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Some Dynamic Inequalities via Diamond Integrals for Function of Several Variables

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Abstract: In this paper, Jensen's inequality and Fubini's Theorem are extended for the function of several variables via diamond integrals of time scale calculus. These extensions are used to generalize Hardy-type inequalities with general kernels via diamond integrals for the function of several variables. Some Hardy Hilbert and Polya Knop type inequalities are also discussed as special cases. Classical and new inequalities are deduced from the main results using special kernels and particular time scales.

Keywords: time scale calculus; diamond integrals; Jensen inequality; hardy inequality

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1. Introduction

Most of the inequalities relative to the integration and differentiation have their analog for sums and differences. Earlier, about thirty two years ago, these results were investigated separately as continuous and discrete cases. In 1988, Stephan Hilger introduced the theory of time scales that provides a platform to deal with discrete and continuous cases together [1,2].

English mathematician G. H. Hardy has introduced the following integral inequality ([3], Theorem 327):

If $p > 1$, $g(r) \geq 0$ and $G(r) = \int_0^r g(s)ds$, then:

$$\int_0^\infty \left(\frac{G}{r}\right)^p dr < \left(\frac{p}{p-1}\right)^p \int_0^\infty g^p dr, \quad (1)$$

unless $g \equiv 0$. The constant is best possible.

Inequality (1) is called Hardy's integral inequality.

In [4], Hardy proved the following generalization of inequality (1).

If $p > 1$, $n \neq 1$, $g(r) \geq 0$ and $R(r)$ is defined by:

$$R(r) := \begin{cases} \int_0^r g(s)ds, & \text{if } n > 1, \\ \int_r^\infty g(s)ds, & \text{if } n < 1, \end{cases}$$

then

$$\int_0^\infty r^{-n} R^p(r) dr < \left\{ \frac{p}{|n-1|} \right\}^p \int_0^\infty r^{-n} (rg(r))^p dr,$$

unless $g \equiv 0$. The constant is the best possible.

In [5], Oguntuase et al. established the following new multidimensional Hardy-type inequality:

Let $1 < p \leq q < \infty$, $0 < b_j \leq \infty$, $a_1, \dots, a_n \in (1, p)$, $j = 1, 2, \dots, n$, $k(\mathbf{r}, \mathbf{s})$ be a locally integrable kernel and $-\infty \leq a < c \leq \infty$. Then,

$$\left(\int_0^{b_1} \cdots \int_0^{b_n} \phi(A_K g(\mathbf{r}))^q u(\mathbf{r}) \frac{d\mathbf{r}}{r_1 \cdots r_n} \right)^{\frac{1}{q}} \leq C \left(\int_0^{b_1} \cdots \int_0^{b_n} \phi^p(g(\mathbf{r})) v(\mathbf{r}) \frac{d\mathbf{r}}{r_1 \cdots r_n} \right)$$

holds. The arithmetic mean operator and general positive kernel are given by:

$$A_K g(\mathbf{r}) := \frac{1}{K(\mathbf{r})} \int_0^{a_1} \cdots \int_0^{a_n} k(\mathbf{r}, \mathbf{s}) g(\mathbf{s}) d\mathbf{v}$$

and

$$K(\mathbf{r}) := \int_0^{a_1} \cdots \int_0^{a_n} k(\mathbf{r}, \mathbf{s}) d\mathbf{s},$$

respectively, where ϕ is a positive convex function.

Some variants of Hardy's inequality are studied via time scale calculus in [6–11]. The study of integration theory for functions of two variables on time scales is started in [12], after that many researchers used these integrals. The study of Hardy type inequalities for different recent settings are found in [9,11,13–18] and the references therein. In [19], Nosheen et al. studied Hardy-type inequalities for functions of several variables with general kernels using the delta-integral.

Assume $(\Omega, \Psi, \nu_\Delta)$ and $(\Theta, Y, \Theta_\Delta)$ are two time scale measure spaces and $H : \Omega \times \Theta \rightarrow \mathbb{R}_+$, such that

$$H(r) := \int_\Theta k(r, s) \Delta s < \infty, \quad r \in \Omega$$

and $\zeta : \Omega \rightarrow \mathbb{R}_+$, such that

$$\omega(r) := \int_\Omega \frac{k(r, s) \zeta(r)}{H(r)} \Delta r < \infty, \quad s \in \Theta.$$

If $\psi : V \rightarrow \mathbb{R}$ is convex and $V \subset \mathbb{R}^m$ is closed convex set, then:

$$\int_\Omega \zeta(r) \psi \left(\frac{\int_\Theta k(r, s) g(s)}{H(r)} \Delta s \right) \Delta r \leq \int_\Theta \omega(s) \psi(g(s)) \Delta s,$$

holds for all Θ_Δ -integrable functions, $g : \Theta \rightarrow \mathbb{R}^m$ and $g(\Theta) \subset V$.

In [20], George A. Anastassiou generalized Hardy-type inequalities with general kernel, using Diamond- α integral, are given as:

Consider a continuous weight function $w : [a, b]_{T_1} \rightarrow \mathbb{R}_+$, and the function:

$$v(s) := \int_a^b \frac{w(r) k(r, s)}{K(r)} \diamond_\alpha r$$

then

$$\int_a^b w(r) \Phi \left(\frac{|g(r)|}{K(r)} \right) \diamond_\alpha r \leq \int_c^d v(s) \Phi(|f(s)|) \diamond_\alpha s.$$

If $\Phi : (0, \infty) \rightarrow \mathbb{R}$ is not necessarily an increasing function and $f > 0$, then:

$$\int_a^b w(x) \Phi \left(\frac{|g(r)|}{K(r)} \right) \diamond_\alpha r \leq \int_c^d v(s) \Phi(|f(s)|) \diamond_\alpha s.$$

In this paper, Hardy-type inequalities via diamond integrals using a multivariable convex function with general kernels are proved.

Firstly, in Section 2, some necessary preliminary facts are recalled and Fubini's Theorem for Diamond Integrals is proved. We have discussed Jensen's Inequality and Hardy-

type inequalities with general kernels for diamond integrals in Section 3. In Section 4, some inequalities with certain kernels are discussed. In Section 5, Hilbert-type and Polya–Knopp type inequalities are proved in two ways. Lastly, some particular cases of Hardy-type inequalities are discussed. Our results generalize the results of [5–7,19].

Time scale $\mathbb{T}$ is a nonempty closed subset of real numbers.

For Example: $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$.

Let $r \in \mathbb{T}$, forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is given as:

$$\sigma(r) := \inf\{v \in \mathbb{T} : v > r\},$$

and backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is

$$\rho(r) := \sup\{v \in \mathbb{T} : v < r\}.$$

Classification of points:

- Point v is a right-scattered point, if $\sigma(v) > v$.
- Point v is a left-scattered point, if $\rho(v) < v$.
- Point v is a isolated point, if $\sigma(v) > v$ and $\rho(v) < v$.
- Point v is a right-dense point, if $v < \sup(\mathbb{T})$ and $\sigma(v) = v$.
- Point v is a left-dense point, if $v > \inf(\mathbb{T})$ and $\rho(v) = v$.
- Point v is a dense point, if $\rho(v) = v = \sigma(v)$.

A mapping $g : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous if it is continuous $\forall v \in \mathbb{T}$, $\sigma(v) = v$ and $\lim_{s \rightarrow v^-} g(s)$ exist (finite) $\forall v \in \mathbb{T}$, $\rho(v) = v$. The set of such functions $g : \mathbb{T} \rightarrow \mathbb{R}$ is represented by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

A mapping g is ld-continuous if it is continuous $\forall v \in \mathbb{T}$, $\sigma(v) = v = \rho(v)$ and $\lim_{s \rightarrow v^+} g(s)$ exists (finite) $\forall v \in \mathbb{T}$, $\sigma(v) = v$, $v \in \mathbb{T}$.

Delta Integral [1]

A mapping $H : \mathbb{T} \rightarrow \mathbb{R}$ is said to be delta antiderivative of $h : \mathbb{T} \rightarrow \mathbb{R}$ if $H^\Delta(v) = h(v)$ holds true $\forall v \in \mathbb{T}^\kappa$. The delta integral of h from d to e is:

$$\int_d^e h(v) \Delta v = H(e) - H(d).$$

Nabla Integral [1]

A mapping $H : \mathbb{T} \rightarrow \mathbb{R}$ is said to be nabla antiderivative of $h : \mathbb{T} \rightarrow \mathbb{R}$ if $H^\nabla(v) = h(v)$ $\forall v \in \mathbb{T}^\kappa$. The nabla integral of h from d to e is

$$\int_d^e g(v) \nabla v = G(e) - G(d).$$

For introduction to time scales calculus see [1].

In [21], real valued function $\gamma(r)$ is given as:

$$\gamma(r) = \lim_{s \rightarrow r} \frac{\sigma(r) - s}{\sigma(r) + 2r - 2s - \rho(r)}.$$

Clearly,

$$\gamma(r) = \begin{cases} \frac{1}{2}, & \text{if } r \text{ is dense,} \\ \frac{\sigma(r) - r}{\sigma(r) - \rho(r)}, & \text{if } r \text{ is not dense,} \end{cases}$$

and $0 \leq \gamma(r) \leq 1$.

In [22], the authors provided the more refined form of diamond- α integrals, which are called diamond integrals, and are of tremendous interest including in the classical case $\mathbb{T} = \mathbb{R}$.

Diamond Alpha Integral ([22])

Consider $l : \mathbb{T} \rightarrow \mathbb{R}$ to be a continuous mapping and $c, d \in \mathbb{T} (c < d)$. The diamond alpha integral of l from c to d is defined by:

$$\int_c^d l(u) \diamond_{\alpha} u = \int_c^d \alpha l(u) \Delta u + \int_c^d (1 - \alpha) l(u) \nabla u,$$

where α is constant and $0 \leq \alpha \leq 1$.

Diamond Integral ([22])

Consider $l : \mathbb{T} \rightarrow \mathbb{R}$ to be a continuous mapping and $c, d \in \mathbb{T} (c < d)$. The diamond(or $\diamond$ -integral) integral of l from c to d is defined by:

$$\int_c^d l(u) \diamond u = \int_c^d \gamma(u) l(u) \Delta u + \int_c^d (1 - \gamma(u)) l(u) \nabla u, \quad 0 \leq \gamma(u) \leq 1$$

if γl is Δ and $(1 - \gamma)l$ is ∇ integrable on $[c, d]_{\mathbb{T}}$.

For additive, multiplicative, reflexive and monotonicity properties of $\diamond$ -integral, see [22].

2. Main Results

Firstly, Jensen's inequality for the diamond integral via the function of several variables is being proved in this section. Then, Hardy-type inequalities with general kernels for the diamond integral are deduced.

Let the m -tuple of functions be $\mathbf{f}(r) = (f_1(r), \dots, f_m(r))$, where $f_k(r)$ are $\diamond$ -integrable for all $k \in \{1, \dots, m\}$. Then, $\int_{\Theta} \mathbf{f}(r) \diamond(r)$ denotes the m -tuple $(\int_{\Theta} f_1(r) \diamond(r), \dots, \int_{\Theta} f_m(r) \diamond(r))$; that is, the $\diamond$ -integral acts on each component of $\mathbf{f}(r)$.

Further, suppose $n \in \mathbb{N}$; define the n -dimensional time scale by the Cartesian product of given time scales $\mathbb{T}_i, i \in \{1, \dots, n\}$, as

$$\Omega^n = \{r = (r_1, r_2, \dots, r_n) : r_i \in \mathbb{T}_i, i \in \{1, \dots, n\}\}.$$

Assume the finite dimensional time scale measure spaces $(\Omega, \Psi, \mu_{\diamond})$ and $(\Theta, Y, \Theta_{\diamond})$, define the product measure space $(\Omega \times \Theta, \Psi \times Y, \mu_{\diamond} \times \Theta_{\diamond})$, where $\Psi \times Y$ is the product σ -algebra generated by $\{H \times G : H \in \Psi, G \in Y\}$ and

$$(\mu_{\diamond} \times \Theta_{\diamond})(H \times G) = \mu_{\diamond}(F) \Theta_{\diamond}(G).$$

2.1. Jensen's Inequality for Diamond Integrals

Theorem 1. Assume $(\Theta, Y, \Theta_{\diamond})$ is time scale measure space and $\psi \in C(K, \mathbb{R})$ is convex, where $K \subseteq \mathbb{R}^m$ is convex and closed. Consider $f_i, i = 1, 2, \dots, m$, are $\Theta_{\diamond}$ -integrable on Θ such that $\mathbf{f}(s) = (f_1(s), f_2(s), \dots, f_m(s)) \in K \quad \forall s \in \Theta$. Let $l : \Theta \rightarrow \mathbb{R}$ be non negative and $\Theta_{\diamond}$ -integrable functions such that $\int_{\Theta} l(s) \diamond s > 0$ then

$$\psi \left(\frac{\int_{\Theta} l(s) \mathbf{f}(s) \diamond s}{\int_{\Theta} l(s) \diamond s} \right) \leq \frac{\int_{\Theta} l(s) \psi(\mathbf{f}(s)) \diamond s}{\int_{\Theta} l(s) \diamond s}.$$

Proof. Let ψ be convex on $K \subseteq \mathbb{R}^m$ for every $\mathbf{x}_0 \in K \exists \mathbf{t} \in \mathbb{R}^m$ (see Theorem 1.31 of [23]) such that:

$$\psi(x) - \psi(x_0) \geq \langle \mathbf{t}, x - x_0 \rangle.$$

Let $\mathbf{t} = (t_1, t_2, \dots, t_m)$. Consider:

$$\begin{aligned}
 \frac{\int_{\Theta} l(s) \psi(\mathbf{f}(s)) \diamond s}{\int_{\Theta} l(s) \diamond s} &= \psi \left(\frac{\int_{\Theta} l(s) \mathbf{f}(s) \diamond s}{\int_{\Theta} l(s) \diamond s} \right) \\
 &= \frac{\int_{\Theta} l(s) \psi(\mathbf{f}(s)) \diamond s}{\int_{\Theta} l(s) \diamond s} - \psi \left(\frac{\int_{\Theta} l(s) \mathbf{f}(s) \diamond s}{\int_{\Theta} l(s) \diamond s} \right) \frac{\int_{\Theta} l(s) \diamond s}{\int_{\Theta} l(s) \diamond s} \\
 &= \frac{\int_{\Theta} l(s) [\psi(\mathbf{f}) - \psi(\frac{\int_{\Theta} l(s) \mathbf{f}(s) \diamond s}{\int_{\Theta} l(s) \diamond s})] \diamond s}{\int_{\Theta} l(s) \diamond s} \\
 &\geq \frac{\int_{\Theta} l(s) \langle \mathbf{t}, \mathbf{f}(s) - (\frac{\int_{\Theta} l(s) \mathbf{f}(s) \diamond s}{\int_{\Theta} l(s) \diamond s}) \rangle \diamond s}{\int_{\Theta} l(s) \diamond s} \\
 &= \frac{\int_{\Theta} l(s) \sum_{i=1}^m t_i [f_i(s) - (\frac{\int_{\Theta} l(s) f_i(s) \diamond s}{\int_{\Theta} l(s) \diamond s})] \diamond s}{\int_{\Theta} l(s) \diamond s} \\
 &= \frac{\int_{\Theta} l(s) t_1 [f_1(s) - (\frac{\int_{\Theta} l(s) f_1(s) \diamond s}{\int_{\Theta} l(s) \diamond s})] \diamond s}{\int_{\Theta} l(s) \diamond s} \\
 &+ \dots + \frac{\int_{\Theta} l(s) t_m [f_m(s) - (\frac{\int_{\Theta} l(s) f_m(s) \diamond s}{\int_{\Theta} l(s) \diamond s})] \diamond s}{\int_{\Theta} l(s) \diamond s} \\
 &= \frac{t_1 \int_{\Theta} l(s) f_1(s) \diamond s - t_1 \frac{\int_{\Theta} l(s) f_1(s) \diamond s}{\int_{\Theta} l(s) \diamond s} \int_{\Theta} l(s) \diamond s}{\int_{\Theta} l(s) \diamond s} \\
 &+ \dots + \frac{t_m \int_{\Theta} l(s) f_m(s) \diamond s - t_m \frac{\int_{\Theta} l(s) f_m(s) \diamond s}{\int_{\Theta} l(s) \diamond s} \int_{\Theta} l(s) \diamond s}{\int_{\Theta} l(s) \diamond s} \\
 &= 0,
 \end{aligned}$$

and hence the proof is complete. $\square$

2.2. Fubini's Theorem for Diamond Integrals

We prove Fubini's Theorem for diamond integrals by taking the procedure used in [20].

Lemma 1. Let function $h : [x, y]_{\mathbb{T}_1} \times [u, v]_{\mathbb{T}_2} \rightarrow \mathbb{R}$ be continuous and γh is Δ and $(1 - \gamma)h$ be ∇ -integrable on $[x, y]_{\mathbb{T}_1} \times [u, v]_{\mathbb{T}_2}$ and $(x, y]_{\mathbb{T}_1} \times (u, v]_{\mathbb{T}_2}$ respectively. Then,

$$\int_u^v \left(\int_x^y h(t, s) \diamond s \right) \diamond t = \int_x^y \left(\int_u^v h(t, s) \diamond t \right) \diamond s,$$

where $0 \leq \gamma \leq 1$.

Proof. By using Fubini's theorem for Δ and ∇ -integrals [20], we find that:

$$\begin{aligned}
 \int_u^v \left(\int_x^y h(t,s) \diamond s \right) \diamond t &= \int_u^v \left(\int_x^y \gamma(s) h(t,s) \Delta s \right) \diamond t \\
 &+ \int_u^v \left(\int_x^y (1 - \gamma(s)) h(t,s) \nabla s \right) \diamond t \\
 &= \int_u^v \gamma(t) \left(\int_x^y \gamma(s) h(t,s) \Delta s \right) \Delta t \\
 &+ \int_u^v (1 - \gamma(t)) \left(\int_x^y \gamma(s) h(t,s) \Delta s \right) \nabla t \\
 &+ \int_u^v \gamma(t) \left(\int_x^y (1 - \gamma(s)) h(t,s) \nabla s \right) \Delta t \\
 &+ \int_u^v (1 - \gamma(t)) \left(\int_x^y (1 - \gamma(s)) h(t,s) \nabla s \right) \nabla t \\
 &= \int_x^y \gamma(s) \left(\int_u^v \gamma(t) h(t,s) \Delta t \right) \Delta s \\
 &+ \int_x^y (1 - \gamma(s)) \left(\int_u^v \gamma(t) h(t,s) \Delta t \right) \nabla s \\
 &+ \int_x^y \gamma(s) \left(\int_u^v (1 - \gamma(t)) h(t,s) \nabla t \right) \Delta s \\
 &+ \int_x^y (1 - \gamma(s)) \left(\int_u^v (1 - \gamma(t)) h(t,s) \nabla t \right) \nabla s \\
 &= \int_x^y \left(\int_u^v \gamma(t) h(t,s) \Delta t \right) \diamond s \\
 &+ \int_x^y \left(\int_u^v (1 - \gamma(t)) h(t,s) \nabla t \right) \diamond s \\
 &= \int_x^y \left(\int_u^v h(t,s) \diamond t \right) \diamond s.
 \end{aligned}$$

The proof is therefore complete. $\square$

Theorem 2. Consider two time scale measure spaces $(\Omega, \Psi, \mu_\diamond)$ and $(\Theta, \Upsilon, \Theta_\diamond)$. Assume $H : \Omega \times \Theta \rightarrow \mathbb{R}_+$ is such that

$$H(r) := \int_\Theta k(r,s) \diamond s < \infty, \quad r \in \Omega$$

and $\eta : \Omega \rightarrow \mathbb{R}_+$ is such that:

$$\omega(s) := \int_\Omega \frac{k(r,s) \eta(r)}{H(r)} \diamond r < \infty, \quad s \in \Theta,$$

if $\psi \in C(K, \mathbb{R})$ where $K \subseteq \mathbb{R}^m$ is closed and convex, then:

$$\int_\Omega \eta(r) \psi \left(\frac{1}{H(r)} \int_\Theta k(r,s) \mathbf{f}(s) \diamond s \right) \diamond r \leq \int_\Theta \omega(s) \psi(\mathbf{f}(s)) \diamond s \quad (2)$$

holds $\forall \Theta_\diamond$ -integrable functions $\mathbf{f} : \Theta \rightarrow \mathbb{R}^m$ such that: $\mathbf{f}(\Theta) \subset K$.

Proof. By using (the Diamond integral version of) Jensen's inequality and the Fubini's Theorem, we have:

$$\begin{aligned}
 \int_{\Omega} \eta(r) &\times \psi\left(\frac{\int_{\Theta} k(r,s) \mathbf{f}(s) \diamond s}{H(r)}\right) \diamond r \\
 &= \int_{\Omega} \eta(r) \psi\left[\frac{1}{H(r)} \left(\int_{\Theta} k(r,s) f_1(s) \diamond s, \dots, \int_{\Theta} k(r,s) f_m(s) \diamond s\right)\right] \diamond r \\
 &\leq \int_{\Omega} \frac{\eta(r)}{H(r)} \left(\int_{\Theta} k(r,s) \psi(\mathbf{f}(s)) \diamond s\right) \diamond r \\
 &= \int_{\Theta} \psi(\mathbf{f}(s)) \left(\int_{\Omega} \frac{k(r,s) \eta(r)}{H(r)} \diamond r\right) \diamond s \\
 &= \int_{\Theta} \omega(s) \psi(\mathbf{f}(s)) \diamond s.
 \end{aligned}$$

□

Remark 1. Inequality (2) holds in the reverse direction, if ψ is concave.

Corollary 1. Assume two time scale measure spaces $(\Omega, \Psi, \mu_{\diamond})$ and $(\Theta, Y, \Theta_{\diamond})$, then the following results hold for all $\Theta_{\diamond}$ -integrable functions $\mathbf{f} : \Theta \rightarrow \mathbb{R}^m$, where $\mathbf{f}(\Theta) \subset K$ on $(0, +\infty)$.

(1) If $\beta > 1$ or $\beta < 0$, then:

$$\int_{\Omega} \eta(r) \left(\frac{\int_{\Theta} k(r,s) \mathbf{f}(s) \diamond s}{H(r)}\right)^{\beta} \diamond r \leq \int_{\Theta} \omega(s) (\mathbf{f}(s))^{\beta} \diamond s. \quad (3)$$

If $\beta \in (0, 1)$, then

$$\int_{\Omega} \eta(r) \left(\frac{\int_{\Theta} k(r,s) \mathbf{f}(s) \diamond s}{H(r)}\right)^{\beta} \diamond r \geq \int_{\Theta} \omega(s) (\mathbf{f}(s))^{\beta} \diamond s. \quad (4)$$

(2)

$$\int_{\Omega} \eta(r) \ln \left(\frac{\int_{\Theta} k(r,s) \mathbf{f}(s) \diamond s}{H(r)}\right) \diamond r \geq \int_{\Theta} \omega(s) \ln(\mathbf{f}(s)) \diamond s. \quad (5)$$

Proof. (1) By using $\psi(x) = x^{\beta}$ and $K = \mathbb{R}_+^m$ in Theorem 2, we get (3). Since, in this case, $\psi(x)$ is convex for all $\beta > 1$, $\beta < 0$.

By using $\psi(x) = x^{\beta}$ and $K = \mathbb{R}_+^m$ in Theorem 2, we get (4). Since $\psi(x)$ is concave for all $\beta \in (0, 1)$.

(2) Here, by using $\psi(x) = \ln(x)$ and $K = \mathbb{R}_+^m$ in Theorem 2 we get (5). Since, in this case, $\psi(x)$ is convex on $(0, +\infty)$.

□

Corollary 2. Let $\phi : [l_1, l'_1) \times \dots \times [l_m, l'_m) \rightarrow \mathbb{R}_+$ be a continuous function and

$$\tilde{L}_j(f_j, \Theta) := L_j^{-1} \left(\frac{\int_{\Theta} k(r,s) L_j f_j(s) \diamond s}{H(r)} \right)$$

$\forall j \in \{1, 2, \dots, m\}$. If $\psi(s_1, \dots, s_m) = \phi(L_1^{-1}(s_1), \dots, L_m^{-1}(s_m))$ is convex, then:

$$\int_{\Omega} \eta(r) \phi\left(\tilde{L}_1(f_1, \Theta), \dots, \tilde{L}_m(f_m, \Theta)\right) \diamond r \leq \int_{\Theta} \omega(s) \phi\left(f_1(s), \dots, f_m(s)\right) \diamond s$$

holds $\forall f_j(\Theta) \subset [l_j, l'_j)$ and continuous monotone functions $L_j : [l_j, l'_j) \rightarrow \mathbb{R}$, where $L_j \circ f_j$ are $\Theta_{\diamond}$ -integrable functions $\forall j \in \{1, \dots, m\}$.

Proof. From Theorem 2

$$\begin{aligned} & \int_{\Omega} \eta(r) \psi \left[\frac{1}{H(r)} \left(\int_{\Theta} k(r, s) f_1(s) \diamond s, \dots, \int_{\Theta} k(r, s) f_m(s) \diamond s \right) \right] \diamond r \\ & \leq \int_{\Omega} \frac{\eta(r)}{H(r)} \left(\int_{\Theta} k(r, s) \psi(f_1(s), \dots, f_m(s)) \diamond s \right) \diamond r. \end{aligned}$$

Replace $f_j(s)$ with $L_j(f_j(s)) \quad \forall \quad j \in \{1, \dots, m\}$

$$\begin{aligned} \Rightarrow & \int_{\Omega} \eta(r) \psi \left[\left(\frac{\int_{\Theta} k(r, s) L_1(f_1(s)) \diamond s}{H(r)} \right), \dots, \left(\frac{\int_{\Theta} k(r, s) L_m(f_m(s)) \diamond s}{H(r)} \right) \right] \diamond r \\ & \leq \int_{\Omega} \frac{\eta(r)}{H(r)} \left(\int_{\Theta} k(r, s) \psi(L_1(f_1(s)), \dots, L_m(f_m(s))) \diamond s \right) \diamond r. \end{aligned}$$

Replace $\psi(s_1, \dots, s_m)$ with $\phi(L_1^{-1}(s_1), \dots, L_m^{-1}(s_m))$

$$\begin{aligned} \Rightarrow & \int_{\Omega} \eta(r) \phi \left[L_1^{-1} \left(\frac{\int_{\Theta} k(r, s) L_1(f_1(s)) \diamond s}{H(r)} \right), \dots, L_m^{-1} \left(\frac{\int_{\Theta} k(r, s) L_m(f_m(s)) \diamond s}{H(r)} \right) \right] \diamond r \\ & \leq \int_{\Omega} \frac{\eta(r)}{H(r)} \left(\int_{\Theta} k(r, s) \phi(L_1^{-1} L_1(f_1(s)), \dots, L_m^{-1} L_m(f_m(s))) \diamond s \right) \diamond r, \end{aligned}$$

so that

$$\begin{aligned} & \int_{\Omega} \eta(r) \phi \left(\tilde{L}_1(f_1, \Theta), \dots, \tilde{L}_m(f_m, \Theta) \right) \diamond r \\ & \leq \int_{\Theta} \omega(s) \phi \left(f_1(s), \dots, f_m(s) \right) \diamond s. \end{aligned}$$

This completes the proof. $\square$

Corollary 3. Let $p > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$. If $\mathbf{f}(x) = (f_1(x), f_2(x))$, then

$$\begin{aligned} & \int_{\Omega} \eta(r) \left(\frac{1}{H(r)} \int_{\Theta} k(r, s) (f_1(s))^p \diamond s \right)^{\frac{1}{p}} \times \left(\frac{1}{H(r)} \int_{\Theta} k(r, s) (f_2(s))^{p'} \diamond s \right)^{\frac{1}{p'}} \diamond r \\ & \geq \int_{\Theta} \omega(s) f_1(s) f_2(s) \diamond s \end{aligned}$$

$\forall \quad \Theta_{\diamond} - \text{integrable functions } f_j : \Theta \rightarrow \mathbb{R}_+, \text{ where } j \in \{1, 2\}.$

Proof. If $m = 2$, $\phi(s_1, s_2) = s_1 s_2$, $L_1(t_1) = (t_1)^p$, $L_2(t_2) = (t_2)^{p'}$, in Corollary 2, then $\psi(s_1, s_2) = s_1^{\frac{1}{p}} s_2^{\frac{1}{p'}}$ is concave in Theorem 2. $\square$

Corollary 4. If $\mathbf{f}(x) = (f_1(x), f_2(x))$, then:

$$\begin{aligned} & \int_{\Omega} \eta(r) \left(\left(\frac{1}{H(r)} \int_{\Theta} k(r, s) (f_1(s))^p \diamond s \right)^{\frac{1}{p}} + \left(\frac{1}{H(r)} \int_{\Theta} k(r, s) (f_2(s))^p \diamond s \right)^{\frac{1}{p}} \right)^p \diamond r \\ & \geq \int_{\Theta} \omega(s) (f_1(s) + f_2(s))^p \diamond s \end{aligned}$$

$\forall \quad \Theta_{\diamond} - \text{integrable functions } f_j : \Theta \rightarrow \mathbb{R}_+, \text{ where } j \in \{1, 2\}.$

Proof. If we use $m = 2$, $\phi(s_1, s_2) = (s_1 + s_2)^p$, $L_1(t_1) = (t_1)^p$ and $L_2(t_2) = (t_2)^p$ in Corollary 2 then $\psi(s_1, s_2) = (s_1^{\frac{1}{p}} + s_2^{\frac{1}{p}})^p$ is concave in Theorem 2. $\square$

Remark 2. If $p < 1$, then reverse inequalities hold in Corollaries 3 and 4.

3. Applications to Special Kernels

Assume that the following hypothesis holds throughout this paper:
Let $\Omega = \Theta = [a_1, b_1]_{\mathbb{T}} \times [a_2, b_2]_{\mathbb{T}} \times \cdots \times [a_n, b_n]_{\mathbb{T}}$ and $\Omega = \Theta = (a_1, b_1]_{\mathbb{T}} \times (a_2, b_2]_{\mathbb{T}} \times \cdots \times (a_n, b_n]_{\mathbb{T}}$ holds for delta and nabla integrals respectively, $0 \leq a_i < b_i < \infty \quad \forall \quad i \in \{1, \dots, n\}$, where $\mathbb{T}$ is an arbitrary time scale. Moreover, let $n \in \mathbb{N}$, $r = (r_1, r_2, \dots, r_n)$, $s = (s_1, s_2, \dots, s_n)$ and $(r, s) = (r_1, r_2, \dots, r_n, s_1, s_2, \dots, s_n)$, where $r_i, s_i \in \mathbb{T}_i \quad \forall \quad i \in \{1, \dots, n\}$.

Corollary 5. Consider $\eta : \Omega \rightarrow \mathbb{R}_+$ and

$$\omega(s) = \int_{s_1}^{b_1} \cdots \int_{s_n}^{b_n} \frac{k(r_1, \dots, r_n, s) \eta(r_1, \dots, r_n)}{H(r_1, \dots, r_n)} \diamond r_1 \cdots \diamond r_n, \quad s \in \Theta.$$

If $U \subset \mathbb{R}^m$ is a closed convex set, $\psi : U \rightarrow \mathbb{R}$ is continuous and convex, then

$$\begin{aligned} & \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \eta(r_1, \dots, r_n) \psi(A_k \mathbf{f}(r_1, \dots, r_n)) \diamond r_1 \cdots \diamond r_n \\ & \leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \omega(s_1, \dots, s_n) \psi(\mathbf{f}(s_1, \dots, s_n)) \times \diamond s_1 \cdots \diamond s_n \end{aligned}$$

$\forall \quad \Theta_{\diamond}$ -integrable functions $\mathbf{f} : \Theta \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Theta) \subset U$, where

$$(A_k \mathbf{f})(r) = \frac{1}{H(r)} \int_{a_1}^{\sigma(r_1)} \cdots \int_{a_n}^{\sigma(r_n)} k(r, s_1, \dots, s_n) \times \mathbf{f}(s_1, \dots, s_n) \diamond s_1 \cdots \diamond s_n.$$

Proof. In Theorem 2, use: $k(r_1, \dots, r_n, s_1, \dots, s_n) = 0$ if $a_i \leq s_i \leq \sigma(r_i)$, $\forall \quad i \in \{1, \dots, n\}$. In this case,

$$H(r) = \int_{a_1}^{\sigma(r_1)} \cdots \int_{a_n}^{\sigma(r_n)} k(r, s_1, \dots, s_n) \diamond s_1 \cdots \diamond s_n.$$

□

Corollary 6. Consider $\eta : \Omega \rightarrow \mathbb{R}_+$ and

$$\omega(s) = \int_{a_1}^{s_1} \cdots \int_{a_n}^{s_n} \frac{k(r_1, \dots, r_n, s) \eta(r_1, \dots, r_n)}{H(r_1, \dots, r_n)} \diamond r_1 \cdots \diamond r_n, \quad s \in \Theta.$$

If $U \subset \mathbb{R}^m$ is a closed convex set, $\psi : U \rightarrow \mathbb{R}$ is continuous and convex, then

$$\begin{aligned} & \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \eta(r_1, \dots, r_n) \psi(A_k \mathbf{f}(r_1, \dots, r_n)) \diamond r_1 \cdots \diamond r_n \\ & \leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \omega(s_1, \dots, s_n) \psi(\mathbf{f}(s_1, \dots, s_n)) \times \diamond s_1 \cdots \diamond s_n \end{aligned}$$

$\forall \quad \Theta_{\diamond}$ -integrable functions, $\mathbf{f} : \Theta \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Theta) \subset U$, where

$$(A_k \mathbf{f})(r) = \frac{1}{H(r)} \int_{\sigma(r_1)}^{b_1} \cdots \int_{\sigma(r_n)}^{b_n} k(r, s_1, \dots, s_n) \times \mathbf{f}(s_1, \dots, s_n) \diamond s_1 \cdots \diamond s_n.$$

Proof. In Theorem 2, use $k(r_1, \dots, r_n, s_1, \dots, s_n) = 0$, if $a_i \leq \sigma(r_i) \leq s_i \quad \forall \quad i \in \{1, \dots, n\}$. Since in this case:

$$H(r) = \int_{\sigma(r_1)}^{b_1} \cdots \int_{\sigma(r_n)}^{b_n} k(r, s_1, \dots, s_n) \diamond s_1 \cdots \diamond s_n.$$

□

Theorem 3. Consider $\eta : \Omega \rightarrow \mathbb{R}_+$ such that:

$$\tilde{\omega}(s) = \int_{s_1}^{b_1} \cdots \int_{s_n}^{b_n} \frac{\eta(r_1, \dots, r_n)}{\int_{a_1}^{\sigma(r_1)} \cdots \int_{a_n}^{\sigma(r_n)} \diamond s_1 \cdots \diamond s_n} \diamond r_1 \cdots \diamond r_n, \quad (6)$$

$s \in \Theta, \forall i \in \{1, \dots, n\}$. If $U \subset \mathbb{R}^m$ is a closed convex set such that $\psi : U \rightarrow \mathbb{R}$ is continuous and convex, then:

$$\begin{aligned} & \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \eta(r_1, \dots, r_n) \psi(\tilde{A}_k \mathbf{f}(r_1, \dots, r_n)) \diamond r_1 \cdots \diamond r_n \\ & \leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \tilde{\omega}(s_1, \dots, s_n) \psi(\mathbf{f}(s_1, \dots, s_n)) \times \diamond s_1 \cdots \diamond s_n \end{aligned}$$

$\forall \Theta_{\diamond}$ -integrable functions $\mathbf{f} : \Theta \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Theta) \subset U$, where

$$\begin{aligned} (\tilde{A}_k \mathbf{f})(r) &= \frac{1}{\int_{a_1}^{\sigma(r_1)} \cdots \int_{a_n}^{\sigma(r_n)} \diamond s_1 \cdots \diamond s_n} \\ &\times \int_{a_1}^{\sigma(r_1)} \cdots \int_{a_n}^{\sigma(r_n)} \tilde{\mathbf{f}}(s_1, \dots, s_n) \diamond s_1 \cdots \diamond s_n. \end{aligned}$$

Proof. The statement follows from Theorem 2 by using:

$$k(r_1, \dots, r_n, s_1, \dots, s_n) = \begin{cases} 1, & \text{if } a_i \leq s_i \leq b_i, \\ 0, & \text{otherwise,} \end{cases}$$

$\forall i \in \{1, \dots, n\}$. Since, in this case,

$$H(r_1, \dots, r_n) = \int_{a_1}^{\sigma(r_1)} \cdots \int_{a_n}^{\sigma(r_n)} \diamond s_1 \cdots \diamond s_n,$$

$\forall i \in \{1, \dots, n\}$. Thus, $A_k = \tilde{A}_k, \omega = \tilde{\omega}$. $\square$

Corollary 7. If $a_i = 0 \forall i \in \{1, \dots, n\}$ and $U \subset \mathbb{R}^m$ is a closed convex set such that $\psi : U \rightarrow \mathbb{R}$ is continuous and convex, $i \in \{1, \dots, n\}$,

$$\begin{aligned} & \int_0^{b_1} \cdots \int_0^{b_n} \psi(A_k \mathbf{f}(r_1, \dots, r_n)) \frac{\diamond r_1 \cdots \diamond r_n}{r_1 \cdots r_n} \\ & \leq \int_0^{b_1} \cdots \int_0^{b_n} \omega(s_1, \dots, s_n) \psi(\mathbf{f}(s_1, \dots, s_n)) \times \diamond s_1 \cdots \diamond s_n, \end{aligned} \quad (7)$$

$\forall \Theta_{\diamond}$ -integrable functions $\mathbf{f} : \Theta \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Theta) \subset U$, where:

$$\begin{aligned} (A_k \mathbf{f})(r_1, \dots, r_n) &= \frac{1}{\int_0^{\sigma(r_1)} \cdots \int_0^{\sigma(r_n)} \diamond s_1 \cdots \diamond s_n} \\ &\times \int_0^{\sigma(r_1)} \cdots \int_0^{\sigma(r_n)} \mathbf{f}(s_1, \dots, s_n) \diamond s_1 \cdots \diamond s_n. \end{aligned}$$

Proof. In Theorem 3, use $\eta(r_1, \dots, r_n) = \frac{1}{r_1 \cdots r_n}$, since in this case:

$$\begin{aligned} \omega(s_1, \dots, s_n) &= \int_{s_1}^{\sigma(b_1)} \cdots \int_{s_n}^{\sigma(b_n)} \frac{1}{r_1 \cdots r_n \times \int_0^{\sigma(r_1)} \cdots \int_0^{\sigma(r_n)} \diamond s_1 \cdots \diamond s_n} \\ &\times \diamond r_1 \cdots \diamond r_n, \end{aligned}$$

$\forall i \in \{1, \dots, n\}$. $\square$

4. Particular Cases

Firstly, in this section, Hilbert-type inequality on time scales are discussed.

Theorem 4. For $q > 1$, define:

$$K_1(r) =: \int_0^\infty \frac{(s/r)^{-1/q}}{r+s} \diamond s$$

and

$$K_2(s) =: \int_0^\infty \frac{(s/r)^{1-1/q}}{r+s} \diamond r,$$

then

$$\begin{aligned} \int_0^\infty \left(\int_0^\infty \frac{(s/r)^{1-1/q} (g_1(s))^p}{r+s} \diamond s \right)^{1/p} \times \left(\int_0^\infty \frac{(s/r)^{1-1/q} (g_2(s))^{p'}}{r+s} \diamond s \right)^{1/p'} \diamond r \\ \geq \int_0^\infty K_2(s) g_1(s) g_2(s) \diamond s \end{aligned} \quad (8)$$

$\forall \Theta_\diamond$ -integrable functions $g_j : \Theta \rightarrow \mathbb{R}_+$, where $j \in \{1, 2\}$.

Proof. Use $\eta(r) = \frac{K_1(r)}{r}$, $H(r) = K_1(r)$ and

$$k(r, s) = \begin{cases} \frac{(s/r)^{-1/q}}{r+s}, & \text{if } r, s, r+s \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

in Corollary 3, to obtain:

$$\begin{aligned} \int_0^\infty \frac{K_1(r)}{r} \left(\frac{1}{K_1(r)} \int_0^\infty \frac{(s/r)^{-1/q} (f_1(s))^p}{r+s} \diamond s \right)^{1/p} \\ \times \left(\frac{1}{K_1(r)} \int_0^\infty \frac{(s/r)^{-1/q} (f_2(s))^{p'}}{r+s} \diamond s \right)^{1/p'} \diamond r \\ \geq \int_0^\infty \omega(s) f_1(s) f_2(s) \diamond s, \end{aligned} \quad (9)$$

where

$$\begin{aligned} \omega(s) &= \int_0^\infty \frac{k(r, s) \eta(r)}{K_1(r)} \diamond r = \int_0^\infty \frac{k(r, s)}{r} \diamond r \\ &= \frac{1}{s} \int_0^\infty \frac{(s/r)^{1-1/q}}{r+s} \diamond r = \frac{K_2(s)}{s}. \end{aligned}$$

Using this value in (9), we obtain:

$$\begin{aligned} \int_0^\infty \left(\int_0^\infty \frac{(s/r)^{-1/q} (f_1(s))^p}{r+s} \diamond s \right)^{1/p} \times \left(\int_0^\infty \frac{(s/r)^{-1/q} (f_2(s))^{p'}}{r+s} \diamond s \right)^{1/p'} \diamond r \\ \geq \int_0^\infty K_2(s) f_1(s) f_2(s) \frac{\diamond s}{s}. \end{aligned}$$

Now, if we replace $f_1(s)$ with $g_1(s)s^{1/p}$ and $f_2(s)$ with $g_2(s)s^{1/p'}$, we obtain:

$$\begin{aligned} \int_0^\infty \left(\int_0^\infty \frac{(s/r)^{1-1/q} (g_1(s))^p}{r+s} \diamond s \right)^{1/p} \times \left(\int_0^\infty \frac{(s/r)^{1-1/q} (g_2(s))^{p'}}{r+s} \diamond s \right)^{1/p'} \diamond r \\ \geq \int_0^\infty K_2(s) g_1(s) g_2(s) \diamond s. \end{aligned}$$

The proof is therefore complete. $\square$

Proof. (Second proof of (8)). Assume the L.H.S of (9), applying Hölder's inequality ([22], Theorem 4) and Fubini's Theorem, we get:

$$\begin{aligned}
 & \int_0^\infty \frac{K_1(r)}{r} \left(\frac{1}{K_1(r)} \int_0^\infty \frac{(s/r)^{-1/q} (f_1(s))^p}{r+s} \diamond s \right)^{1/p} \\
 & \times \left(\frac{1}{K_1(r)} \int_0^\infty \frac{(s/r)^{-1/q} (f_2(s))^{p'}}{r+s} \diamond s \right)^{1/p'} \diamond r \\
 & \geq \int_0^\infty \left(\int_0^\infty \frac{(s/r)^{-1/q} f_1(s) f_2(s)}{r+s} \diamond s \right) \frac{\diamond r}{r} \\
 & = \int_0^\infty f_1(s) f_2(s) \left(\int_0^\infty \frac{(s/r)^{-1/q} \diamond r}{r+s} \right) \diamond s \\
 & = \int_0^\infty \frac{f_1(s) f_2(s)}{s} \left(\int_0^\infty \frac{(s/r)^{1-1/q}}{r+s} \diamond r \right) \diamond s \\
 & = \int_0^\infty K_2(s) f_1(s) f_2(s) \frac{\diamond s}{s}.
 \end{aligned}$$

$\square$

Theorem 5. Under the same assumptions of Theorem 5, the inequality,

$$\begin{aligned}
 & \int_0^\infty \left(\left(\int_0^\infty \frac{(s/r)^{1-1/q} (f_1(s))^p}{r+s} \diamond s \right)^{1/p} \right. \\
 & \left. + \left(\int_0^\infty \frac{(s/r)^{1-1/q} (f_2(s))^{p'}}{r+s} \diamond s \right)^{1/p'} \right)^p \diamond r \\
 & \geq \int_0^\infty K_2(s) (f_1(s) + f_2(s))^p \diamond s.
 \end{aligned} \tag{10}$$

$\forall \Theta_\diamond$ -integrable functions $f_j : \Theta \rightarrow \mathbb{R}_+$, where $j \in \{1, 2\}$.

Proof. Put $\eta(r) = \frac{K_1(r)}{r}$, $H(r) = K_1(r)$ and

$$k(r, s) = \begin{cases} \frac{(s/r)^{-1/q}}{r+s}, & \text{if } r, s, r+s \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

in Corollary 4 to obtain:

$$\begin{aligned}
 & \int_0^\infty \frac{K_1(r)}{r} \left(\left(\frac{1}{K_1(r)} \int_0^\infty \frac{(s/r)^{-1/q} (f_1(s))^p}{r+s} \diamond s \right)^{1/p} \right. \\
 & \left. + \left(\frac{1}{K_1(r)} \int_0^\infty \frac{(s/r)^{-1/q} (f_2(s))^p}{r+s} \diamond s \right)^{1/p} \right)^p \diamond r \\
 & \geq \int_0^\infty \omega(s) (f_1(s) + f_2(s))^p \diamond s.
 \end{aligned} \tag{11}$$

Now, if we replace $f_1(s)$ with $f_1(s)s^{1/p}$ and $f_2(s)$ with $f_2(s)s^{1/p'}$ in (11), we obtain (10). $\square$

Proof. (Second proof of (10)). Assume L.H.S of (11) and applying Minkowski's inequality ([22], Theorem 5) and Fubini's Theorem, we get:

$$\begin{aligned}
 & \int_0^\infty \frac{K_1(r)}{r} \left(\left(\frac{1}{K_1(r)} \int_0^\infty \frac{(s/r)^{-1/q} (f_1(s))^p}{r+s} \diamond s \right)^{1/p} \right. \\
 & + \left. \left(\frac{1}{K_1(r)} \int_0^\infty \frac{(s/r)^{-1/q} (f_2(s))^p}{r+s} \diamond s \right)^{1/p} \right)^p \diamond r \\
 & \geq \int_0^\infty \left(\int_0^\infty \frac{(s/r)^{-1/q} (f_1(s) + f_2(s))^p}{r+s} \diamond s \right) \frac{\diamond r}{r} \\
 & = \int_0^\infty (f_1(s) + f_2(s))^p \left(\int_0^\infty \frac{(s/r)^{-1/q} \diamond r}{r+s} \right) \diamond s \\
 & = \int_0^\infty \frac{(f_1(s) + f_2(s))^p}{s} \left(\int_0^\infty \frac{(s/r)^{1-1/q}}{r+s} \diamond r \right) \diamond s \\
 & = \int_0^\infty \omega(s) (f_1(s) + f_2(s))^p \diamond s.
 \end{aligned}$$

□

Remark 3. For $p < 1$, we have reverse inequalities to (8) and (10).

Now we prove generalized Pólya-Knopp type inequalities.

Corollary 8. Consider (6), $b_1 = \infty$ and $U \subset \mathbb{R}^m$ is a closed convex set such that $\psi : U \rightarrow \mathbb{R}$ is convex, then:

$$\begin{aligned}
 & \int_a^\infty \eta(r) \psi \left(\frac{1}{\int_a^{\sigma(r)} \diamond s} \int_a^{\sigma(r)} \mathbf{f}(s) \diamond s \right) \diamond r \\
 & \leq \int_a^\infty \omega(s) \psi(\mathbf{f}(s)) \diamond s
 \end{aligned}$$

$\forall \quad \Theta_{\diamond}$ -integrable functions $\mathbf{f} : \Theta \rightarrow \mathbb{R}^m$ and $\mathbf{f}(\Theta) \subset U$.

Proof. Using $n = 1$ in Theorem 3, the statement is proved. □

Corollary 9. Assume (6), then

$$\begin{aligned}
 & \int_a^\infty \eta(r) \left(\frac{1}{\int_a^{\sigma(r)} \diamond s} \int_a^{\sigma(r)} (f_1(s))^p \diamond s \right)^{1/p} \\
 & \times \left(\frac{1}{\int_a^{\sigma(r)} \diamond s} \int_a^{\sigma(r)} (f_2(s))^{p'} \diamond s \right)^{1/p'} \diamond r \\
 & \geq \int_a^\infty \omega(s) f_1(s) f_2(s) \diamond s
 \end{aligned}$$

$\forall \quad \Theta_{\diamond}$ -integrable functions $f_i : \Theta \rightarrow \mathbb{R}_+$, $i \in \{1, 2\}$.

Proof. Using $m = 2$ in Corollary 3, statement is proved. □

Corollary 10. Consider (6), then:

$$\int_a^\infty \eta(r) \left(\left(\frac{1}{\int_a^{\sigma(r)} \diamond s} (f_1(s))^p \diamond s \right)^{1/p} \right.$$

$$\begin{aligned}
& + \left(\frac{1}{\int_a^{\sigma(r)} \diamond s} \int_a^{\sigma(r)} (f_2(s))^p \diamond s \right)^{1/p} \diamond r \\
& \geq \int_a^\infty \omega(s) (f_1(s) + f_2(s))^p \diamond s
\end{aligned}$$

$\forall$ Θ -integrable functions $f_i : \Theta \rightarrow \mathbb{R}_+$, where $i \in \{1, 2\}$.

Proof. Using $m = 2$ in Corollary 4, the statement is proved. $\square$

5. Applications

Some applications of dynamic Hardy type inequalities are given in this section.

Theorem 6. Assume $c, d \in \mathbb{T}$, $(1 - \gamma)f : \mathbb{T} \rightarrow \mathbb{R}$ is ld -continuous and $\gamma f : \mathbb{T} \rightarrow \mathbb{R}$ is rd -continuous. If $\mathbb{T}$ contains only isolated points, then:

$$\int_c^d f(v) \diamond v = \sum_{v \in [c, d]} \frac{(\sigma(v) - v)^2 + (v - \rho(v))^2}{\sigma(v) - \rho(v)} f(v)$$

$\forall v \in \mathbb{T}$.

Proof. The statement follows from Theorems 1.79 and 8.48 of [1]:

$$\begin{aligned}
\int_c^d f(v) \diamond v &= \sum_{v \in [c, d)} (\gamma(v)f(v)\mu(v)) + \sum_{v \in (c, d]} ((1 - \gamma(v))f(v)\nu(v)) \\
&= \sum_{v \in [c, d]} \frac{(\sigma(v) - v)^2 + (v - \rho(v))^2}{\sigma(v) - \rho(v)} f(v).
\end{aligned}$$

$\square$

Theorem 7. Assume $c, d \in \mathbb{T}$, rd -continuous function $\gamma f : \mathbb{T} \rightarrow \mathbb{R}$ and ld -continuous function $(1 - \gamma)f : \mathbb{T} \rightarrow \mathbb{R}$. If $\mathbb{T}$ contains only isolated points, then:

$$\int_c^d f(s) \diamond s = \sum_{s \in [c, \sigma(d))} \left(\mu(s)(\gamma(s)f(s) + (1 - \gamma^\sigma(s))f^\sigma(s)) \right)$$

$\forall s \in \mathbb{T}$, where $\gamma^\sigma(s) = \gamma(\sigma(s))$ and $f^\sigma(s) = f(\sigma(s))$.

Proof.

$$\begin{aligned}
\int_c^d f(s) \diamond s &= \int_c^d (\gamma(s)f(s)) \Delta s + \int_c^d ((1 - \gamma(s))f(s)) \nabla s \\
&= \sum_{s \in [c, \sigma(d))} \left(\mu(s)\gamma(s)f(s) \right) + \sum_{s \in [\rho(c), d)} \left(\nu(s)(1 - \gamma(s))f(s) \right) \\
&= \sum_{s \in [c, \sigma(d))} \left(\mu(s)\gamma(s)f(s) \right) + \sum_{s \in [c, \sigma(d))} \left(\mu(s)(1 - \gamma^\sigma(s))f^\sigma(s) \right) \\
&= \sum_{s \in [c, \sigma(d))} \left(\mu(s)(\gamma(s)f(s) + (1 - \gamma^\sigma(s))f^\sigma(s)) \right).
\end{aligned}$$

$\square$

Example 1. If $\mathbb{T}$ contains only isolated points, take $n = 1$, $a_1 = a$, $b_1 = b$ and consider that $\eta : [a, b] \rightarrow \mathbb{R}_+$ is such that:

$$\omega(s) := \sum_{r \in [a, \rho(b)]} \frac{(\gamma(r)\eta(r)\mu(r)k(r,s))}{H(r)} + \sum_{r \in [\sigma(a), b]} \frac{((1-\gamma(r))\eta(r)v(r))k(r,s)}{H(r)}$$

and

$$H(r) := \sum_{s \in [a, \rho(b)]} (\gamma(s)\mu(s)k(r,s)) + \sum_{s \in [\sigma(a), b]} ((1-\gamma(s))v(s)k(r,s)),$$

then, from Theorem 2,

$$\begin{aligned} & \left(\sum_{r \in [a, \rho(b)]} (\gamma(r)\eta(r)\mu(r)) + \sum_{r \in [\sigma(a), b]} ((1-\gamma(r))\eta(r)v(r)) \right) \\ & \times \psi \left(\frac{1}{H(r)} \left\{ \sum_{s \in [a, \rho(b)]} (\gamma(s)\mu(s)(k(r,s)f(s)) + \sum_{s \in [\sigma(a), b]} ((1-\gamma(s))v(s)k(r,s)f(s)) \right\} \right) \\ & \leq \left(\sum_{s \in [a, \rho(b)]} (\gamma(s)\mu(s)\omega(s)\psi(f(s))) + \sum_{s \in [\sigma(a), b]} ((1-\gamma(s))v(s)\omega(s)\psi(f(s))) \right) \end{aligned}$$

for all $f : [a, b] \rightarrow \mathbb{R}$ integrable functions.

Example 2. For $\mathbb{T} = \mathbb{Z}$. If $n = 1$, $a_1 = a$, $b_1 = b$ and consider that $\eta : [a, b] \rightarrow \mathbb{R}_+$ is such that:

$$\omega(s) := \frac{1}{2} \left(\sum_{r=a}^{b-1} \frac{\eta(r)k(r,s)}{H(r)} + \sum_{r=a+1}^b \frac{\eta(r)k(r,s)}{H(r)} \right)$$

and

$$H(r) := \frac{1}{2} \left(\sum_{r=a}^{b-1} k(r,s) + \sum_{r=a+1}^b k(r,s) \right);$$

then, from Theorem 2

$$\begin{aligned} & \frac{1}{2} \left(\sum_{r=a}^{b-1} (\eta(r)) + \sum_{r=a+1}^b \eta(r) \right) \times \psi \left(\frac{1}{2H(r)} \left[\sum_{r=a}^{b-1} (k(r,s)f(s) + \sum_{r=a+1}^b k(r,s)f(s)) \right] \right) \\ & \leq \frac{1}{2} \left(\sum_{r=a}^{b-1} (\omega(s)\psi(f(s))) + \sum_{r=a+1}^b \omega(s)\psi(f(s)) \right) \end{aligned}$$

for all $f : [a, b] \rightarrow \mathbb{R}$ integrable functions.

Example 3. If $\mathbb{T}$ contains only isolated points, from Corollary 9, we get

$$\begin{aligned} & \sum_{r \in [a, \infty)} \frac{\eta(r)(\mu(r))(\gamma(r) - \gamma^\sigma(r) + 1)}{\sum_{t \in [a, \sigma(r))} (\mu(t))(\gamma(t) - \gamma^\sigma(t) + 1)} \\ & \times \left(\sum_{s \in [a, \infty)} (f_1(s))^p (\mu(s))(\gamma(s) - \gamma^\sigma(s) + 1) \right)^{1/p} \\ & \times \left(\sum_{s \in [a, \infty)} (f_2(s))^{p'} (\mu(s))(\gamma(s) - \gamma^\sigma(s) + 1) \right)^{1/p'} \\ & \geq \sum_{s \in [a, \infty)} \omega(s)f_1(s)f_2(s)(\mu(s))(\gamma(s) - \gamma^\sigma(s) + 1), \end{aligned} \quad (12)$$

where

$$\omega(s) = \sum_{r \in [a, \infty)} \frac{\eta(r)(\mu(r))(\gamma(r) - \gamma^\sigma(r) + 1)}{\sum_{t \in [a, \sigma(r))} (\mu(t))(\gamma(t) - \gamma^\sigma(t) + 1)}.$$

Remark 4. In case $\gamma(t) = 1$ for all $t \in \mathbb{T}$, inequality (12) is the same as (51) in [19].

Example 4. If $\mathbb{T}$ contains only isolated points, from Corollary 10, we get

$$\begin{aligned} & \sum_{r \in [a, \infty)} \frac{\eta(r)(\mu(r))(\gamma(r) - \gamma^\sigma(r) + 1)}{\sum_{t \in [a, \sigma(r))} (\mu(t))(\gamma(t) - \gamma^\sigma(t) + 1)} \\ & \times \left(\left(\sum_{s \in [a, \infty)} (f_1(s))^p (\mu(s)) (\gamma(s) - \gamma^\sigma(s) + 1) \right)^{1/p} \right. \\ & + \left. \left(\sum_{s \in [a, \infty)} (f_2(s))^p (\mu(s)) (\gamma(s) - \gamma^\sigma(s) + 1) \right)^{1/p} \right)^p \\ & \geq \sum_{s \in [a, \infty)} \omega(s) (f_1(s) + f_2(s))^p (\mu(s)) (\gamma(s) - \gamma^\sigma(s) + 1), \end{aligned} \quad (13)$$

where

$$\omega(s) = \sum_{r \in [a, \infty)} \frac{\eta(r)(\mu(r))(\gamma(r) - \gamma^\sigma(r) + 1)}{\sum_{t \in [a, \sigma(r))} (\mu(t))(\gamma(t) - \gamma^\sigma(t) + 1)}.$$

Remark 5. In case $\gamma(t) = 1$ for all $t \in \mathbb{T}$, inequality (13) is the same as (52) in [19].

Example 5. If $\mathbb{T} = h\mathbb{N} = \{hm : m \in \mathbb{N} \text{ where } h > 0, a = 1, \text{ and } \eta(r) = 1/\sigma(r)\}$, then

$$\begin{aligned} \gamma(t) &= \frac{h(m+1) - hm}{h(m+1) - h(m-1)} = \frac{hm + h - hm}{hm + h - hm + h} = \frac{h}{2h} = \frac{1}{2}, \\ \gamma^\sigma(t) &= \frac{h(m+2) - h(m+1)}{h(m+2) - hm} = \frac{hm + 2h - hm - h}{hm + 2h - hm} = \frac{h}{2h} = \frac{1}{2} \end{aligned}$$

$\forall t \in \mathbb{T}$. Inequality (12) becomes

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{m(m+1)} & \times \left(\sum_{k=1}^m (f_1(kh))^p \right)^{1/p} \left(\sum_{k=1}^m (f_2(kh))^{p'} \right)^{1/p'} \\ & \geq \sum_{m=1}^{\infty} \frac{1}{m} f_1(mh) f_2(mh). \end{aligned} \quad (14)$$

Inequality (13) becomes

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{m(m+1)} & \times \left(\left(\sum_{k=1}^m (f_1(kh))^p \right)^{1/p} + \left(\sum_{k=1}^m (f_2(kh))^p \right)^{1/p} \right)^p \\ & \geq \sum_{m=1}^{\infty} \frac{1}{m} (f_1(mh) + f_2(mh))^p. \end{aligned} \quad (15)$$

Remark 6. For $h = 1$, inequalities (14) and (15) are the same as (1.7) and (1.9) in ([24], Corollary 1.3).

Example 6. If $\mathbb{T} = q^{\mathbb{N}} = \{q^m : m \in \mathbb{N}, q > 1\}$, $a = q$, and $\eta(r) = \frac{\sigma(r)-a}{\sigma(r)(\mu(r))}$, then

$$\gamma(t) = \frac{q^{k+1} - q^k}{q^{k+1} - q^{k-1}} = \frac{q^k(q-1)}{q^{k-1}(q^2-1)} = \frac{q^k(q-1)}{q^{k-1}(q-1)(q+1)} = \frac{q}{q+1},$$

$$\gamma(\sigma(t)) = \frac{q^{k+2} - q^{k+1}}{q^{k+2} - q^k} = \frac{q^{k+1}(q-1)}{q^k(q^2-1)} = \frac{q^{k+1}(q-1)}{q^k(q-1)(q+1)} = \frac{q}{q+1},$$

$\forall t \in \mathbb{T}$. Inequality (12) takes the form

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{(q-1)}{q^m} &\times \left(\sum_{k=1}^m q^{k-1} (f_1(q^k))^p \right)^{1/p} \left(\sum_{k=1}^m q^{k-1} (f_2(q^k))^{p'} \right)^{1/p'} \\ &\geq \sum_{m=1}^{\infty} \frac{1}{m} f_1(q^m) f_2(q^m). \end{aligned} \quad (16)$$

Inequality (13) takes the form:

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{(q-1)}{q^m} &\times \left(\left(\sum_{k=1}^m q^{k-1} (f_1(q^k))^p \right)^{1/p} + \left(\sum_{k=1}^m q^{k-1} (f_2(q^k))^p \right)^{1/p} \right)^p \\ &\geq \sum_{n=1}^{\infty} (f_1(q^n) f_2(q^n))^p. \end{aligned} \quad (17)$$

Remark 7. Inequalities (16) and (17) are the same as (1.6) and (1.8) in ([24], Corollary 1.3).

If $\mathbb{T} = \mathbb{R}$, then from the definition of the gamma function for dense points $\gamma(s) = \frac{1}{2}$ for all $s \in \mathbb{R}$, we have: $\int_c^d f(s) \diamond s = \int_c^d f(s) ds$.

Corollary 11. Assume that if $\psi \in C(K, \mathbb{R})$, where $K \subseteq \mathbb{R}^m$ is closed and convex then:

$$\int_{\Omega} \eta(r) \psi \left(\frac{\int_{\Theta} k(r, s) \mathbf{f}(s)}{H(r)} ds \right) dr \leq \int_{\Theta} \omega(s) \psi(\mathbf{f}(s)) ds, \quad (18)$$

for all integrable functions $\mathbf{f} : \Theta \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Theta) \subset K$.

Proof. The statement follows from Theorem 2 by taking $\mathbb{T} = \mathbb{R}$, since in this case

$$H(r) = \int_{\Theta} k(r, s) ds < \infty, \quad r \in \Omega$$

and

$$\omega(s) = \int_{\Omega} \frac{k(r, s) \eta(r)}{H(r)} dr < \infty, \quad s \in \Theta.$$

□

Remark 8. Inequality (18) is the same as (8.13) in [23].

Example 7. For $\mathbb{T} = \mathbb{R}$, $b_i = \infty \quad \forall i \in \{1, \dots, n\}$ and with assumptions of Corollary 7, (7) takes the form:

$$\begin{aligned} &\int_0^{\infty} \cdots \int_0^{\infty} \psi(A_k \mathbf{f}(r_1, \dots, r_n)) \frac{dr_1 \cdots dr_n}{r_1 \cdots r_n} \\ &\leq \int_0^{\infty} \cdots \int_0^{\infty} \psi(\mathbf{f}(s_1, \dots, s_n)) \frac{ds_1 \cdots ds_n}{s_1 \cdots s_n}. \end{aligned} \quad (19)$$

Remark 9. Inequality (19) is same as (37) in [19].

Example 8. For $m = 1$ and $\mathbb{T} = \mathbb{R}$, (19) takes the form:

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \psi(A_k f_1(r_1, \dots, r_n)) \frac{dr_1 \cdots dr_n}{r_1 \cdots r_n} \\ & \leq \int_0^\infty \cdots \int_0^\infty \psi(f_1(s_1, \dots, s_n)) \frac{ds_1 \cdots ds_n}{s_1 \cdots s_n}. \end{aligned} \quad (20)$$

Remark 10. Inequality (20) is proved in [25].

Example 9. It is known that:

$$\int_0^\infty \frac{(s/r)^{-1/q}}{r+s} ds = \int_0^\infty \frac{(s/r)^{1-1/q}}{r+s} dr = \frac{\pi}{\sin(\pi/q)},$$

$\forall r, s \in \mathbb{R}_+ = (0, \infty)$ and $q > 1$. If $\mathbb{T} = \mathbb{R}$; from (8), we have:

$$\begin{aligned} & \int_0^\infty \left(\int_0^\infty \frac{(s/r)^{1-1/q} (f_1(s))^p}{r+s} ds \right)^{1/p} \times \left(\int_0^\infty \frac{(s/r)^{1-1/q} (f_2(s))^{p'}}{r+s} ds \right)^{1/p'} dr \\ & \geq \frac{\pi}{\sin(\pi/q)} \int_0^\infty f_1(s) f_2(s) ds. \end{aligned} \quad (21)$$

Remark 11. Inequality (21) is same as (45) in [19].

6. Conclusions

In this research, the study of Jensen's inequality and Hardy's inequality is conducted for functions of several variables involving a generalized class of diamond integrals of time scales calculus. The method is very adequate and can be applied to other inequalities. The outcomes will be the inventions of new inequalities, both integral and discrete types.

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