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Monotone Iterative and Upper–Lower Solution Techniques for Solving the Nonlinear ψ –Caputo Fractional Boundary Value Problem

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Abstract: The objective of this paper is to study the existence of extremal solutions for nonlinear boundary value problems of fractional differential equations involving the ψ –Caputo derivative ${}^C\mathbb{D}_{a+}^{\sigma,\psi} \varrho(t) = \mathcal{V}(t, \varrho(t))$ under integral boundary conditions $\varrho(a) = \lambda \mathbb{I}^{\nu;\psi} \varrho(\eta) + \delta$. Our main results are obtained by applying the monotone iterative technique combined with the method of upper and lower solutions. Further, we consider three cases for $\psi^*(t)$ as t , Caputo, 2^t , $\sqrt{t}$, and Katugampola (for $\rho = 0.5$) derivatives and examine the validity of the acquired outcomes with the help of two different particular examples.

Keywords: extremal solutions; monotone iterative technique; ψ –Caputo fractional derivative; upper and lower solutions

MSC: 26A33; 34A08; 34B18

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1. Introduction

The notion of fractional calculus refers to the last three centuries and it can be described as the generalization of classical calculus to orders of integration and differentiation that are not necessarily integers. Many researchers have used fractional calculus in different scientific areas [1–4].

In the literature, various definitions of the fractional-order derivative have been suggested. The oldest and the most famous ones advocate for the use of the Riemann–Liouville and Caputo settings. One of the most recent definitions of a fractional derivative was delivered by Kilbas et al., where the fractional differentiation of a function with respect to another function in the sense of Riemann–Liouville was introduced [5]. They further defined appropriate weighted spaces and studied some of their properties by using the corresponding fractional integral. In [6], Almeida defined the following new fractional derivative and integrals of a function with respect to some other function:

$$\begin{aligned} \mathbb{D}_{a+}^{\sigma,\psi^*} \varrho(x) &:= \left(\frac{1}{\psi^{*'}(\xi)} \frac{d}{d\xi} \right)^n \mathbb{I}_{a+}^{n-\sigma,\psi^*} \varrho(x) \\ &= \left(\frac{1}{\psi^{*'}(\xi)} \frac{d}{d\xi} \right)^n \int_a^x \frac{\psi^{*'}(\xi)}{\Gamma(n-\sigma)} (\psi^*(x) - \psi^*(\xi))^{n-\sigma-1} \varrho(\xi) d\xi, \end{aligned} \quad (1)$$

where $n = [\sigma] + 1$ and

$$\mathbb{I}_{a+}^{\sigma, \psi^*} \varrho(\mathfrak{x}) := \int_a^{\mathfrak{x}} \frac{\psi'(\xi)}{\Gamma(\sigma)} (\psi^*(\mathfrak{x}) - \psi^*(\xi))^{\sigma-1} \varrho(\xi) d\xi, \quad (2)$$

respectively. He called the fractional derivative the ψ -Caputo fractional operator. In the above definitions, we get the Riemann–Liouville and Hadamard fractional operators whenever we consider $\psi^*(\mathfrak{x}) = \mathfrak{x}$ or $\psi^*(\mathfrak{x}) = \ln \mathfrak{x}$, respectively. Many researchers used this ψ -Caputo fractional derivative (see [7–13] and the references therein). Abdo et al., in [14], investigated the BVP for a fractional differential equation (FDE) involving ψ^* operator and was given as

$$\mathbb{D}_{t_1^+}^{n-\sigma, \psi^*} \varrho(t) = \mathcal{V}(t, \varrho(t)), \quad t \in \iota =: [t_1, t_2],$$

$\varrho_{\psi^*}^{[k]}(t_1) = \varrho_{t_1}^k$, ($k = 0, 1, 2, \dots, n-2$), $\varrho_{\psi^*}^{[n-1]}(t_2) = \varrho_{t_2}$. For more details on the development of the theory of fractional differential equations, one can refer to [15–20]. In order to establish existence theory, researchers have used diverse techniques of nonlinear analysis consisting of fixed-point theory, hybrid fixed-point theory, topological degree theory, and measure of noncompactness [21–24]. However, the use of the monotone iterative technique (MIT) along with the method of upper and lower solutions (u-l solutions) for solving a BVP involving the operator remains rare.

In the present paper, we are interested in the MIT blended with the method of upper and lower solutions to prove the existence of extremal solutions for the following BVP of an FDE involving the operator

$$\begin{cases} {}^C\mathbb{D}_{t_1^+}^{\sigma, \psi^*} \varrho(t) = \mathcal{V}(t, \varrho(t)), & t \in \iota, \\ \varrho(t_1) = \lambda \mathbb{I}^{\nu, \psi^*} \varrho(\eta) + \delta, \end{cases} \quad (3)$$

where ${}^C\mathbb{D}^{\sigma, \psi^*}$ is the operator (1) of order $0 < \sigma, \nu \leq 1$, $\mathbb{I}^{\sigma, \psi^*}$ is the operator (2), the function $\mathcal{V} : [t_1, t_2] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, λ and δ are real constants, and $\eta \in (t_1, t_2)$. It is worth mentioning that the MIT is efficiently used in the literature to investigate the existence of extremal solutions to many applied problems of nonlinear equations [25–38].

The rest of this paper is organized as follows. In Section 2, we recall some preliminary concepts, definitions, and lemmas that will act as prerequisites to proving the main results. The main results are stated and proved in Section 3. Finally, we give numerical examples to illustrate the correctness of the outcome.

2. Preliminaries

Let $\sigma > 0$. The left-sided ψ -Riemann–Liouville fractional integral (l-s- ψ -RLfi) of order σ for an integrable function $\varrho : \iota \rightarrow \mathbb{R}$ with respect to another function $\psi^* : \iota \rightarrow \mathbb{R}$, which is an increasing differentiable function such that $\psi^{*\prime}(t) \neq 0$, ($\forall t \in \iota$), is defined as follows:

$$\mathbb{I}_{t_1^+}^{\sigma, \psi^*} \varrho(t) = \frac{1}{\Gamma(\sigma)} \int_{t_1}^t \psi^{*\prime}(\xi) (\psi^*(t) - \psi^*(\xi))^{\sigma-1} \varrho(\xi) d\xi, \quad (4)$$

where Γ is the classical Euler Gamma function [5,6]. Appendix A Algorithm A1 shows the MATLAB lines for the calculation of the l-s- ψ -RLfi. Let $n \in \mathbb{N}$ and $\psi^*, \varrho \in C^n(\iota, \mathbb{R})$ be two functions such that ψ^* is increasing and $\psi^{*\prime}(t) \neq 0$, ($\forall t \in \iota$). The left-sided ψ -Riemann–Liouville fractional derivative (l-s- ψ -RLfd) of a function ϱ of order σ is defined by

$$\begin{aligned} \mathbb{D}_{t_1^+}^{\sigma, \psi^*} \varrho(t) &= \left(\frac{1}{\psi^{*\prime}(t)} \frac{d}{dt} \right)^n \mathbb{I}_{t_1^+}^{n-\sigma, \psi^*} \varrho(t) \\ &= \frac{1}{\Gamma(n-\sigma)} \left(\frac{1}{\psi^{*\prime}(t)} \frac{d}{dt} \right)^n \int_{t_1}^t \psi^{*\prime}(\xi) (\psi^*(t) - \psi^*(\xi))^{n-\sigma-1} \varrho(\xi) d\xi, \end{aligned} \quad (5)$$

where $n = [\sigma] + 1$ [6]. Appendix A Algorithm A2 shows the MATLAB lines for the calculation of the I-s- ψ -RLfd. In addition, the left-sided ψ -Caputo fractional derivative (I-s- ψ -Cfd) of a function q of order σ is defined by

$${}^C\mathbb{D}_{t_1^+}^{\sigma;\psi^*} q(t) = \mathbb{I}_{t_1^+}^{n-\sigma;\psi^*} \left(\frac{1}{\psi^{*'}(t)} \frac{d}{dt} \right)^n q(t),$$

where $\psi^*, q \in C^n(\iota, \mathbb{R})$ are two functions such that ψ^* is increasing, $\psi^{*'}(t) \neq 0$, ($\forall t \in \iota$), and $n = [\sigma] + 1$ and $n = \sigma$ whenever $\sigma \notin \mathbb{N}$ and $\sigma \in \mathbb{N}$, respectively [6]. To simplify the notation, we use:

$$q_{\psi^*}^{[n]}(t) = \left(\frac{1}{\psi^{*'}(t)} \frac{d}{dt} \right)^n q(t).$$

So,

$${}^C\mathbb{D}_{t_1^+}^{\sigma;\psi^*} q(t) = \begin{cases} \frac{1}{\Gamma(n-\sigma)} \int_t^{\iota} \psi^{*'}(\xi) (\psi^*(t) - \psi^*(\xi))^{n-\sigma-1} q_{\psi^*}^{[n]}(\xi) d\xi, & \sigma \notin \mathbb{N}, \\ q_{\psi^*}^{[n]}(t), & \sigma \in \mathbb{N}. \end{cases} \quad (6)$$

Appendix A Algorithm A3 shows the MATLAB lines for the calculation of ${}^C\mathbb{D}_{t_1^+}^{\sigma;\psi^*} q(t)$. If $q \in C^n(\iota, \mathbb{R})$, then the ψ Cfd of order σ of q is determined as ([6], Theorem 3):

$${}^C\mathbb{D}_{t_1^+}^{\sigma;\psi^*} q(t) = \mathbb{D}_{t_1^+}^{\sigma;\psi^*} \left[q(t) - \sum_{k=0}^{n-1} \frac{q_{\psi^*}^{[k]}(t_1)}{k!} (\psi^*(t) - \psi^*(t_1))^k \right]. \quad (7)$$

Lemma 1 ([8]). Let $\sigma, \nu > 0$, and $q \in L^1(\iota, \mathbb{R})$. Then, $\mathbb{I}_{t_1^+}^{\sigma;\psi^*} \mathbb{I}_{t_1^+}^{\nu;\psi^*} q(t) = \mathbb{I}_{t_1^+}^{\sigma+\nu;\psi^*} q(t)$, ($t \in \iota$). In particular, if $q \in C(\iota, \mathbb{R})$, then $\mathbb{I}_{t_1^+}^{\sigma;\psi^*} \mathbb{I}_{t_1^+}^{\nu;\psi^*} q(t) = \mathbb{I}_{t_1^+}^{\sigma+\nu;\psi^*} q(t)$, ($\forall t \in \iota$).

Lemma 2 ([8]). Let $\sigma > 0$. If $q \in C(\iota, \mathbb{R})$, then ${}^C\mathbb{D}_{t_1^+}^{\sigma;\psi^*} \mathbb{I}_{t_1^+}^{\sigma;\psi^*} q(t) = q(t)$, ($t \in \iota$), and

$$\mathbb{I}_{t_1^+}^{\sigma;\psi^*} \left({}^C\mathbb{D}_{t_1^+}^{\sigma;\psi^*} q(t) \right) = q(t) - \sum_{k=0}^{n-1} \frac{q_{\psi^*}^{[k]}(t_1)}{k!} [\psi^*(t) - \psi^*(t_1)]^k, \quad (t \in \iota), \quad (8)$$

whenever $q \in C^n(\iota, \mathbb{R})$, $n - 1 < \sigma < n$.

Lemma 3 ([5,8]). Let $t > t_1$, $\sigma \geq 0$, and $\nu > 0$. Then,

- (1) $\mathbb{I}_{t_1^+}^{\sigma;\psi^*} (\psi^*(t) - \psi^*(t_1))^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\sigma)} (\psi^*(t) - \psi^*(t_1))^{\nu+\sigma-1}$;
- (2) ${}^C\mathbb{D}_{t_1^+}^{\sigma;\psi^*} (\psi^*(t) - \psi^*(t_1))^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\sigma)} (\psi^*(t) - \psi^*(t_1))^{\nu-\sigma-1}$;
- (3) ${}^C\mathbb{D}_{t_1^+}^{\sigma;\psi^*} (\psi^*(t) - \psi^*(t_1))^k = 0$, ($\forall k \in \{0, \dots, n-1\}$) and $n \in \mathbb{N}$.

3. Main Results

First, we start the following key fixed-point theorem.

Theorem 1 ([16,17]). Consider $\iota \subset \mathcal{O}$ of an ordered Banach space $\mathfrak{B}$ and a nondecreasing mapping $u : \iota \rightarrow \iota$. If each sequence $[u_{q_n}] \subset u(\iota)$ converges whenever $[q_n]$ is a monotone sequence in ι , then the sequence of the u -iteration of t_1 converges to the least fixed point q_* of u , and the sequence of the u -iteration of t_2 converges to the greatest fixed point q_* of u . Moreover, $q_* = \min\{t \in \iota : t \geq ut\}$, and $q_* = \max\{t \in \iota : t \leq ut\}$.

In fact, a function $\varrho \in C(\iota, \mathbb{R})$ is said to be a solution of Equation (3) if ϱ satisfies the equation ${}^C\mathbb{D}_{t_1^+}^{\sigma; \Psi^*} \varrho(t) = \mathcal{V}(t, \varrho(t))$, $(\forall t \in \iota)$ and the condition $\varrho(0) = \lambda \mathbb{I}_{t_1^+}^{\sigma; \Psi^*} \varrho(\eta) + \delta$. Now, we prove the the next key lemma of a solution for problem (3).

Lemma 4. Let $y \in C(\iota, \mathbb{R})$ and $\sigma \in (n-1, n]$, $(n \in \mathbb{N})$; the linear fractional initial value problem

$$\begin{cases} {}^C\mathbb{D}_{t_1^+}^{\sigma; \Psi^*} \varrho(t) = y(t), & t \in \iota, \\ \varrho(t_1) = \lambda \mathbb{I}_{t_1^+}^{\nu; \Psi^*} \varrho(\eta) + \delta, \end{cases} \quad (9)$$

has the following unique solution:

$$\begin{aligned} \varrho(t) &= \mathbb{I}_{t_1^+}^{\sigma; \Psi^*} y(t) + \frac{1}{\Lambda} \left(\lambda \mathbb{I}_{t_1^+}^{\sigma+\nu; \Psi^*} y(\eta) + \delta \right) \\ &= \frac{1}{\Gamma(\sigma)} \int_{t_1}^t \Psi^{*'}(\xi) (\Psi^*(t) - \Psi^*(\xi))^{\sigma-1} y(\xi) d\xi \\ &\quad + \frac{1}{\Lambda} \left[\frac{\lambda}{\Gamma(\sigma)} \int_{t_1}^{\eta} \Psi^{*'}(\xi) (\Psi^*(\eta) - \Psi^*(\xi))^{\sigma-1} y(\xi) d\xi + \delta \right], \end{aligned} \quad (10)$$

where $\Lambda = 1 - \frac{\lambda}{\Gamma(\nu+1)} (\Psi^*(\eta) - \Psi^*(t_1))^{\nu}$.

Proof. Assume that ϱ satisfies (9). Then, Lemma 2 implies that

$$\varrho(t) = \mathbb{I}^{\sigma} y(t) + c_1. \quad (11)$$

The condition of problem (9) implies that $\varrho(0) = c_1$ and

$$\mathbb{I}^{\nu} \varrho(0) = \mathbb{I}^{\sigma+\nu} \sigma(\eta) + c_1 \frac{(\Psi^*(\eta) - \Psi^*(t_1))^{\nu}}{\Gamma(\nu+1)}.$$

Thus,

$$c_1 \left(1 - \frac{\lambda}{\Gamma(\nu+1)} (\Psi^*(\eta) - \Psi^*(t_1))^{\nu} \right) = \lambda \mathbb{I}^{\sigma+\nu; \Psi^*} y(\eta) + \delta.$$

Consequently,

$$c_1 = \frac{1}{\Lambda} \left(\lambda \mathbb{I}^{\sigma+\nu; \Psi^*} y(\eta) + \delta \right).$$

Finally, we obtain the solution (10):

$$\varrho(t) = \mathbb{I}_{t_1^+}^{\sigma; \Psi^*} y(t) + \frac{1}{\Lambda} \left(\lambda \mathbb{I}_{t_1^+}^{\sigma+\nu; \Psi^*} \sigma(\eta) + \delta \right),$$

which completes the proof. $\square$

Lemma 5 (Comparison result). Let $\varrho \in C(\iota, \mathbb{R})$ satisfy the following inequalities:

$$\begin{cases} {}^C\mathbb{D}_{t_1^+}^{\sigma; \Psi^*} \varrho(t) \geq 0, & t \in \iota, \\ \varrho(0) \geq \lambda \mathbb{I}^{\nu; \Psi^*} \varrho(\eta). \end{cases} \quad (12)$$

Then, $\varrho(t) \geq 0$, $\forall t \in \iota$, where $0 < \sigma \leq 1$ is fixed.

Proof. Lemma 4 implies that the problem (9) has the following unique solution:

$$\varrho(t) = \mathbb{I}_{t_1^+}^{\sigma; \Psi^*} (y(t) + \frac{1}{\Lambda} \left(\lambda \mathbb{I}_{t_1^+}^{\sigma+\nu; \Psi^*} y(\eta) + \delta \right)). \quad (13)$$

Moreover, Lemma 5 follows from (13). $\square$

Definition 1. A function $\underline{q}_0 \in C(\iota, \mathbb{R})$ and $\bar{q}_0 \in C(J, \mathbb{R})$ is said to be a lower solution (l-solution) and upper solution (u-solution) of problem (3) if it satisfies

$$\begin{cases} {}^C\mathbb{D}_{t_1^+}^{\sigma;\Psi^*} \underline{q}_0(t) \geq \mathcal{V}(t, \underline{q}_0), \\ \underline{q}_0(0) \geq \lambda \mathbb{I}^{\nu;\Psi^*} \underline{q}_0(\eta) + \delta, \end{cases} \quad \begin{cases} {}^C\mathbb{D}_{t_1^+}^{\sigma;\Psi^*} \bar{q}_0(t) \leq \mathcal{V}(t, \bar{q}_0), \\ \bar{q}_0(0) \leq \lambda \mathbb{I}^{\nu;\Psi^*} \bar{q}_0(\eta) + \delta, \end{cases} \quad (14)$$

$\forall t \in \iota$, respectively.

Theorem 2. Consider the function $\mathcal{V} \in C(\iota \times \mathbb{R}, \mathbb{R})$ and the following assumptions:

(H1) $\exists \underline{q}_0, \bar{q}_0 \in C(\iota, \mathbb{R})$ such that $\underline{q}_0$ and $\bar{q}_0$ are the l-solution and u-solution of problem (3), respectively, with $\underline{q}_0(t) \leq \bar{q}_0(t)$, ($\forall t \in \iota$);

(H2) $\exists$ a function $\mathcal{V} \in C(\iota, \mathbb{R})$ such that $\mathcal{V}(t, \bar{q}) - \mathcal{V}(t, \underline{q}) \geq 0$, for $\underline{q}_0 \leq \underline{q} \leq \bar{q} \leq \bar{q}_0$;

(H3) $\lambda > 0$ and $\lambda(\psi^*(\eta) - \psi^*(t_1))^{\nu} < \Gamma(\nu + 1)$.

Then, there exist monotone iterative sequences (mis) $\{\underline{q}_n\}$ and $\{\bar{q}_n\}$ that converge uniformly on the interval ι to the extremal solutions $\underline{q}_*, \bar{q}_* \in [\underline{q}_0, \bar{q}_0]$, respectively, of BVP (3), where

$$\underline{q}_0 \leq \underline{q}_1 \leq \dots \leq \underline{q}_n \leq \dots \leq \underline{q}_* \leq \bar{q}_* \leq \dots \leq \bar{q}_n \leq \dots \leq \bar{q}_1 \leq \bar{q}_0.$$

Proof. First, for any $\underline{q}_0(t), \bar{q}_0(t) \in C(\iota, \mathbb{R})$, we consider the BVPs of fractional order

$$\begin{cases} {}^C\mathbb{D}_{t_1^+}^{\sigma;\Psi^*} \underline{q}_{n+1}(t) = \mathcal{V}(t, \underline{q}_n(t)), \\ \underline{q}_{n+1}(0) = \lambda \mathbb{I}^{\nu;\Psi^*} \underline{q}_n(\eta) + \delta, \end{cases} \quad (15)$$

and

$$\begin{cases} {}^C\mathbb{D}_{t_1^+}^{\sigma;\Psi^*} \bar{q}_{n+1}(t) = \mathcal{V}(t, \bar{q}_n(t)), \\ \bar{q}_{n+1}(0) = \lambda \mathbb{I}^{\nu;\Psi^*} \bar{q}_n(\eta) + \delta, \end{cases} \quad (16)$$

$\forall t \in \iota$. Now, Lemma 4 implies that (15) and (16) have the following unique solutions:

$$\underline{q}_{n+1}(t) = \mathbb{I}_{t_1^+}^{\sigma;\Psi^*} \mathcal{V}(t, \underline{q}_n(t)) + \frac{1}{\Lambda} \left(\lambda \mathbb{I}_{t_1^+}^{\sigma+\nu;\Psi^*} \mathcal{V}(\eta, \underline{q}_n(\eta)) + \delta \right), \quad (17)$$

and

$$\bar{q}_{n+1}(t) = \mathbb{I}_{t_1^+}^{\sigma;\Psi^*} \mathcal{V}(t, \bar{q}_n(t)) + \frac{1}{\Lambda} \left(\lambda \mathbb{I}_{t_1^+}^{\sigma+\nu;\Psi^*} \mathcal{V}(\eta, \bar{q}_n(\eta)) + \delta \right), \quad (18)$$

for $t \in \iota$. Then, we structure the proof as follows. For any $\bar{h} \in [\underline{q}_0, \bar{q}_0]$, define an operator $\mathcal{F}$ with $\mathcal{F}(\bar{h}) = \underline{q}(\bar{h})$. As a first step, we show that the operator $\mathcal{F} : [\underline{q}_0, \bar{q}_0] \rightarrow [\underline{q}_0, \bar{q}_0]$. Let $\underline{q}_1 = \mathcal{F}\underline{q}_0$, $v_1 = \mathcal{F}v_0$. Then, $\underline{q}_1, \bar{q}_1$ are well defined and satisfy

$$\begin{cases} {}^C\mathbb{D}_{t_1^+}^{\sigma;\Psi^*} \underline{q}_1(t) = \mathcal{V}(t, \underline{q}_0(t)), \quad t \in \iota, \\ \underline{q}_1(t_1) = \lambda \mathbb{I}^{\nu;\Psi^*} \underline{q}_0(\eta) + \delta, \end{cases} \quad (19)$$

and

$$\begin{cases} {}^C\mathbb{D}_{t_1^+}^{\sigma;\Psi^*} \bar{q}_1(t) = \mathcal{V}(t, \bar{q}_0(t)), \quad t \in \iota, \\ \bar{q}_1(t_1) = \lambda \mathbb{I}^{\nu;\Psi^*} \bar{q}_0(\eta) + \delta. \end{cases} \quad (20)$$

We set $\underline{\mu}(t) = \underline{\varrho}_1(t) - \underline{\varrho}_0(t)$. From (15) and Definition 1, we get

$${}^c\mathbb{D}_{t_1^+}^{\sigma;\Psi^*} \underline{\mu}(t) \geq 0.$$

Again, since $\underline{\mu}(t_1) = \lambda \mathbb{I}^{\nu;\Psi^*} \underline{\mu}(\eta)$, by Lemma 5, $\underline{\mu}(t) \geq 0$, $(\forall t \in \iota)$. That is,

$$\underline{\varrho}_0(t) \leq \mathcal{F} \underline{\varrho}_0(t) = \underline{\varrho}_1(t).$$

Similarly, using the definition of the upper solution, we can show that $\bar{\varrho}_1 = \mathcal{F} \bar{\varrho}_0(t) \leq \bar{\varrho}_0(t)$, $(t \in \iota)$. Now, let $\bar{\mu}(t) = \bar{\varrho}_1(t) - \underline{\varrho}_1(t)$. From (15), (16), and (H2), we have

$${}^c\mathbb{D}_{t_1^+}^{\sigma;\Psi^*} \bar{\mu}(t) = \mathcal{V}(t, \bar{\varrho}_0(t)) - \mathcal{V}(t, \underline{\varrho}_0(t)) \geq 0.$$

Therefore,

$$\bar{\mu}(t_1) = \bar{\varrho}_1(t_1) - \underline{\varrho}_1(t_1) = \lambda \mathbb{I}^{\nu;\Psi^*} \bar{\varrho}(\eta).$$

Moreover, $\bar{\mu}(t) \geq 0$ from Lemma 5. Thus, $\mathcal{F} \underline{\varrho}_0 \leq \mathcal{F} \bar{\varrho}_0$. This, together with $\underline{\varrho}_0 \leq \mathcal{F} \underline{\varrho}_0$ and $\mathcal{F} \bar{\varrho}_0 \leq \bar{\varrho}_0$, implies that $\mathcal{F}$ is nondecreasing,

$$\mathcal{F} : [\underline{\varrho}_0, \bar{\varrho}_0] \rightarrow [\underline{\varrho}_0, \bar{\varrho}_0],$$

and $\underline{\varrho}_0 \leq \mathcal{F} \underline{\varrho} \leq \bar{\varrho}_0$ for any $\underline{\varrho} \in [\underline{\varrho}_0, \bar{\varrho}_0]$. In consequence, $\mathcal{F}[\underline{\varrho}_0, \bar{\varrho}_0] \subset [\underline{\varrho}_0, \bar{\varrho}_0]$ and

$$\|\mathcal{F} \underline{\varrho}\| \leq \max \left\{ \|\underline{\varrho}_0\|, \|\bar{\varrho}_0\| \right\} := d.$$

Let $\{\underline{\varrho}_n\}$ be an *mis* in $[\underline{\varrho}_0, \bar{\varrho}_0]$. Then, $\underline{\varrho}_0 \leq \mathcal{F} \underline{\varrho}_n \leq \bar{\varrho}_0$ and $\|\mathcal{F} \underline{\mu}_n\| \leq d$. For any $(t, \varrho) \in \iota \times [-d, d]$, there exists a positive constant L_0 such that $|\mathcal{V}(t, \varrho)| \leq L_0$. Then, for any $t_1, t_2 \in \iota$ with $t_1 \leq t_2$, we obtain

$$\begin{aligned} |\mathcal{F} \varrho(t) - \mathcal{F} \varrho(\hat{t})| &= \left| \frac{1}{\Gamma(\sigma)} \int_{t_1}^t \Psi^{*\prime}(\xi) (\Psi^*(t) - \Psi^*(\xi))^{\sigma-1} \mathcal{V}(\xi, \varrho(\xi)) d\xi \right. \\ &\quad \left. - \frac{1}{\Gamma(\sigma)} \int_{t_1}^{\hat{t}} \Psi^{*\prime}(\xi) (\Psi^*(\hat{t}) - \Psi^*(\xi))^{\sigma-1} \mathcal{V}(\xi, \varrho(\xi)) d\xi \right| \\ &\leq \frac{1}{\Gamma(\sigma)} \left| \int_{t_1}^{\hat{t}} \Psi^{*\prime}(\xi) \left[(\Psi^*(t) - \Psi^*(\xi))^{\sigma-1} \right. \right. \\ &\quad \left. \left. - (\Psi^*(\hat{t}) - \Psi^*(\xi))^{\sigma-1} \right] \mathcal{V}(\xi, \varrho(\xi)) d\xi \right| \\ &\quad + \frac{1}{\Gamma(\sigma)} \left| \int_{\hat{t}}^{t_1} \Psi^{*\prime}(\xi) (\Psi^*(t) - \Psi^*(\xi))^{\sigma-1} \mathcal{V}(\xi, \varrho(\xi)) d\xi \right| \\ &\leq \frac{L_0}{\Gamma(\sigma+1)} \left| (\Psi^*(t) - \Psi^*(t_1))^\sigma - (\Psi^*(\hat{t}) - \Psi^*(t_1))^\sigma \right. \\ &\quad \left. - (\Psi^*(t) - \Psi^*(\hat{t}))^\sigma + (\Psi^*(t) - \Psi^*(\hat{t}))^\sigma \right|, \end{aligned}$$

which converges to zero as $\hat{t} \rightarrow t$. Let us observe that for $t, \hat{t} \in J$,

$$|\mathcal{F}q(t) - \mathcal{F}q(\hat{t})| \rightarrow 0,$$

when $\hat{t} \rightarrow t$. Thus, $\{\mathcal{F}q_n\}$ is equicontinuous on all J . So, $\mathcal{F}$ is relatively compact on $[\underline{q}_0, \bar{q}_0]$. Hence, the Arzelà–Ascoli theorem implies that $\mathcal{F}$ is compact on $[\underline{q}_0, \bar{q}_0]$, and so,

$$\{\mathcal{F}q_n\} \subset \mathcal{F}([\underline{q}_0, \bar{q}_0]),$$

converges. On the other hand, Theorem 1 implies that the sequence of the $\mathcal{F}$ -iteration of $\underline{q}_0$ and $\bar{q}_0$ converges to the least and the greatest fixed points $\underline{q}_*$ and $\bar{q}^*$ of $\mathcal{F}$, respectively. This, in turn, implies that problem (3) has extremal solutions $\underline{q}_*, \bar{q}^* \in [\underline{q}_0, \bar{q}_0]$, which can be obtained with the corresponding iterative sequences $(\underline{q}_n)$ and $(\bar{q}_n)$ defined in (17) and (18), respectively. Furthermore, we have

$$\underline{q}_0 \leq \underline{q}_1 \leq \dots \leq \underline{q}_n \leq \dots \leq \underline{q}_* \leq \bar{q}^* \leq \dots \leq \bar{q}_n \leq \dots \leq \bar{q}_1 \leq \bar{q}_0.$$

This completes the proof. $\square$

4. Some Relevant Examples

Example 1. Consider the following problem:

$$\begin{cases} {}^C\mathbb{D}_{0+}^{\frac{1}{4}, \Psi^*} q(t) = 2t - q^2(t) + 1, & t \in \iota := [0, 1], \\ q(0) = \frac{3}{4}\mathbb{I}_{0+}^{\frac{1}{2}} q\left(\frac{2}{3}\right) + \frac{1}{2}, \end{cases} \quad (21)$$

where

$$\sigma = \frac{1}{4} \in \iota, \nu = \frac{1}{2} \in (0, 1], \lambda = \frac{3}{4} > 0, \eta = \frac{2}{3} \in (0, 1), \delta = \frac{1}{2}, t_1 = 0, t_2 = 1,$$

and $\mathcal{V} : \iota \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\mathcal{V}(t, q) = 2t - q^2 + 1,$$

for $t \in \iota, q \in \mathbb{R}$. We take $\underline{q}_0(t) = 0$ as the lower solution and $\bar{q}_0(t) = 1 + 2t$ as the upper solution of problem (21), and we take $\underline{q}_0 \leq \bar{q}_0$ for $t \in \iota$. So, (H1) of Theorem 2 holds. Now, we consider three cases for Ψ^* :

$$\psi_1^*(t) = t, \quad \psi_2^*(t) = 2^t, \quad \psi_3^*(t) = \sqrt{t}.$$

Note that $\psi_1^*(t) = t$ and $\psi_3^*(t) = \sqrt{t}$ give the Caputo and Katugampola (for $\rho = 0.5$) derivatives in this example.

With the data provided in Tables 1 and 2, we can see from assumption (H3) that

$$\frac{\lambda}{\Gamma(\nu+1)} (\psi^*(\eta) - \psi^*(a))^\nu = \frac{\frac{3}{4}}{\Gamma\left(\frac{1}{2}+1\right)} \left(\psi^*\left(\frac{2}{3}\right) - \psi^*(0) \right)^{\frac{1}{2}}$$

$$= \begin{cases} \frac{\frac{3}{4}}{\Gamma\left(\frac{3}{2}\right)} \left(\frac{2}{3}\right)^{\frac{1}{2}} \approx 0.690988 < 1, & \psi^*(t) = t. \\ \frac{\frac{3}{4}}{\Gamma\left(\frac{3}{2}\right)} \left(2^{\frac{2}{3}} - 1\right)^{\frac{1}{2}} \approx 0.648610 < 1, & \psi^*(t) = 2^t. \\ \frac{\frac{3}{4}}{\Gamma\left(\frac{3}{2}\right)} \left(\sqrt{\frac{2}{3}}\right)^{\frac{1}{2}} \approx 0.764704 < 1, & \psi^*(t) = \sqrt{t}. \end{cases}$$

Tables 1 and 2 show these results. One can see the 2D line plots of $\underline{q}_n(t)$ and $\bar{q}_n(t)$ for the $\psi_1^*(t) = t$ Caputo derivative, $\psi_2^*(t) = 2^t$, and the $\psi_3^*(t) = \sqrt{t}$ Katugampola derivative (for $\rho = 0.5$) in Figure 1a,b. In addition, assumption (H2) is clearly satisfied.

Table 1. Numerical results of $\underline{q}_n$ for $\psi^*(t) = t, 2^t, \sqrt{t}$ in Example 1.

n	t	$\underline{q}_n$		
		$\psi^*(t) = t$	$\psi^*(t) = 2^t$	$\psi^*(t) = \sqrt{t}$
1	0.00000	6.16427	1.42292	9.07593
2	0.06250	4.28485	1.48618	6.19447
3	0.12500	6.94358	2.16988	10.07951
4	0.18750	3.08762	1.32280	4.40385
5	0.25000	7.27414	2.49829	10.38117
6	0.31250	1.44744	0.99029	2.03591
7	0.37500	6.79598	2.59341	9.54727
8	0.43750	1.96779	1.26042	2.73167
9	0.50000	7.91490	3.16083	10.95387
10	0.56250	3.00121	0.16361	4.18637
11	0.62500	0.40120	0.71785	0.62015
12	0.68750	6.91232	3.19814	9.34005
13	0.75000	1.12735	0.60722	1.66134
14	0.81250	6.93050	3.44249	9.21394
15	0.87500	2.52051	0.25975	3.59983
16	0.93750	5.59257	3.21189	7.26441
17	1.00000	1.08675	1.70043	1.16584

Table 2. Numerical results of $\bar{q}_n$ for $\psi^*(t) = t, 2^t, \sqrt{t}$ in Example 1.

n	t	$\bar{q}_n$		
		$\psi^*(t) = t$	$\psi^*(t) = 2^t$	$\psi^*(t) = \sqrt{t}$
1	0.00000	06.1643	01.4229	09.0759
2	0.06250	06.7849	01.9923	09.9536
3	0.12500	04.4538	01.4135	07.1562
4	0.18750	11.6960	02.3487	18.5273
5	0.25000	99.2723	01.1392	01.5203×10^2
6	0.31250	51.8545×10^2	02.5625	79.2912×10^2
7	0.37500	13.8157×10^6	01.0813	21.1237×10^6
8	0.43750	97.9661×10^{12}	02.7994	01.4979×10^{14}
9	0.50000	49.2583×10^{26}	03.4989×10^{12}	75.3138×10^{26}
10	0.56250	12.4534×10^{54}	88.4591×10^{38}	19.0406×10^{54}
11	0.62500	79.5978×10^{108}	05.6540×10^{94}	01.2170×10^{110}
12	0.68750	32.5184×10^{218}	02.3099×10^{204}	49.7193×10^{218}
13	0.75000	NaN	NaN	NaN
14	0.81250	NaN	NaN	NaN
15	0.87500	NaN	NaN	NaN
16	0.93750	NaN	NaN	NaN
17	1.00000	NaN	NaN	NaN

Thus, by Theorem 2, it follows that problem (21) has extremal solutions $\underline{q}_*, \bar{q}^* \in [0, 1 + 2t]$, which can be found by means of the iterative sequences $\{\underline{q}_n\}$ and $\{\bar{q}_n\}$ defined by (17) and (18), respectively, as follows:

$$\begin{aligned}\underline{q}_{n+1}(t) &= \mathbb{I}_{t_1^+}^{\sigma; \psi^*} \mathcal{V}(t, \underline{q}_n(t)) + \frac{1}{\Lambda} \left(\lambda \mathbb{I}_{t_1^+}^{\sigma+\nu; \psi^*} \mathcal{V}(\eta, \underline{q}_n(\eta)) + \delta \right) \\ &= \mathbb{I}_{0^+}^{\frac{1}{4}; \psi^*} \left(2t - (\underline{q}_n(t))^2 + 1 \right) + \frac{1}{\Lambda} \left[\frac{3}{4} \mathbb{I}_{t_1^+}^{\frac{3}{4}; \psi^*} \left(\frac{4}{3} - (\underline{q}_n(\eta))^2 + 1 \right) + \frac{1}{2} \right]\end{aligned}$$

and

$$\begin{aligned}\bar{\varrho}_{n+1}(t) &= \mathbb{I}_{t_1^+}^{\sigma; \Psi^*} \mathcal{V}(t, \bar{\varrho}_n(t)) + \frac{1}{\Lambda} \left(\lambda \mathbb{I}_{t_1^+}^{\sigma+v; \Psi^*} \mathcal{V}(\eta, \bar{\varrho}_n(\eta)) + \delta \right) \\ &= \mathbb{I}_{0^+}^{\frac{1}{4}; \Psi^*} \left(2t - (\bar{\varrho}_n(t))^2 + 1 \right) + \frac{1}{\Lambda} \left[\frac{3}{4} \mathbb{I}_{t_1^+}^{\frac{3}{4}; \Psi^*} \left(\frac{4}{3} - (\bar{\varrho}_n(\eta))^2 + 1 \right) + \frac{1}{2} \right]\end{aligned}$$

(a) $\varrho_n(t)$ for $t \in (0, 1)$

(b) $\bar{\varrho}_n(t)$ for $t \in (0, 0.7)$

Figure 1. Graphical representation of $\varrho_n(t)$ and $\bar{\varrho}_n(t)$ in (a) and (b) respectively for the $\psi_1^*(t) = t$ Caputo derivative, $\psi_2^*(t) = 2^t$, and the $\psi_3^*(t) = \sqrt{t}$ Katugampola derivative (for $\rho = 0.5$) in Example 1.

One can see the 2D line plots of $\varrho_n(t)$ and $\bar{\varrho}_n(t)$ for the $\psi_1^*(t) = t$ Caputo derivative, $\psi_2^*(t) = 2^t$, and the $\psi_3^*(t) = \sqrt{t}$ Katugampola derivative (for $\rho = 0.5$) in Figure 2a–c. Appendix A Algorithm A4 shows how to calculate $\varrho_n(t)$ and $\bar{\varrho}_n(t)$ for $t \in \mathcal{I}$.

(a) $\psi^* = t$ (Caputo-type generalized derivative) (b) $\psi^* = 2^t$ (c) $\psi^* = \sqrt{t}$ Katugampola generalized derivative (for $\rho = 0.5$)

Figure 2. Graphical representation of $\underline{\varrho}_n(t)$ and $\overline{\varrho}_n(t)$ in (a), (b) and (c) respectively for the $\psi_1^*(t) = t$ Caputo derivative, $\psi_2^*(t) = 2^t$, and the $\psi_3^*(t) = \sqrt{t}$ Katugampola derivative (for $\rho = 0.5$) and $t \in (0, 0.07)$ in Example 1.

Example 2. Consider the following problem:

$$\begin{cases} {}^C\mathbb{D}_{0+}^{\frac{3}{5}, \psi^*} \varrho(t) = 2t - \varrho^2(t) + 1, & t \in \iota := [0.1, 1.5], \\ \varrho(0) = \frac{5}{6} \mathbb{I}_{0+}^{\frac{2}{3}} \varrho(\frac{3}{7}) + \frac{4}{5}, \end{cases} \quad (22)$$

where

$$\sigma = \frac{3}{5} \in (0, 1], \nu = \frac{2}{3}, \lambda = \frac{5}{6}, \eta = \frac{6}{7} \in (0, 1), \delta = \frac{9}{11}, t_1 = 0.1, t_2 = 1.5,$$

and $\mathcal{V} : \iota \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\mathcal{V}(t, \varrho) = 0.6t - \varrho^2 + 8.5,$$

for $t \in \iota$, $\varrho \in \mathbb{R}$. We take $\underline{\varrho}_0(t) = 0.1$ as the l -solution and $\bar{\varrho}_0(t) = 1.1 + \sqrt{t}$ as the u -solution of problem (21), and we take $\underline{\varrho}_0 \leq \bar{\varrho}_0$ for $t \in \jmath$. So, (H1) of Theorem 2 holds. Now, we consider three cases for ψ^* :

$$\psi_1^*(t) = t, \quad \psi_2^*(t) = 2^t, \quad \psi_3^*(t) = \sqrt{t}.$$

Note that $\psi_1^*(t) = t$ and $\psi_3^*(t) = \sqrt{t}$ give the Caputo and Katugampola (for $\rho = 0.5$) derivatives in this example. These results are plotted in Figure 3a,b.

(a) $\underline{\varrho}_n(t)$ for $t \in (0.1, 1.5)$

(b) $\bar{\varrho}_n(t)$ for $t \in (0.1, 1.5)$

Figure 3. Graphical representation of $\underline{\varrho}_n(t)$ and $\bar{\varrho}_n(t)$ in (a) and (b) respectively for the $\psi_1^*(t) = t$ Caputo derivative, $\psi_2^*(t) = 2^t$, and the $\psi_3^*(t) = \sqrt{t}$ Katugampola derivative (for $\rho = 0.5$) in Example 2.

With the data provided, we can see from assumption (H3) that

$$\frac{\lambda}{\Gamma(\nu+1)}(\psi^*(\eta) - \psi^*(t_1))^\nu = \frac{\frac{5}{6}}{\Gamma(\frac{2}{3}+1)}\left(\psi^*\left(\frac{6}{7}\right) - \psi^*(0.1)\right)^{\frac{2}{3}}$$

$$= \begin{cases} \frac{\frac{5}{6}}{\Gamma(\frac{2}{3})}\left(\frac{6}{7} - 0.1\right)^{\frac{1}{2}} \approx 0.766841 < 1, & \psi^*(t) = t. \\ \frac{\frac{4}{3}}{\Gamma(\frac{3}{2})}\left(2^{\frac{2}{3}} - 1\right)^{\frac{1}{2}} \approx 0.755000 < 1, & \psi^*(t) = 2^t. \\ \frac{\frac{4}{3}}{\Gamma(\frac{3}{2})}\left(\sqrt{\frac{2}{3}}\right)^{\frac{1}{2}} \approx 0.663661 < 1, & \psi^*(t) = \sqrt{t} \end{cases}$$

Tables 3 and 4 show these results. One can see the 2D line plots of $\underline{\varrho}_n(t)$ and $\bar{\varrho}_n(t)$ for the $\psi_1^*(t) = t$ Caputo derivative, $\psi_2^*(t) = 2^t$, and the $\psi_3^*(t) = \sqrt{t}$ Katugampola derivative (for $\rho = 0.5$) in Figure 3a,b. Further, assumption (H2) is clearly satisfied. Thus, by Theorem 2, it follows that problem (22) has extremal solutions ϱ_* , $\varrho^* \in [0.1, 1.1 + \sqrt{t}]$, which can be found by means of the iterative sequences $\{\underline{\varrho}_n\}$ and $\{\bar{\varrho}_n\}$ defined by (17) and (18), respectively, as follows:

$$\begin{aligned} \underline{\varrho}_{n+1}(t) &= \mathbb{I}_{t_1^+}^{\sigma; \psi^*} \mathcal{V}(t, \underline{\varrho}_n(t)) + \frac{1}{\Lambda} \left(\lambda \mathbb{I}_{t_1^+}^{\sigma+\nu; \psi^*} \mathcal{V}(\eta, \underline{\varrho}_n(\eta)) + \delta \right) \\ &= \mathbb{I}_{t_1^+}^{\sigma; \psi^*} \left(2t - (\underline{\varrho}_n(t))^2 + 1 \right) + \frac{1}{\Lambda} \left[\lambda \mathbb{I}_{t_1^+}^{\sigma+\nu; \psi^*} \left(2\eta - (\underline{\varrho}_n(\eta))^2 + 1 \right) + \delta \right] \end{aligned}$$

and

$$\begin{aligned} \underline{\varrho}_{n+1}(t) &= \mathbb{I}_{t_1^+}^{\sigma; \psi^*} \mathcal{V}(t, \bar{\varrho}_n(t)) + \frac{1}{\Lambda} \left(\lambda \mathbb{I}_{t_1^+}^{\sigma+\nu; \psi^*} \mathcal{V}(\eta, \bar{\varrho}_n(\eta)) + \delta \right) \\ &= \mathbb{I}_{t_1^+}^{\sigma; \psi^*} \left(2t - (\bar{\varrho}_n(t))^2 + 1 \right) + \frac{1}{\Lambda} \left(\lambda \mathbb{I}_{t_1^+}^{\sigma+\nu; \psi^*} \left(2\eta + (\bar{\varrho}_n(\eta))^2 + 1 \right) + \delta \right). \end{aligned}$$

One can see the 2D line plots of $\underline{\varrho}_n(t)$ and $\bar{\varrho}_n(t)$ for the $\psi_1^*(t) = t$ Caputo derivative, $\psi_2^*(t) = 2^t$, $\psi_3^*(t) = \sqrt{t}$, and the Katugampola derivative (for $\rho = 0.5$) in Figures 3a,b. Appendix A Algorithm A4 shows how to calculate $\underline{\varrho}_n(t)$ and $\bar{\varrho}_n(t)$ for $t \in \iota$.

Table 3. Numerical results of $\underline{\varrho}_n$ for $\psi^*(t) = t, 2^t, \sqrt{t}$ in Example 2.

n	$\underline{\varrho}_n$			
	t	$\psi^*(t) = t$	$\psi^*(t) = 2^t$	$\psi^*(t) = \sqrt{t}$
1	0.10000	23.0887	21.4299	12.7468
2	0.22500	01.7027×10^2	01.3640×10^2	01.0319×10^2
3	0.35000	01.3185×10^4	99.6965×10^2	82.2118×10^2
4	0.47500	68.0783×10^6	49.0307×10^6	42.9176×10^6
5	0.60000	17.1314×10^{14}	11.8461×10^{14}	10.8451×10^{14}
6	0.72500	01.0361×10^{30}	69.1495×10^{28}	65.6348×10^{28}
7	0.85000	36.4361×10^{58}	23.5622×10^{58}	23.0527×10^{58}
8	0.97500	04.3535×10^{118}	02.7357×10^{118}	02.7475×10^{118}
9	1.10000	06.0253×10^{236}	03.6879×10^{236}	03.7901×10^{236}
10	1.22500	NaN	NaN	NaN
11	1.35000	NaN	NaN	NaN
12	1.47500	NaN	NaN	NaN

Table 4. Numerical results of $\bar{q}_n$ for $\psi^*(t) = t, 2^t, \sqrt{t}$ in Example 2.

n	$\bar{q}_n$			
	t	$\psi^*(t) = t$	$\psi^*(t) = 2^t$	$\psi^*(t) = \sqrt{t}$
1	0.10000	23.0887	21.4299	12.7468
2	0.22500	01.1374×10^2	90.2639	69.2844
3	0.35000	58.6017×10^2	43.0855×10^2	37.2431×10^2
4	0.47500	13.6661×10^6	09.1617×10^6	08.9616×10^6
5	0.60000	71.6666×10^{12}	41.3606×10^{12}	49.1158×10^{12}
6	0.72500	20.3287×10^{26}	08.4297×10^{26}	15.0160×10^{26}
7	0.85000	02.0382×10^{54}	35.0156×10^{52}	01.6593×10^{54}
8	0.97500	03.2721×10^{108}	06.0417×10^{106}	02.7695×10^{108}
9	1.10000	10.7866×10^{216}	17.9871×10^{212}	08.9489×10^{216}
10	1.22500	NaN	NaN	NaN
11	1.35000	NaN	NaN	NaN
12	1.47500	NaN	NaN	NaN

5. Conclusions

In this study, we investigated the existence of solutions for a nonlinear FDE in the frame of the ψ –Caputo derivative with integral boundary conditions. To prove the main theorems, the monotone iterative and the upper–lower solution techniques in the sense of the ψ –Caputo fractional operator were used. Based on certain conditions, we constructed *mis* that uniformly converged to the extremal solutions of BVP. The results were tested by constructing two equations corresponding to BVP (3). Different values for ψ , such as the $\psi^*(t) = t$, Caputo, $\psi^*(t) = 2^t$, $\psi^*(t) = \sqrt{t}$, and Katugampola (for $\rho = 0.5$) derivatives and the upper and lower solutions, were examined and illustrated for the purpose of verification. We conclude that the results reported in this paper are of great significance for the relevant audience and can be applied to different types of fractional differential problems.

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Appendix A. Supporting Informations

Algorithm A1 MATLAB lines for the calculation of $\mathbb{I}_{t_1}^{\sigma;\Psi} \varrho(t)$ in Equation (4).

```

1: LSfractionalintegral
Require:  $t_1, \sigma, \psi, \varrho, t$ 
2: syms  $v\ e$ ;
3:  $E = \text{int}(\text{subs}(\text{diff}(\psi, v), v, e) * (\text{eval}(\text{subs}(\psi, v, t)) - \text{subs}(\psi, v, e))^{(\sigma-1)} * \text{subs}(\varrho, v, e), t_1, t);$ 
4:  $\text{mathbbI} = \text{round}(1/\gamma(\sigma) * \text{eval}(E), 8);$ 
5: return  $\text{mathbbI}$ 
```

Algorithm A2 MATLAB lines for the calculation of $\mathbb{D}_{t_1}^{\alpha;\Psi} \varrho(t)$ in Equation (5).

```

1: LSfractionalderivative
Require:  $t_1, \sigma, \psi, \varrho, t$ 
2: syms  $v\ e\ t$ ;
3:  $n = \text{floor}(\alpha) + 1;$ 
4:  $E = \text{int}(\text{subs}(\text{diff}(\psi, v), v, e) * (\text{subs}(\psi, v, t) - \text{subs}(\psi, v, e))^{(n-\sigma-1)} * \text{subs}(\varrho, v, e), e, t_1, t);$ 
5:  $F = \text{diff}(E, t, n);$ 
6:  $F = (1/\text{subs}(\text{diff}(\psi, v), v, t))^n * F;$ 
7:  $F = \text{round}(1/\gamma(n - \sigma), 8) * F;$ 
8:  $\text{mathbbD} = F;$ 
9: return  $\text{mathbbD}$ 
```

Algorithm A3 MATLAB lines for the calculation of ${}^C\mathbb{D}_{t_1}^{\sigma;\Psi} \varrho(t)$ in Equation (6).

```

1: LSCaputofractionalderivative
Require:  $t_1, \sigma, \psi, \varrho, t$ 
2: syms  $v\ e$ ;
3:  $n = \text{floor}(\sigma) + 1;$ 
4: if  $\text{fix}(\alpha) == \sigma$  then
5:    $F = \text{eval}(\text{subs}(\text{diff}(\varrho, v, n), v, t));$ 
6:    $E = F;$ 
7: else
8:    $F = \text{int}(\text{subs}(\text{diff}(\psi, v), v, e) * (\text{eval}(\text{subs}(\psi, v, t)) - \text{subs}(\psi, v, e))^{(n-\sigma-1)} * \text{eval}(\text{subs}(\text{diff}(\varrho, v, n), v, e)), t_1, t);$ 
9:    $E = 1/\gamma(n - \sigma) * F;$ 
10: end if
11:  $\text{mathbbD} = E;$ 
12: return  $\text{mathbbD}$ 
```

Algorithm A4 MATLAB lines for the calculation of $u_n(t)$ and $v_n(t)$ in Example 1.**Require:** $t_1, \sigma, \psi, \varrho, t$

```

1: syms  $v\ e$ ;
2: clear;
3: format long;
4: syms  $v\ e\ \vartheta$ ;
5:  $\alpha = 1/4$ ;  $\beta = 1/2$ ;  $\lambda = 3/4$ ;  $\eta = 2/3$ ;  $\delta = 1/2$ ;
6:  $a = 0$ ;  $b = 1$ ;
7:  $\psi = v$ ;  $\varrho = v$ ;
8:  $n = \text{floor}(\alpha) + 1$ ;
9:  $\Lambda = 1 - \lambda/\gamma(\beta + 1) * (\text{eval}(\text{subs}(\psi, v, \eta)) - \text{eval}(\text{subs}(\psi, v, a)))^\beta$ ;
10:  $u_0 = 0$ ;  $v_0 = 1 + 2 * v$ ;
11:  $\psi = [\text{sym}(v) \text{sym}(2^v) \text{sym}(v^{1/2})]$ ;
12:  $h = 4$ ;  $c = 1$ ;
13: for  $i = 1$  to 3 do
14:    $t = a$ ;
15:    $r = 1$ ;
16:    $\text{MatrixResults}(r, c) = r$ ;
17:    $\text{MatrixResults}(r, c + 1) = t$ ;
18:    $\text{MatrixResults}(r, c + 2) = \text{round}(1 - \lambda/\gamma(\beta + 1) * (\text{eval}(\text{subs}(\psi(i), v, \eta)) - \text{eval}(\text{subs}(\psi(i), v, a)))^\beta, 6)$ ;
19:    $\text{MatrixResults}(r, c + 3) = 1 - 0 + 2 * t$ ;
20:    $\text{MatrixResults}(r, c + 4) = 1 - 0 + 2 * \eta$ ;
21:    $\text{MatrixResults}(r, c + 5) = \text{round}(\text{abs}(\text{LSfractionalintegral}(a, \alpha, \psi(i), \text{MatrixResults}(r, c + 3), t) + 1/\text{MatrixResults}(r, c + 2) * (\lambda * (\text{LSfractionalintegral}(a, \alpha + \beta, \psi(i), \text{MatrixResults}(r, c + 4), \eta)) + \delta))), 6)$ ;
22:    $\text{MatrixResults}(r, c + 6) = 1 - (1 + 2 * t) + 2 * t$ ;
23:    $\text{MatrixResults}(r, c + 7) = 1 - (1 + 2 * \eta) + 2 * \eta$ ;
24:    $\text{MatrixResults}(r, c + 8) = \text{round}(\text{abs}(\text{LSfractionalintegral}(a, \alpha, \psi(i), \text{MatrixResults}(r, c + 3), t) + 1/\text{MatrixResults}(r, c + 2) * (\lambda * (\text{LSfractionalintegral}(a, \alpha + \beta, \psi(i), \text{MatrixResults}(r, c + 4), \eta)) + \delta))), 6)$ ;
25:    $t = t + 1/2^h$ ;
26:    $r = r + 1$ ;
27: while  $(t \leq b)$  do
28:    $\text{MatrixResults}(r, c) = r$ ;
29:    $\text{MatrixResults}(r, c + 1) = t$ ;
30:    $\text{MatrixResults}(r, c + 2) = \text{round}(1 - \lambda/\gamma(\beta + 1) * (\text{eval}(\text{subs}(\psi(i), v, \eta)) - \text{eval}(\text{subs}(\psi(i), v, a)))^\beta, 6)$ ;
31:    $\text{MatrixResults}(r, c + 3) = 1 - (\text{MatrixResults}(r - 1, c + 3))^2 + 2 * t$ ;
32:    $\text{MatrixResults}(r, c + 4) = 1 - (\text{MatrixResults}(r - 1, c + 3))^2 + 2 * \eta$ ;
33:    $\text{MatrixResults}(r, c + 5) = \text{round}(\text{abs}(\text{LSfractionalintegral}(a, \alpha, \psi(i), \text{MatrixResults}(r, c + 3), t) + 1/\text{MatrixResults}(r, c + 2) * (\lambda * (\text{LSfractionalintegral}(a, \alpha + \beta, \psi(i), \text{MatrixResults}(r, c + 4), \eta)) + \delta))), 6)$ ;
34:    $\text{MatrixResults}(r, c + 6) = 1 - (\text{MatrixResults}(r - 1, c + 6))^2 + 2 * t$ ;
35:    $\text{MatrixResults}(r, c + 7) = 1 - (\text{MatrixResults}(r - 1, c + 7))^2 + 2 * \eta$ ;
36:    $\text{MatrixResults}(r, c + 8) = \text{round}(\text{abs}(\text{LSfractionalintegral}(a, \alpha, \psi(i), \text{MatrixResults}(r, c + 6), t) + 1/\text{MatrixResults}(r, c + 2) * (\lambda * (\text{LSfractionalintegral}(a, \alpha + \beta, \psi(i), \text{MatrixResults}(r, c + 7), \eta)) + \delta))), 6)$ ;
37:    $t = t + 1/2^h$ ;
38:    $r = r + 1$ ;
39: end while
40:    $c = c + 9$ ;
41: end for
42: return  $\text{MatrixResults}$ 

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