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Quantum Walks in Hilbert Space of Lévy Matrices: Recurrences and Revivals

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Abstract: The quantum evolution of wave functions controlled by the spectrum of Lévy random matrices is considered. An analytical treatment of quantum recurrences and revivals in the Hilbert space is performed in the framework of a theory of almost periodic functions. It is shown that the statistics of quantum recurrences in the Hilbert space of quantum systems is sensitive to the statistics of the corresponding quantum spectrum. In particular, it is shown that both the Poisson energy level statistics and the Brody distribution correspond to the power law of the quantum recurrences, while the Wigner–Dyson and Lévy–Smirnov statistics of the energy spectra are responsible for the exponential statistics of the quantum returns of the wave function.

Keywords: Lévy matrix; Hilbert space; Poincaré recurrences; statistics of quantum spectrum; almost periodic functions; quantum revivals

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1. Introduction

The introduction of Lévy processes in quantum mechanics by means of fractional–integral operators [1,2] is a natural procedure also supported by the experimental realization of a fractional harmonic oscillator by means of optical Airy beams [3]. Apparently, the implementation of Lévy matrices (LM)s [4] leads to essential extension of the consideration of the Lévy processes in many body quantum systems with long-range interactions [4], as well as nonlinear systems [5]. These interactions are described by matrix elements $H_{i,j}$, which are independent random variables distributed by the power law

$$P(H) \equiv P(H_{i.j}) \sim \mu |H_{i,j}|^{-\mu-1}, \tag{1}$$

where $0 < \mu < 2$. When $\mu \leq 0$, then $P(H)$ cannot be normalized, while for $\mu \geq 2$, the distribution has a finite variance and corresponds to the Gaussian orthogonal ensemble (GOE) case. Such matrices have been introduced and called "Lévy matrices" in Reference [4], where an Anderson delocalization–localization transition from the GOE to the Poisson distribution was proposed and observed for $\mu < 1$ as a function of energies as well; see discussion in Reference [6–8]. It should be noted that such a situation takes place also in dynamical systems such as quantum chaos, where the quantum spectrum follows either chaotic or regular dynamics of corresponding classical counterparts, e.g., [9–12]. In particular, in the semiclassical limit, the quantum spectrum follows the classical dynamics. Namely, for integrable systems, the uncorrelated spectrum is distributed according to the Poisson statistics [13,14]

$$P^{(P)}(\Delta) = \frac{1}{\Delta_0} \exp(-\Delta/\Delta_0), \tag{2}$$

where Δ_0 is the mean level spacing. By contrast, in quantum counterparts of chaotic systems, the quantum spectrum is strongly repelled, and the level spacing is described by the Wigner–Dyson statistics [14–16]

$$P^{(WD)}(\Delta) = C_\beta(\Delta_0)\Delta^\beta \exp(-\Delta^2/\Delta_0^2), \tag{3}$$

where $C_\beta(\Delta_0)$ is the normalization constant and $\beta = 1, 2, 4$ for the orthogonal, unitary, and symplectic Gaussian ensembles, GOE, GUE, and GSE, respectively. Note that the properties of the LMs are well studied for systems with long-range interactions, including Lévy–Smirnov statistics (see discussions, e.g., in References [4,17–20]) and theory of random matrices in quantum chaos [9–12].

In this paper, we apply the theory (namely properties) of the LMs to study statistics of quantum (Poincaré) recurrences, as the return probabilities in the Hilbert space of the LMs. This approach can be considered also as a quantum analogy of classical Poincaré recurrences in classical systems with chaotic and regular dynamics [21]. Our main interest here is the investigation of the statistics of the quantum recurrences (QR)s and the average characteristics of recurrent times.

Return probabilities are a specific realization of the first passage problem, which is an important characteristic in random walk theory, including random search theory [22]. The same role belongs to Poincaré recurrences in dynamical systems. In particular, Poincaré recurrences reflex the topology of the phase space of dynamical systems and segregate the return statistics of regular and chaotic regions which can coexist [21]. That is, this sensitivity is reflected in different statistics of the topological structure of the phase space. Namely, for the chaotic systems with a uniform mixing property, the distribution is exponential [23] $P(\tau) = \frac{1}{\tau_{rec}} \exp(-\tau/\tau_{rec})$ with the mean recurrence time $\tau_{rec} = \int_0^\infty \tau P(\tau) d\tau \propto 1/h_0$, which is finite and inversely proportional to the metric entropy h_0. In systems with nonuniform mixing, the distribution of recurrences is algebraic in the large recurrence times and asymptotic: $P(\tau) \sim 1/\tau^\gamma, \quad (\tau \to \infty)$, where γ is the recurrence exponent [21]. Another important property of the phase space topology is the Kac lemma, which states that the mean recurrence time is finite, $\tau_{rec} < \infty$ for the area preserving and bounded dynamics [24].

Albeit, the classical methodology fails in the quantum system, because of the absence of trajectories, and a straightforward relation between statistical properties of the quantum spectrum and statistics of quantum recurrences has been established in preliminary studies [25]. In turn, as admitted above, according to quantum chaos, e.g., [11,12], this also relates to the topology of classical trajectories in phase space either chaotic or regular [25]. It is worth be mentioning that, for systems with chaotic, or stochastic dynamics, a sequence of recurrence times $\{t\}_{rec} = \{t_1, t_2, \dots\}_{rec}$ is a stochastic process with properties that depend on both the type of the dynamics and a noise nature. One can expect a similar process in the quantum case without confusing this situation with the phenomenon of periodic revivals of the wave functions. In the latter case, a truncation of the energy expansion near some level n_0 is possible, namely, $E_n = E_{n_0} + E'_{n_0}(n - n_0) + E''_{n_0}(n - n_0)^2 + \mathrm{O}[(n - n_0)^3]$, e.g., review [26] and references therein. This situation is considered separately in Section 4. However, both cases can be considered as quantum walks in Hilbert space. It is shown here, that the situation depends on statistical properties of the spectrum of the LMs (see Appendix A), which are also functions of μ and the energy E of the quantum system. In particular, we study the recurrence time statistics for the GOE, the Poisson and the Brody (of sparse matrices) distributions of energy levels [4,9,19].

The Lévy matrices are the real symmetric matrices $\hat{H}$ of size N with independent and identically distributed elements $H_{i,j}$ according to the asymptotic distribution given by Equation (1). Some properties of the LMs are presented in Appendix A. The distribution of the eigenvalues is determined by the trace $T(z)$ of the resolvent $\hat{R}(z) = (z - \hat{H})^{-1}$. Then, the density of states $\rho(z)$ is given by the imaginary part of the trace as follows:

$$T(z) = \mathrm{tr}\big[\hat{\mathrm{R}}(\mathrm{z})\big] = \mathrm{N}^{-1} \sum_{\mathrm{j}=1}^{\mathrm{N}} \mathrm{R}_{\mathrm{i,i}}(\mathrm{z}) \,, \tag{4a}$$

$$\rho(z) = \frac{1}{\pi} \lim_{\epsilon \to 0} \lim_{N \to \infty} \mathrm{Im}[T(z - i\epsilon)] \,. \tag{4b}$$

As is shown in Reference [4], when the variance of the matrix elements $\bar{H}^2_{i,j} = \sigma N$ is finite, then the density of states obeys the semicircle law: $\rho(z) = (2\pi)^{-1}\sqrt{4 - z^2/\sigma}$

that corresponds to the GOE of $\hat{H}$ with a possible transition to the Poisson distribution. When the variance is divergent, the density of states corresponds to the Lévy statistics; see Appendix A.

Mobilizing the standard notion of recurrences for the evolution of a finite length vector $\mathbf{C} = (C_1, \ldots, C_N)$ in the Hilbert space, a distance between any vectors $\mathbf{C}^a$ and $\mathbf{C}^b$ is defined as follows:

$$d_{ab}^2 = |\mathbf{C}^a - \mathbf{C}^b|^2 = \sum_{j=1}^{N} |C_j^a - C_j^b|^2 . \tag{5}$$

Exploration of this heuristic definition can show how quantum walks in the Hilbert space reflect the topology of the classical phase space [25]. However, considering QRs for the LMs, one extends this consideration to pure quantum processes, which have no analogy in the classical topology of phase space.

2. Quantum Recurrences

In this section, we consider the unitary evolution of an initial wave function Ψ_0, according to the evolution operator $\hat{U}(t)$ with the Hamiltonian $\hat{H}$, such that $\hat{U}(t)\psi_k = e^{-iE_k t}\psi_k$, where E_k is the energy spectrum of the Hamiltonian. Correspondingly, the wave function at time t reads

$$\Psi(t) = \hat{U}(t)\Psi_0 = \sum_k a_k \exp(-iE_k t)\psi_k . \tag{6}$$

This also defines the evolution of the distance (5) in the Hilbert space

$$d^2(t) = |\Psi(t) - \Psi_0|^2 = \sum_k |a_k|^2 |e^{-iE_k t} - 1|^2 . \tag{7}$$

According to the exact analysis in the theory of almost periodic functions [27,28], expression (7) is the squared translation function. By definition [28], the translation function is

$$v_f(\tau) = \sup_{-\infty<t<\infty} |f(t+\tau) - f(t)| , \tag{8}$$

where τ is the translation time. Therefore, $d^2(t) = v_\Psi^2(t) = d^2(t+\tau)$. All possible values of τ for which $d^2(\tau) < \epsilon^2$ form a set of translation numbers, which is denoted $\mathcal{E} = \mathcal{E}\{\epsilon, \Psi(t)\}$. Therefore, for the QRs, the set $\mathcal{E}$ is determined by the condition

$$d^2(\tau) = \sum_k |a_k|^2 |e^{-iE_k\tau} - 1|^2 < \epsilon^2 . \tag{9}$$

Here, without restriction of generality, we set $t = 0$.

To proceed, we take into account that the wave function is normalized $\sum_k |a_k|^2 = 1$; therefore, there exists an integer N [29,30] such that

$$\sum_{k=N+1}^{\infty} |a_k|^2 < \epsilon^2 \ll 1 , \tag{10}$$

This expression justifies the finiteness of the summation in Equation (9), which now reads with the well-defined N

$$d^2(\tau) = \sum_{k=1}^{N} |a_k|^2 |e^{-iE_k\tau} - 1|^2 < \epsilon^2 . \tag{11}$$

Following the theory of almost periodic functions [28], let all the translation numbers belong to a set $\mathcal{E} = \mathcal{E}\{\epsilon^2, \Psi(t)\}$, which is determined by Equation (9). Then, all numbers $\tau \in \mathcal{E}$ of the set $\mathcal{E}$ satisfy the following N Diophantine inequalities [28] (Theorem 2, page 53)

$$|e^{-iE_k\tau} - 1|^2 < \delta_1^2 , \tag{12}$$

where $\delta_1 = \max|e^{-iE_k\tau} - 1|$ is the maximum of the modulus. Substituting Equation (12) in Equation (9), one obtains that

$$\sum_{k=1}^{N} |a_k|^2 |e^{-iE_k\tau} - 1|^2 < \delta_1^2 \sum_{k=1}^{N} |a_k|^2 < \epsilon^2 . \tag{13}$$

Therefore, the normalization condition yields $\delta_1 \sim \epsilon \ll 1$. Note that according to the rigorous analysis, $\delta_1 < \frac{\epsilon}{3}$ [28]. Rewriting Equation (12) in the sine-function form

$$|e^{-iE_k\tau} - 1|^2 = 4\sin^2\left(\frac{E_k\tau}{2}\right) < \delta_1^2 \sim \epsilon^2 , \tag{14}$$

where the argument can be taken by modulus 2π, one arrives at the expression

$$|E_k\tau - 2\pi n_k| < \epsilon/2 . \tag{15}$$

where integer numbers n_k relate to the energies E_k. Equations (14) and (15) are equivalent to

$$E_k\tau = 2\pi n_k + \eta_k , \quad |\eta_k| < \epsilon/2 . \tag{16}$$

From these expressions, one can also define n_k, considering the level spacing $\Delta_k = E_k - E_{k+1}$ of the ordered spectrum $E_1 < E_2, \cdots < E_N < E_{N+1}$ for $k = 1, 2, \ldots, N$. The r.h.s. of the equality in Equation (14) can be rewritten by means of Equation (16) as follows:

$$\begin{aligned}\sin^2\left(\frac{E_k\tau}{2}\right) &= \sin^2\left[\frac{1}{2}(E_k\tau + E_{k+1}\tau) - E_{k+1}\tau)\right] = \sin^2\left[\frac{1}{2}(\Delta_k\tau + E_{k+1}\tau)\right] \\ &= \sin^2\left[\frac{1}{2}(\Delta_k\tau \pm \epsilon/2)\right] < \frac{\epsilon^2}{4} ,\end{aligned} \tag{17}$$

where we used

$$\begin{aligned}|\Delta_k\tau|(\text{mod } 2\pi) &= |E_k\tau - E_{k+1}\tau|(\text{mod } 2\pi) = |2\pi(n_k - n_{k+1}) + \eta_k - \eta_{k+1}|_{2\pi} \\ &= |\eta_k - \eta_{k+1}| < |\eta_k| + |\eta_{k+1}| < \epsilon ,\end{aligned}$$

which is valid for each k. These expressions also yield the following N Diophantine inequalities

$$\left|\sin\left[\frac{\Delta_k\tau}{2}\right]\right| < \epsilon . \tag{18}$$

Eventually, one obtains that the translation times τ of QRs are determined by a new set of the N Diophantine inequalities, related to the level spacing Δ_k as follows:

$$|\Delta_k\tau - 2\pi m_k| < 2\epsilon , \tag{19}$$

where m_k are integers and correspondingly $\tau \in \mathcal{E}\{2\epsilon, \Psi(t)\}$.

Equation (19) yields the structure of the translations, which is

$$\tau = 2\pi\frac{\tilde{m}(\{\Delta_k\})}{\Delta_k} . \tag{20}$$

Note that while it is the same value for each fixed k, defined by Δ_k in denominator, $\tilde{m}(\{\Delta_k\})$ in numerator is a function of all N random variables Δ_k, such that

$$|\tilde{m}(\{\Delta_k\}) - m_k| < \frac{\epsilon}{\pi} , \quad k = 1, \ldots, N . \tag{21}$$

These quantum walks correspond to independent random processes for every trial of the returning/recurrence in the dynamics of the wave function in the Hilbert space. The set of translations–recurrences $\mathcal{E}\{2\epsilon, \Psi(t)\}$ is constructed by the system of N Diophantine inequalities (15), (18), and (19).

One should recognize that the translation times τ and correspondingly $\tilde{m}(\{\Delta_k\})$ are random values defined quite implicitly. However, their averaged values

$$\langle\tau\rangle = \langle\tilde{m}(\{\Delta_k\})\rangle_N = \int \tilde{m}(\{\Delta_k\})P(\{\Delta_k\})d^N\{\Delta_k\} < \infty$$

with the corresponding level spacing statistics $P(\{\Delta_k\})$ are well-defined values according to the Kac lemma [24,31]. This also means that $N-1$ dimensional integrals $\langle\tilde{m}(\{\Delta_k\})\rangle_{N-1}$ are well defined, and their explicit form is observed and ensured by the validity of the Kac lemma. These integrations are discussed and dealt with in Section 3, where the relation between the form of the level spacing statistics of the LMs and statistics of QRs is established.

3. Statistics of Quantum Recurrences

The recurrent property of random walks can be specified by their distribution function $\rho_{QR}(\tau)$ of QRs. To find the distribution function $\rho_{QR}(\tau)$, we determine the mean value of the translation numbers and the mean squared translation numbers. An important property used here is the Kac lemma [24] for the Poincare recurrences and its quantum generalization for the QRs [31] on the finiteness of the recurrent times τ. Therefore, although the recurrence times τ, described by Equations (20) and (21), are extremely large values, the mean value of the QR times is however finite. This property relates to the spectral statistics with the density of states $\rho(E)$ (or the level spacing distribution $P(\Delta)$) of the LMs, which in its turn, depends on the statistical properties of the matrix elements of the LMs, namely on the finiteness of the variance of the matrix elements of the LMs [4].

Therefore, for a finite N, the mean recurrent time (or translation number) reads

$$\langle\tau\rangle = \int \rho_{QR}(\tau)\tau d\tau < \infty\,, \tag{22}$$

where $\rho_{QR}(\tau)$ is the distribution function of the QR times. Equation (22) is also the expression of the quantum Kac lemma that sounds that for every spectral statistic of the LMs, the averaged recurrence times are finite values. Since the recurrence time is the function of the spectrum according to Equation (20), its averaged value can be defined by the level spacing distribution of the LMs that yield

$$\langle\tau\rangle = \int \tau(\{\Delta\})P(\{\Delta\})\prod_{k=1}^{N} d\Delta_k\,, \tag{23}$$

where $P(\{\Delta\})$ is a many-dimensional joint level spacing distribution function.

3.1. Poisson Distribution

We start the calculation of the averaged values of the translation time $\langle\tau\rangle$ from the Poisson statistics (2), which is the simplest form of the level spacing distributions. From another point of view, its knowledge is important to understand the structure of the recurrent times, which is the same for all LMs. In this case, the sequence of levels E_j is an uncorrelated random set, e.g., [11], and the joint distribution $P^{(P)}(\{\Delta\})$ is a product of distributions (2). Thus, substituting Equation (20) into Equation (23), we have

$$\langle\tau\rangle^{(P)} = 2\pi\int_0^{\infty} \frac{\tilde{m}(\{\Delta_k\})}{\Delta_k}\prod_{j=1}^{N} P^{(P)}(\Delta_j)d\Delta_j\,. \tag{24}$$

It is worth noting that although the Poisson statistics takes place only for the energies related to the localization states, the limits of the integration for the level spacing are determined by the infinite interval $\Delta_j \in [0, \infty)$. Performing integration of $\tilde{m}(\{\Delta_k\})$ with respect to $N-1$ variables Δ_j, besides Δ_k, we obtain $\langle\tilde{m}(\Delta)\rangle$, which is the function of only one variable $\Delta_k \equiv \Delta$. Another important condition for the integration (24) is the Kac lemma, which states that the integral is finite: $\langle\tau\rangle^{(P)} < \infty$. This, eventually, imposes the condition for the lower limit $\Delta \to 0$ due to the singular-pole behavior of the integrand, which according to the Kac lemma reads $\langle\tilde{m}(\Delta)\rangle \sim M\Delta^{\gamma}$ with $0 < \gamma \ll 1$ and $M \gg 1$. Taking this condition into account, one obtains that the integral in Equation (24) is the Gamma function $\Gamma(\gamma)$. Indeed, it reads

$$\langle\tau\rangle^{(P)} = \frac{2\pi}{\Delta_0}\int_0^{\infty}\frac{\langle\tilde{m}(\Delta)\rangle}{\Delta}e^{-\frac{\Delta}{\Delta_0}}d\Delta \sim \frac{2\pi M}{\Delta_0}\int_0^{\infty}\Delta^{\gamma-1}e^{-\frac{\Delta}{\Delta_0}}d\Delta = 2\pi M\Delta_0^{\gamma-1}\Gamma(\gamma)\,. \tag{25}$$

Note that for $\gamma \to 0$, the mean recurrent time diverges. Therefore, by suggesting a reasonable structure of the recurrent times in the form $\tau = 2\pi\Delta_k^{\gamma-1}M(\{\Delta_k\})$ with $M(\{\Delta_k\})$ being singular in the vicinity of $\Delta \to 0$ not stronger than $\prod_{l\neq k}\Delta_l^{-\delta_l}$, one obtains the following estimation of the QRs times

$$\tau \propto \prod_{l=1}^{N}\Delta_l^{-\delta_l} \sim \Delta^{-N\delta}\,, \tag{26}$$

where $0 < \delta, \delta_l < 1$.

Calculations of the second moment and the variance show that these are divergent values, $\langle\tau^2\rangle^{(P)} = \infty$. Therefore, the recurrent times are distributed according to the power law

$$\rho_{QR}^{(P)}(\tau) \sim \left(\frac{\tau_0}{\tau_0+\tau}\right)^{\alpha},\quad 2<\alpha<3\,, \tag{27}$$

where τ_0 is a characteristic time scale that is taken in such a way that $\int\rho_{QR}^{(P)}(\tau)\tau d\tau = 2\pi M\Gamma(\gamma)$.

3.2. Gaussian Orthogonal Ensemble

One can easily observe that for the Wigner–Dyson distribution (3), the second moment and the variance are finite values. This fact results from the correlations between the levels E_j. Our interest however is in the GOE with $\beta = 1$. Then, the joint distribution of levels for the GOE reads (see for example [11])

$$P^{(GOE)}(\{E\}) = C(A)\times\prod_{k<l}^{1\ldots N}|E_k - E_l|\exp\left(-A\sum_{k=1}^{N}E_k^2\right), \tag{28}$$

where A fixes the unit of energy (for example, it can be the mean squared level spacing, as in Equation (3)) and $C(A)$ is a normalization constant. Let us estimate the second moment for the GOE and show that it is finite (in this case, the variance is finite as well). From Equations (26) and (28), we arrive at the integral

$$\begin{aligned}\langle\tau^2\rangle^{(GOE)} &= \int_0^{\infty}\tau^2\rho_{QR}(\tau)d\tau \\ &= \tilde{C}\int_{-\infty}^{\infty}\prod_{k<l}^{1\ldots N}|E_k-E_l|\exp\left(-A\sum_{k=1}^{N}E_k^2\right)\times\prod_{r\neq s}|E_r-E_{r+1}|^{-2\delta_l}|E_s-E_{s+1}|^{2\gamma-2}d^NE \\ &\equiv \tilde{C}\int_{-\infty}^{\infty}|E_s-E_{s+1}|^{2\gamma-2}\mathcal{F}(\{E_j\})d^NE\,,\end{aligned} \tag{29}$$

Here, for brevity sake, we define the rest of the integrand in Equation (29) by $\mathcal{F}(\{E_j\})$ and $d^NE \equiv \prod_{j=1}^{N}dE_j$, and $\tilde{C} = (2\pi)^2C_1(A)$. Rewriting this integration in the form of an additional integration with the Dirac δ function and using the definition $\Delta = E_{s+1} - E_s$, one obtains

$$\langle\tau^2\rangle^{(GOE)} = \int_0^\infty d\Delta|\Delta|^{2\gamma-2}\int_{-\infty}^{\infty}\delta(\Delta+E_s-E_{s+1})\mathcal{F}(\{E_j\})d^NE \equiv \int_0^\infty d\Delta|\Delta|^{2\gamma-2}P^{(GOE)}(\Delta)\,. \quad (30)$$

As discussed in the literature, e.g., References [11,12,15], the level spacing distribution for $N\times N$ random matrices can be well approximated by 2×2 random matrix distribution. Therefore, integration in the N dimensional energy space can be reduced to integration with the GOE in Equation (3) with $\beta=1$. Therefore, following this standard approach, one arrives at the following integral

$$\langle\tau^2\rangle^{(GOE)} \sim \pi MC_1(\Delta_0)\int_0^\infty \Delta^{2\gamma-1}e^{-\frac{\Delta^2}{\Delta_0^2}}d\Delta = \pi MC_1(\Delta_0)\Delta_0^{2\gamma-2}\Gamma(\gamma)\,. \quad (31)$$

The existence of the first and the second moments for the Gaussian recurrent process means that the distribution of the recurrent times (as some "trapping" times outside the ϵ_1-cone) is well approximated by exponential, e.g., [32]

$$\rho_{QR}^{(GOE)}(\tau) \sim \frac{1}{\tau_0}\exp\Big(-\tau/\tau_0\Big)\,, \quad (32)$$

where τ_0 now is the averaged recurrence time:

$$\tau_0 = 2\pi MC_1(\Delta_0)\int_0^\infty \Delta^{\beta-1+\gamma}e^{-\frac{\Delta^2}{\Delta_0^2}}d\Delta = 2\pi MC_1(\Delta_0)\Delta_0^{\beta+\gamma}\Gamma\left(\frac{\beta+\gamma}{2}\right). \quad (33)$$

3.3. Brody Distribution for Sparse Matrices

The Brody distribution [33] can be considered as in intermediate case between the Poisson and Wigner–Dyson level spacing statistics. Although it is not proven that it belongs to an LM ensemble [19], it is suitable to describe the spectral statistical of quantum Hamiltonian systems in the regime of transition between integrability and chaos of corresponding classical counterparts [9]. The Brody distribution reads

$$P^{(B)}(\Delta) = a\Delta^\beta\exp\Big(-b\Delta^{1+\beta}\Big)\,, \quad a=(1+\beta)b\,, \quad b=\left[\Gamma\left(\frac{2+\beta}{1+\beta}\right)\right]^{\beta+1}\,, \quad (34)$$

where b is the bandwidth of the banded LM, while the level repulsion parameter β now ranges as $\beta\in(0,1)$. Although it has a simple analytic form, it still has no rigorous physical justification. A detailed discussion of the issue with respect to the energy level statistics and localization in sparse banded random matrix ensembles can be found in Reference [9].

Taking into account Equation (34), integration for the second moment in Equation (31) now is divergent, while for the mean value, we obtain

$$\langle\tau\rangle^{(B)} \sim (1+\beta)b\int_0^\infty \Delta^{\gamma-1+\beta}e^{-b\Delta^{1+\beta}}d\Delta = b^{\frac{1-\gamma}{1+\beta}}\Gamma\left(\frac{\gamma+\beta}{1+\beta}\right). \quad (35)$$

Therefore, the recurrent times are distributed according to the power law by analogy with Equation (27):

$$\rho_{QR}^{(B)}(\tau) \sim \left(\frac{\tau_0}{\tau_0+\tau}\right)^\alpha\,, \quad 2<\alpha<3\,, \quad (36)$$

where τ_0 is a characteristic time scale.

3.4. A Comment on Lévy–Smirnov Distribution

The spectrum for the Lévy–Smirnov distribution of the LMs was studied in Reference [17]. In this case, the Lévy–Smirnov ensemble is described by the distribution

$$P^{(LS)}(E) \sim \prod_{i=1}^{N} \left(\frac{e^{-N/E_i}}{E_i^2 N} \right) \prod_{i>j}^{1\ldots N} (E_i - E_j)^2 , \quad E_i \geq 0 . \tag{37}$$

Properties of the LMs are briefly discussed in Appendix A, where the Lévy–Smirnov distribution for the LMs spectrum is defined in Equation (A3), which reads

$$\tilde{L}(E_k; 1/2) = \frac{1}{2\sqrt{\pi E_k^3}} \exp\left(-\frac{1}{4E_k} \right) , \qquad E_k > 0 . \tag{38}$$

The correct structure of QRs in the energy space E_k is determined by Equation (15): $|E_k \tau - 2\pi n_k| < \epsilon/2$ where n_k are integer numbers, which corresponds to the energies $E_k > 0$. Then, Equation (20) for the recurrent times $\tau = \tau(\Delta)$ can be used, as well. However, there is no any reasonably simple expression for the level spacing distribution, which makes it possible to treat the problem. Therefore we make a crude approximation, accounting for the level spacing correlation. Following Reference [17] and performing the variable change $E_k = 1/x_k$ in Equation (37), one obtains Equation (37) in the form of the chiral GUE [17] as follows:

$$\prod_i dx_i e^{-Nx_i} \prod_{i>j} (x_i - x_j)^2 . \tag{39}$$

Then, the average characteristics of the QR times are

$$\langle \tau \rangle^{(LS)} \sim \int_0^\infty \prod_i dx_i e^{-Nx_i} \prod_{i>j} (x_i - x_j)^{2-\delta} < \infty , \tag{40a}$$

$$\langle \tau^2 \rangle^{(LS)} \sim \int_0^\infty \prod_i dx_i e^{-Nx_i} \prod_{i>j} (x_i - x_j)^{2-2\delta} < \infty . \tag{40b}$$

The existence of the first and the second moments ensures the exponential distribution of the recurrent times.

4. Quantum Revivals

Let us consider a continuous time quantum walk of a wave packet with revivals. In this case, instead of the distance (7), our main concern is the autocorrelation functions of the form

$$\mathcal{R}(t) = \langle \Psi_0 | \Psi(t) \rangle = \Psi_0 \hat{U}(t) \Psi_0 = \sum_n |a_n|^2 \exp(-iE_n t/\tilde{h}) , \tag{41}$$

which determines the probability density $|\mathcal{R}(t)|$ to find a wave packet in the initial state after time t and $\tilde{h}$ is a dimensionless Planck constant. If, however, the dynamics of this localized wave packet has an energy spectrum E_n, which is tightly spread around the quantum number, n_0, then the spectrum can be approximated by polynomials as follows [26]:

$$E_n \equiv E(n) \approx E(n_0) + E'(n_0)\Delta n + \frac{E''(n_0)}{2} \Delta n^2 + \ldots , \qquad \Delta n \equiv n - n_0 , \tag{42}$$

where the expansion is truncated. Therefore, this restricted quantum dynamics is analogous to a periodic quantum dynamics determined by a quantum nonlinear oscillator with the Hamiltonian

$$\hat{\mathcal{H}}_0 = \tilde{h}\omega \hat{a}^\dagger \hat{a} + \kappa \tilde{h}^2 \left(\hat{a}^\dagger \hat{a} \right)^2 , \tag{43}$$

where we use the Fock representation (the occupation-number representation) $\hat{a}^\dagger\hat{a}|\Delta n\rangle = \Delta n|\Delta n\rangle$, and a linear frequency ω and nonlinearity κ are related to the coefficients of the expansion (42), while creation and annihilation operators commute according to $[\hat{a},\hat{a}^\dagger] = 1$.

In particular, one can consider this process in the form of stability of wave functions with respect to a small variation, ε of the spectrum that is called fidelity of the wave functions, which is a measure of quantum reversibility [34] and it is also known as "Loschmidt echo" [35]. Note also that the fidelity amplitudes can be directly measured in Ramsey-type interferometry experiments, e.g., [36].

For the initial condition, we consider a coherent state basis, which is the eigenfunction of the annihilation operator $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. It is also defined as a superposition of the Fock states $|n\rangle$ as follows:

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_0^\infty \frac{\alpha^n}{\sqrt{n!}}|n\rangle\,. \tag{44}$$

Here, we just replaced $|\Delta n\rangle$ with $|n\rangle$. The evolution of the coherent states is due to the nonlinear oscillators $\hat{\mathcal{H}}_0$ and $\hat{\mathcal{H}}_\epsilon$. For the latter Hamiltonian, there are two possibilities of the perturbation. The first one is $\omega \to \omega + \epsilon$, while the second is $\kappa \to \kappa + \tilde{\epsilon}$ with the "dimensionality" relation $[\epsilon] = [\tilde{h}\tilde{\epsilon}]$. Therefore, we have two possibilities for the correlation function $\mathcal{R}_\epsilon(t) = \langle\alpha|e^{i\hat{\mathcal{H}}_\epsilon t - i\hat{\mathcal{H}}_0 t}|\alpha\rangle$, which yields

$$\mathcal{R}_\epsilon(t) = e^{-|\alpha|^2}\sum_0^\infty \frac{|\alpha|^2 n}{n!}e^{i\epsilon n t} = \exp\Big[|\alpha|^2\Big(e^{i\epsilon t} - 1\Big)\Big]\,, \tag{45a}$$

$$\mathcal{R}_{\tilde{\epsilon}}(t) = e^{-|\alpha|^2}\sum_0^\infty \frac{|\alpha|^2 n}{n!}e^{i\tilde{h}\tilde{\epsilon}n^2 t}\,. \tag{45b}$$

The first result (45a) leads to known golden rule decay of the fidelity of the wave functions $|\mathcal{R}_\epsilon(t)| \sim e^{-|\alpha|^2\epsilon^2 t^2}$ [37] for $\epsilon t \ll 1$. The second expression in Equation (45b) can be evaluated in the framework of the Schrödinger equation consideration as follows: Let us define the new function $\mathcal{M}(x,\tau) = e^x\mathcal{R}_{\tilde{\epsilon}}(t)$, where $x = |\alpha|^2$ and $\tau = \tilde{h}\tilde{\epsilon}t$ (here τ should not be confused with the translation time in Sections 2 and 3). Then, we have from Equation (45b)

$$-i\partial_\tau\mathcal{M} = (x\partial_x)^2\mathcal{M}\,, \quad \mathcal{M}(\tau = 0) = e^x\,, \quad x \geq 0\,. \tag{46}$$

Taking into account that for any entire function $f(x)$ the dilatation operator $x\partial_x$ acts as follows $e^{-i\tau x\partial_x}f(x) = f(e^{-i\tau}x)$, and one obtains from Equation (46) a formal solution for the correlation function in the form

$$\mathcal{R}_{\tilde{\epsilon}}(t) = \sqrt{i\tau/4\pi}\int d\xi e^{-i\tau\frac{\xi^2}{4}}\exp\Big[-x(1 - e^{-i\tau\xi})\Big]\,. \tag{47}$$

Performing integration in the stationary phase approximation, we obtain again the golden rule decay, $\mathcal{R}_{\tilde{\epsilon}}(t) \sim e^{-(\xi_0\tilde{h}\tilde{\epsilon}t)^2/2}$ for $\xi_0\tilde{h}\tilde{\epsilon}t \ll 1$, where ξ_0 is defined from the equation $\xi = 2\cos(\tilde{h}\tilde{\epsilon}t\xi)$.

5. Conclusions

In the present research, we focus on a quantum evolution of wave functions, which is controlled by the spectrum of Lévy random matrices. An analytical treatment of quantum recurrences and revivals in the Hilbert space is performed in the framework of the theory of almost periodic functions. In this case, the analytical expression for the return time τ as a function of the level spacing Δ is obtained: $\tau = \tau(\Delta)$. It is shown that the statistics of quantum recurrences in the Hilbert space of quantum systems is sensitive to the statistics of the corresponding quantum spectrum. In particular, it is shown that both the Poisson energy level statistics and the Brody distribution correspond to the power law distribution of the quantum recurrences, while the GOE and Lévy–Smirnov statistics of the energy spectra are responsible for the exponential statistics of the quantum return times of the wave functions.

Along with the unitary evolution of the wave function, which is completely controlled by the spectrum of the Lévy matrices, the Kac lemma and its quantum generalization play an important role in the observation of the analytical form of the return time statistics as the function of the spectrum. The statement that the mean return time is finite, applied to the Poisson level spacing distribution, yields the explicit expression for the return time, $\tau(\Delta) \sim \Delta^{\gamma-1}$, which results from $N-1$ dimensional integration in the N dimensional energy space. Since the Kac lemma is the only restriction for the return times, the analytical form of $\tau(\Delta)$ is universal and used for all statistics of the Lévy matrices for the analytical estimation of the statistics of the quantum recurrences in the Hilbert space. It should be admitted that the mean recurrent times and corresponding statistics of the quantum recurrences are sensitive to the statistics of the corresponding quantum spectrum in spite of the universal form of $\tau(\Delta)$. The essential difference in the statistics of the quantum recurrences in the Hilbert space for the chaotic–delocalized systems and integrable–localized systems results from the essential difference between the level statistics of the Lévy matrices. In turn, it also depends on the integrability of the corresponding dynamics of the classical counterparts. It should be stressed that this statement is valid for both Lévy matrices with $\mu \in (0, 2)$ and "exponential" matrices with $\mu > 2$; see Equation (1). The parameter $\mu = 2$ separates also corresponding physical phenomena described by the matrices. The typical examples belonging to the "exponential" matrices (with Poisson and GUE) is discussed below.

The quantum dynamics is described by the almost periodic wave functions [27–30]; however, the quantum walks in the Hilbert space are random, and the returning times are functions of the level spacing Δ, which are random variables with different distributions. An important property of integrable systems is that the quantum walks establish revivals of wave functions in the Hilbert space. Apparently, this situation is suitable for the Poisson ensemble and relates to the expansion (42) (probably, this situation can be also realized for the Brody distribution). The situation changes dramatically, when the expansion of the energy (42) cannot be performed. For example, for the Hamiltonian $\hat{\mathcal{H}}_\epsilon$ of the form

$$\hat{\mathcal{H}}_\epsilon = \hat{\mathcal{H}}_0 - \tilde{h}\epsilon(\hat{a} + \hat{a}^\dagger)\sum_l \delta(t - lT)\,, \tag{48}$$

where T is a period of the train of delta kicks and ϵ now is an amplitude of the perturbation. This model has been suggested to observe the Ehrenfest time on the order of $\ln(1/\tilde{h})$, which specifies the time scale of the quantum-to-classical correspondence, firstly observed in Reference [38] with further studies in quantum chaos [39–46]. In this case, the correlation function $\mathcal{R}_\epsilon(t)$ describes the Loschmidt echo [45,47,48] with the exponential decay, and it reads

$$\mathcal{R}_\epsilon(t) = \langle\alpha|\exp\left\{i\int_0^t [\hat{\mathcal{H}}_\epsilon(\tau) - i\hat{\mathcal{H}}_0(\tau)]d\tau\right\}|\alpha\rangle \propto e^{-\Lambda t}\,. \tag{49}$$

This decay of $\mathcal{R}_\epsilon(t)$ in Equation (49) is determined by the classical Lyapunov exponent Λ that reflexes the classical nature and is independent of the $\tilde{h}$, while it is valid on the Ehrenfest time scale, which depends on $\tilde{h}$. In this case, there are no revivals, and quantum recurrences are determined by the GUE ensemble due to the Hamiltonian $\hat{\mathcal{H}}_\epsilon$.

Considering this quantum dynamics with a relaxation process [46], we take into account that the frequency $\omega_\varsigma = \Omega - i\varsigma/2$ can be a complex value in the Hamiltonian (48), which determines the effective frequency $\omega = [\Omega^2 + \varsigma^2/4]^{1/2}$ in the presence of a finite width of the levels, $\varsigma/2$. In this case, the correlation function decays exponentially fast as well, according to Equation (49) on the Ehrenfest time scale, which now reads $\sim \ln(1/\tilde{h})/(\Lambda - \varsigma T)$, while the corresponding classical counterpart is a strange attractor [46,49]. In the limit $\Lambda - \varsigma T \to +0$, the Ehrenfest time can be extremely large.

It is also worth noting that the quantum Kac lemma is proven for open quantum systems [31]. In this connection, the fidelity for mixed quantum states [50] can be an interesting issue for future exploration of the geometry of quantum phase transitions, in particular quantum phase transitions and nonequilibrium dissipative phase transitions [51].

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Appendix A. Lévy Matrices

Theory of Lévy matrices (LM)s was suggested in Reference [4]. It supposes that random matrix elements are determined by the Lévy distribution. We follow discussions of the issue in References [52–54], related to the Lévy stable distribution and properties of the spectra of the LMs [4].

Appendix A.1. Lévy Distributions

Lévy distribution $L(x;\alpha,\beta)$ is defined by its Fourier transform

$$L(x;\alpha,\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{L}(k;\alpha,\beta)e^{ikx}dk\,, \tag{A1a}$$

$$\tilde{L}(k;\alpha,\beta) = \exp\{-|k|^{\alpha}[1 - i\beta\mathrm{sign}(k)\tan(\pi\alpha/2)]\}\,, \tag{A1b}$$

where $0 < \alpha \leq 2$ and $\beta \in [-1\,,1]$. When the skewness parameter $\beta = 0$, the distribution is symmetrical. We used here two subclasses, which are mostly popular in applications [54]. These are: (i) the symmetrical stable distribution formed by the stable characteristic function

$$\tilde{L}(k;\alpha,0) \equiv \tilde{L}(k;\alpha) = e^{-|k|^{\alpha}}\,. \tag{A2}$$

In particular, $L(x:2) = (2\sqrt{\pi})^{-1}e^{-x^2/4}$ is a Gaussian distribution, when all moments are finite, while $L(x,1) = [\pi(1+x^2)]^{-1}$ is a Cauchy distribution, when all moments diverge.

(ii) Another example is the Lévy–Smirnov distribution:

$$\tilde{L}(x;1/2) = \frac{1}{2\sqrt{\pi x^3}}\exp\left(-\frac{1}{4x}\right), \qquad x > 0\,. \tag{A3}$$

In general cases, the characteristic function (A2) corresponds to the Lévy distribution, which is determined by the Fox H-function [55].

$$e^{-|k|^{\alpha}} = H_{0,1}^{1,0}\left[|k|^{\alpha}\,\middle|\, \begin{matrix} - \\ (0,1) \end{matrix}\right]. \tag{A4}$$

Then, the Mellin-cosine transformation [56,57] yields

$$\begin{aligned} L(x;\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|^{\alpha}}e^{ikx}dx = \frac{\pi}{|x|}H_{2,2}^{1,1}\left[|x|^{\alpha}\,\middle|\, \begin{matrix} (1.1),(1,\alpha/2) \\ (1,\alpha),(1,\alpha/2) \end{matrix}\right] \\ = (\pi\alpha)^{-1}\sum_{l=0}^{\infty}(-1)^l\frac{x^{2l}}{(2l)!}\Gamma(1/\alpha + 2l\alpha)\,, \end{aligned} \tag{A5}$$

where the expansion of the Fox H-function results from the integration [52].

Note that the Fox H function is defined in terms of the Mellin–Barnes integral [55,58],

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \middle| \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array}\right] = \frac{1}{2\pi i}\int_{\Omega} \Theta(s) z^{-s} ds\,, \tag{A6}$$

where

$$\Theta(s) = \frac{\left\{\prod_{j=1}^{m} \Gamma(b_j + sB_j)\right\}\left\{\prod_{j=1}^{n} \Gamma(1 - a_j - sA_j)\right\}}{\left\{\prod_{j=m+1}^{q} \Gamma(1 - b_j - sB_j)\right\}\left\{\prod_{j=n+1}^{p} \Gamma(a_j + sA_j)\right\}}\,, \tag{A7}$$

with $0 \leq n \leq p$, $1 \leq m \leq q$ and $a_i, b_j \in C$, while $A_i, B_j \in R+$, for $i = 1, \ldots, p$, and $j = 1, \ldots, q$. The contour Ω starting at $c - i\infty$ and ending at $c + i\infty$, separates the poles of the functions $\Gamma(b_j + sB_j)$, $j = 1, \ldots, m$ from those of the function $\Gamma(1 - a_i - sA_i)$, $i = 1, \ldots, n$.

Appendix A.2. Spectrum of the Lévy Matrices

In Reference [4], the LMs are the real symmetric matrices $\hat{H}$ of size N with independent and identically distributed (iid) elements $H_{i,j}$ according to Equation (A1), while asymptotically given by Equation (1). The distribution of the eigenvalues is determined by the resolvent $\hat{R}(z) = (z - \hat{H})^{-1}$, namely by the trace $T(z) = \mathrm{tr}\big[\hat{\mathrm{R}}(\mathrm{z})\big] = \mathrm{N}^{-1}\sum_{\mathrm{j}=1}^{\mathrm{N}} \mathrm{R}_{\mathrm{i,i}}(\mathrm{z})$. Then, the density of the states $\rho(z)$ is given by the imaginary part of the trace as follows:

$$\rho(z) = \frac{1}{\pi}\lim_{\epsilon\to 0}\lim_{N\to\infty} \mathrm{Im}[T(z - i\epsilon)]\,. \tag{A8}$$

Matrix elements $R_{i,j}(z)$ can be expressed by means of a Gaussian integral over auxiliary fields ϕ_i as follows [4,59]:

$$\begin{aligned} Z_N &= \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp\left[-2^{-1}\sum_{k,l=1}^{N} K_{k,l}(z)\phi_k\phi_l - \sum_k h_k\phi_k\right] \\ &= \prod_{n=1}^{N}\int_{-\infty}^{\infty} d\xi_n \exp\left(-2^{-1}K_n(z)\xi_n^2 - \bar{h}_n\xi_n\right) = \prod_{n=1}^{N}\sqrt{2\pi K_n^{-1}}\exp\left(2^{-1}\bar{h}_n K_n^{-1}\bar{h}_n\right) \\ &= \sqrt{(2\pi)^N / \det\hat{K}}\exp\left(2^{-1}\sum_{k,l=1}^{N} h_k K_{k,l}^{-1} h_l\right). \end{aligned} \tag{A9}$$

Here, diagonalization of the quadratic form, takes place according to the unitary transformation

$$\phi^T\hat{K}\phi = \phi^T\hat{U}^{-1}\hat{U}\hat{K}\hat{U}^{-1}\hat{U}\phi = \sum_n K_n\xi_n^2,$$

where $\hat{U} = \|e_1, \ldots, e_N\|$, and e_n are eigenfunctions of the LM $\hat{H}e_n = E_n e_n$ and $K_n = z - E_n$ are elements of the diagonal matrix similar to $(z - \hat{H})$. The Jacobian of the transformation is unity. Taking into account that $R_{k,l}(z) = K_{k,l}^{-1}(z)$, we have

$$R_{i,j}(z) = \frac{1}{Z_N}\frac{\partial^2}{\partial h_i \partial h_j} Z_N\big|_{h_i,h_j=0}\,. \tag{A10}$$

After generating this $N \times N$ matrix, a new row and a symmetric column are added to the LM $\hat{H}$, which is called $H_{0,i}$ in Reference [4]. The size of the matrix is $N + 1$, and according to Equation (A10) where $h_k = 2^{-1}[K_{0,k} + K_{k,0}]\phi_0 = K_{k,0}\phi_0$, we have that $R_{0,0}^{N+1}(z)$ corresponds to the expression

$$R_{0,0}^{N+1}(z) = \frac{1}{Z_{N+1}} \int d\phi_0 \phi_0^2 e^{-2^{-1}K_{0,0}(z)\phi_0^2} \int \prod_{i=1}^{N} d\phi_i \exp\left[-2^{-1} \sum_{k,l=1}^{N} K_{k,l}(z)\phi_k\phi_l - \sum_k h_k\phi_k\right]$$
$$= \frac{1}{Z_{N+1}} \int d\phi_0 \phi_0^2 e \exp\exp\left(-2^{-1}K_{0,0}(z)\phi_0^2 + 2^{-1} \sum_{k,l=1}^{N} h_k K_{k,l}^{-1} h_l\right). \quad \text{(A11)}$$

This eventually yields the recursion relation [4] as follows:

$$z - \frac{1}{R_{0,0}^{N+1}(z)} = \sum_{i=1}^{N} H_{i,0}^2 R_{ii}^N(z)\,. \quad \text{(A12)}$$

As shown in Reference [4], the properties of the density of states and the corresponding statistics of the LMs depend on the variance of the matrix elements $H_{0,i}$ in Equation (A12). In particular, when the variance $\bar{H}_{0,i}^2 = \sigma N$ is finite, then the density of states obeys the semicircle law: $\rho(z) = (2\pi)^{-1}\sqrt{4 - z^2/\sigma}$ which corresponds with the GOE of $\hat{H}$. By contrast, when the variance is divergent, the density of states corresponds to the Lévy statistics.

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