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Nonexistence of Global Positive Solutions for p-Laplacian Equations with Non-Linear Memory

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Abstract: The Cauchy problem in $\mathbb{R}^d$, $d \geq 1$, for a non-local in time p-Laplacian equations is considered. The nonexistence of nontrivial global weak solutions by using the test function method is obtained.

Keywords: Cauchy problem; p-Laplacian equation; global nonexistence

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Citation: Kirane, M.; Fino, A.Z.; Kerbal, S. Nonexistence of Global Positive Solutions for p-Laplacian Equations with Non-Linear Memory. *Fractal Fract.* **2021**, *5*, 189. https://doi.org/10.3390/fractalfract5040189

Academic Editors: Vassili Kolokoltsov and Jozsef Lorinczi

Received: 20 September 2021
Accepted: 25 October 2021
Published: 28 October 2021

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1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} u_t - \mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = \displaystyle\int_0^t (t-s)^{-\gamma} u(s)^q\, ds, & t > 0, x \in \mathbb{R}^d, \\ u \geq 0, & t > 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $p, q > 1$, $d \geq 1$, $0 < \gamma < 1$, and $u_0 \in L^2_{loc}(\mathbb{R}^d)$. We are interested in the nonexistence of nontrivial global weak solutions.

The study of nonexistence of global solutions for nonlinear parabolic equations was started by Fujita [1]; he studied the Cauchy problem

$$\begin{cases} u_t - \Delta u = u^p, & p > 1, t > 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x) \geq 0, & x \in \mathbb{R}^d; \end{cases} \tag{2}$$

and found that

(a) there is no nontrivial global solution of (2) whenever $p < 1 + 2/d$.

(b) whereas (2) admits a global solution whenever $p > 1 + 2/d$ and $u_0(x) \leq \delta e^{-|x|^2}$, $(0 < \delta \ll 1)$.

For the limiting case $p = 1 + 2/d$, it is shown by Hayakawa [2] for $d = 1, 2$ and by Kobayashi, Sirao and Tanaka [3] for $d \geq 1$ that (2) has no global solution $u(x,t)$ satisfying $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} < \infty$ for $t \geq 0$. Weissler in [4] proved that if (2) has no global solution $u(x,t)$ satisfying $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} < \infty$ for $t > 0$ and some $q \in [1, +\infty)$ whenever $p = 1 + 2/d$. Therefore, the value

$$p_F = 1 + 2/d \tag{3}$$

is the limiting exponent of (2); it is called Fujita's exponent.

Since Fujita's paper, a sizeable number of extensions in many directions have been published. Recently, Cazenave, Dickstein and Weissler [5] extended the results of Fujita to the non-local in time heat equation

$$\begin{cases} w_t - \Delta w = \int_0^t (t-s)^{-\vartheta} |w(s)|^{p-1} w(s)\, ds, & t>0, x \in \mathbb{R}^d,\ p>1, \\ w(0,x) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{4}$$

where $w_0 \in C_0(\mathbb{R}^d)$ and $\vartheta \in (0,1)$. In this case, the value of the critical Fujita exponent is

$$p_c = \max\left\{\frac{1}{\vartheta}, 1 + \frac{2(2-\vartheta)}{(d-2(1-\vartheta))_+}\right\}. \tag{5}$$

For the p-Laplacian equation

$$\begin{cases} w_t - \operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) = w^q, & t>0, x\in\mathbb{R}^d, p>2, q>1, \\ w \ge 0, & t>0, x \in \mathbb{R}^d, \\ w(0,x) = w_0(x) \ge 0, & x \in \mathbb{R}^d, \end{cases} \tag{6}$$

Zhao [6] and Mitidieri and Pohozaev [7] obtained the critical exponent

$$q^* = p - 1 + p/d; \tag{7}$$

In fact, Zhao [6] proved that if $p-1 < q < q^*$, then the Cauchy problem (6) has no nontrivial global solution, however if $q > q^*$ and $w_0(x)$ is small enough, then (6) admits a global solution. Mitidieri and Pohozaev [7] completed the study by proving the nonexistence of nontrivial global solution in the case $q \le q^*$ and for all $p > 2d/(d+1)$. Andreucci and Tedeev [8,9] obtained similar results by considering doubly singular parabolic equations.

The test function method was used to prove the nonexistence of global solutions. This method was introduced by Baras and Kersner in [10] and developed by Zhang in [11] and Pohozaev and Mitidieri in [7], it was also used by Kirane et al. in [12].

Here, we are concerned with the non-existence of nontrivial global solutions of (1); inspired by [7], we choose a suitable test function in the weak formulation of the problem. Our main result is

Theorem 1. *Let $u_0 \in L^2_{loc}(\mathbb{R}^d)$, $0<\gamma<1$, $p>1$, $d\ge 1$, $q > \max\{1, p-1\}$. Suppose that*

$$\begin{cases} q \le \max\left\{\frac{p-1}{\gamma}; q_c\right\}, & \text{if } p < \frac{d}{1-\gamma}, \\ q < \infty, & \text{if } p \ge \frac{d}{1-\gamma}, \end{cases} \tag{8}$$

where

$$q_c := p - 1 + \frac{(d+p)(1-\gamma)(p-1) + p - d(1-\gamma)}{d-(1-\gamma)p},$$

then no nontrivial global weak solutions exist for problem (1).

Remark 1. *Note that (8) can be seen as follows*

$$q \le \max\left\{\frac{p-1}{\gamma}; p-1+\frac{(d+p)(1-\gamma)(p-1)+p-d(1-\gamma)}{(d-(1-\gamma)p)_+}\right\},$$

whence

- *when $p=2$ and $\gamma \to 1$, then $q_c \to p_F$ where p_F is defined in (3).*

- *when $p = 2$, then $q_c = p_c$ where p_c is defined in (5).*
- *when $\gamma \to 1$, and $p > 2d/(d+1)$, then $q_c \to q^*$ where q^* is defined in (7).*

Remark 2. *The same result can be obtained for the problem*

$$\begin{cases} u_t - divA(t,x,u,\nabla u) = \displaystyle\int_0^t (t-s)^{-\gamma} u(s)^q\, ds, & t > 0, x \in \mathbb{R}^d, \\ u \geq 0, & t > 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

where $q > 1, 0 < \gamma < 1, d \geq 1, u_0 \in L^2_{loc}(\mathbb{R}^d)$, and $A : \mathbb{R}^*_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Carathéodory function, where one assumes the existence of $c_1, c_2 > 0$ and $p > 1$ such that

$$\begin{cases} (A(t,x,u,w), w) \geq c_1\, |w|^p, \\ \\ |A(t,x,u,w)| \leq c_2\, |w|^{p-1}, \end{cases} \tag{9}$$

for all $(t,x,u,w) \in \mathbb{R}^*_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d$. For the study of the non-existence of global solution, the following condition is needed

$$(A(t,x,u,w), w) \geq c_3\, |A(t,x,u,w)|^{\frac{p}{p-1}}, \quad \text{for all } (t,x,u,w) \in \mathbb{R}^*_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^d, \tag{10}$$

which is a general case of (9).

Some words on the structure of the paper: Some definitions and properties on the fractional integrals and derivatives are recalled in Section 2. Section 3, is reserved to the proof of the main result (Theorem 1).

2. Preliminaries

Definition 1. *A function $\mathcal{A} : [a,b] \to \mathbb{R}$, $-\infty < a < b < \infty$, is said to be absolutely continuous if and only if there exists $\psi \in L^1(a,b)$ such that*

$$\mathcal{A}(t) = \mathcal{A}(a) + \int_a^t \psi(s)\, ds.$$

$AC[a,b]$ denotes the space of these functions. Moreover,

$$AC^2[a,b] := \{\varphi : [a,b] \to \mathbb{R} \,/\, \varphi' \in AC[a,b]\}.$$

Definition 2. *Let $f \in L^1(c,d)$, $-\infty < c < d < \infty$. The Riemann–Liouville left-and right-sided fractional integrals are defined by*

$$I^\alpha_{c|t} f(t) := \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{-(1-\alpha)} f(s)\, ds, \quad t > c,\ \alpha \in (0,1), \tag{11}$$

and

$$I^\alpha_{t|d} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^d (s-t)^{-(1-\alpha)} f(s)\, ds, \quad t < d,\ \alpha \in (0,1) \tag{12}$$

where Γ is the Euler gamma function.

Definition 3. *Let $f \in AC[c,d]$, $-\infty < c < d < \infty$. The Riemann–Liouville left-and right-sided fractional derivatives are defined by*

$$D^\alpha_{c|t} f(t) := \frac{d}{dt} I^{1-\alpha}_{c|t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_c^t (t-s)^{-\alpha} f(s)\, ds, \quad t > c,\ \alpha \in (0,1), \tag{13}$$

and

$$D^{\alpha}_{t|d} f(t) := -\frac{d}{dt} I^{1-\alpha}_{t|d} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^d (s-t)^{-\alpha} f(s)\, ds, \quad t < d,\ \alpha \in (0,1). \tag{14}$$

Proposition 1 ([13] ((2.64), p. 46)). *Let $\alpha \in (0,1)$ and $-\infty < c < d < \infty$. The fractional integration by parts formula*

$$\int_c^d f(t) D^{\alpha}_{c|t} g(t)\, dt \ = \ \int_c^d g(t) D^{\alpha}_{t|d} f(t)\, dt, \tag{15}$$

is satisfied for every $f \in I^{\alpha}_{t|d}(L^p(c,d))$, $g \in I^{\alpha}_{c|t}(L^q(c,d))$ such that $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$, $p, q > 1$, where

$$I^{\alpha}_{c|t}(L^q(0,T)) := \left\{ f = I^{\alpha}_{c|t} h,\ h \in L^q(c,d) \right\},$$

and

$$I^{\alpha}_{t|d}(L^p(c,d)) := \left\{ f = I^{\alpha}_{t|d} h,\ h \in L^p(c,d) \right\}.$$

Remark 3. *A simple sufficient condition for functions f and g to satisfy (15) is that $f, g \in C[c,d]$, such that $D^{\alpha}_{t|d} f(t), D^{\alpha}_{c|t} g(t)$ exist at every point $t \in [c,d]$ and are continuous.*

Proposition 2 ([13] (Chapter 1)). *For $0 < \alpha < 1$, $-\infty < c < d < \infty$, we have the following identities*

$$D^{\alpha}_{c|t} I^{\alpha}_{c|t} \varphi(t) = \varphi(t),\ a.e.\ t \in (c,d), \quad \textit{for all } \varphi \in L^r(c,d), 1 \le r \le \infty, \tag{16}$$

and

$$-\frac{d}{dt} D^{\alpha}_{t|d} \varphi = D^{1+\alpha}_{t|d} \varphi, \quad \varphi \in AC^2[c,d], \tag{17}$$

where $D := \frac{d}{dt}$.

Given $T > 0$, let us define the function w_1 by

$$w_1(t) = (1 - t/T)^{\sigma}, \quad t \in [0,T],\ \sigma \gg 1 \tag{18}$$

The following properties concerning the functions w_1 will be used later on.

Lemma 1 ([13] ((2.45), p. 40)). *Let $T > 0$, $0 < \alpha < 1$. For all $t \in [0,T]$, we have*

$$D^{\alpha}_{t|T} w_1(t) = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-\alpha)} T^{-\alpha} (1 - t/T)^{\sigma-\alpha}, \tag{19}$$

and

$$D^{1+\alpha}_{t|T} w_1(t) = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\alpha)} T^{-(1+\alpha)} (1 - t/T)^{\sigma-\alpha-1}. \tag{20}$$

Any constant will be denoted by C.

3. Global Nonexistence

In this section, the proof of Theorem 1 will be presented.

We set

$$Q_T = (0,T) \times \mathbb{R}^d, \qquad \mathcal{B}_{T,B} = (0,T) \times \Omega(B),$$

where $\Omega(B) := \{x \in \mathbb{R}^n;\ |x| < 2B\}$, for $T, B > 0$.

Definition 4 (Weak solution)**.**
Let $0 < \gamma < 1$*, and* $T > 0$*. A function*

$$u \in C([0,T], L^2_{loc}(\mathbb{R}^d)) \cap L^{2q}((0,T), L^{2q}_{loc}(\mathbb{R}^d)) \cap L^p((0,T), W^{1,p}_{loc}(\mathbb{R}^d)) \cap W^{1,2}((0,T), L^2_{loc}(\mathbb{R}^d)),$$

is said to be a weak solution of (1) if $u_{|_{t=0}} = u_0$ *and the following formulation*

$$\Gamma(\delta)\int_{Q_T} I^\delta_{0|t}(u^q)\varphi\,dx\,dt = \int_{Q_T} |\nabla u|^{p-2}\nabla u \nabla\varphi\,dx\,dt + \int_{Q_T} u_t\varphi\,dx\,dt, \tag{21}$$

holds for all $\varphi \in L^2(0,\infty; L^2(\mathbb{R}^d)) \cap L^p(0;\infty, W^{1,p}(\mathbb{R}^d))$ *such that* $supp\varphi \subset [0,T]\times\mathbb{R}^d$ *is compact, where* $I^\delta_{0|t}$ *is defined in* (11) *and* $\delta = 1-\gamma$*. We denote the lifespan for the weak solution by*

$$T_w(u_0) := \sup\{T \in (0,\infty];\ \textit{there exists a unique weak solution } u \textit{ to } (1)\}.$$

Moreover, if $T > 0$ *can be arbitrary chosen, i.e.,* $T_w(u_0) = \infty$*, then* u *is called a global weak solution of (1).*

Proof of Theorem 1. Let $u \geq 0$ be a global weak solution of (1), then

$$\Gamma(\delta)\int_{Q_T} I^\delta_{0|t}(u^q)\varphi\,dx\,dt = \int_{Q_T} |\nabla u|^{p-2}\nabla u \nabla\varphi\,dx\,dt + \int_{Q_T} u_t\varphi\,dx\,dt,$$

for all $T > 0$ and all $\varphi \in L^2(0,\infty; L^2(\mathbb{R}^d)) \cap L^p(0;\infty, W^{1,p}(\mathbb{R}^d))$ such that supp$\varphi \subset [0,T]\times\mathbb{R}^d$ is compact, where $\delta = 1-\gamma$. Let

$$\begin{aligned} \varphi(x,t) &= D^\delta_{t|T}(\tilde\varphi(x,t)) := D^\delta_{t|T}\Big(\varphi_1^\ell(x)\varphi_2(t)\Big) \\ \text{with}\quad \varphi_1(x) &:= \Phi(|x|/B),\ \varphi_2(t) := \left(1-\frac{t}{T}\right)_+^\eta, \end{aligned} \tag{22}$$

where $\ell, \eta \gg 1$ and $\Phi \in C^\infty(\mathbb{R}_+)$ be

$$\Phi(s) = \begin{cases} 1, & 0 \leq s \leq 1, \\ \searrow & 1 \leq s \leq 2, \\ 0, & s \geq 2. \end{cases}$$

Then, using the integration-by-parts formula, we get

$$\begin{aligned} \Gamma(\delta)\int_{\mathcal{B}_{T,B}} I^\delta_{0|t}(u^q) D^\delta_{t|T}\tilde\varphi\,dx\,dt \quad &+ \quad \int_{\Omega(B)} u_0(x) D^\delta_{t|T}\tilde\varphi(0,x)\,dx \\ &= \int_{\mathcal{B}_{T,B}} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx\,dt - \int_{\mathcal{B}_{T,B}} u\,\partial_t\varphi\,dx\,dt. \end{aligned}$$

From (15) and Lemma 1, we conclude that

$$\begin{aligned} \int_{\mathcal{B}_{T,B}} D^\delta_{0|t} I^\delta_{0|t}(u^q)\tilde\varphi\,dx\,dt \quad &+ \quad C\,T^{-\delta}\int_{\Omega(B)} u_0(x)\varphi_1^\ell(x)\,dx \\ &= C\int_{\mathcal{B}_{T,B}} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx\,dt - C\int_{\mathcal{B}_{T,B}} u\,\partial_t\varphi\,dx\,dt. \end{aligned}$$

Moreover, using (16), $u_0 \geq 0$ and $u \geq 0$, it follows

$$\int_{\mathcal{B}_{T,B}} u^q\tilde\varphi\,dx\,dt \leq C\int_{\mathcal{B}_{T,B}} |\nabla u|^{p-1}|\nabla\varphi|\,dx\,dt + C\int_{\mathcal{B}_{T,B}} u|\partial_t\varphi|\,dx\,dt =: I_1 + I_2. \tag{23}$$

Next, by introducing $\tilde{\varphi}^{1/q}\tilde{\varphi}^{-1/q}$ in I_2 and applying the following Young's inequality

$$AB \leq \frac{1}{8}A^q + C(q)B^{q'}, \quad A \geq 0,\ B \geq 0,\ q + q' = qq', \quad C(q) = \frac{(q-1)8^{1/(q-1)}}{q^{q/(q-1)}},$$

we get

$$I_2 \leq \frac{1}{8}\int_{\mathcal{B}_{T,B}} u^q \tilde{\varphi}\,dx\,dt + C\int_{\mathcal{B}_{T,B}} \varphi_1^{\ell}\varphi_2^{-\frac{1}{q-1}}\left(D_{t|T}^{1+\delta}\varphi_2\right)^{q'} dx\,dt. \tag{24}$$

In order to obtain a similar estimation on I_1, let $\alpha < 0$ be an auxiliary constant such that $\alpha > \max\{-1, 1-p, 1-\frac{q}{p-1}\}$ and let

$$u_\epsilon(x,t) = u(x,t) + \epsilon, \quad \epsilon > 0.$$

By taking $\varphi_\epsilon(x,t) = u_\epsilon^\alpha(x,t)\varphi(x,t)$ as a test function where φ is given in (22), and using the fact that u is a weak solution, we obtain

$$\Gamma(\delta)\int_{\mathcal{B}_{T,B}} I_{0|t}^{\delta}(u^q)u_\epsilon^\alpha\varphi\,dx\,dt = \int_{\mathcal{B}_{T,B}} |\nabla u|^{p-2}\nabla u\nabla(u_\epsilon^\alpha\varphi)\,dx\,dt + \int_{\mathcal{B}_{T,B}} u_t\,u_\epsilon^\alpha\varphi\,dx\,dt.$$

Using the integration-by-parts formula, we get

$$\begin{aligned}\Gamma(\delta)\int_{\mathcal{B}_{T,B}} I_{0|t}^{\delta}(u^q)u_\epsilon^\alpha\varphi\,dx\,dt &= \int_{\mathcal{B}_{T,B}} |\nabla u|^{p-2}\nabla u\nabla(u_\epsilon^\alpha\varphi)\,dx\,dt\\ &- \int_{\mathcal{B}_{T,B}} u\partial_t(u_\epsilon^\alpha\varphi)\,dx\,dt - \int_{\Omega(B)} u_0(x)u_\epsilon^\alpha(x,0)\varphi(x,0)\,dx.\end{aligned}$$

Then, as

$$\nabla(u_\epsilon^\alpha\varphi) = \alpha u_\epsilon^{\alpha-1}\varphi\nabla u + u_\epsilon^\alpha\nabla\varphi \quad \text{and} \quad \partial_t(u_\epsilon^\alpha\varphi) = \alpha u_\epsilon^{\alpha-1}\varphi\partial_t u + u_\epsilon^\alpha\partial_t\varphi,$$

it comes

$$\begin{aligned}&\Gamma(\delta)\int_{\mathcal{B}_{T,B}} I_{0|t}^{\delta}(u^q)u_\epsilon^\alpha\varphi\,dx\,dt\\ &= \alpha\int_{\mathcal{B}_{T,B}} |\nabla u|^p u_\epsilon^{\alpha-1}\varphi\,dx\,dt + \int_{\mathcal{B}_{T,B}} |\nabla u|^{p-2}\nabla u\nabla\varphi\,u_\epsilon^\alpha\,dx\,dt\\ &\quad - \alpha\int_{\mathcal{B}_{T,B}} uu_\epsilon^{\alpha-1}\varphi\partial_t u\,dx\,dt - \int_{\mathcal{B}_{T,B}} uu_\epsilon^\alpha\partial_t\varphi\,dx\,dt - \int_{\Omega(B)} u_0(x)u_\epsilon^\alpha(x,0)\varphi(x,0)\,dx.\end{aligned} \tag{25}$$

Whereupon,

$$\begin{aligned}J_1 &:= \int_{\mathcal{B}_{T,B}} uu_\epsilon^{\alpha-1}\varphi\partial_t u\,dx\,dt\\ &= \int_{\mathcal{B}_{T,B}} (u_\epsilon - \epsilon)u_\epsilon^{\alpha-1}\varphi\partial_t u\,dx\,dt\\ &= \int_{\mathcal{B}_{T,B}} u_\epsilon^\alpha\varphi\partial_t u_\epsilon\,dx\,dt - \epsilon\int_{\mathcal{B}_{T,B}} u_\epsilon^{\alpha-1}\varphi\partial_t u_\epsilon\,dx\,dt\\ &= \frac{1}{\alpha+1}\int_{\mathcal{B}_{T,B}} \partial_t(u_\epsilon^{\alpha+1})\varphi\,dx\,dt - \frac{\epsilon}{\alpha}\int_{\mathcal{B}_{T,B}} \partial_t(u_\epsilon^\alpha)\varphi\,dx\,dt\\ &= -\frac{1}{\alpha+1}\int_{\mathcal{B}_{T,B}} u_\epsilon^{\alpha+1}\partial_t\varphi\,dx\,dt - \frac{1}{\alpha+1}\int_{\Omega(B)} u_\epsilon^{\alpha+1}(x,0)\varphi(x,0)\,dx\\ &\quad + \frac{\epsilon}{\alpha}\int_{\mathcal{B}_{T,B}} u_\epsilon^\alpha\partial_t\varphi\,dx\,dt + \frac{\epsilon}{\alpha}\int_{\Omega(B)} u_\epsilon^\alpha(x,0)\varphi(x,0)\,dx.\end{aligned}$$

Similarly,

$$\begin{aligned} J_2: &= \int_{\mathcal{B}_{T,B}} u u_\epsilon^\alpha \partial_t \varphi \, dx\, dt \\ &= \int_{\mathcal{B}_{T,B}} (u_\epsilon - \epsilon) u_\epsilon^\alpha \partial_t \varphi \, dx\, dt \\ &= \int_{\mathcal{B}_{T,B}} u_\epsilon^{\alpha+1} \partial_t \varphi \, dx\, dt - \epsilon \int_{\mathcal{B}_{T,B}} u_\epsilon^\alpha \partial_t \varphi \, dx\, dt, \end{aligned}$$

and

$$\begin{aligned} J_3: &= \int_{\Omega(B)} u_0(x) u_\epsilon^\alpha(x,0)\varphi(x,0)\, dx \\ &= \int_{\Omega(B)} (u_\epsilon(x,0) - \epsilon) u_\epsilon^\alpha(x,0)\varphi(x,0)\, dx \\ &= \int_{\Omega(B)} u_\epsilon^{\alpha+1}(x,0)\varphi(x,0)\, dx - \epsilon \int_{\Omega(B)} u_\epsilon^\alpha(x,0)\varphi(x,0)\, dx. \end{aligned}$$

By J_1,J_2, and J_3, it follows from (25) that

$$\begin{aligned} &\Gamma(\delta) \int_{\mathcal{B}_{T,B}} I^\delta_{0|t}(u^q) u_\epsilon^\alpha \varphi \, dx\, dt = \alpha \int_{\mathcal{B}_{T,B}} |\nabla u|^p u_\epsilon^{\alpha-1} \varphi \, dx\, dt + \\ &\int_{\mathcal{B}_{T,B}} |\nabla u|^{p-2} \nabla u \nabla \varphi \, u_\epsilon^\alpha \, dx\, dt - \frac{1}{\alpha+1} \int_{\mathcal{B}_{T,B}} u_\epsilon^{\alpha+1} \partial_t \varphi \, dx\, dt \\ &- \frac{1}{\alpha+1} \int_{\Omega(B)} u_\epsilon^{\alpha+1}(x,0)\varphi(x,0)\, dx, \end{aligned}$$

then

$$\begin{aligned} &\Gamma(\delta) \int_{\mathcal{B}_{T,B}} I^\delta_{0|t}(u^q) u_\epsilon^\alpha \varphi \, dx\, dt + |\alpha| \int_{\mathcal{B}_{T,B}} |\nabla u|^p u_\epsilon^{\alpha-1} \varphi \, dx\, dt \\ &= \int_{\mathcal{B}_{T,B}} |\nabla u|^{p-2} \nabla u \nabla \varphi \, u_\epsilon^\alpha \, dx\, dt - \frac{1}{\alpha+1} \int_{\mathcal{B}_{T,B}} u_\epsilon^{\alpha+1} \partial_t \varphi \, dx\, dt \\ &\quad - \frac{1}{\alpha+1} \int_{\Omega(B)} u_\epsilon^{\alpha+1}(x,0)\varphi(x,0)\, dx \\ &\le \int_{\mathcal{B}_{T,B}} |\nabla u|^{p-1} |\nabla \varphi| u_\epsilon^\alpha \, dx\, dt + \frac{1}{\alpha+1} \int_{\mathcal{B}_{T,B}} u_\epsilon^{\alpha+1} |\partial_t \varphi| \, dx\, dt \\ &= \int_{\mathcal{B}_{T,B}} \Big(|\nabla u|^{p-1} u_\epsilon^{\frac{(\alpha-1)(p-1)}{p}} \varphi^{\frac{p-1}{p}}\Big)\Big(|\nabla \varphi| u_\epsilon^{\alpha - \frac{(\alpha-1)(p-1)}{p}} \varphi^{-\frac{p-1}{p}}\Big) \, dx\, dt \\ &\quad + \frac{1}{\alpha+1} \int_{\mathcal{B}_{T,B}} u_\epsilon^{\alpha+1} |\partial_t \varphi| \, dx\, dt \\ &\le \frac{|\alpha|}{2} \int_{\mathcal{B}_{T,B}} |\nabla u|^p u_\epsilon^{\alpha-1} \varphi \, dx\, dt + C(\alpha) \int_{\mathcal{B}_{T,B}} u_\epsilon^{p-1+\alpha} |\nabla \varphi|^p \varphi^{1-p} \, dx\, dt \\ &\quad + \frac{1}{\alpha+1} \int_{\mathcal{B}_{T,B}} u_\epsilon^{\alpha+1} |\partial_t \varphi| \, dx\, dt, \end{aligned}$$

where Young's inequality

$$AB \le \frac{|\alpha|}{2} A^{\frac{p}{p-1}} + C(\alpha,p)\, B^p, \qquad A \ge 0,\ B \ge 0, \quad C(\alpha,p) = \frac{(2(p-1))^{p-1}}{|\alpha|^{p-1} p^p} \tag{26}$$

has been used. Consequently, as $u \ge 0$, we conclude that

$$\int_{\mathcal{B}_{T,B}} |\nabla u|^p u_\epsilon^{\alpha-1} \varphi \, dx\, dt \le C \int_{\mathcal{B}_{T,B}} u_\epsilon^{p-1+\alpha} |\nabla \varphi|^p \varphi^{1-p} \, dx\, dt + C \int_{\mathcal{B}_{T,B}} u_\epsilon^{\alpha+1} |\partial_t \varphi| \, dx\, dt.$$

Young's inequality and the last inequality, allow to get

$$\begin{aligned}
&\int_{\mathcal{B}_{T,B}} |\nabla u|^{p-1} |\nabla \varphi| \, dx \, dt \\
&= \int_{\mathcal{B}_{T,B}} \left(|\nabla u|^{p-1} u_\epsilon^{\frac{(\alpha-1)(p-1)}{p}} \varphi^{\frac{p-1}{p}} \right) \left(u_\epsilon^{\frac{(1-\alpha)(p-1)}{p}} \varphi^{\frac{1-p}{p}} |\nabla \varphi| \right) dx \, dt \\
&\leq \int_{\mathcal{B}_{T,B}} |\nabla u|^p u_\epsilon^{\alpha-1} \varphi \, dx \, dt + C \int_{\mathcal{B}_{T,B}} u_\epsilon^{(1-\alpha)(p-1)} \varphi^{1-p} |\nabla \varphi|^p \, dx \, dt \\
&\leq C \int_{\mathcal{B}_{T,B}} u_\epsilon^{p-1+\alpha} |\nabla \varphi|^p \varphi^{1-p} \, dx \, dt + C \int_{\mathcal{B}_{T,B}} u_\epsilon^{\alpha+1} |\partial_t \varphi| \, dx \, dt \\
&\quad + C \int_{\mathcal{B}_{T,B}} u_\epsilon^{(1-\alpha)(p-1)} \varphi^{1-p} |\nabla \varphi|^p \, dx \, dt.
\end{aligned}$$

Applying Fatou's and Lebesgue's theorems, as $\epsilon \to 0$, we get

$$\begin{aligned}
I_1 \quad \leq \quad & C \int_{\mathcal{B}_{T,B}} u^{p-1+\alpha} |\nabla \varphi|^p \varphi^{1-p} \, dx \, dt + C \int_{\mathcal{B}_{T,B}} u^{\alpha+1} |\partial_t \varphi| \, dx \, dt \\
& + C \int_{\mathcal{B}_{T,B}} u^{(1-\alpha)(p-1)} \varphi^{1-p} |\nabla \varphi|^p \, dx \, dt =: K_1 + K_2 + K_3.
\end{aligned}$$

Now, we use the following Young's inequality

$$AB \leq \frac{1}{8} A^{q_1} + C(q_1) B^{q_1'}, \quad A \geq 0, \; B \geq 0, \; q_1 + q_1' = q_1 q_1', \quad C(q_1) = \frac{(q_1 - 1) 8^{1/(q_1 - 1)}}{q_1^{q_1/(q_1-1)}},$$

with $q_1 = q/(p-1+\alpha)$, the fact that $\nabla(\varphi_1^\ell) = \ell \varphi_1^{\ell-1} \nabla \varphi_1$, and the conditions that $q > \max\{1, p-1\}$, $\alpha < 0$ to get estimations of K_1, K_2 and K_3; we have

$$\begin{aligned}
K_1 \quad = \quad & \int_{\mathcal{B}_{T,B}} (u^{p-1+\alpha} \tilde{\varphi}^{(p-1+\alpha)/q})(C \, \tilde{\varphi}^{-(p-1+\alpha)/q} |\nabla \varphi|^p \varphi^{1-p}) \, dx \, dt \\
\leq \quad & \frac{1}{8} \int_{\mathcal{B}_{T,B}} u^q \tilde{\varphi} \, dx \, dt \\
& + C \int_{\mathcal{B}_{T,B}} \varphi_1^{\ell - \frac{pq}{q-p+1-\alpha}} |\nabla \varphi_1|^{\frac{pq}{q-p+1-\alpha}} \varphi_2^{-\frac{p-1+\alpha}{q-p+1-\alpha}} \left(D_{t|T}^\delta \varphi_2 \right)^{\frac{q}{q-p+1-\alpha}} dx \, dt,
\end{aligned}$$

$$\begin{aligned}
K_2 \quad = \quad & \int_{\mathcal{B}_{T,B}} (u^{\alpha+1} \tilde{\varphi}^{(\alpha+1)/q})(C \, \tilde{\varphi}^{-(\alpha+1)/q} |\partial_t \varphi|) \, dx \, dt \\
\leq \quad & \frac{1}{8} \int_{\mathcal{B}_{T,B}} u^q \tilde{\varphi} \, dx \, dt + C \int_{\mathcal{B}_{T,B}} \varphi_1^\ell \varphi_2^{-\frac{\alpha+1}{q-1-\alpha}} \left(D_{t|T}^{1+\delta} \varphi_2 \right)^{\frac{q}{q-1-\alpha}} dx \, dt,
\end{aligned}$$

and

$$\begin{aligned}
K_3 \quad = \quad & \int_{\mathcal{B}_{T,B}} (u^{(1-\alpha)(p-1)} \tilde{\varphi}^{(1-\alpha)(p-1)/q})(C \, \tilde{\varphi}^{-(1-\alpha)(p-1)/q} \varphi^{1-p} |\nabla \varphi|^p) \, dx \, dt \\
\leq \quad & \frac{1}{8} \int_{\mathcal{B}_{T,B}} u^q \tilde{\varphi} \, dx \, dt \\
& + C \int_{\mathcal{B}_{T,B}} \varphi_1^{\ell - \frac{pq}{q-(1-\alpha)(p-1)}} |\nabla \varphi_1|^{\frac{pq}{q-(1-\alpha)(p-1)}} \varphi_2^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} \left(D_{t|T}^\delta \varphi_2 \right)^{\frac{q}{q-(1-\alpha)(p-1)}} dx \, dt.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} I_1 &\leq \frac{3}{8}\int_{\mathcal{B}_{T,B}} u^q \tilde{\varphi}\, dx\, dt \\ &+ C\int_{\mathcal{B}_{T,B}} \varphi_1^{\ell-\frac{pq}{q-p+1-\alpha}} |\nabla \varphi_1|^{\frac{pq}{q-p+1-\alpha}} \varphi_2^{-\frac{p-1+\alpha}{q-p+1-\alpha}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-p+1-\alpha}} dx\, dt \\ &\quad + C\int_{\mathcal{B}_{T,B}} \varphi_1^{\ell}\varphi_2^{-\frac{\alpha+1}{q-1-\alpha}} \left(D^{1+\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-1-\alpha}} dx\, dt \\ &\quad + C\int_{\mathcal{B}_{T,B}} \varphi_1^{\ell-\frac{pq}{q-(1-\alpha)(p-1)}} |\nabla \varphi_1|^{\frac{pq}{q-(1-\alpha)(p-1)}} \varphi_2^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-(1-\alpha)(p-1)}} dx\, dt. \end{aligned}$$

Using the estimates of I_1 and I_2 into (23), we obtain

$$\begin{aligned} &\int_{\mathcal{B}_{T,B}} u^q \tilde{\varphi}\, dx\, dt \\ &\leq C\int_{\mathcal{B}_{T,B}} \varphi_1^{\ell}\varphi_2^{-\frac{1}{q-1}} \left(D^{1+\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-1}} dx\, dt \\ &\quad + C\int_{\mathcal{B}_{T,B}} \varphi_1^{\ell-\frac{pq}{q-p+1-\alpha}} |\nabla \varphi_1|^{\frac{pq}{q-p+1-\alpha}} \varphi_2^{-\frac{p-1+\alpha}{q-p+1-\alpha}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-p+1-\alpha}} dx\, dt \\ &\quad + C\int_{\mathcal{B}_{T,B}} \varphi_1^{\ell}\varphi_2^{-\frac{\alpha+1}{q-1-\alpha}} \left(D^{1+\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-1-\alpha}} dx\, dt \\ &\quad + C\int_{\mathcal{B}_{T,B}} \varphi_1^{\ell-\frac{pq}{q-(1-\alpha)(p-1)}} |\nabla \varphi_1|^{\frac{pq}{q-(1-\alpha)(p-1)}} \varphi_2^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-(1-\alpha)(p-1)}} dx\, dt. \quad (27) \end{aligned}$$

At this stage, we choose $B = T^{\theta}$, $\theta > 0$. Taking α small enough and passing to $s = T^{-1}t$, $y = T^{-\theta}x$, we get from (27) that

$$\int_0^T \int_{\Omega(T^{\theta})} |u|^q \tilde{\varphi}\, dx\, dt \leq C\, T^{-\frac{(\delta+1)q}{q-1}+d\theta+1} + C\, T^{-\frac{(\delta+\theta p)q}{p-p+1}+d\theta+1}. \quad (28)$$

If all exponents of T are negative, by taking $T \longrightarrow +\infty$ and using the dominated convergence theorem, we conclude that $u = 0$. In order to ensure the negativity of the exponents of T, it is sufficient to require

$$q < \frac{1+d\theta}{(d\theta-\delta)_+} =: q_1(\theta) \quad \text{and} \quad q < \frac{(p-1)(1+d\theta)}{(d\theta-\theta p+1-\delta)_+} =: q_2(\theta).$$

which is equivalent to

$$q < \max_{\theta>0} \min\{q_1(\theta), q_2(\theta)\}. \quad (29)$$

To take into consideration $q_1(\theta)$ and $q_2(\theta)$, we first look at $(d\theta-\delta)_+$ and $(d\theta - \theta p + 1 - \delta)_+$ and try to compare them in terms of θ, i.e, to compare between δ/d and $(1-\delta)/((p-d)_+)$; this requires to study two cases: $p \geq d/\delta$ and $p < d/\delta$.

(a) **If** $p \geq d/\delta$**:** In this case we have $(1-\delta)/(p-d) \leq \delta/d$. As $q_1(\theta)$ is non-increasing and $q_2(\theta)$ is non-decreasing as a function of θ, (29) can be read as

$$q < \max\left\{ \max_{\theta\in[\frac{\delta}{d},+\infty[} q_1(\theta),\ \max_{\theta\in]0,\frac{1-\delta}{p-d}]} q_2(\theta) \right\} = +\infty.$$

(b) **If** $p < d/\delta$**:** Technical calculations lead us to to distinguish 3 cases.

(i) **Case** $p > d$**.** In this case we have $\delta/d < (1-\delta)/(p-d)$. So, (29) can be read as

$$q < \max\left\{\max_{\theta\in[\theta_0,+\infty[} q_1(\theta), \max_{\theta\in]0,\theta_0]} q_2(\theta)\right\} = q_1(\theta_0) = q_2(\theta_0) = q_c = \max\left\{\frac{p-1}{1-\delta}, q_c\right\},$$

with

$$\theta_0 := \frac{\delta p + 1 - 2\delta}{dp + p - 2d} > 0,$$

using that $q_1(\theta)$ is non-increasing and $q_2(\theta)$ is non-decreasing.

(ii) **Case** $2d/(d+1) < p \leq d$**.** In this case, (29) can be read as

$$q < \max\left\{\max_{\theta\in[\theta_0,+\infty[} q_1(\theta), \max_{\theta\in]0,\theta_0]} q_2(\theta)\right\}. \tag{30}$$

As $p_2(\theta)$ is non-decreasing when $p \geq d\delta$, and non-increasing when $p \leq d\delta$, we can see that (30) is equivalent to

$$q < \max\left\{\frac{p-1}{1-\delta}, q_c\right\}.$$

(iii) **Case of** $1 < p \leq 2d/(d+1)$**:** In this case, (29) can be read as

$$q < \max\left\{\max_{\theta\in]0,+\infty[} q_2(\theta)\right\} = q_2(0) = \frac{p-1}{1-\delta} = \max\left\{\frac{p-1}{1-\delta}, q_c\right\}.$$

Finally, to get similar results in the critical case

$$q = \max\left\{\frac{p-1}{1-\delta}, q_c\right\},$$

we choose $B = R^{-\theta}T^{\theta}$, where $1 \ll R < T$ is such that T and R do not go simultaneously to infinity. Moreover, due to the calculation made above, a positive constant D independent of T exists such that

$$\int_0^{\infty}\int_{\mathbb{R}^d} |u|^q \, dx\, dt \leq D,$$

leading to

$$\int_0^T \int_{\Delta(B)} |u|^q \tilde{\varphi}\, dx\, dt \to 0 \quad \text{as} \quad T \to \infty, \tag{31}$$

where $\Delta(B) := \{x \in \mathbb{R}^d;\ B < |x| < 2B\}$. Repeating a similar calculation as in the subcritical case, $q < \max\{\frac{p-1}{1-\delta}, q_c\}$, and using Hölder's inequality instead of Young's one in K_1 and K_3, we get

$$I_2 \leq \frac{1}{3}\int_{\mathcal{B}_{T,B}} u^q \tilde{\varphi}\, dx\, dt + C\int_{\mathcal{B}_{T,B}} \varphi_1^{\ell}\varphi_2^{-1/(q-1)}\left(D_{t|T}^{1+\delta}\varphi_2\right)^{q'} dx\, dt,$$

$$K_1 \leq C\left(\int_0^T\int_{\Delta(B)} u^q\tilde{\varphi}\, dx\, dt\right)^{\frac{p-1+\alpha}{q}}$$
$$\left(\int_{\mathcal{B}_{T,B}} \varphi_1^{\frac{\ell(q-p+1-\alpha)-pq}{q-p+1-\alpha}} |\nabla\varphi_1|^{\frac{pq}{q-p+1-\alpha}} \varphi_2^{-\frac{p-1+\alpha}{q-p+1-\alpha}} \left(D_{t|T}^{\delta}\varphi_2\right)^{\frac{q}{q-p+1-\alpha}} dx\, dt\right)^{\frac{q-p+1-\alpha}{q}},$$

$$K_2 \leq \frac{1}{3}\int_{\mathcal{B}_{T,B}} u^q\tilde{\varphi}\, dx\, dt + C\int_{\mathcal{B}_{T,B}} \varphi_1^{\ell}\varphi_2^{-\frac{\alpha+1}{q-1-\alpha}}\left(D_{t|T}^{1+\delta}\varphi_2\right)^{\frac{q}{q-1-\alpha}} dx\, dt,$$

and

$$K_3 \leq C\left(\int_0^T \int_{\Delta(B)} u^q \tilde{\varphi}\, dx\, dt\right)^{\frac{(1-\alpha)(p-1)}{q}}$$
$$\left(\int_{\mathcal{B}_{T,B}} \varphi_1^{\frac{\ell(q-(1-\alpha)(p-1))-pq}{q-(1-\alpha)(p-1)}} |\nabla \varphi_1|^{\frac{pq}{q-p+1-\alpha}} \varphi_2^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-(1-\alpha)(p-1)}}\, dx\, dt\right)^{\frac{q-(1-\alpha)(p-1)}{q}}.$$

Whereupon

$$\begin{aligned}
&\int_{\mathcal{B}_{T,B}} u^q \tilde{\varphi}\, dx\, dt \\
&\leq\; C \int_{\mathcal{B}_{T,B}} \varphi_1^{\ell} \varphi_2^{-1/(q-1)} \left(D^{1+\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-1}} dx\, dt \\
&+C\left(\int_0^T \int_{\Delta(B)} u^q \tilde{\varphi}\, dx\, dt\right)^{\frac{p-1}{q}} \\
&\quad\left(\int_{\mathcal{B}_{T,B}} \varphi_1^{\frac{\ell(q-p+1)-pq}{q-p+1}} |\nabla \varphi_1|^{\frac{pq}{q-p+1}} \varphi_2^{-\frac{p-1}{q-p+1}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-p+1}} dx\, dt\right)^{\frac{q-p+1}{q}} \\
&+C \int_{\mathcal{B}_{T,B}} \varphi_1^{\ell} \varphi_2^{-\frac{1}{q-1}} \left(D^{1+\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-1}} dx\, dt \\
&+C\left(\int_0^T \int_{\Delta(B)} u^q \tilde{\varphi}\, dx\, dt\right)^{\frac{p-1}{q}} \\
&\quad\left(\int_{\mathcal{B}_{T,B}} \varphi_1^{\frac{\ell(q-p+1)-pq}{q-p+1}} |\nabla \varphi_1|^{\frac{pq}{q-p+1}} \varphi_2^{-\frac{p-1}{q-p+1}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-p+1}} dx\, dt\right)^{\frac{q-p+1}{q}}.
\end{aligned} \tag{32}$$

Taking into account that $q = \max\{\frac{p-1}{1-\delta}, q_c\}$ and $s = T^{-1}t,\ y = R^{\theta}T^{-\theta}x$, we get

$$\int_{\mathcal{B}_{T,B}} |u|^q \tilde{\varphi}\, dx\, dt \leq C\, R^{-d\theta} + CR^{-d\theta+\frac{pq\theta}{q-p+1}} \left(\int_0^T \int_{\Delta(B)} u^q \tilde{\varphi}\, dx\, dt\right)^{\frac{p-1}{q}}$$

Taking the limit when $T \to \infty$, and using (31), we obtain

$$\int_0^{\infty} \int_{\mathbb{R}^d} |u|^q\, dx\, dt \leq C\, R^{-d\theta}.$$

Finally, letting $R \to \infty$, it comes that $u = 0$. The proof of Theorem 1 is complete. □

Author Contributions: Conceptualization, M.K. and A.Z.F.; methodology, M.K. and A.Z.F.; validation, M.K., A.Z.F. and S.K.; formal analysis, M.K., A.Z.F. and S.K.; investigation, M.K., A.Z.F. and S.K.; writing—original draft preparation, M.K., A.Z.F. and S.K.; writing, M.K., A.Z.F. and S.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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