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Wasserstein Bounds in the CLT of the MLE for the Drift Coefficient of a Stochastic Partial Differential Equation

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Abstract: In this paper, we are interested in the rate of convergence for the central limit theorem of the maximum likelihood estimator of the drift coefficient for a stochastic partial differential equation based on continuous time observations of the Fourier coefficients $u_i(t),\ i = 1, \dots, N$ of the solution, over some finite interval of time $[0, T]$. We provide explicit upper bounds for the Wasserstein distance for the rate of convergence when $N \to \infty$ and/or $T \to \infty$. In the case when T is fixed and $N \to \infty$, the upper bounds obtained in our results are more efficient than those of the Kolmogorov distance given by the relevant papers of Mishra and Prakasa Rao, and Kim and Park.

Keywords: parameter estimation; stochastic partial differential equations; rate of normal convergence of the MLE; Wasserstein distance

MSC: 62F12; 60F05; 60G15; 60H15; 60H07

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1. Introduction

Consider the process $\{u(t, x), 0 < x < 1, 0 \le t \le T\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as a solution to the stochastic partial differential equation

$$du(t, x) = \theta \Delta u(t, x) + dW_Q(t, x) \tag{1}$$

with initial and boundary conditions

$$u(0, x) = f(x), \quad f \in L^2([0, 1]),$$
$$u(t, 0) = u(t, 1) = 0, \quad 0 \le t \le T,$$

where $\Delta = \frac{\partial^2}{\partial x^2}$ and $\theta > 0$ is an unknown parameter, whereas Q is the covariance operator for the Wiener process $W_Q(t, x)$, so that

$$W_Q(t, x) = Q^{1/2} W(t, x),$$

with $W(t, x)$ being a cylindrical Brownian motion in $L^2([0, 1])$. It is a standard fact (see, e.g., [1]) that, given Q is nuclear,

$$dW_Q(t, z) = \sum_{i=1}^{\infty} q_i^{1/2} e_i(x)\, dW_i(t),$$

where $W_1, W_2, \dots$ are independent standard Brownian motions and $\{e_i, i = 1, 2, \dots\}$ is a complete orthonormal system in $L^2([0, 1])$, which consists of eigenvectors of Q. We denote q_i as the eigenvalue corresponding to e_i. For simplicity, we consider a special covariance operator $Q = (1 - \Delta)^{-1}$ and a complete orthonormal system $e_i := \sin i\pi x,\ i = 1, 2, \dots$

with $\lambda_i = (\pi i)^2$, $i = 1, 2, \ldots$ In this case, the corresponding eigenvalues $\{e_i,\ i = 1, 2, \ldots\}$ are $q_i := (1+\lambda_i)^{-1}$, $i = 1, 2 \ldots$, that is,

$$Qe_i = q_i e_i = (1+\lambda_i)^{-1} e_i,\ i = 1, 2 \ldots.$$

We define a solution $u(t,x)$ to the problem (1) as a formal sum (see [1])

$$u(t,x) = \sum_{i=1}^{\infty} u_i(t) e_i(x), \quad i = 1, 2 \ldots,$$

where the Fourier coefficient $u_i(t)$, $i = 1, 2 \ldots$ follows the dynamics of Ornstein–Uhlenbeck processes as follows:

$$du_i(t) = -\lambda_i \theta u_i(t) dt + \frac{1}{\sqrt{\lambda_i + 1}} dW_i(t), \tag{2}$$

with initial condition

$$u_i(0) = v_i.$$

Here, the v_i, $i = 1, 2 \ldots$ are determined by

$$f(x) = \sum_{i=0}^{\infty} v_i e_i(x), \qquad v_i = \int_0^1 f(x) e_i(x) dx, \quad i = 1, 2 \ldots.$$

It can be shown (see [1]) that $u(t,x)$ belongs to $L^2([0,T] \times \Omega;\ L^2([0,1]))$ together with its derivative in x. It vanishes at 0 and 1 and its norm in $L^2([0,1])$ is continuous in t. In addition, $u(t,x)$ is the only solution to (1) with the above properties. Let Π^N be the finite dimensional subspace of $L^2(\Omega)$ generated by $\{e_1, \ldots, e_N\}$. The likelihood ratio of the projection of the solution $u(t,x)$ onto the subspace Π^N (see [2,3])

$$u^N(t,x) = \sum_{i=1}^{N} u_i(t) e_i(x)$$

can be expressed as follows:

$$\begin{aligned} &\frac{dP^N_{\theta_0+\theta}}{dP^N_{\theta_0}} \left(u^N\right) \\ &= \exp\left\{ -\sum_{i=1}^{N} \lambda_i(\lambda_i+1) \left[\theta \int_0^T u_i(t)(du_i(t) + \theta_0 \lambda_i u_i(t) dt) + \frac{1}{2}\theta^2 \lambda_i \int_0^T u_i^2(t) dt \right] \right\}, \end{aligned}$$

where P_θ denotes the probability measure on $C([0,T])$ generated by the u^N.

By maximizing the log likelihood ratio with respect to the parameter θ, we obtain the Maximum Likelihood Estimator (MLE) $\widehat{\theta}_{N,T}$ for θ based on u^N as follows:

$$\widehat{\theta}_{N,T} = -\frac{\sum_{i=1}^{N} \lambda_i \sqrt{1+\lambda_i} \int_0^T u_i(s) du_i(s)}{\sum_{i=1}^{N} \lambda_i^2 (1+\lambda_i) \int_0^T u_i^2(s) ds}, \quad N \geq 1, T > 0. \tag{3}$$

Moreover, using (2) and (3), we can write

$$\theta - \widehat{\theta}_{N,T} \quad = \quad \frac{\sum_{i=1}^{N} \lambda_i \sqrt{1+\lambda_i} \int_0^T u_i(s) dW_i(s)}{\sum_{i=1}^{N} \lambda_i^2 (1+\lambda_i) \int_0^T u_i^2(s) ds}, \quad N \geq 1, T > 0. \tag{4}$$

Recently, several papers provided explicit upper bounds for the Kolmogorov distance for the rate of convergence for the central limit theorem of estimators for coefficients in stochastic Gaussian models, see, e.g., [4–8].

The purpose of this paper is to derive upper bounds of the Wasserstein distance for the rate of convergence of the distribution of the MLE $\widehat{\theta}_{N,T}$ when $N \to \infty$ and/or $T \to \infty$. Upper bounds of the Kolmogorov distance for the central limit theorem of the MLE $\widehat{\theta}_{N,T}$, as $N \to \infty$ and T fixed, are provided in [4,9]. Let us describe what is proved in this direction. In [9], Mishra and Prakasa Rao proved that there exists a constant $C_{\theta,T}$ depending on $\theta, \|f\|^2_{L^2((0,1))}$ and T such that, for any $\gamma > 0$ and $N \geq N_0$, depending on θ and T,

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\varphi_N(\theta)}\left(\widehat{\theta}_{N,T}-\theta\right)\leq z\right)-\mathbb{P}(Z\leq z)\right| \leq C_{\theta,T}\frac{\left(1+\sqrt{T}\right)N^{3+\gamma}}{TN^3+\sum_{k=1}^N k^4 v_k^2}+\frac{3}{N^{\frac{\gamma}{2}}}, \quad (5)$$

where $Z \sim \mathcal{N}(0,1)$ denotes a standard normal random variable and the normalizing factor $\varphi_N(\theta)$ is

$$\varphi_N(\theta) = \frac{1}{2\theta}\sum_{i=1}^N \lambda_i(\lambda_i+1)\left\{v_i^2\left(1-e^{-2\theta\lambda_i T}\right)+\frac{T}{\lambda_i+1}\right\}.$$

Moreover, in ([9], Remark 4.4), Mishra and Prakasa Rao proved that, if $\sum_{k=1}^N k^4 v_k^2 \geq g(N) = O(N^5)$, then the upper bound given by (5) is of order $O(N^{\gamma-2}) + O\left(N^{-\gamma/2}\right)$, and, in such case, the upper bound can be obtained to be of order $O\left(N^{-2/3}\right)$ by choosing $\gamma = 4/3$. However, if $\sum_{k=1}^N k^4 v_k^2 \leq g(N) = O(N^3)$ (for example, $f = 0$ i.e., $v_i = 0$ for all $i = 1, 2, \ldots$), then the upper bound in (5) is given by

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\varphi_N(\theta)}\left(\widehat{\theta}_{N,T}-\theta\right)\leq z\right)-\mathbb{P}(Z\leq z)\right| \leq C_{\theta,T}N^{\gamma}. \quad (6)$$

In this case, we notice that the upper bound of the Kolmogorov distance given by (6) does not show that the normal approximation of the MLE $\widehat{\theta}_{N,T}$ holds. Hence, the sharp upper bound is needed to prove the normal approximation through the Kolmogorov distance. This problem has been solved by Kim and Park in [4], where they improved the bound in (5) to that converging to zero when $N \to \infty$ and T fixed, by using techniques based the combination Malliavin calculus and Stein's method. More precisely, they proved, in the case when $f = 0$, that, for sufficiently large N, there exists a constant $C_{\theta,T}$ depending on θ and T such that

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\sqrt{\varphi_N(\theta)}\left(\widehat{\theta}_{N,T}-\theta\right)\leq z\right)-\mathbb{P}(Z\leq z)\right| \leq C_{\theta,T}\frac{1}{N},$$

where the normalizing factor $\varphi_N(\theta)$ is given by $\varphi_N(\theta) = \frac{T}{2\theta}\sum_{i=1}^N \lambda_i$.

The goal of this paper is to provide Berry–Esseen bounds in Wasserstein distance for the MLE $\widehat{\theta}_{N,T}$ when $N \to \infty$ and/or $T \to \infty$. Let us first recall that the estimator $\widehat{\theta}_{N,T}$ is strongly consistent and asymptotically normal in three asymptotic regimes: for the two cases $N \to \infty$ and T fixed, and $T \to \infty$ and N fixed, see, for instance, [10] and references therein, and for the case when both $N, T \to \infty$, see [11]. However, the study of the asymptotic distribution of an estimator is not very useful in general for practical purposes unless the rate of convergence is known. To the best of our knowledge, no results of Berry–Esseen bounds are known for MLE $\widehat{\theta}_{N,T}$ in terms of Wasserstein distance when $N \to \infty$ and/or $T \to \infty$. Recall that, if X, Y are two real-valued integrable random variables, then the Wasserstein distance between the law of X and the law of Y is given by

$$d_W(X,Y) := \sup_{f\in Lip(1)} |\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]|,$$

where $Lip(1)$ is the set of all Lipschitz functions with Lipschitz constant $\leqslant 1$.

In what follows, in order to simplify the notation, we set $u(0,x) = f(x) = 0$ and hence $u_i(0) = 0$ for all $i \geq 1$. The following are the main results of this paper.

- Case 1: $N \to \infty$ and T fixed. Then, there exists a positive constant $C_{\theta,T}$ depending only on θ and T such that, for every $N \geq 1$,
$$d_W\left(N^{\frac{3}{2}}\left(\theta - \widehat{\theta}_{N,T}\right), \mathcal{N}\left(0, \frac{6\theta}{\pi^2 T}\right)\right) \quad \leq \quad \frac{C_{\theta,T}}{N^{\frac{3}{2}}}. \tag{7}$$
In particular, as $N \to \infty$,
$$N^{\frac{3}{2}}\left(\theta - \widehat{\theta}_{N,T}\right) \xrightarrow{law} \mathcal{N}\left(0, \frac{6\theta}{\pi^2 T}\right).$$
- Case 2: $T \to \infty$ and N fixed. Then, there exists a positive constant $C_{\theta,N}$ depending only on θ and N such that, for every $T \geq 1$,
$$d_W\left(\sqrt{T}\left(\theta - \widehat{\theta}_{N,T}\right), \mathcal{N}\left(0, \frac{2\theta}{\sum_{i=1}^N \lambda_i}\right)\right) \quad \leq \quad \frac{C_{\theta,N}}{\sqrt{T}}. \tag{8}$$
In particular, as $T \to \infty$,
$$\sqrt{T}\left(\theta - \widehat{\theta}_{N,T}\right) \xrightarrow{law} \mathcal{N}\left(0, \frac{2\theta}{\sum_{i=1}^N \lambda_i}\right).$$
- Case 3: $N \to \infty$ and $T \to \infty$. Then, there exists a positive constant C_θ depending only on θ such that, for every $N \geq 1$ and $T \geq 1$,
$$d_W\left(\sqrt{T}N^{\frac{3}{2}}\left(\theta - \widehat{\theta}_{N,T}\right), \mathcal{N}\left(0, \frac{6\theta}{\pi^2}\right)\right) \quad \leq \quad \frac{C_\theta}{\sqrt{T}N^{\frac{3}{2}}}. \tag{9}$$
In particular, as $N, T \to \infty$,
$$\sqrt{T}N^{\frac{3}{2}}\left(\theta - \widehat{\theta}_{N,T}\right) \xrightarrow{law} \mathcal{N}\left(0, \frac{6\theta}{\pi^2}\right).$$

Remark 1. *Note that, in Case 1, $N \to \infty$ and T fixed, we obtained the upper bound $O(1/N^{\frac{3}{2}})$ in Wasserstein distance for normal approximation of the MLE $\widehat{\theta}_{N,T}$, while the upper bound in Kolmogorov distance obtained by Kim and Park [4] is $O(1/N)$.*

The paper is organized as follows: Section 2 contains some preliminaries presenting the tools needed from the analysis on Wiener space, including Wiener chaos calculus and Malliavin calculus. In Section 3, we derive upper bounds for the rate of convergence of the distribution of the MLE $\widehat{\theta}_{N,T}$ when $N \to \infty$ and/or $T \to \infty$, see Theorem 1. We also included in this section a lemma that plays an important role in the proof of Theorem 1.

2. Preliminaries

In this section, we recall some elements from the analysis on Wiener space and the Malliavin calculus for Gaussian processes that we will need in the paper. For more details, we refer the reader to [12,13]. Let $\mathcal{H} := L^2([0,T])$ and let $\{W(\varphi), \varphi \in \mathcal{H}\}$ be a Wiener process that is a centered Gaussian family of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbf{E}(W(\varphi)W(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$. In this case, we denote $W_t := W(1_{[0,t]})$ and $\int_0^T \varphi(s) dW(s) =: W(\varphi)$ for every $\varphi \in \mathcal{H}$.

The Wiener chaos $\mathcal{H}_p$ of order p is defined as the closure in $L^2(\Omega)$ of the linear span of the random variables $H_p(W(\varphi))$, where $\varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1$ and H_p is the Hermite polynomial of degree p.

- **Multiple Wiener–Itô integral.** The multiple Wiener stochastic integral I_p with respect to W of order p is defined as an isometry between the Hilbert space $\mathcal{H}^{\odot p} =$

$L^2_{sym}\left([0,T]^p\right)$ (symmetric tensor product) equipped with the norm $\sqrt{p!}\|\cdot\|_{\mathcal{H}^{\otimes p}}$ and the Wiener chaos of order p under $L^2(\Omega)$'s norm, that is, the multiple Wiener stochastic integral of order p:

$$I_p : \left(\mathcal{H}^{\odot p}, \sqrt{p!}\|\cdot\|_{\mathcal{H}^{\otimes p}}\right) \longrightarrow \left(\mathcal{H}_p, L^2(\Omega)\right)$$

is a linear isometry defined by $I_p(f^{\otimes p}) = H_p(W(f))$.

- **The Wiener chaos expansion.** Let $F \in L^2(\Omega)$; then, there exists a unique sequence of functions f_p in $\mathcal{H}^{\odot p}$ such that

$$F = \mathbf{E}[F] + \sum_{p=1}^{\infty} I_p(f_p),$$

where the terms $I_p(f_p)$ are all mutually orthogonal in $L^2(\Omega)$ and

$$\mathbf{E}\left[I_p(f_p)^2\right] = p!\|f_p\|^2_{\mathcal{H}^{\otimes p}}.$$

- **Product formula and contractions.** Let $p,\ q \geq 1$ be integers and $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$; then,

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r} I_{p+q-2r}(f\widetilde{\otimes}_r g), \tag{10}$$

where $f \otimes_r g$ is the contraction of f and g of order r, which is an element of $\mathcal{H}^{\otimes(p+q-2r)}$ defined by

$$\begin{aligned}(f \otimes_r g)&(s_1,\ldots,s_{p-r},t_1,\ldots,t_{q-r})\\ &:= \int_{[0,T]^{p+q-2r}} f(s_1,\ldots,s_{p-r},u_1,\ldots,u_r)g(t_1,\ldots,t_{q-r},u_1,\ldots,u_r)\,du_1\cdots du_r.\end{aligned}$$

Its symmetrization is denoted by $f\widetilde{\otimes}_r g$, where the symmetrization $\tilde{f}$ of a function f is defined by $\tilde{f}(x_1,\ldots,x_p) = \frac{1}{p!}\sum_{\sigma} f(x_{\sigma(1)},...,x_{\sigma(p)})$ where the sum runs over all permutations σ of $\{1,...,p\}$. The special case for $p = q = 1$ in (10) is particularly handy, and can be written in its symmetrized form:

$$I_1(f)I_1(g) = 2^{-1}I_2(f\otimes g + g\otimes f) + \langle f, g\rangle_{\mathcal{H}}. \tag{11}$$

where $f \otimes g$ means the tensor product of f and g.

- **Hypercontractivity property in Wiener chaos**. Fix $q \geq 1$. For any $p \geq 2$, there exists $c_{p,q}$ depending only on p and q such that, for every $F \in \oplus_{l=1}^{q}\mathcal{H}_l$,

$$\left(\mathbf{E}\left[|F|^p\right]\right)^{1/p} \leqslant c_{p,q}\left(\mathbf{E}\left[|F|^2\right]\right)^{1/2}. \tag{12}$$

It should be noted that the constants $c_{p,q}$ above are known with some precision when $F \in \mathcal{H}_q$: by ([12], Corollary 2.8.14), $c_{p,q} = (p-1)^{q/2}$.

- **Optimal fourth moment theorem**. Let Z denote the standard normal law. Let a sequence $X : X_n \in \mathcal{H}_q$, such that $EX_n = 0$ and $Var[X_n] = 1$, and assume X_n converges to a normal law in distribution, which is equivalent to $\lim_n E\left[X_n^4\right] = 3$ (this equivalence, proved originally in [14], is known as the *fourth moment theorem*). Then, we have the optimal estimate for total variation distance $d_{TV}(X_n, Z)$, known as the optimal 4th moment theorem, proved in [15]. This optimal estimate also holds with

Wasserstein distance $d_W(X_n, Z)$, see ([16], Remark 2.2), as follows: there exist two constants $c, C > 0$ depending only on the sequence X but not on n, such that

$$c\max\left\{E\left[X_n^4\right]-3,\left|E\left[X_n^3\right]\right|\right\} \leqslant d_W(X_n,Z) \leqslant C\max\left\{E\left[X_n^4\right]-3,\left|E\left[X_n^3\right]\right|\right\}. \quad (13)$$

Moreover, we recall that, for a standardized random variable X, i.e., with $\mathbb{E}[X]=0$ and $\mathbb{E}\left[X^2\right]=1$, the third and fourth cumulants are, respectively,

$$\kappa_3(X) := \mathbb{E}\left[X^3\right], \quad \kappa_4(X) := \mathbb{E}\left[X^4\right]-3.$$

Fix $T \geq 1$ and an integer $N \geq 1$. Recall that, if $\mathcal{H} = L^2\left([0,T],\mathbb{R}^N\right)$ and $W = (W_1, W_2, \ldots, W_N)$ with $W_1, W_2, \ldots, W_N$ are independent standard Brownian motions; then, for every $h = \left(h^1, \ldots, h^N\right) \in \mathcal{H}$, the multiple integral $I_1(h)$ is defined by

$$I_1(h) := I_1^W(h) = \sum_{i=1}^N I_1^{W_i}(h^i) = \sum_{i=1}^N \int_0^T h_s^i dW_i(s), \quad (14)$$

and

$$\|h\|_{\mathcal{H}}^2 = \sum_{i=1}^N \int_0^T (h_s^i)^2 ds.$$

Moreover, if $g \in \mathcal{H}^{\otimes 2}$, then the third and fourth cumulants for $I_2(g)$ satisfy the following (see (6.2) and (6.6) in [17], respectively):

$$k_3(I_2(g)) = E[(I_2(g))^3] = 8\langle g, g \otimes_1 g\rangle_{\mathcal{H}^{\otimes 2}} \quad (15)$$

and

$$\begin{aligned} |k_4(I_2(g))| &= 16\left(\|g\otimes_1 g\|_{\mathcal{H}^{\otimes 2}}^2 + 2\|g\widetilde{\otimes}_1 g\|_{\mathcal{H}^{\otimes 2}}^2\right) \\ &\leq 48\|g\otimes_1 g\|_{\mathcal{H}^{\otimes 2}}^2. \end{aligned} \quad (16)$$

Throughout the paper, $Z \sim \mathcal{N}(0,1)$ denotes a standard normal random variable, while $\mathcal{N}(\mu,\sigma^2)$ denotes a normal variable with mean μ and variance σ^2.

3. Berry–Esseen Bounds for the MLE

Recall that, in what follows, in order to simplify the notation, we set $u(0,x) = f(x) = 0$ and hence $u_i(0) = 0$ for all $i \geq 1$. In this case, since the Equation (2) is linear, it is immediate to solve it explicitly; one then gets the following formula:

$$u_i(t) = \frac{e^{-\theta\lambda_i t}}{\sqrt{1+\lambda_i}} \int_0^t e^{\theta\lambda_i s} dW_i(s), \quad i = 1, \ldots, N. \quad (17)$$

Let us introduce the following sequences:

$$S_{N,T} := \sum_{i=1}^N \lambda_i\sqrt{1+\lambda_i} \int_0^T u_i(s) dW_i(s), \quad N \geq 1, T > 0, \quad (18)$$

and

$$\varphi_{N,T} := \frac{T}{2\theta}\sum_{i=1}^N \lambda_i = \frac{\pi^2 T}{2\theta}\frac{N(N+1)(2N+1)}{6}, \quad N \geq 1, T > 0. \quad (19)$$

Combining (4) and (18), we have, for every $N \geq 1, T > 0$,

$$\theta - \widehat{\theta}_{N,T} = \frac{\sum_{i=1}^{N} \lambda_i \sqrt{1+\lambda_i} \int_0^T u_i(s) dW_i(s)}{\sum_{i=1}^{N} \lambda_i^2 (1+\lambda_i) \int_0^T u_i^2(s) ds} = \frac{S_{N,T}}{\langle S_{N,T} \rangle}. \tag{20}$$

Using (14), we can write

$$S_{N,T} = \frac{1}{2} I_2(f_{N,T}) = \frac{1}{2} \sum_{i=1}^{N} I_2^{W_i}(f^i), \tag{21}$$

where

$$f_{N,T} := \left(f^1, \ldots, f^N\right), \quad f^i(s,t) := \lambda_i e^{-\theta\lambda_i |t-s|} 1_{[0,T]^2}(s,t), \quad i = 1, \ldots, N.$$

On the other hand, using the product formula (11),

$$\begin{aligned} \left(\int_0^s e^{\theta\lambda_i r} dW_i(r) \right)^2 &= I_2^{W_i}\left(e^{\theta\lambda_i(u+v)} 1_{[0,s]^2}(u,v) \right) + \int_0^s e^{2\theta\lambda_i r} dr \\ &= I_2^{W_i}\left(e^{\theta\lambda_i(u+v)} 1_{[0,s]^2}(u,v) \right) + \frac{1}{2\theta\lambda_i}\left(e^{2\theta\lambda_i s} - 1 \right). \end{aligned}$$

Thus, for every $i = 1, \ldots, N$,

$$\begin{aligned} u_i^2(s) &= \frac{e^{-2\theta\lambda_i s}}{\lambda_i + 1} \left(\int_0^s e^{\theta\lambda_i r} dW_i(r) \right)^2 \\ &= \frac{e^{-2\theta\lambda_i s}}{\lambda_i + 1} I_2^{W_i}\left(e^{\theta\lambda_i(u+v)} 1_{[0,s]^2}(u,v) \right) + \frac{1}{2\theta\lambda_i(\lambda_i+1)} \left(1 - e^{-2\theta\lambda_i s} \right). \end{aligned}$$

This and the linearity of $I_2^{W_i}$ imply

$$\begin{aligned} \int_0^T u_i^2(s) ds &= \frac{1}{\lambda_i + 1} I_2^{W_i}\left(e^{\theta\lambda_i(u+v)} \int_{u \vee v}^T e^{-2\theta\lambda_i s} ds \right) + \frac{1}{2\theta\lambda_i(\lambda_i+1)} \left(T + \frac{e^{-2\theta\lambda_i T} - 1}{2\theta\lambda_i} \right) \\ &= I_2^{W_i}\left(\frac{e^{-\theta\lambda_i |u-v|}}{2\theta\lambda_i(\lambda_i+1)} 1_{[0,T]^2}(u,v) \right) - I_2^{W_i}\left(\frac{e^{-2\theta\lambda_i T} e^{\theta\lambda_i(u+v)}}{2\theta\lambda_i(\lambda_i+1)} 1_{[0,T]^2}(u,v) \right) \\ &\quad + \frac{1}{2\theta\lambda_i(\lambda_i+1)} \left(T + \frac{e^{-2\theta\lambda_i T} - 1}{2\theta\lambda_i} \right). \end{aligned}$$

Consequently,

$$\begin{aligned} \langle S_{N,T} \rangle &= \sum_{i=1}^{N} \lambda_i^2 (1+\lambda_i) \int_0^T u_i^2(s) ds \\ &= \sum_{i=1}^{N} \lambda_i \left[I_2^{W_i}\left(\frac{e^{-\theta\lambda_i |u-v|}}{2\theta} 1_{[0,T]^2}(u,v) \right) - I_2^{W_i}\left(\frac{e^{-2\theta\lambda_i T} e^{\theta\lambda_i(u+v)}}{2\theta} 1_{[0,T]^2}(u,v) \right) \right. \\ &\qquad \left. + \frac{1}{2\theta} \left(T + \frac{e^{-2\theta\lambda_i T} - 1}{2\theta\lambda_i} \right) \right] \\ &= I_2\left(\frac{1}{2\theta} f_{N,T} \right) - I_2(h_{N,T}) + \delta_{N,T} \\ &= \frac{S_{N,T}}{\theta} - I_2(h_{N,T}) + \delta_{N,T}, \end{aligned} \tag{22}$$

where

$$h_{N,T} := \left(h^1,\ldots,h^N\right), \quad h^i(u,v) := \frac{\lambda_i e^{-2\theta\lambda_i T} e^{\theta\lambda_i(u+v)}}{2\theta} 1_{[0,T]^2}(u,v),$$

and

$$\begin{aligned}\delta_{N,T} &:= \frac{T}{2\theta}\sum_{i=1}^{N}\lambda_i + \frac{1}{4\theta^2}\sum_{i=1}^{N}\left(e^{-2\theta\lambda_i T}-1\right)\\ &= \varphi_{N,T} + \frac{1}{4\theta^2}\sum_{i=1}^{N}\left(e^{-2\theta\lambda_i T}-1\right).\end{aligned}$$

According to (20)–(22), we can write, for every $N \geq 1, T > 0$,

$$\theta - \widehat{\theta}_{N,T} = \frac{\frac{1}{2}I_2(f_{N,T})}{\frac{1}{2\theta}I_2(f_{N,T}) - I_2(h_{N,T}) + \delta_{N,T}}. \tag{23}$$

Lemma 1. *Let $\alpha > 0$ and $V_t = e^{-\alpha t}\int_0^t e^{\alpha s}dW_s$, where $\{W_t, t \geq 0\}$ is a Wiener process. Let $\mathcal{F}$ denote the sigma-field generated by W, that is, $\mathcal{F}_t = \sigma\{W_u, u \leq t\}$. Then, for every $0 \leq a < b$,*

$$\int_a^b E\left(V_t^2 \mid \mathcal{F}_a^W\right)dt \geq \mu_\alpha(b-a) := \frac{b-a}{2} + \frac{e^{-2\alpha(b-a)}-1}{4\alpha} > 0, \tag{24}$$

where the function $\mu_\alpha(x)$ is increasing and hence $\mu_\alpha(x) > \mu_\alpha(0) = 0$ for all $x > 0$.

Furthermore, for every $p, T_0 > 0$, there exists a positive constant C_{θ,T_0} depending only on θ and T_0 such that

$$\sup_{T\geq T_0} \mathbb{E}\left[\left(\frac{1}{\varphi_{N,T}}\sum_{i=1}^{N}\lambda_i^2(1+\lambda_i)\int_0^T u_i^2(s)ds\right)^{-p}\right] < C_{\theta,T_0} < \infty, \tag{25}$$

where the processes u_i, $i = 1\ldots,N$ are given by (17).

Proof. We will use similar arguments as in ([16], Proposition 6.3). Let $0 \leq a < b$. Using the fact that, for every $t > a$, $\int_a^t e^{-\alpha(t-u)}dW_u$ is independent of $\mathcal{F}_a^W$, we have

$$\begin{aligned}\int_a^b \mathbb{E}\left(V_t^2 \mid \mathcal{F}_a^W\right)dt &= \int_a^b E\left(\left(\int_0^a e^{-\alpha(t-u)}dW_u\right)^2 \mid \mathcal{F}_a^W\right)dt\\ &\quad + \int_a^b \mathbb{E}\left(\left(\int_a^t e^{-\alpha(t-u)}dW_u\right)^2 \mid \mathcal{F}_a^W\right)dt\\ &\geq \int_a^b \mathbb{E}\left(\left(\int_a^t e^{-\alpha(t-u)}dW_u\right)^2\right)dt\\ &= \int_a^b\int_a^t e^{-2\alpha(t-u)}dudt\\ &= \int_a^b \frac{1-e^{-2\alpha(t-a)}}{2}dt\\ &= \frac{b-a}{2} + \frac{e^{-2\alpha(b-a)}-1}{4\alpha}\\ &= \mu_\alpha(b-a).\end{aligned}$$

Moreover, since $\mu_\alpha'(x) = \frac{1-e^{-2\alpha x}}{2} > 0$ for all $x > 0$, the function $\mu_\alpha(x)$ is increasing. Thus, the proof (24) is complete.

Let us now prove (25). Fix $p, T_0 > 0$, and let m be a positive integer such that $\frac{m}{2p} > 1$. Using Hölder's inequality, we have, for all $x_1,\ldots,x_m \geq 0$,

$$\left(\sum_{k=1}^{m} x_k\right)^{-p} = \left(\int_0^\infty e^{-t\sum_{k=1}^{m} x_k}dt\right)^p \leq \prod_{k=1}^{m}\left(\int_0^\infty e^{-mtx_k}dt\right)^{\frac{p}{m}} = m^{-p}\prod_{k=1}^{m} x_k^{-\frac{p}{m}}.$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_0^T u_i^2(s)ds\right)^{-p}\right] \\
&= \mathbb{E}\left[\left(\sum_{k=1}^{m}\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m} u_i^2(s)ds\right)^{-p}\right] \\
&\leq m^{-p}\mathbb{E}\left[\prod_{k=1}^{m}\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m} u_i^2(s)ds\right)^{-p/m}\right].
\end{aligned} \tag{26}$$

Using the fact that if $X \geq 0$, almost surely, $\mathbb{E}(X \mid \mathcal{F}) = \int_0^\infty P(X \geq x \mid \mathcal{F})dx$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m} u_i^2(s)ds\right)^{-p/m} \mid \mathcal{F}^W_{(k-1)T/m}\right] \\
&= \int_0^\infty \mathbb{P}\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m} u_i^2(s)ds \leq x^{-m/p} \mid \mathcal{F}^W_{(k-1)T/m}\right)dx \\
&\leq 1+\int_1^\infty \mathbb{P}\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m} u_i^2(s)ds \leq x^{-m/p} \mid \mathcal{F}^W_{(k-1)T/m}\right)dx.
\end{aligned} \tag{27}$$

Applying Carbery–Wright Inequality, there is a universal constant $c > 0$ such that, for any $\varepsilon > 0$, we can write

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m} u_i^2(s)ds \leq \varepsilon \mid \mathcal{F}^W_{(k-1)T/m}\right) \\
&\leq \frac{c\sqrt{\varepsilon}}{\mathbb{E}\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m} u_i^2(s)ds \mid \mathcal{F}^W_{(k-1)T/m}\right)}.
\end{aligned} \tag{28}$$

Using (24) for $\alpha = \theta\lambda_i$ and the fact that, for any fixed $x > 0$, the function $y \longrightarrow \mu_y(x)$ is increasing on $(0, \infty)$. Moreover, since $\frac{\mu(x)}{x}$ is positive and continuous on $(0, \infty)$ and $\frac{\mu_{\theta\lambda_1}(x)}{x} \to 0$ as $x \to \infty$, we have $0 < \sup_{x \geq \frac{T_0}{m}} \frac{\mu_{\theta\lambda_1}(x)}{x} < \infty$. Combining these facts, we get, for every $T \geq T_0$, that

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m} u_i^2(s)ds \mid \mathcal{F}^W_{(k-1)T/m}\right) &\geq \sum_{i=1}^{N}\frac{\lambda_i^2}{\varphi_{N,T}}\mu_{\theta\lambda_i}\left(\frac{T}{m}\right) \\
&\geq \sum_{i=1}^{N}\frac{2\theta\lambda_i^2}{T\sum_{i=1}^{N}\lambda_i}\mu_{\theta\lambda_i}\left(\frac{T}{m}\right) \\
&\geq \lambda_1\mu_{\theta\lambda_1}\left(\frac{T}{m}\right)\sum_{i=1}^{N}\frac{2\theta\lambda_i}{T\sum_{i=1}^{N}\lambda_i} \\
&= \frac{2\theta\lambda_1}{T}\mu_{\theta\lambda_1}\left(\frac{T}{m}\right) \\
&\geq \frac{2\theta\lambda_1}{m}\sup_{x\geq\frac{T_0}{m}}\frac{\mu_{\theta\lambda_1}(x)}{x} > 0.
\end{aligned} \tag{29}$$

Therefore, combining (27)–(29), we deduce that, for every $T \geq T_0$,

$$\mathbb{E}\left[\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m}u_i^2(s)ds\right)^{-p/m} \mid \mathcal{F}^W_{(k-1)T/m}\right] \leq 1+\frac{cm}{2\theta\lambda_1\sup_{x\geq\frac{T_0}{m}}\frac{\mu(x)}{x}}\int_1^\infty x^{-\frac{m}{2p}}dx.$$

Thus,

$$\gamma_{m,T_0} := \sup_{T\geq T_0, 1\leq k\leq m}\mathbb{E}\left[\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m}u_i^2(s)ds\right)^{-p/m} \mid \mathcal{F}^W_{(k-1)T/m}\right] < \infty. \quad (30)$$

Consequently, it follows from (26) and (30) that, for all $T \geq T_0$,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_0^T u_i^2(s)ds\right)^{-p}\right] \\
\leq\; & m^{-p}\mathbb{E}\left[\prod_{k=1}^{m}\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m}u_i^2(s)ds\right)^{-p/m}\right] \\
=\; & m^{-p}\mathbb{E}\Bigg[\prod_{k=1}^{m-1}\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m}u_i^2(s)ds\right)^{-p/m} \\
&\times\mathbb{E}\left\{\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(m-1)T/m}^{T}u_i^2(s)ds\right)^{-p/m} \mid \mathcal{F}_{(m-1)T/m}\right\}\Bigg] \\
=\; & m^{-p}\gamma_{m,T_0}\mathbb{E}\left[\prod_{k=1}^{m-1}\left(\sum_{i=1}^{N}\frac{\lambda_i^2(1+\lambda_i)}{\varphi_{N,T}}\int_{(k-1)T/m}^{kT/m}u_i^2(s)ds\right)^{-p/m}\right] \\
\leq\; & m^{-p}(\gamma_{m,T_0})^m < \infty,
\end{aligned}$$

which completes the proof of (25). □

Theorem 1. *Suppose that $\theta > 0$. Let $\widehat{\theta}_{N,T}$ be the MLE given by (3), and let $\varphi_{N,T}(\theta)$ be the normalizing factor given by (19). Then, there exists a positive constant C_θ depending only on θ such that, for any integer $N \geq 1$ and any real number $T \geq 1$,*

$$d_W\left(\sqrt{\varphi_{N,T}}\left(\theta-\widehat{\theta}_{N,T}\right), Z\right) \leq \frac{C_\theta}{\sqrt{\varphi_{N,T}}}, \quad (31)$$

where Z is standard Normal law.

Consequently, the estimates (7)–(9) are obtained.

Proof. It follows from (21) that

$$\begin{aligned}
\mathbb{E}[S_{N,T}^2] = \mathbb{E}[(\frac{1}{2}I_2(f_{N,T}))^2] &= \frac{1}{2}\|f_{N,T}\|^2_{\mathcal{H}^{\otimes 2}} = \frac{1}{2}\sum_{i=1}^{N}\lambda_i^2\int_0^T\int_0^T e^{-2\theta\lambda_i|t-s|}dsdt \\
&= \sum_{i=1}^{N}\lambda_i^2\int_0^T\int_0^t e^{-2\theta\lambda_i(t-s)}dsdt \\
&= \sum_{i=1}^{N}\lambda_i^2\int_0^T\int_0^t e^{-2\theta\lambda_i x}dxdt \\
&= \sum_{i=1}^{N}\lambda_i^2\int_0^T \frac{1-e^{-2\theta\lambda_i T}}{2\theta\lambda_i}dt \\
&= \varphi_{N,T} + \frac{1}{4\theta^2}\sum_{i=1}^{N}\left(e^{-2\theta\lambda_i T}-1\right) = \delta_{N,T}.
\end{aligned}$$

Thus,

$$\mathbb{E}\left[\left(\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}\right)^2\right] = \frac{\delta_{N,T}}{\varphi_{N,T}} = 1 + \frac{1}{4\theta^2\varphi_{N,T}}\sum_{i=1}^{N}\left(e^{-2\theta\lambda_i T}-1\right). \tag{32}$$

Combining (19) and (32), we get

$$\left|\mathbb{E}\left[\left(\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}\right)^2\right]-1\right| = \left|\frac{\delta_{N,T}}{\varphi_{N,T}}-1\right| \leq \frac{3}{2\theta\pi^2 TN^2}. \tag{33}$$

Notice also that, from (32), we have

$$\mathbb{E}\left[\left(\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}\right)^2\right] \leq 1. \tag{34}$$

Moreover, since $\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}$ belongs to $\mathcal{H}_2$, it follows from (12) and (34) that

$$\left\|\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}\right\|_{L^4(\Omega)} \leq 3\left\|\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}\right\|_{L^2(\Omega)} \leq 3. \tag{35}$$

On the other hand, since

$$\begin{aligned}
\|h_{N,T}\|^2_{\mathcal{H}} &= \sum_{i=1}^{N}\frac{\lambda_i^2}{4\theta^2}e^{-4\theta\lambda_i T}\int_0^T\int_0^T e^{2\theta\lambda_i(u+v)}dudv \\
&= \sum_{i=1}^{N}\frac{1}{16\theta^4}\left(1-e^{-2\theta\lambda_i T}\right)^2 \\
&\leq \frac{N}{16\theta^4},
\end{aligned}$$

we obtain

$$\mathbb{E}\left[\left(\frac{I_2(h_{N,T})}{\varphi_{N,T}}\right)^2\right] \leq \frac{9}{2\theta^2\pi^4T^2N^5}. \tag{36}$$

Therefore, using (19), (22), (33), (34) and (36), there exists a positive constant C_θ depending only on θ such that, for every $N, T \geq 1$,

$$\begin{aligned}
\left\| \frac{\langle S_{N,T}\rangle}{\varphi_{N,T}} - 1 \right\|_{L^2(\Omega)} &\leq \left\| \frac{S_{N,T}}{\theta \varphi_{N,T}} \right\|_{L^2(\Omega)} + \left\| \frac{I_2(h_{N,T})}{\varphi_{N,T}} \right\|_{L^2(\Omega)} + \left| \frac{\delta_{N,T}}{\varphi_{N,T}} - 1 \right| \\
&\leq \frac{1}{\theta\sqrt{\varphi_{N,T}}} + \frac{3}{\sqrt{2}\theta\pi^2 T N^{5/2}} + \frac{3}{2\theta\pi^2 T N^2} \\
&\leq \frac{C_\theta}{\sqrt{\varphi_{N,T}}}. \qquad (37)
\end{aligned}$$

Using (15), we have

$$\begin{aligned}
k_3\left(\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}\right) &= k_3\left(\frac{I_2(f_{N,T})}{2\sqrt{\varphi_{N,T}}}\right) = \mathbb{E}\left[\left(\frac{I_2(f_{N,T})}{2\sqrt{\varphi_{N,T}}}\right)^3\right] = \sum_{i=1}^N \mathbb{E}\left[\left(\frac{I_2(f_N^i)}{2\sqrt{\varphi_{N,T}}}\right)^3\right] \\
&= \frac{1}{\varphi_{N,T}^{3/2}} \sum_{i=1}^N \left\langle f_N^i, f_N^i \otimes_1 f_N^i \right\rangle_{\mathcal{H}^{\otimes 2}} \\
&= \frac{1}{\varphi_{N,T}^{3/2}} \sum_{i=1}^N \lambda_i^3 \int_{[0,T]^3} e^{-\theta\lambda_i|x_1-x_2|} e^{-\theta\lambda_i|x_2-x_3|} e^{-\theta\lambda_i|x_3-x_1|} dx_1 dx_2 dx_3 \\
&= \frac{3!}{\varphi_{N,T}^{3/2}} \sum_{i=1}^N \lambda_i^3 \int_0^T dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 e^{-\theta\lambda_i(2x_3-2x_1)} \\
&= \frac{3}{\theta\varphi_{N,T}^{3/2}} \sum_{i=1}^N \lambda_i^2 \int_0^T \left(\frac{1-e^{-2\theta\lambda_i x_3}}{2\theta\lambda_i} - x_3 e^{-2\theta\lambda_i x_3} \right) dx_3 \\
&\leq \frac{3T}{2\theta^2 \varphi_{N,T}^{3/2}} \sum_{i=1}^N \lambda_i = \frac{3}{\theta\sqrt{\varphi_{N,T}}}. \qquad (38)
\end{aligned}$$

Using (16), straightforward calculations lead to

$$\begin{aligned}
\left| k_4\left(\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}\right) \right| &= \left| k_4\left(\frac{I_2(f_{N,T})}{2\sqrt{\varphi_{N,T}}}\right) \right| \\
&\leq \frac{3}{\varphi_{N,T}^2} \sum_{i=1}^N \| f_N^i \otimes_1 f_N^i \|^2_{\mathcal{H}^{\otimes 2}} \\
&= \frac{3}{\varphi_{N,T}^2} \sum_{i=1}^N \int_{[0,T]^4} f_N^i(x_1,x_2) f_N^i(x_2,x_3) f_N^i(x_3,x_4) f_N^i(x_4,x_1) dx_1 dx_2 dx_3 dx_4 \\
&= \frac{3}{\varphi_{N,T}^2} \sum_{i=1}^N 4!\lambda_i^4 \int_0^T dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 e^{-\theta\lambda_i(2x_4-2x_1)} \\
&\leq \frac{18}{\theta^2 \varphi_{N,T}}. \qquad (39)
\end{aligned}$$

Combining (13), (38) and (39), there exists a positive constant C_θ depending only on θ such that, for every $N, T \geq 1$,

$$d_W\left(\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}, Z\right) \leq \frac{C_\theta}{\theta\sqrt{\varphi_{N,T}}}. \qquad (40)$$

It follows from (20) that

$$\sqrt{\varphi_{N,T}}\left(\theta - \widehat{\theta}_{N,T}\right) = \frac{S_{N,T}/\sqrt{\varphi_{N,T}}}{\langle S_{N,T}\rangle/\varphi_{N,T}}. \qquad (41)$$

On the other hand, from (25), we have

$$\left\| \frac{\varphi_{N,T}}{\langle S_{N,T} \rangle} \right\|_{L^4(\Omega)} \leq C_\theta. \tag{42}$$

Using (35), (37) and (40)–(42), there exists a positive constant C_θ depending only on θ such that, for every $N, T \geq 1$,

$$\begin{aligned}
d_W\left(\sqrt{\varphi_{N,T}}\left(\theta - \widehat{\theta}_{N,T}\right), Z\right) &= d_W\left(\frac{S_{N,T}/\sqrt{\varphi_{N,T}}}{\langle S_{N,T}\rangle/\varphi_{N,T}}, Z\right) \\
&\leq d_W\left(\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}, Z\right) + d_W\left(\frac{S_{N,T}/\sqrt{\varphi_{N,T}}}{\langle S_{N,T}\rangle/\varphi_{N,T}}, \frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}\right) \\
&\leq d_W\left(\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}, Z\right) + \mathbb{E}\left|\frac{S_{N,T}/\sqrt{\varphi_{N,T}}}{\langle S_{N,T}\rangle/\varphi_{N,T}} - \frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}\right| \\
&\leq d_W\left(\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}, Z\right) + \left\|\frac{S_{N,T}/\sqrt{\varphi_{N,T}}}{\langle S_{N,T}\rangle/\varphi_{N,T}}\right\|_{L^2(\Omega)} \left\|1 - \frac{\langle S_{N,T}\rangle}{\varphi_{N,T}}\right\|_{L^2(\Omega)} \\
&\leq d_W\left(\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}, Z\right) + \left\|\frac{S_{N,T}}{\sqrt{\varphi_{N,T}}}\right\|_{L^4(\Omega)} \left\|\frac{\varphi_{N,T}}{\langle S_{N,T}\rangle}\right\|_{L^4(\Omega)} \left\|1 - \frac{\langle S_{N,T}\rangle}{\varphi_{N,T}}\right\|_{L^2(\Omega)} \\
&\leq \frac{C_\theta}{\sqrt{\varphi_{N,T}}}.
\end{aligned}$$

Therefore, the desired result is obtained. □

In this paper, we are interested in the rate of convergence for the central limit theorem of the maximum likelihood estimator of the drift coefficient for a stochastic partial differential equation based on continuous time observations of the Fourier coefficients $u_i(t),\ i = 1, \ldots, N$ of the solution, over some finite interval of time $[0, T]$. We provide explicit upper bounds for the Wasserstein distance for the rate of convergence when $N \to \infty$ and/or $T \to \infty$. In the case when T is fixed and $N \to \infty$, the upper bounds obtained in our results are more efficient than those of the Kolmogorov distance given by Mishra and Prakasa Rao [9] and Kim and Park [4].

4. Conclusions

To conclude, in this paper, we provide a rate of convergence for the central limit theorem of the MLE $\widehat{\theta}_{N,T}$ of the drift coefficient for the stochastic partial differential Equation (1) based on continuous time observations of the Fourier coefficients $u_i(t),\ i = 1, \ldots, N$ of the solution, over some finite interval of time $[0, T]$. The novelty of our approach is that it allows, comparing with the literature on the rate of convergence for $\widehat{\theta}_{N,T}$ discussed in [4,9], for improving the upper bound for the Wasserstein distance for the rate of convergence of the MLE $\widehat{\theta}_{N,T}$ when $N \to \infty$ and/or $T \to \infty$. More precisely,

- if $N \to \infty$ and T fixed, then there exists a positive constant $C_{\theta,T}$ depending only on θ and T such that, for every $N \geq 1$,
$$d_W\left(N^{\frac{3}{2}}\left(\theta - \widehat{\theta}_{N,T}\right), \mathcal{N}\left(0, \frac{6\theta}{\pi^2 T}\right)\right) \leq \frac{C_{\theta,T}}{N^{\frac{3}{2}}}.$$
- If $T \to \infty$ and N is fixed, then there exists a positive constant $C_{\theta,N}$ depending only on θ and N such that, for every $T \geq 1$,
$$d_W\left(\sqrt{T}\left(\theta - \widehat{\theta}_{N,T}\right), \mathcal{N}\left(0, \frac{2\theta}{\sum_{i=1}^N \lambda_i}\right)\right) \leq \frac{C_{\theta,N}}{\sqrt{T}}.$$
- If $N \to \infty$ and $T \to \infty$, then there exists a positive constant C_θ depending only on θ such that, for every $N \geq 1$ and $T \geq 1$,

$$d_W\left(\sqrt{T}N^{\frac{3}{2}}\left(\theta-\widehat{\theta}_{N,T}\right),\mathcal{N}\left(0,\frac{6\theta}{\pi^2}\right)\right) \quad\leq\quad \frac{C_\theta}{\sqrt{T}N^{\frac{3}{2}}}.$$

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