb{S}_1$. At first, for each $\phi \in \overline{\Omega}$ and for all $t \in A_j$, we have

$$\begin{aligned} \Theta_{\Psi}^{-1}(I - \Phi)W\phi(t) &= \Theta_{\Psi}^{-1}(W\phi(t) - \Phi W\phi(t)) \\ &= I_{T_{j-1}}^{u_j} \left[g(t, \phi(t)) - \frac{u_j}{(T_j - T_{j-1})^u} \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} g(s, \phi(s)) ds \right] \\ &= \frac{1}{\Gamma(u_j)} \int_{T_{j-1}}^t (t - s)^{u_j-1} g(s, \phi(s)) ds \\ &\quad - \frac{t^{u_j}}{(T_j - T_{j-1})^{u_j} \Gamma(u_j)} \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} g(s, \phi(s)) ds. \end{aligned}$$

Further, for each $\phi \in \overline{\Omega}$ and for all $t \in A_j$, we get

$$\begin{aligned} |\Theta_{\Psi}^{-1}(I - \Phi)W\phi(t)| &\leq \frac{2}{\Gamma(u_j)} \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} |g(s, \phi(s)) - g(t, 0)| ds \\ &\quad + \frac{2}{\Gamma(u_j)} \int_{T_{j-1}}^{T_j} (T_j - s)^{u_j-1} |g(t, 0)| ds \\ &\leq [g^* + MKT_{j-1}^{-\delta}] \frac{2(T_j - T_{j-1})^{u_j}}{\Gamma(u_j + 1)} := B_1. \end{aligned}$$

so

$$\|\Theta_{\Psi}^{-1}(I - \Phi)W\phi\|_{E_j} \leq B_1,$$

which gives the uniform boundedness of $\Theta_{\Psi}^{-1}(I - \Phi)W(\overline{\Omega})$ in $\mathbb{S}_1$.

To prove the equi-continuity of $\Theta_{\Psi}^{-1}(I - \Phi)W(\overline{\Omega})$, notice that for $T_{j-1} \leq t_1 \leq t_2 \leq T_j$ and $\phi \in \overline{\Omega}$, we get

$$\begin{aligned} |\Theta_{\Psi}^{-1}(I - \Phi)W\phi(t_2) - \Theta_{\Psi}^{-1}(I - \Phi)W\phi(t_1)| &\leq \frac{g^* + T_{j-1}^{-\delta} MK}{\Gamma(u_j)} \left[\int_{t_1}^{t_2} (t_2 - s)^{u_j-1} ds \right. \\ &\quad \left. + \int_{T_{j-1}}^{t_1} |(t_2 - s)^{u_j-1} - (t_1 - s)^{u_j-1}| ds \right] + \left[\frac{T_{j-1}^{-\delta} MK + g^*}{\Gamma(u_j + 1)} \right] (t_2^{u_j} - t_1^{u_j}). \end{aligned}$$

The right-hand side of above inequality tends to zero as $t_1 \rightarrow t_2$. Thus, $\Theta_{\Psi}^{-1}(I - \Phi)W(\overline{\Omega})$ is equicontinuous in $\mathbb{S}_1$. On the basis of the Ascoli-Arzelà theorem, $L_{\Psi}^{-1}(I - \Phi)W(\overline{\Omega})$ is relatively compact. In accordance with the steps 1 to 3, we can follow that W is Θ -compact in $\overline{\Omega}$.

Claim 5. There exists $\epsilon > 0$ (not depending on λ) so that if

$$\Theta(\phi) - W(\phi) = -\lambda[\Theta(\phi) + W(-\phi)], \quad \lambda \in (0, 1], \quad (12)$$

then $\|\phi\|_{\mathbb{S}_1} \leq \epsilon$. By the condition (AS2) and for each $\phi \in \mathbb{S}_1$ satisfying (12), we get

$$\Theta(\phi) - W(\phi) = -\lambda\Theta(\phi) - \lambda W(-\phi).$$

So

$$\Theta(\phi) = \frac{1}{1+\lambda}W(\phi) - \frac{\lambda}{1+\lambda}W(-\phi). \quad (13)$$

By (13), and for all $t \in A_j$, we get

$$\phi(t) = \frac{1}{1+\lambda}\Theta_{\Psi}^{-1}W\phi(t) - \frac{\lambda}{1+\lambda}\Theta_{\Psi}^{-1}W(-\phi(t)),$$

and so we estimate

$$\begin{aligned} |\phi(t)| &\leq \frac{1}{(1+\lambda)\Gamma(u_j)} \int_{T_{j-1}}^t (t-s)^{u_j-1} |g(s, \phi(s)) - g(s, 0)| ds \\ &\quad + \frac{\lambda}{(1+\lambda)\Gamma(u_j)} \int_{T_{j-1}}^t (t-s)^{u_j-1} |g(s, -\phi(s)) - g(s, 0)| ds \\ &\quad + \frac{g^*(T_j - T_{j-1})^{u_j}}{(1+\lambda)\Gamma(u_j+1)} + \frac{\lambda g^*(T_j - T_{j-1})^{u_j}}{(1+\lambda)\Gamma(u_j+1)} \\ &\leq \left(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} \right) \frac{T_{j-1}^{-\delta}(T_j - T_{j-1})^{u_j}}{\Gamma(u_j+1)} (K\|\phi\|_{E_j}) \\ &\quad + \left(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} \right) \frac{g^*(T_j - T_{j-1})^{u_j}}{\Gamma(u_j+1)} \\ &= \frac{KT_{j-1}^{-\delta}(T_j - T_{j-1})^{u_j}}{\Gamma(u_j+1)} \|\phi\|_{E_j} + \frac{g^*(T_j - T_{j-1})^{u_j}}{\Gamma(u_j+1)}. \end{aligned}$$

Hence,

$$\|\phi\|_{E_j} \leq \left(g^* + KT_{j-1}^{-\delta} \|\phi\|_{E_j} \right) \frac{(T_j - T_{j-1})^{u_j}}{\Gamma(u_j+1)}, \quad (14)$$

and so

$$\|\phi\|_{\mathbb{S}_1} \leq \frac{g^*}{\frac{\Gamma(u_j+1)}{(T_j - T_{j-1})^{u_j}} - KT_{j-1}^{-\delta}} := \epsilon.$$

Claim 6. There exists a bounded and open set $\Omega \subset \mathbb{S}_1$ such that

$$\Theta(\phi) - W(\phi) \neq -\lambda[\Theta(\phi) + W(-\phi)],$$

for all $\phi \in \partial\Omega$ and all $\lambda \in (0, 1]$.

By the condition (AS2) and Claim 5, there exists $\epsilon > 0$ (independent of λ) such that if ϕ solves

$$\Theta(\phi) - W(\phi) = -\lambda[\Theta(\phi) + W(-\phi)], \quad \lambda \in (0, 1],$$

then $\|\phi\|_{\mathbb{S}_1} \leq \epsilon$. Consequently, if

$$\Omega = \{\phi \in \mathbb{S}_1 : \|\phi\|_{\mathbb{S}_1} < B\}, \quad (15)$$

then from the condition (AS2), it is immediately obtained that the set Ω introduced by (15), is symmetric, $0 \in \Omega$, and $\mathbb{S}_1 \cap \overline{\Omega} = \overline{\Omega} \neq \emptyset$.

Furthermore, it is obtained that

$$\Theta(\phi) - W(\phi) \neq -\lambda[\Theta(\phi) - W(-\phi)],$$

for all $\phi \in \partial\Omega = \{\phi \in \mathbb{S}_1 : \|\phi\|_{\mathbb{S}_1} = B\}$ and for all $\lambda \in (0, 1]$, where $B > \epsilon$. This together with Theorem 1 imply that the equivalent constant order resonance FBVP (6) has at least one solution, and this completes the proof. $\square$

Now, we complete our deduction on the existence property for solutions of the given Caputo FBVP of variable order (1).

Theorem 3. Let the conditions (AS1) and (AS2) be satisfied for all $j \in \mathbb{N}_1^n$. Then, the given Caputo FBVP of variable order (1) has at least a solution in $C(A, \mathbb{R})$.

Proof. We know that for all $j \in \mathbb{N}_1^n$, and according to Theorem 2, the equivalent constant order resonance FBVP (6) has at least one solution $\tilde{\phi}_j \in E_j$. For each $j \in \mathbb{N}_1^n$, and on the existing subintervals, define

$$\phi_j = \begin{cases} 0, & t \in [0, T_{j-1}], \\ \tilde{\phi}_j, & t \in A_j. \end{cases}$$

In such a case, $\phi_j \in C([0, T_j], \mathbb{R})$ satisfies the integral equation (5) for $t \in A_j$, which means that $\phi_j(0) = 0, \phi_j(T_j) = \tilde{\phi}_j(T_j) = 0$ and satisfies (5) for $t \in A_j, j \in \mathbb{N}_1^n$. Therefore, the piecewise function

$$\phi(t) = \begin{cases} \phi_1(t), & t \in A_1, \\ \phi_2(t), & t \in A_2, \\ \dots\dots\dots \\ \phi_n(t), & t \in A_n = [0, T], \end{cases}$$

is a solution to the given Caputo FBVP of variable order (1) in $C(A, \mathbb{R})$. $\square$

4. Example

Example 2. Consider the following FBVP (based on the given Caputo FBVP of variable order (1)) as follows

$$\begin{cases} {}^c D_{0+}^{u(t)} \phi(t) = \frac{\sin \phi(t) - \phi(t) \cos t}{5\sqrt{1+t}}, & t \in A := [0, 2], \\ \phi(0) = \phi(2). \end{cases} \quad (16)$$

Let

$$g(t, \phi) = \frac{\sin \phi - \phi \cos t}{5\sqrt{1+t}}, \quad (t, \phi) \in [0, 2] \times [0, +\infty),$$

and

$$u(t) = \begin{cases} \frac{7}{5}, & t \in A_1 := [0, 1], \\ \frac{3}{2}, & t \in A_2 := [1, 2]. \end{cases} \quad (17)$$

In this case,

$$\begin{aligned} t^{\frac{1}{2}} |g(t, \phi_1) - g_1(t, \phi_2)| &= \left| \frac{t^{\frac{1}{2}} (\sin \phi_1 - \phi_1 \cos t)}{5\sqrt{1+t}} - \frac{t^{\frac{1}{2}} (\sin \phi_2 - \phi_2 \cos t)}{5\sqrt{1+t}} \right| \\ &\leq \frac{1}{5} \sqrt{\frac{t}{1+t}} (|\sin \phi_1 - \sin \phi_2| + |\cos t| |\phi_1 - \phi_2|) \\ &\leq \frac{2}{5} |\phi_1 - \phi_2|. \end{aligned}$$

By (17) and (6), on every subintervals A_1 and A_2 , two auxiliary constant order resonance FBVPs are considered as

$$\begin{cases} {}^c D_{0+}^{\frac{7}{5}} \phi(t) = \frac{\sin \phi(t) - \phi(t) \cos t}{5\sqrt{1+t}}, & t \in A_1, \\ \phi(0) = \phi(1), \end{cases} \quad (18)$$

and

$$\begin{cases} {}^c D_{1+}^{\frac{3}{2}} \phi(t) = \frac{\sin \phi(t) - \phi(t) \cos t}{5\sqrt{1+t}}, & t \in A_2, \\ \phi(1) = \phi(2). \end{cases} \quad (19)$$

Evidently, the condition (AS2) is satisfied for $j = 1$ with $\delta = \frac{1}{2}$ and $K = \frac{2}{5}$, and

$$0 < K = \frac{2}{5} < \min \left\{ 1, \frac{\Gamma(u_j + 1)}{(T_j - T_{j-1})^{u_j}} \right\} = \min \left\{ 1, \Gamma\left(\frac{12}{5}\right) \right\} = 1.$$

According to Theorem 2, the constant order resonance FBVP (18) has a solution like $\tilde{\phi}_1 \in E_1$. Next, the condition (AS2) is also valid for $j = 2$ with $\delta = \frac{1}{2}$ and $K = \frac{2}{5}$, and

$$0 < K = \frac{2}{5} < \min \left\{ 1, \frac{\Gamma(u_j + 1)}{(T_j - T_{j-1})^{u_j}} \right\} = \min \left\{ 1, \Gamma\left(\frac{5}{2}\right) \right\} = 1.$$

According to Theorem 2, the constant order resonance FBVP (19) has a solution like $\tilde{\phi}_2 \in E_2$. Then, by Theorem 3, the given Caputo FBVP of variable order (16) has a solution as

$$\phi(t) = \begin{cases} \tilde{\phi}_1(t), & t \in A_1, \\ \phi_2(t), & t \in A_2, \end{cases}$$

where

$$\phi_2(t) = \begin{cases} 0, & t \in A_1, \\ \tilde{\phi}_2(t), & t \in A_2, \end{cases}$$

and this shows the correctness of our results.

5. Conclusions

In this paper, a theoretical study was done for the given Caputo BVP of variable order (1) at resonance. To conduct this research, we defined some generalized subintervals as a partition of the main interval, and then on each subinterval, the piecewise constant functions were defined. With the help of these notions, we converted the given variable-order system to a constant-order system at resonance. In this case, we implemented the conditions of the Mawhin's continuation theorem for proving the existence criterion for solutions of the corresponding BVP. Finally, an example was simulated numerically to show the correctness of our results. This technique on a variable-order BVP is new and determines the novelty of this work compared with other limited published papers in the form of variable orders. In relation to next studies, we aim to work on hidden-memory variable order systems and analyze the qualitative behaviors of their solutions such as existence, stability, and numerical solutions.

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Abbreviations

The following abbreviations are used in this manuscript:

BVP Boundary Value Problem

References

1. Gomez-Aguilar, J.F. Analytical and numerical solutions of a nonlinear alcoholism model via variable-order fractional differential equations. *Phys. A Stat. Mech. Its Appl.* **2018**, *494*, 52–75. [\[CrossRef\]](#)
2. Almeida, R.; Tavares, D.; Torres, D.F.M. *The Variable-Order Fractional Calculus of Variations*; Springer: Cham, Switzerland, 2019.
3. Sun, H.G.; Chang, A.; Zhang, Y.; Chen, W. A review on variable-order fractional differential equations: Mathematical foundations, Physical models, numerical methods and applications. *Fract. Calc. Appl. Anal.* **2019**, *22*, 27–59. [\[CrossRef\]](#)
4. Abdeljawad, A.; Agarwal, R.P.; Karapinar, E.; Kumari, P.S. Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-Metric Space. *Symmetry* **2019**, *11*, 686. [\[CrossRef\]](#)
5. Abbas, M.I.; Ragusa, M.A. On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain function. *Symmetry* **2021**, *13*, 264. [\[CrossRef\]](#)
6. Boutiara, A.; Adjimi, N.; Benbachir, M.; Abdo, M.S. Analysis of a fractional boundary value problem involving Riesz-Caputo fractional derivative. *Adv. Theory Nonlinear Anal. Its Appl.* **2022**, *6*, 14–27.
7. Ntouyas, S.K.; Etemad, S.; Tariboon, J. Existence results for multi-term fractional differential inclusions. *Adv. Differ. Equ.* **2015**, *2015*, 140. [\[CrossRef\]](#)
8. Rezapour, S.; Ntouyas, S.K.; Iqbal, M.Q.; Hussain, A.; Etemad, S.; Tariboon, J. An analytical survey on the solutions of the generalized double-order φ -integrodifferential equation. *J. Funct. Spaces*, **2021**, *2021*, 6667757. [\[CrossRef\]](#)
9. Adiguzel, R.S.; Aksoy, U.; Karapinar, E.; Erhan, I.M. On the solution of a boundary value problem associated with a fractional differential equation. *Math. Methods Appl. Sci.* **2020**. [\[CrossRef\]](#)
10. Afshari, H.; Karapinar, E. A discussion on the existence of positive solutions of the boundary value problems via ψ -Hilfer fractional derivative on b-metric spaces. *Adv. Differ. Equ.* **2020**, *2020*, 616. [\[CrossRef\]](#)
11. Etemad, S.; Rezapour, S.; Samei, M.E. α - ψ -contractions and solutions of a q -fractional differential inclusion with three-point boundary value conditions via computational results. *Adv. Differ. Equ.* **2020**, *2020*, 218. [\[CrossRef\]](#)
12. Boutiara, A.; Etemad, S.; Hussain, A.; Rezapour, S. The generalized U-H and U-H stability and existence analysis of a coupled hybrid system of integro-differential IVPs involving φ -Caputo fractional operators. *Adv. Differ. Equ.* **2021**, *2021*, 95. [\[CrossRef\]](#)
13. Zhang, S.; Hu, L. The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order. *AIMS Math.* **2020**, *5*, 2923–2943. [\[CrossRef\]](#)
14. Sousa, J.V.d.C.; de Oliveira, E.C. Two new fractional derivatives of variable order with non-singular kernel and fractional differential equation. *Comput. Appl. Math.* **2018**, *37*, 5375–5394. [\[CrossRef\]](#)
15. Benkerrouche, A.; Souid, M.S.; Etemad, S.; Hakem, A.; Agarwal, P.; Rezapour, S.; Ntouyas, S.K.; Tariboon, J. Qualitative study on solutions of a Hadamard variable order boundary problem via the Ulam-Hyers-Rassias stability. *Fractal Fract.* **2021**, *5*, 108. [\[CrossRef\]](#)
16. Kaabar, M.K.A.; Refice, A.; Souid, M.S.; Martínez, F.; Etemad, S.; Siri, Z.; Rezapour, S. Existence and U-H-R stability of solutions to the implicit nonlinear FBVP in the variable order settings. *Mathematics* **2021**, *9*, 1693. [\[CrossRef\]](#)
17. Zhang, S.; Hu, L. Unique existence result of approximate solution to initial value problem for fractional differential equation of variable order involving the derivative arguments on the half-axis. *Mathematics* **2019**, *7*, 286. [\[CrossRef\]](#)
18. Zhang, S.; Li, S.; Hu, L. The existence and uniqueness result of solutions to initial value problems of nonlinear diffusion equations involving with the conformable variable derivative. *Revista de la Real Academia de Ciencias Exactas Físicas y Naturales Serie A Matemáticas* **2019**, *113*, 1601–1623. [\[CrossRef\]](#)
19. Bouazza, Z.; Etemad, S.; Souid, M.S.; Rezapour, S.; Martínez, F.; Kaabar, M.K.A. A study on the solutions of a multiterm FBVP of variable order. *J. Funct. Spaces* **2021**, *2021*, 9939147. [\[CrossRef\]](#)

20. Gaines, R.E.; Mawhin, J.L. *Coincidence Degree and Nonlinear Differential Equations*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1977.
21. Mawhin, J. Topological degree and boundary value problems for nonlinear differential equations. In *Topological Methods for Ordinary Differential Equations*; Furi, M., Zecca, P., Eds.; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1993; Volume 1537.
22. Feng, W.; Webb, J.R.L. Solvability of three point boundary value problems at resonance. *Nonlinear Anal. Theory Methods Appl.* **1997**, *30*, 3227–3238. [[CrossRef](#)]
23. Guezane-Lakoud, A.; Frioui, A. Third order boundary value problem with integral condition at resonance. *Theory Appl. Math. Comput. Sci.* **2013**, *3*, 56–64.
24. Mawhin, J.; Ward, J.R., Jr. Periodic solutions of some forced Lienard differential equations at resonance. *Archiv. Math.* **1983**, *41*, 337–351. [[CrossRef](#)]
25. Infante, G.; Zima, M. Positive solutions of multi-point boundary value problems at resonance. *Nonlinear Anal. Theory Methods Appl.* **2008**, *69*, 2458–2465. [[CrossRef](#)]
26. Umarov, S.; Steinberg, S. Variable order differential equations with piecewise constant order-function and diffusion with changing modes. *Z. Anal. Ihre Anwendungen* **2009**, *28*, 431–450. [[CrossRef](#)]
27. Zheng, X.; Wang, H. A hidden-memory variable-order time-fractional optimal control model: Analysis and approximation. *SIAM J. Control Optim.* **2021**, *59*, 1851–1880. [[CrossRef](#)]
28. Lorenzo, C.F.; Hartley, T.T. Initialization, conceptualization, and application in the generalized (fractional) calculus. *Crit. Rev. Biomed. Eng.* **2007**, *35*, 447–553. [[CrossRef](#)] [[PubMed](#)]
29. Samko, S.G. Fractional integration and differentiation of variable order. *Anal. Math.* **1995**, *21*, 213–236. [[CrossRef](#)]
30. Samko, S.G.; Ross, B. Integration and differentiation to a variable fractional order. *Integral Transform. Spec. Funct.* **1993**, *1*, 277–300. [[CrossRef](#)]
31. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
32. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999; Volume 198.
33. Zhang, S.; Sun, S.; Hu, L. Approximate solutions to initial value problem for differential equation of variable order. *J. Fract. Calc. Appl.* **2018**, *9*, 93–112.
34. An, J.; Chen, P. Uniqueness of solutions to initial value problem of fractional differential equations of variable-order. *Dyn. Syst. Appl.* **2019**, *28*, 607–623.
35. Zhang, S. The uniqueness result of solutions to initial value problems of differential equations of variable-order. *Revista de la Real Academia de Ciencias Exactas Físicas y Naturales Serie A Matemáticas* **2018**, *112*, 407–423. [[CrossRef](#)]
36. Mawhin, J. *Topological Degree Methods in Nonlinear Boundary Value Problems*; CBMS Regional Conference Series in Mathematics; American Mathematical Society: Providence, RI, USA, 1979; Volume 40.
37. O'Regan, D.; Cho, Y.J.; Chen, Y.Q. *Topological Degree Theory and Application*; Chapman and Hall/CRC: Boca Raton, FL, USA, 2006; Volume 10.