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Article

Solving a Higher-Dimensional Time-Fractional Diffusion Equation via the Fractional Reduced Differential Transform Method

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Abstract: In this study, exact and approximate solutions of higher-dimensional time-fractional diffusion equations were obtained using a relatively new method, the fractional reduced differential transform method (FRDTM). The exact solutions can be found with the benefit of a special function, and we applied Caputo fractional derivatives in this method. The numerical results and graphical representations specified that the proposed method is very effective for solving fractional diffusion equations in higher dimensions.

Keywords: fractional reduced differential transform method; fractional calculus; time-fractional diffusion equations; Caputo derivative

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1. Introduction

Fractional calculus is a generalization of integration and differentiation to noninteger-order fundamental operator ${}_a\mathbf{D}_t^{\alpha}$ where a and t are the bounds of the operation and $\alpha \in \mathbf{R}$; this notation was designed by Harold T. Davis. Diverse definitions for fractional derivatives have been proposed such as Riemann–Liouville, Caputo, Hadamard, Erdélyi–Kober, Grünwald–Letnikov, Marchaud, and Riesz, to name a few. The three greatest regular definitions for the universal fractional differintegral are the Caputo, the Riemann–Liouville, and the Grünwald–Letnikov definition [1–3].

In this study, we used the Caputo fractional derivative; the binary significant explanations for that are the initial conditions for fractional-order differential equations in a form connecting only the limit values of integer-order derivatives at the lower terminal initial time [3]. Similarly, the fractional derivative of a constant function is zero.

Up to now, there have been diverse methods for solving fractional differential equations using different definitions of fractional derivatives. Let us indicate some of these applications: Almeida et al. [4] applied fractional differential equations for modeling particular real phenomena, though Bulut et al. [5] considered the nonlinear time-fractional Burgers equation via the improved Bernoulli subequation function method. Atangana et al. [6] considered an advection–dispersion model with a fractional order and fractal dimension. Alshammari et al. [7] proposed residual power series (RPS) to find the numerical solution of a class of fractional Bagley–Torvik problems (FBTP) arising in a Newtonian fluid. Similarly, Yépez-Martínez et al. [8] solved the nonlinear coupled spacetime-fractional mKdV partial differential equation using Feng's first integral method. The definition of the beta fractional derivative to find exact and approximate solutions of time-fractional diffusion equations in different dimensions was modified in [9,10]. Youssri [11] adopted the spectral Tau method for solving the nonlinear Riccati initial-value problem with a new generalized Caputo FF derivative. Youssri et al. [12] presented the numerical solutions of the fractional pantograph differential equations (FPDEs) using generalized Lucas polynomials (GLPs). Abd-Elhameed et al. [13] presented an explicit formula that approximates the fractional derivatives of Chebyshev polynomials of the first kind in the Caputo sense. Abd-Elhameed and Youssri [14] derived novel formulae for the high-order derivatives of Chebyshev polynomials of the fifth kind.

Keskin and Oturanc [15] suggested the fractional reduced differential transform method (FRDTM). The FRDTM is one of the best common methods for solving fractional partial differential equations as the FRDTM is a generalization of the reduced differential transform method (RDTM), which in turn is a generalization of the differential transform method (DTM) for solving different types of differential equations. Scholars frequently try to find different methods to simplify the resulting solutions and decrease the solution steps, making the progress of mathematical techniques required to complete the greatest consequences. The applicability of the FRDTM to some diverse categories of fractional differential equations has been obtainable as follows: Mukhtar et al. [16] applied the FRDTM to solve nonlinear fractional Burgers equations in different dimensions. Gupta [17] offered the approximate analytical solutions of the Benney–Lin equation with a fractional time derivative. Srivastava et al. [18] used the FRDTM to obtain the exact solution of a mathematical model for the generalized time-fractional-order biological population model, for multiterm time-fractional diffusion equations. Abuasad et al. [19] suggested a modified method of the FRDTM, and the FRDTM approximate solution of the time-fractional Korteweg–de Vries equation was offered by Ebenezer et al. [20].

An application of the FRDTM to a system of linear and nonlinear fractional partial differential equations was organized by Singh [21]. Rawashdeh [22] applied the FRDTM to solve nonlinear fractional partial differential equations such as the spacetime-fractional Burgers' equations and the time-fractional Cahn–Allen equations. Singh and Kumar [23] used the FRDTM to find approximate solutions of time-fractional-order multidimensional Navier–Stokes equations. The fractional Helmholtz equations were considered to find exact and approximate solutions via the FRDTM by Abuasad et al. [24]. Furthermore, Abuasad et al. [25] found approximate solutions of the fractional SSIS epidemic model using the fractional multistep differential transformed method.

The main benefits of the FRDTM are that it can be applied to diverse types of linear and nonlinear PFDEs in different dimensions. The multistep differential transform method (MsDTM) can overcome the central complications of the differential transform method (DTM) and reduced differential transform method (RDTM), which are that the achieved series solution often converges in the real irrelevant space and the range of convergence is a precise slow procedure or completely divergent given a wider space [25]. Abdou [26] used the FRDTM to develop a scheme to study the numerical solution of time-fractional nonlinear evolution equations under initial conditions. Comparing the DTM, RDTM, FRDTM, and MsDTM, we found that the DTM is an upgraded method of the Taylor series method, which requests extra computational work for large orders, and it decreases the size of the computational domain [27]. Meanwhile, the RDTM is simpler than the DTM, and the whole number of calculations needed in the RDTM is less than that in the DTM [28]. The FRDTM is an improved method of the RDTM for fractional-order derivatives. The MsDTM can overcome the difficulties of the DTM and RDTM, where the series solutions often converge in a very insignificant space and the range of convergence is a long procedure or completely divergent in a long time span.

The importance of this research lies in finding the exact and approximate solutions for higher-dimensional time-fractional diffusion equations using a relatively new method and comparing the exact solutions of nonfractional diffusion equations with the approximate solutions for different values of the fractional derivatives. The novel aspect of this research is the explanation of the FRDTM with simple and sequential steps so that every researcher can apply and understand the method directly without referring to other literature. The other distinctive aspect of this research is to find a general formula for the solutions, which reach the exact solution in the first and second examples, and to compare the nonfractional exact solutions with the approximate fractional solutions using figures in two and three dimensions. The central physical purpose of accepting and studying diffusion equations of fractional orders is to define the phenomena of anomalous diffusion, usually met in transport processes through complex and/or disordered media with fractal supports [29]. The time-fractional diffusion Equations (12)–(14) can be interpreted as a model of the diffusion of a particle under the action of the external force.

This paper is organized as follows: After giving simple properties and definitions of the fractional derivative in Section 2, we introduce the proposed method in Section 3. Section 4 presents the exact and approximate solutions of three examples of time-fractional diffusion equations. Section 5 is the conclusion.

2. Preliminaries and Fractional Derivative Order

The unique functions of mathematical physics are found to be very useful for finding solutions of initial- and boundary-value problems governed by partial differential equations and fractional differential equations, and they play a significant and exciting role as solutions of fractional-order differential equations [30]. Many special functions have attracted the attention of researchers, such as the Wright function, the error function, and the Millin–Ross function. In this paper, our attention is focused on only two types of these special functions: the Mittag–Leffler function and the Gamma function. We used the Mittag–Leffler function since after finding the solution in a compact form, we can write the exact solution by using the definition of the Mittag–Leffler function, while the Gamma function is an essential part of the definition of fractional derivatives.

2.1. Mittag–Leffler Function

The Mittag–Leffler (M-L) function is named after a Swedish mathematician who defined and studied it in 1903. The M-L function is a straight generalization of the exponential function e^x. The one-parameter M-L function in powers series is given by the formula [3]:

$$E_\gamma(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\gamma k + 1)}, \quad (\gamma > 0). \tag{1}$$

For selected integer values of γ, we obtain:

$$\begin{aligned} &E_0(x) = \frac{1}{1-z}, \quad E_1(x) = e^x, \\ &E_2(x) = \cosh(\sqrt{x}). \end{aligned}$$

In powers series, the two-parameter M-L function is defined by:

$$E_{\gamma,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\gamma k + \beta)}, \quad (\gamma > 0,\ \beta > 0). \tag{2}$$

For special choices of the parameters γ and β, we obtain the famous traditional functions:

$$\begin{aligned} &E_{1,1}(x) = E_1(x) = e^x, \quad E_{1,2}(x) = \frac{e^x - 1}{x}, \\ &E_{2,1}(x^2) = \cosh(x), \quad E_{2,2}(x^2) = \frac{\sinh(x)}{x}. \end{aligned}$$

2.2. Caputo Fractional Derivative

Let $a \in \mathbb{R}$, then the (left-sided) Caputo fractional derivative $\left({}^cD^{\alpha}_{a+}y\right)(x)$ (the small c represents the Caputo derivative) of order $\alpha \in \mathbb{R}^+$ is well defined as [31]:

$$({}^cD^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}}\mathrm{d}t\,, \tag{3}$$

for $(n-1 < \alpha < n;\ x \geq a)$, $n \in \mathbb{N}$ and $\Gamma(x)$ is the Gamma function. For the ease of presentation, we symbolize the Caputo fractional derivative as $\mathbf{D}^{\alpha}_{x} f(x)$.

3. Fractional Reduced Differential Transform Method for *n+1* Variables

This section gives the basic definitions and properties of the FRDTM [16,18,32,33]. Consider a function $f(t, x_1, x_2, \ldots, x_n)$ to be analytical and continuously differentiable with respect to $(n+1)$ variables in the domain of interest, such that:

$$f(t, x_1, x_2, \ldots, x_n) = m_1(x_1)m_2(x_2)\cdots m_n(x_n)h(t). \tag{4}$$

Then, from the properties of the DTM and motivated by the components of the form $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}t^{\alpha j}$, we write the general solution function $f(t, x_1, x_2, \ldots, x_n)$ as an infinite linear combination of such components:

$$\begin{aligned} f(t, x_1, x_2, \ldots, x_n) &= \sum_{i_1=0}^{\infty} m_1(i_1)x_1^{i_1}\sum_{i_2=0}^{\infty} m_2(i_2)x_2^{i_2}\cdots\sum_{i_n=0}^{\infty} m_n(i_n)x_n^{i_n}\sum_{j=0}^{\infty} h(j)t^{\alpha j} \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_n=0}^{\infty}\sum_{j=0}^{\infty} F(i_1, i_2, \ldots, i_n, j)x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}t^{\alpha j}\,, \end{aligned} \tag{5}$$

where $F(i_1, i_2, \ldots, i_n, j) = m_1(i_1)m_2(i_2)\cdots m_n(i_n)h(j)$ is referred to as the spectrum of $f(t, x_1, x_2, \ldots, x_n)$. Furthermore, the lowercase $f(t, x_1, x_2, \ldots, x_n)$ is used for the original function, while its fractional reduced transformed function is represented by the uppercase $F_k(x_1, x_2, \ldots, x_n)$, which is called the T-function.

3.1. Step 1: Finding the Fractional Reduced Transformed Function

Let $f(t, x_1, x_2, \ldots, x_n)$ be analytical and continuously differentiable with respect to $n+1$ variables $t, x_1, x_2, \ldots, x_n$ in the domain of interest, then the FRDTM in n dimensions of $f(t, x_1, x_2, \ldots, x_n)$ is given by:

$$F_k(x_1, x_2, \ldots, x_n) = \frac{1}{\Gamma(k\alpha+1)}[\mathbf{D}^{\alpha k}_t(f(t, x_1, x_2, \ldots, x_n))]_{t=t_0}\,, \tag{6}$$

where $k = 0, 1, 2, \cdots$, with the time-fractional derivative.

3.2. Step 2: Finding the Inverse of the Fractional Reduced Transformed Function

The inverse FRDTM of $F_k(x_1, x_2, \ldots, x_n)$ is defined by:

$$f(t, x_1, x_2, \ldots, x_n) := \sum_{k=0}^{\infty} F_k(x_1, x_2, \ldots, x_n)(t-t_0)^{k\alpha}. \tag{7}$$

From (6) and (7), we have:

$$f(t, x_1, x_2, \ldots, x_n) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}[\mathbf{D}^{\alpha k}_t(f(t, x_1, x_2, \ldots, x_n))]_{t=t_0}(t-t_0)^{k\alpha}.$$

In particular, for $t_0 = 0$, the above equation becomes:

$$f(t, x_1, x_2, \ldots, x_n) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} [\mathbf{D}_t^{\alpha k}(f(t, x_1, x_2, \ldots, x_n))]_{t=0} t^{k\alpha}. \tag{8}$$

3.3. Step 3: Finding the Approximate Solution

The inverse transformation of the set of values $\{F_k(x_1, x_2, \ldots, x_n)\}_{k=0}^{m}$ gives an approximate solution as:

$$\tilde{f}_m(t, x_1, x_2, \ldots, x_n) = \sum_{k=0}^{m} F_k(x_1, x_2, \ldots, x_n) t^{\alpha k}, \tag{9}$$

where m is the order of the approximate solution.

3.4. Step 4: Finding the Exact Solution

The exact solution using the FRDTM is given by:

$$f(t, x_1, x_2, \ldots, x_n) = \lim_{m \to \infty} \tilde{f}_m(t, x_1, x_2, \ldots, x_n). \tag{10}$$

In Table 1, we provide certain properties of the FRDTM, where $\delta(k-m)$ is defined by:

$$\delta(k-m) = \begin{cases} 1, & k = m \\ 0, & k \neq m\,, \end{cases} \tag{11}$$

where $f = f(t, x_1, x_2 \ldots, x_n), u = u(t, x_1, x_2 \ldots, x_n)$, $F_k = F_k(x_1, x_2 \ldots, x_n)$, and $U_k = U_k(x_1, x_2 \ldots, x_n)$.

We prove Property 3 from Table 1 in two dimensions; other proofs of the properties can be found in [33–36].

If $w(x,y) = \mathbf{D}_x^{n\alpha} u(x,y)$, then $W_k(y) = \dfrac{\Gamma(\alpha(k+n)+1)}{\Gamma(k\alpha+1)} U(k+n)(y)$.

From Equation (6), we have:

$$\begin{aligned} W_k(y) &= \frac{1}{\Gamma(k\alpha+1)} [\mathbf{D}_x^{\alpha k}(\mathbf{D}_x^{n\alpha} u(x,y))]_{x=x_0}, \\ &= \frac{1}{\Gamma(k\alpha+1)} [\mathbf{D}_x^{\alpha(k+n)} u(x,y))]_{x=x_0}, \\ &= \frac{\Gamma((k+n)\alpha+1)}{\Gamma((k+n)\alpha+1)\Gamma(k\alpha+1)} [\mathbf{D}_x^{\alpha(k+n)} u(x,y))]_{x=x_0}, \\ &= \frac{\Gamma((k+n)\alpha+1)}{\Gamma(k\alpha+1)} U(k+n)(y). \end{aligned}$$

Table 1. Fundamental operations of the FRDTM for $n+1$ variables.

Original Function	Transformed Function
1. $f = c_1 u \pm c_2 v$	$F_{\alpha k} = c_1 U_{\alpha k} \pm c_2 V_{\alpha k}$
2. $f = uv$	$F_{\alpha k} = \sum_{i=0}^{k} U_{\alpha i} V_{\alpha(k-i)}$
3. $f = \mathbf{D}_t^{m\alpha} u$	$F_{\alpha k} = \dfrac{\Gamma(\alpha(k+m)+1)}{\Gamma(k\alpha+1)} U_{\alpha(k+m)}$

Table 1. *Cont.*

Original Function	Transformed Function
4. $f = \frac{\partial^h u}{\partial x_i^h}$	$F_{\alpha k} = \frac{\partial^h U_{\alpha k}}{\partial x_i^h}, i = 1, 2, \ldots, n$
5. $f = x_i^m t^r$	$F_{k\alpha} = x_i^m \delta(\alpha k - r), i = 1, 2, \ldots, n$
6. $f = x_i^m t^r u$	$F_{\alpha k} = x_i^m \sum_{i=0}^{k} \delta(\alpha i - r) U_{\alpha(k-r)}, i = 1, 2, \ldots, n$

4. Numerical Examples

The purpose of this paper is to apply the FRDTM to find exact and approximate solutions for time-fractional diffusion equations in two, three, and four dimensions. The time-fractional diffusion equation is gained from the standard diffusion equation by consistently changing the first-order time derivative with a specified fractional derivative.

To show the effectiveness of the proposed method for finding exact and approximate solutions, we apply the FRDTM in two-, three-, and four-dimensional time-fractional diffusion equations.

$$\frac{\partial^\beta f(X,t)}{\partial t^\beta} = D\Delta f(X,t) - \nabla \cdot \mathbf{W}(X) f(X,t)\,,\ \ 0 < \beta \leq 1,\ \ D > 0, \tag{12}$$

subject to initial and boundary conditions:

$$f(X,0) = \phi(X),\ X \in \Omega, \tag{13}$$

$$f(X,t) = \varphi(X,t),\ X \in \partial\Omega,\ t \geq 0. \tag{14}$$

Here, $\partial^\beta/\partial t^\beta(\cdot)$ is the m-β-derivative of order β. Δ is the Laplace operator. ∇ is the Hamilton operator. $\Omega = [0, L_1] \times [0, L_2] \times \cdots \times [0, L_d]$ is the spatial domain of the problem. d is the dimension of the space, $X = (x_1, x_2, \cdots, x_d)$. $\partial\Omega$ is the boundary of Ω. $f(X,t)$ denotes the probability density function of finding a particle at X in time t. The positive constant D depends on the temperature, the friction coefficient, the universal gas constant, and lastly, the Avogadro constant. $\mathbf{W}(X)$ is the external force. Equation (12) can be interpreted as a model of the diffusion of a particle under the action of the external force $\mathbf{W}(X)$ [37]. In this section, we establish the applicability of the proposed method through test examples.

4.1. Example 1: Two-Dimensional Time-Fractional Diffusion Equations

Let $D = 1, \mathbf{W}(X) = -1,\ \Omega = [0,1]$, then Equation (12) can be written as:

$$\frac{\partial^\beta f(x,t)}{\partial t^\beta} = \frac{\partial^2 f(x,t)}{\partial x^2} + \frac{\partial f(x,t)}{\partial x}\,,\ \ 0 < \beta \leq 1. \tag{15}$$

The initial condition is given by:

$$f(x,0) = e^x,\ x \in [0,1]. \tag{16}$$

Applying the appropriate properties given in Table 1 to Equation (15), we obtain the following recurrence relation:

$$F_{k+1}(x) = \frac{\Gamma(k\beta+1)}{\Gamma(\beta(k+1)+1)} \left(\frac{\partial^2 f(k-1)}{\partial x\,\partial x} + \frac{\partial f(k-1)}{\partial x} \right), \tag{17}$$

for $k = 0, 1, 2, \cdots$. From (17), we find the inverse transform coefficients of $x^{k\alpha}$ as:

$$
\begin{aligned}
F_0 &= e^x, \\
F_1 &= \frac{2e^x}{\Gamma(\beta+1)}, \\
F_2 &= \frac{4e^x}{\Gamma(2\beta+1)}, \\
F_3 &= \frac{8e^x}{\Gamma(3\beta+1)}, \cdots,
\end{aligned}
$$

or in general,

$$F_k = \frac{2^k e^x}{\Gamma(1+k\beta)}, \quad \text{where } k \geq 0. \tag{18}$$

After a small number of iterations, the differential inverse transform of $\{F_k(y)\}_{k=0}^{\infty}$ will provide the resulting series solution:

$$
\begin{aligned}
f(x,t) &= \sum_{k=0}^{\infty} F_k(x) t^{k\beta} \\
&= e^x + \frac{2e^x}{\Gamma(\beta+1)} t^{\beta} + \frac{4e^x}{\Gamma(2\beta+1)} t^{2\beta} \\
&\quad + \frac{8e^x}{\Gamma(3\beta+1)} t^{3\beta} + \cdots,
\end{aligned}
$$

which can be written in compact form,

$$f(x,t) = e^x \sum_{k=0}^{\infty} \frac{(2t^{\beta})^k}{\Gamma(1+k\beta)}. \tag{19}$$

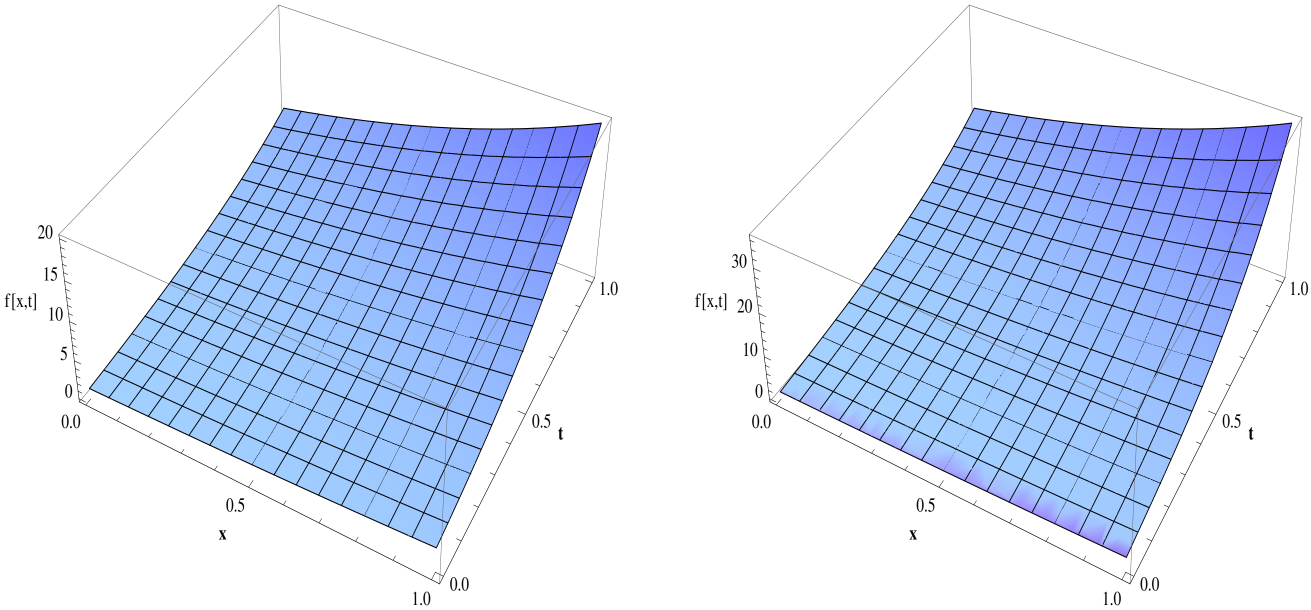
By the M-L function, we find the exact solution of Equation (15) subject to (16):

$$f(x,t) = e^x E_{\beta}(2t^{\beta}), \tag{20}$$

where $0 < \beta \leq 1$ and $E_{\beta}(z)$ is the one-parameter M-L function (1), and this is exactly the same solution obtained using the FVHPIM with the modified Riemann–Liouville derivative [37]. In comparison with the approximate solution obtained by the HDM via the modified beta derivative equation, the FRDTM gives the direct exact solution with simple computations [9]. In the case of $\beta = 1$, we have $E_1(2t) = e^{2t}$. Then, the exact solution of nonfractional Equation (15) when $\beta = 1$ is:

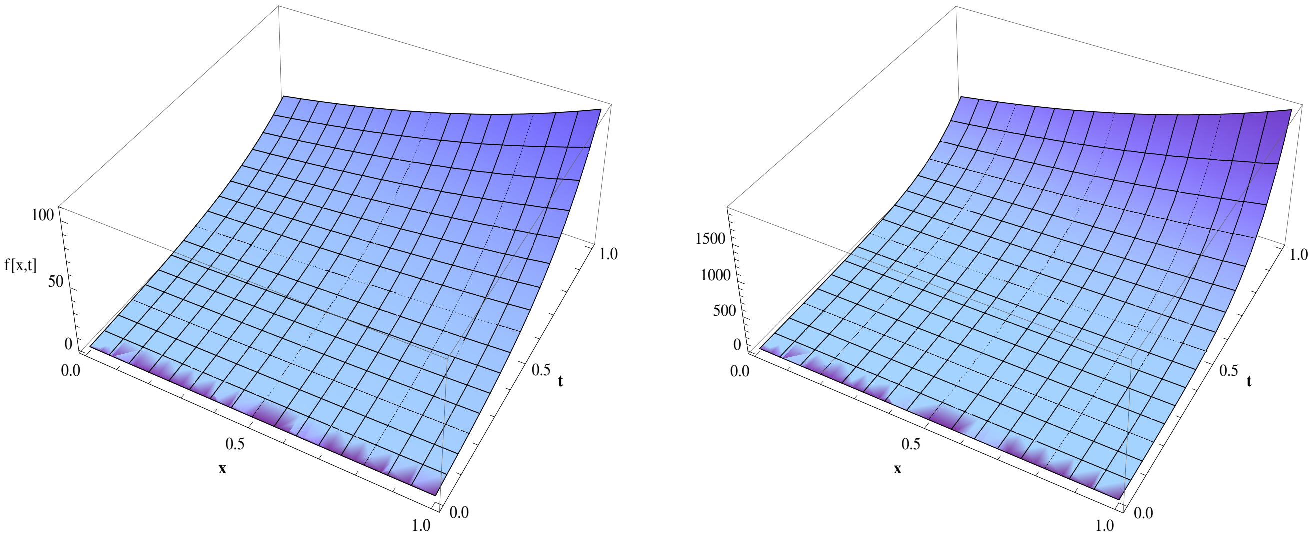
$$f(x,t) = e^{x+2t}. \tag{21}$$

Figure 1 shows the exact solution of nonfractional order and the three-dimensional plot for the approximate solution by the FRDTM of the fractional order ($\beta = 0.8$), while Figure 2 depicts the approximate solutions of the fractional orders ($\beta = 0.6, 0.4$). In these four figures, it is interesting to note that with a constant range for both variables t and x, the lower the value of the fractional order, the greater the value of the approximate solutions is at the highest value of the variable t. Figure 3 depicts solutions in two-dimensional plots for different values of β. Through this figure, we can notice that the lower the fractional order, the more the approximate solutions move away from the exact solution of nonfractional order, and their value increases with the value of the variable t being constant. Figure 4 shows solutions in two-dimensional plots for different values of x. In this figure, we notice, with a constant value of the nonfractional order β, that the greater the values of the variable x, the greater the values of the approximate solutions are with a constant range of the variable t.



(**a**) (**b**)

Figure 1. The FRDTM solutions $f(x,t)$: (**a**) $\beta = 1$ and (**b**) $\beta = 0.8$.



(**a**) (**b**)

Figure 2. The FRDTM solutions $f(x,t)$: (**a**) $\beta = 0.6$ and (**b**) $\beta = 0.4$.

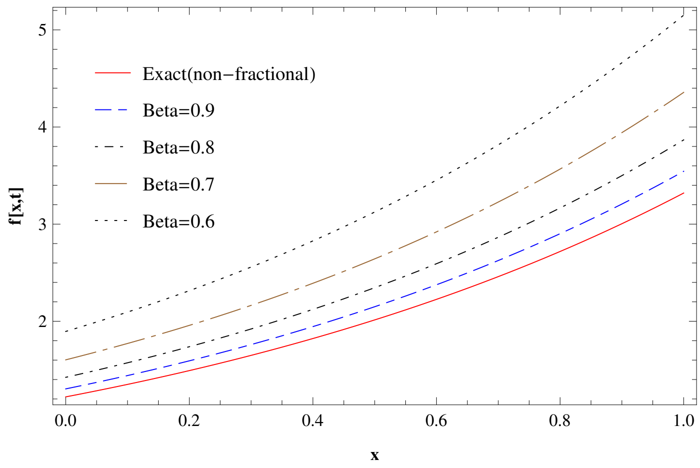


Figure 3. The FRDTM solutions $f(x,t)$ for (exact (nonfractional)), $\beta = 0.8, 0.6, 0.4$; $x \in [0,1]$ and $t = 0.1$.

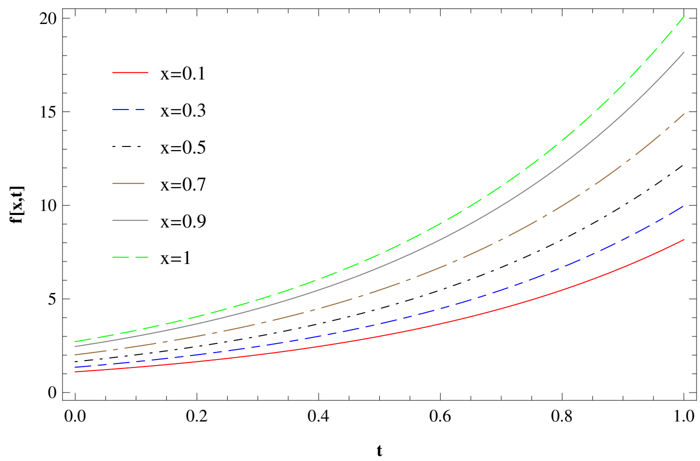


Figure 4. The FRDTM solutions $f(x,t)$ for different values of x; $\beta = 1$ and $t \in [0,1]$.

4.2. Example 2: Three-Dimensional Time-Fractional Diffusion Equations

Let $D = 1$, $\Omega = [0,1] \times [0,1]$, $\mathbf{W} = -(x,y)$ in Equation (12), then we have the following TFDE:

$$\frac{\partial^\beta f(x,y,t)}{\partial t^\beta} = \frac{\partial^2 f(x,y,t)}{\partial x^2} + \frac{\partial^2 f(x,y,t)}{\partial y^2} + x\frac{\partial f(x,y,t)}{\partial x} + y\frac{\partial f(x,y,t)}{\partial y} + 2f(x,y,t)\,, \quad (22)$$

with initial condition:

$$f(x,y,0) = x + y. \quad (23)$$

Using the suitable properties from Table 1 for Equation (22), we achieve the next recurrence relation:

$$F_{k+1}(x,y) = \frac{\Gamma(k\beta+1)}{\Gamma(\beta(k+1)+1)}\left(\frac{\partial^2 w(k)}{\partial x\,\partial x} + \frac{\partial^2 f(k)}{\partial y\,\partial y} + x\frac{\partial f(k)}{\partial x} + y\frac{\partial f(k)}{\partial y} + 2f(k)\right), \quad (24)$$

where $k = 0, 1, 2, \cdots$. The inverse transform coefficients of $t^{k\beta}$ are as follows:

$$
\begin{aligned}
F_0 &= x + y\,, \\
F_1 &= \frac{3(x+y)}{\Gamma(\beta+1)}\,, \\
F_2 &= \frac{9(x+y)}{\Gamma(2\beta+1)}\,, \\
F_3 &= \frac{27(x+y)}{\Gamma(3\beta+1)}\,, \cdots .
\end{aligned}
$$

More generally,

$$
U_k = (x+y)\frac{(3)^k}{\Gamma(1+k\beta)}\,. \tag{25}
$$

Again, if we continue in the same manner, and after a few iterations, the differential inverse transform of $\{F_k(x,y)\}_{k=0}^{\infty}$ will give the following series solution:

$$
\begin{aligned}
f(x,y,t) &= \sum_{k=0}^{\infty} F_k(x,y)t^{k\beta} \\
&= (x+y) + \frac{3(x+y)}{\Gamma(\beta+1)}t^{\beta} + \frac{9(x+y)}{\Gamma(2\beta+1)}t^{2\beta} \\
&\quad + \frac{27(x+y)}{\Gamma(3\beta+1)}t^{3\beta} + \cdots .
\end{aligned}
$$

In compact form,

$$
f(x,y,t) = (x+y)\sum_{k=0}^{\infty}\frac{(3t^{\beta})^k}{\Gamma(1+k\beta)}\,, \tag{26}
$$

and using the M-L function, we obtain the exact solution:

$$
f(x,y,t) = (x+y)\,E_{\beta}(3t^{\beta}), \tag{27}
$$

where $0 < \beta \leq 1$ and $E_{\beta}(z)$ is the one-parameter M-L function (1), which is exactly the same result obtained using the FVHPIM via the m-R-L derivative [37]. In the case of $\beta = 1$, $E_1(3t) = e^{3t}$, the exact solution of the nonfractional Equation (22) is:

$$
u(x,y) = (x+y)e^{3t}. \tag{28}
$$

Figure 5 shows the exact solution of nonfractional order and the three-dimensional plot of the approximate solution of the FRDTM ($\beta = 0.9$), while Figure 6 depicts the approximate solutions for ($\beta = 0.7, 0.5$). Figure 7 depicts solutions in two-dimensional plots for different values of β. Figure 8 shows solutions in two-dimensional plots for different values of x.

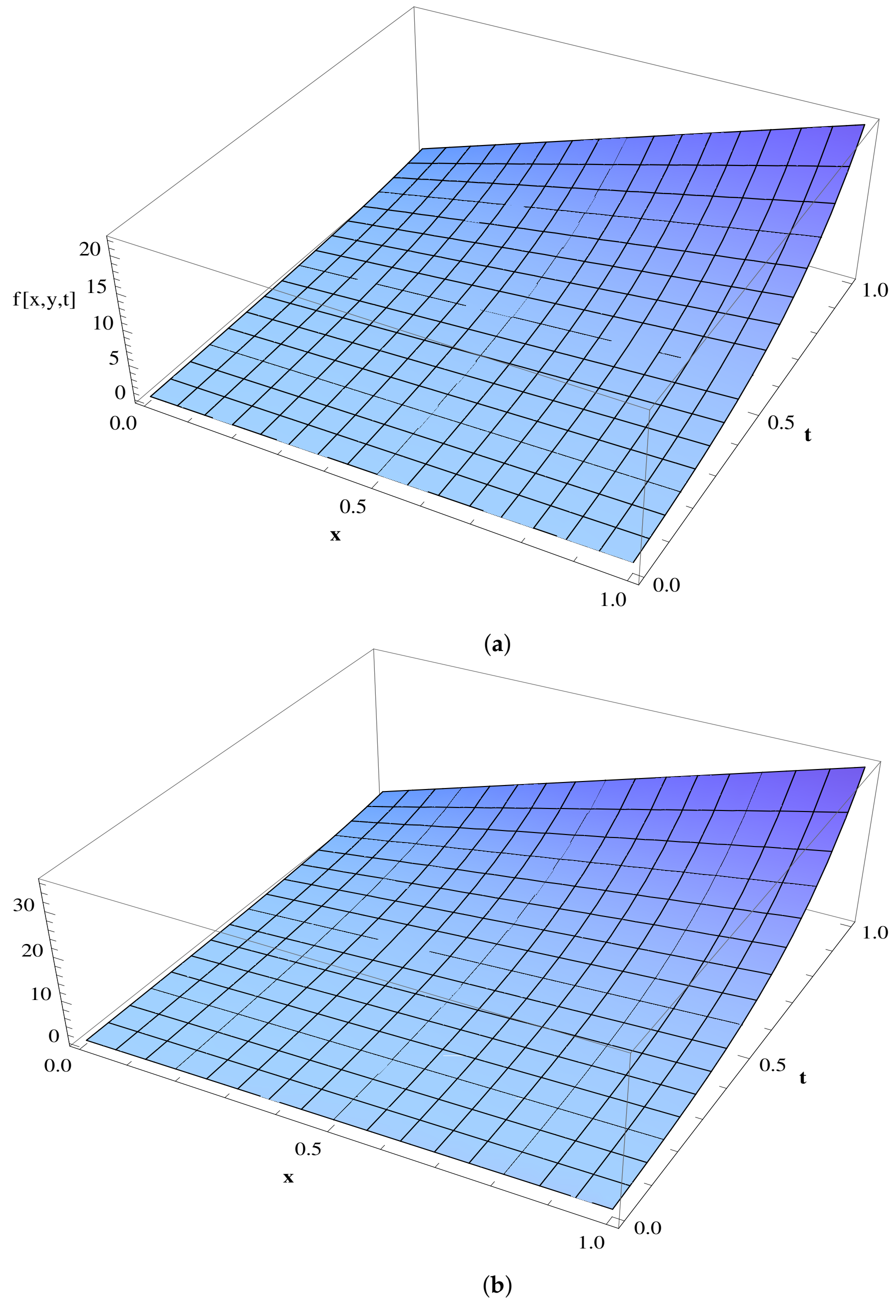
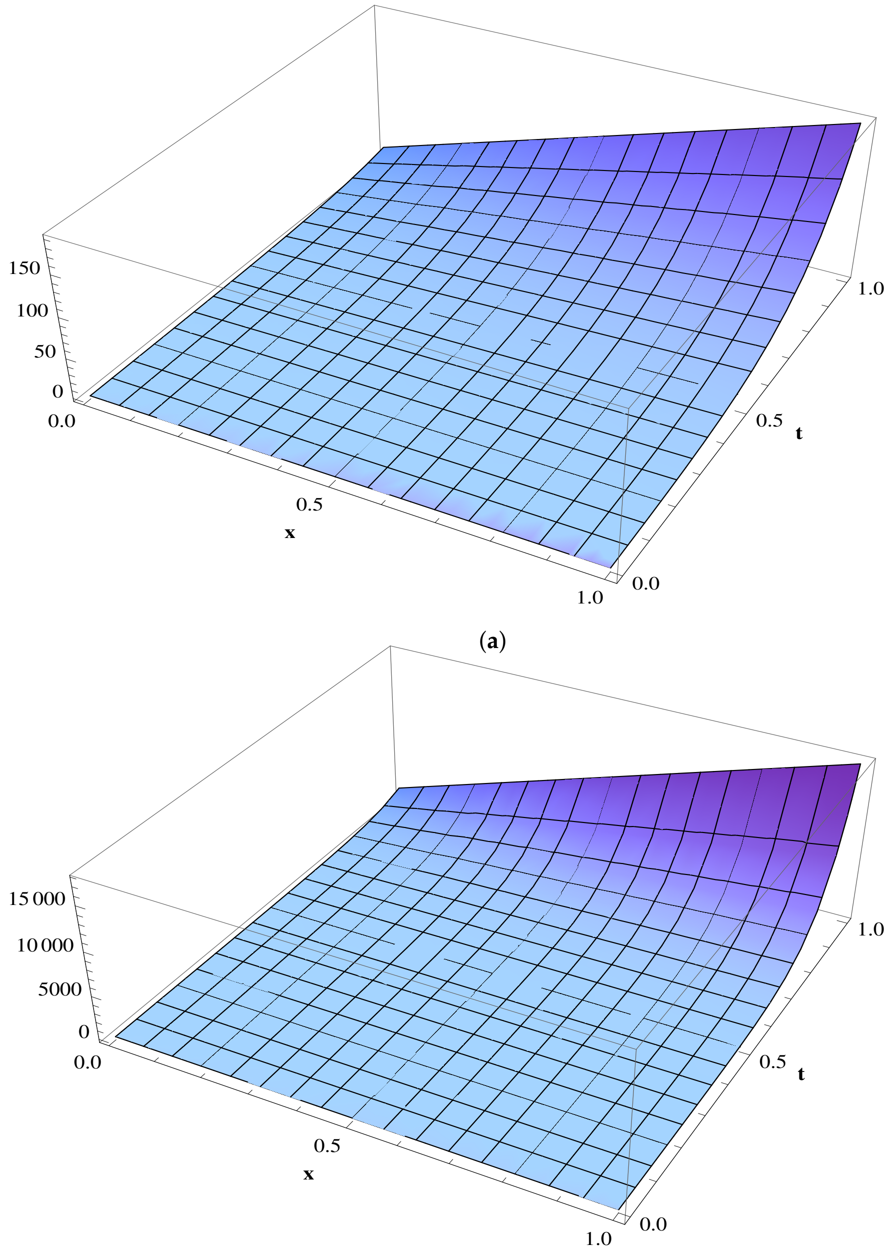


Figure 5. The FRDTM solutions $f(x,y,t)$: (**a**) (exact solution: nonfractional) $\beta = 1$ and (**b**) $\beta = 0.9$.



(**a**)

(**b**)

Figure 6. The FRDTM solutions $f(x, y, t)$: (**a**) $\beta = 0.7$ and (**b**) $\beta = 0.5$.

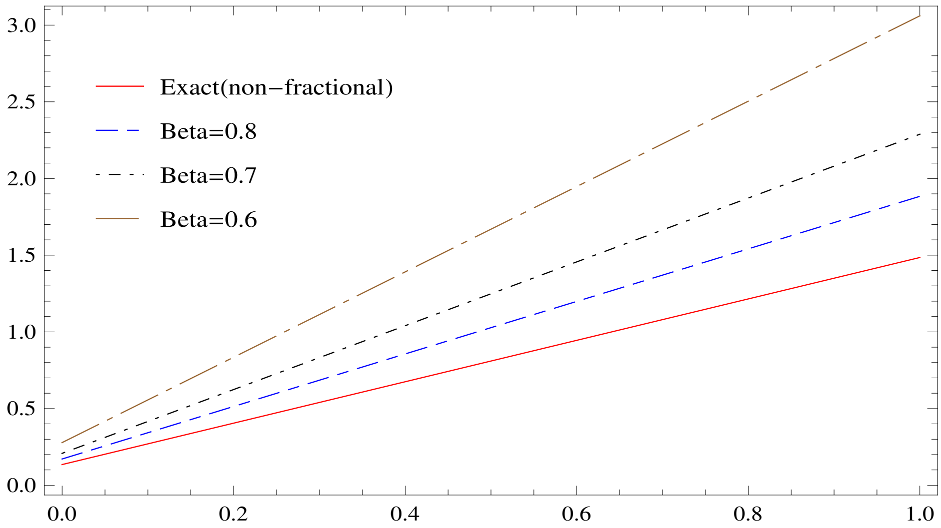


Figure 7. The FRDTM solutions $f(x,y,t)$ for $\beta = 1$ (exact (nonfractional)), $0.8, 0.7, 0.6$; $x \in [0,1]$; $t = 0.1$, and $y = 0.1$.

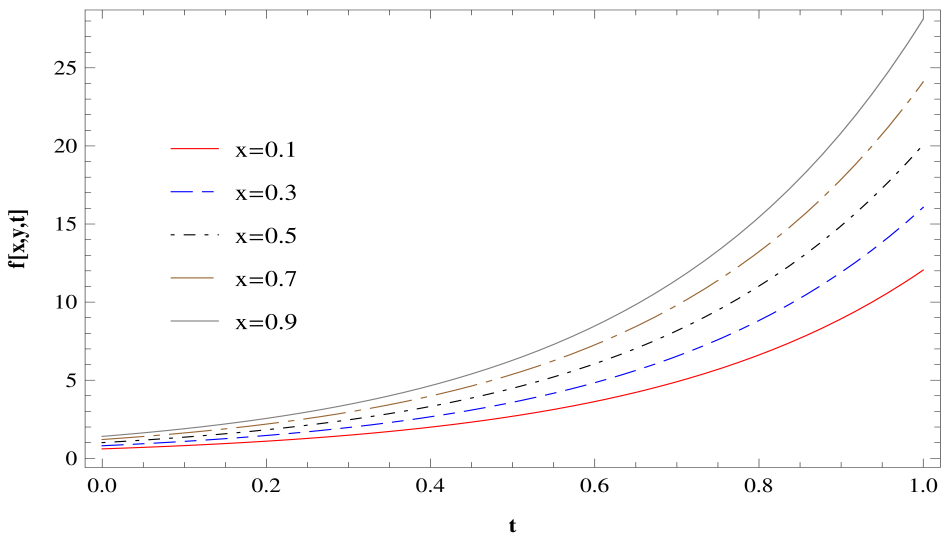


Figure 8. The FRDTM solutions $f(x,y,t)$ for different values of x; $\beta = 1$; $t \in [0,1]$, and $y = 0.5$.

4.3. Example 3: Four-Dimensional Time-Fractional Diffusion Equations

Let $D = 1$, $\Omega = [0,1] \times [0,1] \times [0,1]$, $\mathbf{F}(x,y,z) = -(x,y,z)$ in Equation (12), then we have the following TFDE:

$$\frac{\partial^\beta u(x,y,z,t)}{\partial t^\beta} = \Delta u(x,y,z,t) + x\frac{\partial u(x,y,z,t)}{\partial x} + y\frac{\partial u(x,y,z,t)}{\partial y} + z\frac{\partial u(x,y,z,t)}{\partial z} + 3u(x,y,z,t), \quad 0 < \beta \le 1, \tag{29}$$

with the initial condition,

$$u(x,y,z,0) = (x+y+z)^2. \tag{30}$$

Using the appropriate properties from Table 1 for Equation (29), we obtain the following recurrence relation:

$$F_{k+1}(x,y,z) = \frac{\Gamma(k\beta+1)}{\Gamma(\beta(k+1)+1)}\Big(\frac{\partial^2 w(k)}{\partial x\,\partial x} + \frac{\partial^2 f(k)}{\partial y\,\partial y} + \frac{\partial^2 w(k)}{\partial z\,\partial z} + x\frac{\partial f(k)}{\partial x} + y\frac{\partial f(k)}{\partial y} + z\frac{\partial f(k)}{\partial z} + 3f(k)\Big), \tag{31}$$

where $k = 0, 1, 2, \cdots$. The inverse transform coefficients of $t^{k\beta}$ are as follows:

$$\begin{aligned}
F_0 &= (x+y+z)^2, \\
F_1 &= \frac{5(x+y+z)^2+6}{\Gamma(\beta+1)}, \\
F_2 &= \frac{25(x+y+z)^2+48}{\Gamma(2\beta+1)}, \\
F_3 &= \frac{125(x+y+z)^2+294}{\Gamma(3\beta+1)}, \\
F_4 &= \frac{625(x+y+z)^2+1632}{\Gamma(4\beta+1)}, \\
F_5 &= \frac{3125(x+y+z)^2+8646}{\Gamma(5\beta+1)}, \cdots.
\end{aligned}$$

If we continue in the same manner, and after a few iterations, the differential inverse transform of $\{F_k(x,y,z)\}_{k=0}^{\infty}$ will give the following series solution:

$$\begin{aligned}
f(x,y,z,t) &= \sum_{k=0}^{\infty} F_k(x,y,z)t^{k\beta} \\
&= (x+y+z)^2 + \frac{5(x+y+z)^2+6}{\Gamma(\beta+1)}t^{\beta} + \frac{25(x+y+z)^2+48}{\Gamma(2\beta+1)}t^{2\beta} \\
&\quad + \frac{125(x+y+z)^2+294}{\Gamma(3\beta+1)}t^{3\beta} + \cdots.
\end{aligned}$$

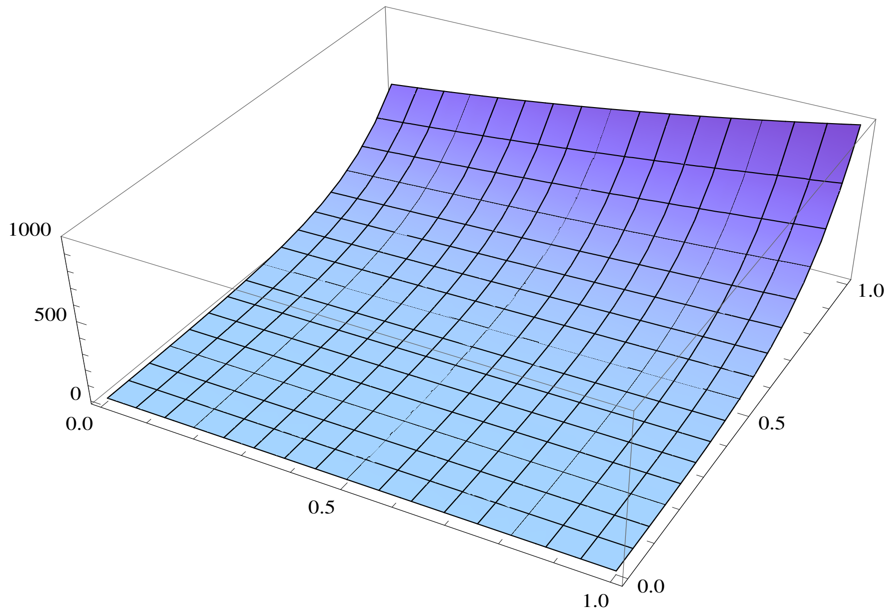
In the case of $\beta = 1$, the tenth-order approximate solution of nonfractional Equation (29) is given by:

$$\begin{aligned}
f_{10}(x,y,Z,t) = &\frac{t^{10}(9765625(x+y+z)^2+29119728)}{3628800} \\
&+\frac{t^{9}(1953125(x+y+z)^2+5800326)}{362880} \\
&+\frac{t^{8}(390625(x+y+z)^2+1152192)}{40320} \\
&+\frac{t^{7}(78125(x+y+z)^2+227814)}{5040} \\
&+\frac{1}{720}t^6\Big(15625(x+y+z)^2+44688\Big) \\
&+\frac{1}{120}t^5\Big(3125(x+y+z)^2+8646\Big) \\
&+\frac{1}{24}t^4\Big(625(x+y+z)^2+1632\Big) \\
&+\frac{1}{6}t^3\Big(125(x+y+z)^2+294\Big) \\
&+\frac{1}{2}t^2\Big(25(x+y+z)^2+48\Big) \\
&+t\Big(5(x+y+z)^2+6\Big) + (x+y+z)^2.
\end{aligned} \tag{32}$$

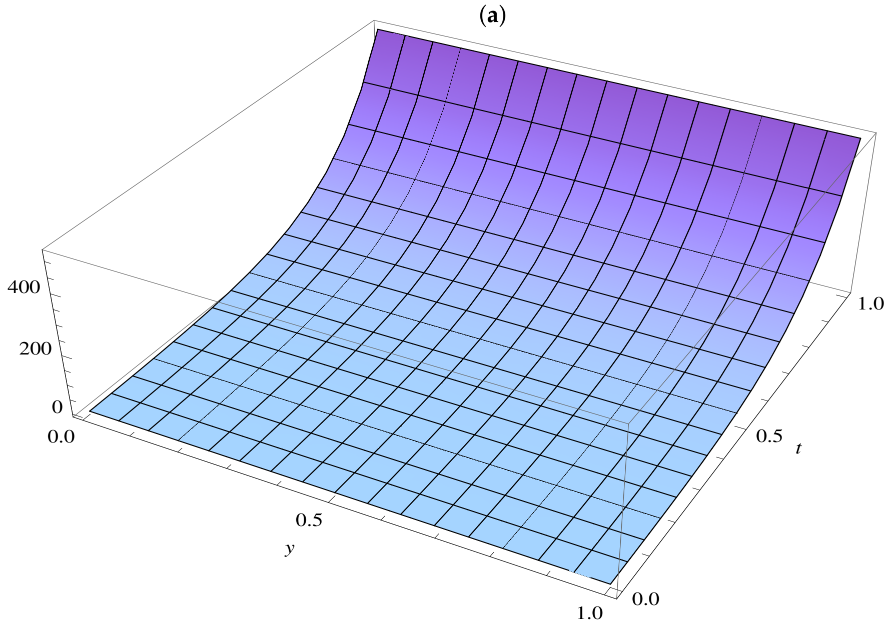
Equation (29) has been solved using the FVHPIM via m-R-L derivative [37], and the exact solution is:

$$u(x,y,z,t) = ((3 + (x+y+z)^2))E_\beta(5t^\beta) - 3E_\beta(3t^\beta). \tag{33}$$

Figure 9 shows the exact solution of nonfractional order and the three-dimensional plot of the approximate solution of the FRDTM ($\beta = 1$), while Figure 10 depicts the approximate solutions for ($\beta = 0.9, 0.7$). Figure 11 depicts solutions in two-dimensional plots for different values of β. Figure 12 shows solutions in two-dimensional plots for different values of x.



(**a**)



(**b**)

Figure 9. (**a**) (Exact solution: nonfractional) and (**b**) FRDTM $\beta = 1$.

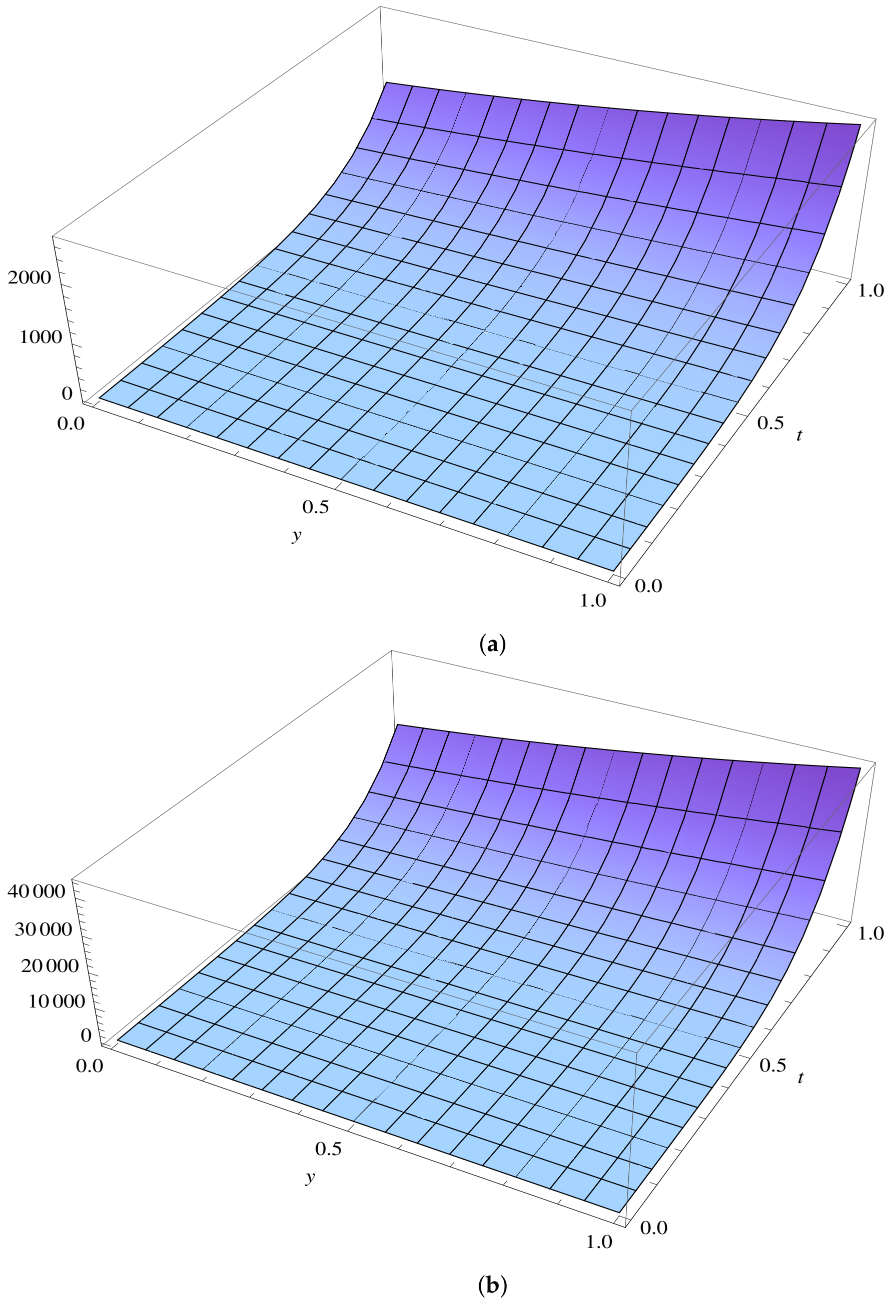


Figure 10. FRDTM solutions $f(x, y, z, t)$: (**a**) $\beta = 0.9$ and (**b**) $\beta = 0.7$.

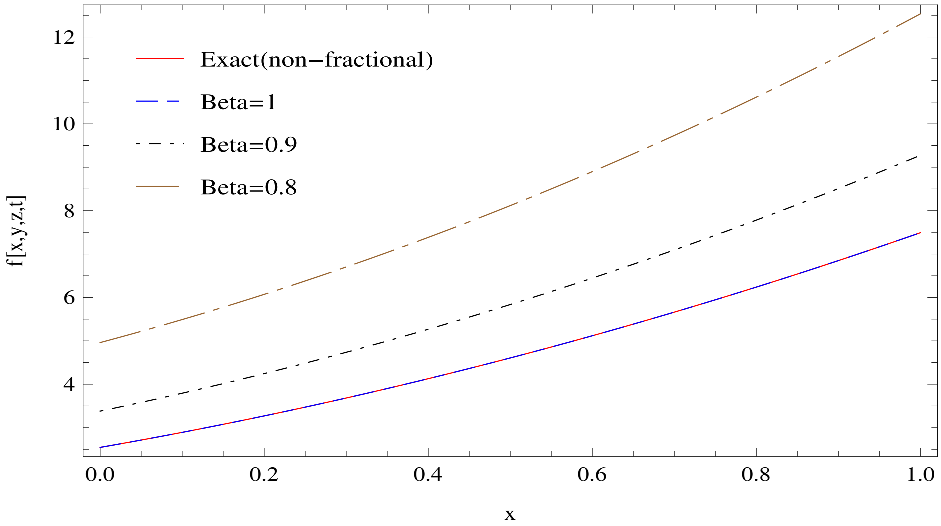


Figure 11. The FRDTM solutions $f(x, y, z, t)$ for $\beta = 1, 0.9, 0.8, 0.7$ and the exact (nonfractional) solution; $x \in [0, 1]$; $t = 0.1$, $z = 0.5$, and $y = 0.5$.

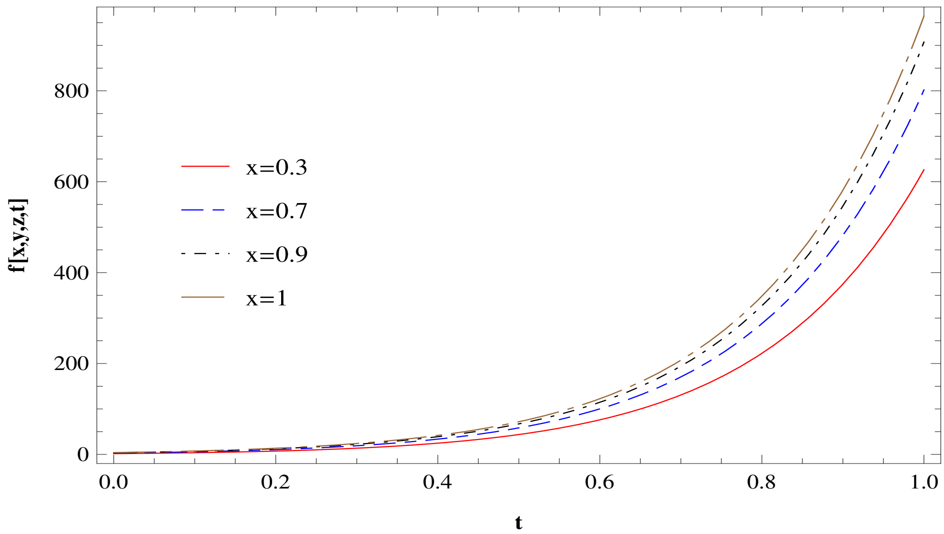


Figure 12. The FRDTM solutions $f(x, y, z, t)$ for different values of x; $\beta = 1$; $t \in [0, 1]$, $z = 0.5$, and $y = 0.5$.

5. Conclusions

Finding an exact solution is often considered difficult in most cases. By applying the FRDTM in Sections 4.1 and 4.2, we were able to find exact solutions in the case of the two- and three-dimensional time-fractional diffusion equations, then we plotted the approximate solutions for different values of the fractional-order β in the three- and two-dimensional time-fractional diffusion equation, and we also depicted the approximate solutions for different values of x. An approximate solution in the four-dimensional time-fractional diffusion equation was found in Section 4.3, and we compared it with the exact solution of a nonfractional differential equation, then we plotted the approximate solutions for different values of the fractional-order β in three- and two-dimensions. Furthermore, we depicted the approximate solutions for different values of x. The graphical representations of the exact and approximate solutions showed the power of the FRDTM for solving different dimensions of the time-fractional diffusion equation. The computations of this paper were carried out by using the computer package of Mathematica 9.

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