fractal and fractional

Article

Approximating Real-Life BVPs via Chebyshev Polynomials' First Derivative Pseudo-Galerkin Method

Mohamed Abdelhakem [1,2,*,†], Toqa Alaa-Eldeen [3,†], Dumitru Baleanu [4,5,6,†], Maryam G. Alshehri [7,†] and Mamdouh El-Kady [1,2]

1 Department of Mathematics, Faculty of Science, Helwan University, Cairo 11795, Egypt; mamdouh_elkady@cic-cairo.com
2 Department of Basic Science, School of Engineering, Canadian International College, New Cairo 11835, Egypt
3 High Institute for Civil and Architectural Engineering of 15 Mayo, Cairo 11744, Egypt; Toqa_alaaeldeen@science.helwan.edu.eg
4 Department of Mathematics, Cankaya University, 06790 Ankara, Turkey; dumitru@cankaya.edu.tr
5 Institute of Space Sciences, 077125 Magurele-Bucharest, Romania
6 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40447, Taiwan
7 Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk 71491, Saudi Arabia; mgalshehri@ut.edu.sa
* Correspondence: mabdelhakem@yahoo.com
† These authors contributed equally to this work.

Citation: Abdelhakem, M.; Alaa-Eldeen, T.; Baleanu, D.; Alshehri, M.G.; El-Kady, M. Approximating Real-Life BVPs via Chebyshev Polynomials' First Derivative Pseudo-Galerkin Method. *Fractal Fract.* **2021**, *5*, 165. https://doi.org/10.3390/fractalfract5040165

Academic Editors: Burcu Gürbüz and Arran Fernandez

Received: 16 July 2021
Accepted: 9 October 2021
Published: 12 October 2021

Abstract: An efficient technique, called pseudo-Galerkin, is performed to approximate some types of linear/nonlinear BVPs. The core of the performance process is the two well-known weighted residual methods, collocation and Galerkin. A novel basis of functions, consisting of first derivatives of Chebyshev polynomials, has been used. Consequently, new operational matrices for derivatives of any integer order have been introduced. An error analysis is performed to ensure the convergence of the presented method. In addition, the accuracy and the efficiency are verified by solving BVPs examples, including real-life problems.

Keywords: Chebyshev polynomials' first derivative; pseudo-Galerkin; weighted residual methods; error analysis; Lane–Emden; population model; MHD

1. Introduction

BVPs are used to model various problems in some fields, such as economics, biology, and engineering [1–5]. Due to the importance of ODEs, significant research work has been carried out about these problems [6–11]. In most instances, the exact solution of some ODEs cannot be obtained analytically, and numerical methods are considered as the way to obtain it.

Numerical methods are the set finite element, finite difference, spectral methods, etc., of all derivatives' approximation solutions that lead to the exact values. Spectral methods (SMs) are considered a class of techniques that are worked in applied mathematics to obtain numerical solutions for many various problems in various fields. Many applications are treated by spectral methods, which obtain better results [12,13]: time–space with sub-diffusion and super-diffusion, Abel's integral equations, and the multi-dimensional fractional Rayleigh–Stokes problem in fluids have all been solved by spectral methods. The consideration of SMs in approximating computations has been taken in the last few decades. SMs have been established to be an identical suitable tool to obtain the numerical solution of ODEs [14–16]. It deals with ODEs by stating these equations in terms of a series of unknown constants and smooth functions. The main idea of spectral methods is to use that set of tested functions, which are also known as expansion or basis approximating functions. Being very smooth, global, and orthogonal appear to be vital properties of these polynomials. It is worth mentioning that the spectral method has been presented in [17] and fifth kind Chebyshev [18], and is still being used and developed in [19]. For more recent work about this polynomial, kindly refer to [20,21]. The approximation of spectral methods depends on the type of its basis function. A numerical method is called stable if the error function does not increase with respect to time. The SMs involve three methods types, called the Galerkin [22], Tau [23], and collocation (pseudo-spectral) methods (SCM) [24–27]. These methods can be used to investigate operational matrices for the derivatives [28]. SCMs are obtained when the test functions in the variational formulation are Dirac functions based on a pre-determined set of collocation points. The follow-on system approximates the derivatives by differentiating a universal interpolant constructed through the collocation points. The set of collocation points is related to the basis functions as the nodes of quadrature formulae, which are used in the subtraction of spectral coefficients from the grid values. Recently, a developing method was raised as a collection between Galerkin and the collocation methods [29].

As mentioned, the core of the SMs is the choice of the orthogonal polynomials. The author in [30] introduced the idea of using the derivative of the orthogonal polynomials. Recently, the first complete contribution for the Legendre's derivative was introduced in [31]. Thus, we will continue this novelty in this work by investigating Chebyshev's derivative as a new base function. Consequently, the new operational matrices for differentiation have been constructed.

According to the SMs' point of view, the orthogonal polynomials have to satisfy the initial/boundary condition in the Galerkin method. While it is not a must in the collocation method, the expansion's constants are determined in terms of the unknown function. As a new trend, the authors in [16] introduced a mix between the Galerkin and collocation methods, called the pseudo-Galerkin method. Therefore, the pseudo-Galerkin method with Chebyshev's derivative will be used.

The frame of this paper is systematized as follows. In Section 2, some preliminaries and notations are presented for subsequent growths, and some differential and forms of Chebyshev polynomials (CHPs) are described. The description of the first derivative of the Chebyshev polynomials method (FDCHPs) and reviewing the algorithm of the method given are displayed in Section 3. Section 4 talks about the error analysis of the shown method. The numerical results of applying the obvious system on different problems are covered in Section 5. Finally, a brief conclusion is shown in Section 6.

2. Preliminaries

In this section, some important relations and properties of the CHPs and FDCHPs are introduced and presented. The relations that follow are the recurrence relations for the CHPs, $T_n(s)$, and its derivatives, $T^{'}_n(s)$, of degree n, [32]:

$$T_n(s) = 2sT_{n-1}(s) - T_{n-2}(s)\,, \tag{1}$$

$$2T_n(s) = \frac{1}{n+1}T^{'}_{n+1}(s) - \frac{1}{n-1}T^{'}_{n-1}(s)\,, \tag{2}$$

with $n = 2,3,...$, where $T_0(s) = 1$ and $T_1(s) = s$. The CHPs (FDCHPs) form a complete orthogonal set on the interval [-1,1] w.r.t. the weighting function $w(s) = \frac{1}{\sqrt{1-s^2}}$ $\left(\hat{w}(s) = \sqrt{1-s^2}\right)$ [33]:

$$\int_{-1}^{1} T_n(s)T_m(s)w(s)ds = \begin{cases} 0 & n \neq m\,, \\ \frac{\pi}{2} & n = m \neq 0\,, \\ \pi & n = m = 0\,, \end{cases} \tag{3}$$

$$\int_{-1}^{1} T'_n(s)T'_m(s)\hat{w}(s)ds = \begin{cases} 0 & n \neq m\,, \\ \frac{n^2\pi}{2} & n = m \neq 0\,. \end{cases} \tag{4}$$

Furthermore, $T_n^{'}(s)$ can be generated as:

$$T_n^{'}(s) = n\frac{sin(n\theta)}{sin(\theta)}, \tag{5}$$

where $s = cos(\theta)$.

The CHPs and FDCHPs can be expanded in a power series as [32,33]:

$$T_n(s) = \frac{n}{2}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n-k-1)!(2s)^{n-2k}}{(n-2k)!(k)!}, \tag{6}$$

where $n \geq 1$ and $\lfloor \frac{n}{2} \rfloor$ is the integer part of $n/2$.

This explicit representation allows us to derive many useful formulae respective to the CHPs [33]:

$$T_n^{'}(s) = 2n\sum_{k=0}^{n-1}{}' \quad T_k(s)\,; \qquad (n-k)\,odd, \tag{7}$$

where $\sum'$ refers to halving the first term when $(n-1)$ is an even number [32], and its inversion formula is given explicitly by:

$$s^n = 2^{1-n}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}{}' \binom{n}{k} T_{n-2k}(s)\,. \tag{8}$$

3. Chebyshev Polynomials' First Derivatives

This section is divided into two subsections. The first section's main target is investigating a higher-order operational matrix for derivatives in terms of our novel base functions, Chebyshev polynomials' first derivatives. Consequently, this investigated matrix will be used via the presented method to design an algorithm for solving some types of problems.

3.1. Chebyshev Polynomials' First Derivatives Operational Matrix

In this subsection, some important and novel relations will be investigated. The recurrence relation of the FDCHPs, $T_{n+1}^{'}(s)$, of degree n, denoted by ${}^{\mathcal{D}}T_n(s)$, is:

$${}^{\mathcal{D}}T_n(s) = 2s\ {}^{\mathcal{D}}T_{n-1}(s) + 2T_n(s) - {}^{\mathcal{D}}T_{n-2}(s)\,, \qquad n = 2,3,4,\ldots. \tag{9}$$

The set $\{ {}^{\mathcal{D}}T_n(s) : n = 0,1,2,...\}$ form an orthogonal set as follows:

$$\int_{-1}^{1} {}^{\mathcal{D}}T_n(s)\, {}^{\mathcal{D}}T_m(s)\hat{w}(s)ds = \begin{cases} 0 & n \neq m\,, \\ \frac{(n+1)^2\pi}{2} & n = m\,, \end{cases} \tag{10}$$

where $\hat{w}(s) = \sqrt{1-s^2}$.

Theorem 1. *The FDCHPs can be approximated in terms of their variable s by the following formula:*

$${}^{\mathcal{D}}T_n(s) = \sum_{k=0}^{n+1} k\, C_{r_k}^{n+1}\, s^{k-1}\,; \qquad (n-k+1)even\,, \tag{11}$$

where $n \geq 1$, $C_{r_k}^{n} = (-1)^{r_k}\, 2^{k-1}\left[2\binom{r_k+1}{r_k} - \binom{r_k+k-1}{r_k}\right]$, *and its inversion formula is given explicitly by:*

$$s^{n+1} = \frac{2^{-(n+1)}}{n+2}\sum_{k=0}^{n+1}\binom{n+2}{\frac{n-k+1}{2}}\, {}^{\mathcal{D}}T_k(s)\,, \tag{12}$$

where $r_k = \frac{n-k+1}{2}$ *and* $(n-k+1)$ *is even.*

Proof. The first part of the theorem is gained by deriving Equation (6), whereas the result of the second part is gained by deriving Equation (8). □

Lemma 1. *Let n be any non-negative integer. The moment formulae for the FDCHPs are given explicitly by:*

$$s\ {}^{\mathcal{D}}T_n(s) = \frac{n+1}{2} \sum_{k=n-1}^{n+1} \frac{1}{k+1}\ {}^{\mathcal{D}}T_k(s); \qquad k \neq n\,. \tag{13}$$

Proof. The result can be proved based on the substituting of Equation (9) into the recurrence relation (12) with some lengthy steps. □

By generalising this lemma by following some sequence mathematical induction steps, we obtain the next corollary.

Corollary 1. *Let m and n be any two non-negative integers. The moments formula for the FDCHPs are given explicitly by,*

$$s^m\ {}^{\mathcal{D}}T_n(s) = \frac{n+1}{(m+n+1)2^m} \sum_{k=0}^{m+n} \binom{m+n+1}{\frac{m+n-k}{2}}\ {}^{\mathcal{D}}T_k(s)\,, \tag{14}$$

where $m > n+1\,,$

$$s^m\ {}^{\mathcal{D}}T_n(s) = \frac{n+1}{2^m} \sum_{k=0}^{m+n} \frac{1}{k+1} \binom{m}{\frac{m+n-k}{2}}\ {}^{\mathcal{D}}T_k(s)\,, \tag{15}$$

where $m \leq n+1\,.$

To establish the derivatives' operational matrix, a relationship between the polynomial's derivative and the polynomial itself, as in the following theorem, needs to be investigated.

Theorem 2. *The FDCHPs can be represented in terms of their original polynomials as,*

$$\frac{d\ {}^{\mathcal{D}}T_n(s)}{d\,s} = 2\,(n+1) \sum_{L=0}^{n-1} {}^{\mathcal{D}}T_L(s)\,; \qquad (n-L)\ \textit{odd}, \tag{16}$$

Proof. The desired result is obtained by deriving Equation (7). □

Corollary 2. *The first derivative operational matrix,* $\mathbf{D}\left[{}^{\mathcal{D}}\mathbf{T}(s)\right]$*, of* ${}^{\mathcal{D}}\mathbf{T}(s)$*, can be written as,*

$$\mathbf{D}\left[{}^{\mathcal{D}}\mathbf{T}(s)\right] = \mathcal{H} \cdot {}^{\mathcal{D}}\mathbf{T}(s)\,, \tag{17}$$

where,
${}^{\mathcal{D}}\mathbf{T}(s) = \left({}^{\mathcal{D}}T_0(s), {}^{\mathcal{D}}T_1(s), \cdots, {}^{\mathcal{D}}T_r(s)\right)^T$, $\mathbf{D}\left[{}^{\mathcal{D}}\mathbf{T}(s)\right] = \left(\frac{d\,{}^{\mathcal{D}}T_0(s)}{d\,s}, \frac{d\,{}^{\mathcal{D}}T_1(s)}{d\,s}, \cdots, \frac{d\,{}^{\mathcal{D}}T_r(s)}{d\,s}\right)^T$,
and $\mathcal{H} = (\mathfrak{H}_{i\,j})_{i,j=1}^{r+1}$ *is a* $(r+1) \times (r+1)$ *lower triangular matrix defined as,*

$$\mathfrak{H}_{i\,j} = \begin{cases} 2\,(i+1)\,, & i > j,\ \ (i-j)\ \textit{odd}, \\ 0, & \textit{otherwise}\,. \end{cases} \tag{18}$$

Corollary 3. *The* p^{th} *derivative of* ${}^{\mathcal{D}}\mathbf{T}(s)$ *is:*

$$\mathbf{D}^p\left[{}^{\mathcal{D}}\mathbf{T}(s)\right] = \mathcal{H}^p \cdot {}^{\mathcal{D}}\mathbf{T}(s)\,. \tag{19}$$

In order to avoid the lack of accuracy due to the multiplication process for the matrix, we investigate the elements of the matrix directly in the following theorem.

Theorem 3. *Let p be any positive integer. Then, the FDCHPs' p^{th}-derivative operational matrix, $\mathcal{H}^p$, may be written as $\mathcal{H}^p = \left(\mathfrak{H}_{ij}^{(p)}\right)_{i,j=1}^{r+1}$, such that:*

$$\mathfrak{H}_{ij}^{(p)} = 2^p \frac{(i+1)}{(p-1)!} \begin{cases} \left(\frac{i-j-p+2}{2}\right)_{p-1} \left(\frac{i+j-p+4}{2}\right)_{p-1}, & i > j, \ \ (i-j-p) \ even, \\ 0, & otherwise\,. \end{cases} \tag{20}$$

Proof. By using the mathematical induction with the aide of Corollaries 2 and 3. □

3.2. Chebyshev's Derivative Pseudo-Galerkin Method

The presented method will be pseudo-Galerkin as one of the residual methods. This method is a mix between the Galerkin method and the collocation method.

The used collocation points, $s_i\,; \quad 0 \le i \le n$, in that method are the normal Chebyshev–Gauss–Lobatto points, $s_i = cos(\frac{\pi i}{n})\,; \quad 0 \le i \le n$, or the equidistant points $s_i = -1 + \frac{2}{N} i\,; \quad 0 \le i \le n$.

Consider the BVPs:

$$h\Big(f_m(s)\,u^{(m)}(s)\,,\, f_{m-1}(s)\,u^{(m-1)}(s)\,,\, f_{m-2}(s)\,u^{(m-2)}(s)\,, \\ ,\ldots \quad , f_0(s)u(s)\big) = 0; \qquad -1 \le s \le 1\,, \tag{21}$$

with the initial/boundary conditions:

$$\begin{cases} u(-1) = \alpha_0\,, \quad u(1) = \beta_0\,, \\ u^{(1)}(-1) = \alpha_1\,, \quad u^{(1)}(1) = \beta_1\,, \\ \vdots \\ u^{(q)}(-1) = \alpha_q\,, \quad u^{(q)}(1) = \beta_q\,, \end{cases} \tag{22}$$

where $\{\alpha_i\}_0^q$ and $\{\beta_i\}_0^q$ are constants. The number of boundary conditions is equal to the order of the problem.

Furthermore, $u(s)$ can be approximated as:

$$u(s) \approx u_n(s) = \sum_{k=0}^{n} A_k\, {}^{\mathcal{D}}T_k(s)\,, \tag{23}$$

where A_k denotes constants.

In addition, ${}^{\mathcal{D}}T_k(s_i)$ forms a square matrix. i.e., the elements of this matrix are the values of the first order derivatives of CHPs at s_i.

The derivatives of the differential equation can be presented, according to Equations (19) and (20) as the following:

$$\frac{d^m u}{ds^m} = \sum_{k=0}^{n} \sum_{j=0}^{k-m} A_k \mathfrak{H}_{ij}^{(m)} \left({}^{\mathcal{D}}T_j(s)\right), \tag{24}$$

where m is the order of derivatives. At $m = 0$, Equation (24) becomes equivalent to Equation (23).

Substituting from Equation (24) into Equation (21) yields:

$$h\left(f_m(s)\sum_{k=0}^{n}\sum_{j=0}^{k-m}A_k\mathfrak{H}_{ij}^{(m)}\ {}^{\mathcal{D}}T_j(s)\,,f_{m-1}(s)\sum_{k=0}^{n}\sum_{j=0}^{k-m+1}A_k\mathfrak{H}_{ij}^{(m-1)}\ {}^{\mathcal{D}}T_j(s),\right.$$
$$\left.\dots,f_0(s)\sum_{k=0}^{n}\sum_{j=0}^{k}A_k\mathfrak{H}_{ij}\ {}^{\mathcal{D}}T_j(s)\right)=0;$$
$$-1\leq s\leq 1\,, \tag{25}$$

such that:

$$\begin{cases} \sum_{k=0}^{n}\sum_{j=0}^{k}A_k\mathfrak{H}_{ij}\ {}^{\mathcal{D}}T_j(-1)=\alpha_0\,, & \sum_{k=0}^{n}\sum_{j=0}^{k}A_k\mathfrak{H}_{ij}\ {}^{\mathcal{D}}T_j(1)=\beta_0\,,\\ \sum_{k=0}^{n}\sum_{j=0}^{k-1}A_k\mathfrak{H}_{ij}^{(1)}\ {}^{\mathcal{D}}T_j(-1)=\alpha_1\,, & \sum_{k=0}^{n}\sum_{j=0}^{k-1}A_k\mathfrak{H}_{ij}^{(1)}\ {}^{\mathcal{D}}T_j(1)=\beta_1\,,\\ \vdots & \\ \sum_{k=0}^{n}\sum_{j=0}^{k-q}A_k\mathfrak{H}_{ij}^{(q)}\ {}^{\mathcal{D}}T_j(-1)=\alpha_q\,, & \sum_{k=0}^{n}\sum_{j=0}^{k-q}A_k\mathfrak{H}_{ij}^{(q)}\ {}^{\mathcal{D}}T_j(1)=\beta_q\,. \end{cases} \tag{26}$$

Equation (25) with Equation (26) can be collocated in order to construct an algebraic equations system for unknowns $\{A_k\}_0^m$. Mainly, the linearity of the constructed algebraic system depends on the formulation of the given BVP, Equations (21) and (22). In the case of non-linearity, the non-linear algebraic system may be treated by numerous methods, such as the Newton method, secant method, or SOR–Steffensen–Newton method, for finding the solution of systems of non-linear equations [34]. Finally, A_k can be approximated by solving the previous system to obtain the expanded approximate solution, which depends on the FDCHPs. Algorithm 1 represented the solution's steps where any software, such as Matlab or Mathematica, can be programmed.

Algorithm 1: Algorithm steps for solving ODE via FDCHPs pseudo-Galerkin

Step 1: Input : $n\in\mathbb{N}$;
Step 2: Select the $\{s_i\}_{i=0}^{i=n}$ (collocation or the equidistant points);
Step 3: Build the base function matrix $\{{}^{\mathcal{D}}T_j(s_i)\}_{i,j=0}^{i,j=n}$ using Equation (9);
Step 4: Construct the ${}^{\mathcal{D}}T_n(s)$ derivative's matrices
$\{D^p\,{}^{\mathcal{D}}T_j(s_i)\}_{i,j=0}^{i,j=n}, p=1,2,3,\cdots$ using Equation (19);
Step 5: Expand the ODE as shown in Equation (25) using steps **3** and **4**;
Step 6: Solve the previous system to obtain the $\{\mathcal{A}_k\}_0^n$;
Step 7: Substitute from step **6** into Equation (23) to obtain the approximate solution.

---

4. Error Analysis

Throughout this section, we need to ensure the convergence and the boundness of the spectral expansion before proceeding to the numerical computations. Finally, the stability's order has been determined. The following lemma will be needed for the investigation of the error analysis and convergence.

Lemma 2 ([35])**.** *Let $u(s)$ be a smooth continuous function, such that:*
(i) $u(k)=a_k$,
(ii) $u(s)$ is positive and decreasing for $n\leq s$,
(iii) $\sum a_n$ convergent, and $R_n=\sum_{k=n+1}^{\infty}a_k$, then

$$R_n\leq\int_n^{\infty}u(s)ds. \tag{27}$$

Theorem 4. *Let $u(s)$ be a continuous function on the interval $[-1,1]$, which can be expanded in terms of FDCHPs, Equation (23), $|u^{(r)}(s)| < M$, where r is a positive integer. Then:*

$$|A_k| \leq \frac{2M}{k^{r+1}}, \qquad k > 2. \tag{28}$$

Proof. According to the expansion Equation (23), and the orthogonality of FDCHPs, Equation (10):

$$A_k = \frac{1}{\gamma_k} \int_{-1}^{1} u(s) \, {}^{\mathcal{D}}T_k(s) \sqrt{1-s^2} \quad ds, \tag{29}$$

where $\gamma_k = \frac{(k+1)^2\pi}{2}$. Furthermore, Equation (5) can be applied to solve the integration to obtain:

$$A_k = -\frac{1}{(k+1)\pi} \int_0^{\pi} u(s)[\cos{(k+2)\theta} - \cos{k\theta}] \quad d\theta, \tag{30}$$

with integrating by parts:

$$A_k = \frac{2}{(k+1)\pi} \int_0^{\pi} u^{'}(s) \left[\frac{\sin{(k+2)\theta}}{k+2} - \frac{\sin{k\theta}}{k} \right] \quad d\theta. \tag{31}$$

By taking the maximum values for $\sin(k+1)\theta$ and $\sin(k\theta)$, it leads to $|A_k| \leq \frac{2M}{k^2}$, and by taking the integration by parts for r-1 times:

$$A_k = -\frac{1}{(k+1)\pi} \int_0^{\pi} u^{(r)}(s)[\varphi_i(\theta) - \varphi_j(\theta)] \quad d\theta, \tag{32}$$

where $\varphi_i(\theta)$ and $\varphi_j(\theta)$ are two trigonometric polynomials in $sin(\theta)$ and $cos(\theta)$. After noting that $|sin(\theta)|, |cos(\theta)| \leq 1$, and after some influences, the required result is obtained. □

Theorem 5. *Let $u(s)$ be a continuous function that satisfies Theorem 4. Then:*

$$|u(s) - u_n(s)| \lesssim O\left(\frac{1}{n^{r-2}}\right). \tag{33}$$

Proof. By using the expansion Equation (23):

$$|u(s) - u_n(s)| = |\sum_{k=n+1}^{\infty} A_k \, {}^{\mathcal{D}}T_k(s)| \leq |\sum_{k=n+1}^{\infty} A_k|. \tag{34}$$

Then, by applying Lemma (2) and the result of Theorem 4:

$$|u(s) - u_n(s)| = \int_n^{\infty} A(k) \, dk \lesssim O(\frac{1}{n^{r-2}}), \qquad r > 2. \tag{35}$$

□

Corollary 4. *Let $u(s)$ be a function that satisfies the assumptions of Theorem 5. Then, the step stability of two successive approximations u_n and u_{n+1} of the function u satisfies:*

$$|u_n - u_{n+1}| \lesssim O\left(\frac{1}{n^{r-2}}\right). \tag{36}$$

In the next section, the FDCHPs pseudo-Galerkin method will be tested through several numerical examples. These will show the efficiency, accuracy, and reliable application of the method. The results ensure the theoretical aspects of the error analysis section.

5. Results

In this section, five examples are solved to demonstrate the efficiency and permanency of the planned technique. These examples include well-known BVPs and real life applications, such as the Lane–Emden equation, population model, and fluid problem.

Example 1. *Consider the following 2^{th} order Lane–Emden Equation [36–38]:*

$$u^{''}(s) + \frac{2}{s}u^{'}(s) + u^{m}(s) = 0\,, \quad 0 \le m \le 5\,,$$

with the following initial conditions, $u(0) = 1$, $u^{'}(0) = 0$, exact solutions $u(s) = 1 - \frac{s^2}{6}$ for $m = 0$, $u(s) = \frac{\sin(s)}{s}$ for $m = 1$, and $u(s) = (1 + \frac{s^2}{3})^{-\frac{1}{2}}$ for $m = 5$. This equation has physical importance for the value of s when $u(s) = 0$, and this value for s is approximately 2.5 when $m = 0$, and about 3.1 for $m = 1$. These cases will be discussed as follows:

$\mathbf{m = 0}$*, and* $\mathbf{s} \in [\mathbf{0}, \mathbf{2.5}]$*: Shifting form $[0, 2.5]$ to the domain of FDCHPs $[-1, 1]$, using the expansion (23): $u(s) \approx u_2(s) = \sum_{k=0}^{2} A_k\, {}^{\mathcal{D}}T_k(s)$, and applying FDCHPs pseudo-Galerkin to obtain the system:*

$$128A_1 + 384A_2 = -25, \quad 64A_1 + 768A_2 = -25, \tag{37}$$

$$A_0 - 4A_1 + 9A_2 = 1, \quad 4A_1 - 24A_2 = 0\,, \tag{38}$$

which yields to $A_0 = 259/384$, $A_1 = -25/192$, and $A_2 = -25/1152$. Consequently, $u(s) = \frac{71}{96} - \frac{50}{96}s - \frac{25}{96}s^2$, which is the exact solution for $s \in [-1, 1]$.

$\mathbf{m = 1}$*, and* $\mathbf{s} \in [\mathbf{0}, \mathbf{3.1}]$*: Tables 1 and 2 represent the point-wise absolute error (PW-AE) for $s \in [0, 1]$ and $s \in [0, 3.1]$, respectively. The author in [38] obtained 10^{-13} at $n = 16$ for $s \in [0, 1]$. At the same time, we achieved a better accuracy, 10^{-16}, with a greater efficiency $n = 12$. For $s \in [0, 3.1]$, we achieve a greater efficiency by using $n = 15$ against the method in [38]. These results show the privilege of the FDCHPs pseudo-Galerkin method.*

Table 1. PW-AE of Example 1 for $m = 1$.

s	**FDCHPs Pseudo-Galerkin** $\boldsymbol{n = 12}$	**[38]** $\boldsymbol{n = 16}$
0.0	2.22×10^{-16}	-
0.1	0	6.24×10^{-13}
0.2	1.11×10^{-16}	-
0.3	1.11×10^{-16}	-
0.4	0	-
0.5	1.11×10^{-16}	5.82×10^{-13}
0.6	0	-
0.7	0	-
0.8	0	-
0.9	0	-

Table 2. PW-AE of Example 1 for $m = 1$.

s	FDCHPs Pseudo-Galerkin $n = 15$	[38] $n = 30$
0.0	4.44×10^{-16}	0
0.1	2.88×10^{-15}	5.48×10^{-15}
0.2	5.77×10^{-15}	6.02×10^{-15}
0.3	6.77×10^{-15}	6.41×10^{-15}
0.4	6.99×10^{-15}	5.97×10^{-15}
0.5	7.54×10^{-15}	6.05×10^{-15}
0.6	7.66×10^{-15}	6.42×10^{-15}
0.7	7.32×10^{-15}	5.71×10^{-15}
0.8	7.43×10^{-15}	5.16×10^{-15}
0.9	7.32×10^{-15}	5.67×10^{-15}
1.0	7.10×10^{-15}	6.15×10^{-15}
1.5	5.88×10^{-15}	5.22×10^{-15}
2.0	4.05×10^{-15}	3.24×10^{-15}
2.5	2.30×10^{-15}	1.19×10^{-15}
3.0	6.45×10^{-16}	3.20×10^{-16}
3.1	4.59×10^{-16}	9.02×10^{-16}

$\mathbf{m} = \mathbf{5}$, *and* $\mathbf{s} \in [\mathbf{0}, \mathbf{1}]$: *FDCHPs pseudo-Galerkin method obtained* 9.91×10^{-13} *at* $n = 15$ *and* 2.67×10^{-15} *at* $n = 19$ *as a MAE, while the MAE is* 1.73×10^{-13} *at* $n = 16$ *in [38]. Figure 1 shows the stability of the error.*

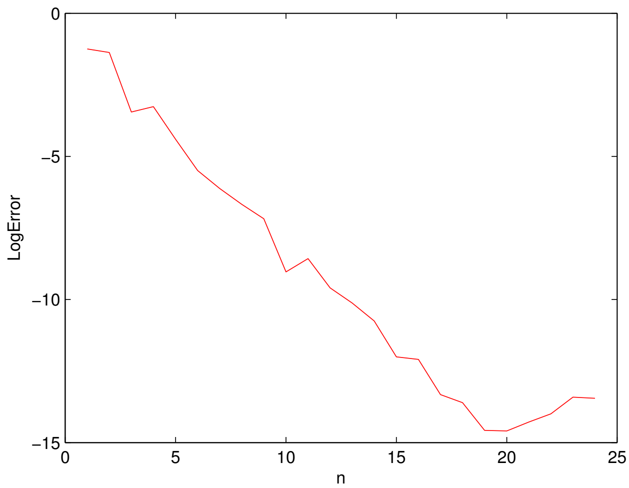

Figure 1. LogError for Example 1 at $m = 5$.

Example 2. *Consider the following* 2^{nd} *order Lane–Emden Equation [39]:*

$$\begin{cases} u''(s) + \frac{1}{s}u'(s) = -e^{u(s)}; \quad 0 \le s \le 1 \\ \qquad u'(0) = 0\,,\; u(1) = 0. \end{cases} \tag{39}$$

The exact solution is $u(s) = 2\log\frac{4-2\sqrt{2}}{(3-2\sqrt{2})s^2+1}$. *The PW-AE obtained using the FDCHPs pseudo-Galerkin method and methods in [40,41] were presented in Table 3 for different values of* n. *The best MAE is shown in Table 4, with a comparison with other methods. As we noted, the presented method reached the double precision. i.e., almost the exact solution for* $n = 18$. *This shows the privilege of the FDCHPs pseudo-Galerkin against the methods represented in [39–41].*

Table 3. PW-AE of Example 2.

s	FDCHPs Pseudo-Galerkin $n = 11$	$n = 14$	$n = 18$	[40] $n = 14$	[41] $n = 14$	[41] $n = 28$
0.0	3.34×10^{-10}	1.81×10^{-13}	4.44×10^{-16}	6.72×10^{-08}	-	-
0.1	2.19×10^{-10}	1.11×10^{-13}	3.89×10^{-16}	6.69×10^{-08}	2.05×10^{-10}	5.99×10^{-13}
0.2	1.59×10^{-10}	8.07×10^{-14}	5.55×10^{-16}	7.87×10^{-09}	2.44×10^{-10}	5.69×10^{-13}
0.3	1.22×10^{-10}	6.23×10^{-14}	3.86×10^{-16}	6.92×10^{-09}	2.64×10^{-10}	4.70×10^{-13}
0.4	9.51×10^{-11}	4.82×10^{-14}	3.33×10^{-16}	2.87×10^{-08}	2.74×10^{-10}	3.83×10^{-13}
0.5	7.27×10^{-11}	3.67×10^{-14}	3.61×10^{-16}	7.40×10^{-10}	2.77×10^{-10}	3.17×10^{-13}
0.6	5.38×10^{-11}	2.73×10^{-14}	5.55×10^{-17}	6.32×10^{-08}	2.11×10^{-10}	3.37×10^{-13}
0.7	3.72×10^{-11}	1.86×10^{-14}	2.50×10^{-16}	6.95×10^{-08}	1.57×10^{-10}	3.66×10^{-13}
0.8	2.23×10^{-11}	1.11×10^{-14}	1.25×10^{-16}	3.38×10^{-09}	1.07×10^{-10}	3.49×10^{-13}
0.9	8.80×10^{-12}	4.36×10^{-15}	4.86×10^{-17}	7.85×10^{-08}	5.72×10^{-11}	1.98×10^{-13}
1.0	1.42×10^{-16}	9.18×10^{-17}	8.28×10^{-17}	6.63×10^{-08}	-	-

Table 4. The best MAE for Example 2.

Method	Best MAE
[39] "$n = 8$"	6.35×10^{-07}
[40] "$n = 14$"	6.32×10^{-08}
[41] "$n = 28$"	1.98×10^{-13}
[42] "$n = 512$"	9.72×10^{-15}
FDCHPs Pseudo-Galerkin "$n = 18$"	5.55×10^{-16}

Example 3. *Consider the integral Equation [43,44]:*

$$u(s) = e^s - \int_0^s (s - x)u(x)dx; \qquad 0 \leq s \leq 1, \tag{40}$$

with the exact solution $u(s) = \frac{1}{2}(e^s + cos(s) + sin(s))$. *Equation (40) represents the population model such that,* $u(s)$ *is the number of female births,* e^s *is the contribution of birth due to female already present at time* s*, and* $s - x$ *is the net maternity function of females class age* x *at time* s*. Equation (40) can be converted to the following BVP:*

$$u''(s) + u(s) = e^s; \qquad 0 \leq s \leq 1, \tag{41}$$

with the initial conditions $u(0) = 1$, $u'(0) = 1$. *The best MAE for the obtained results was shown in Table 5. The comparisons with other methods showed the efficiency of the presented method. Furthermore, Figure 2 illustrates the stability of the solution through the domain interval* $[0, 1]$*, for different values of* n.

Table 5. The best MAE for Example 3.

Method	Best MAE
[43]	2.14×10^{-14}
[44]	1.25×10^{-15}
FDCHPs Pseudo-Galerkin	4.44×10^{-16}

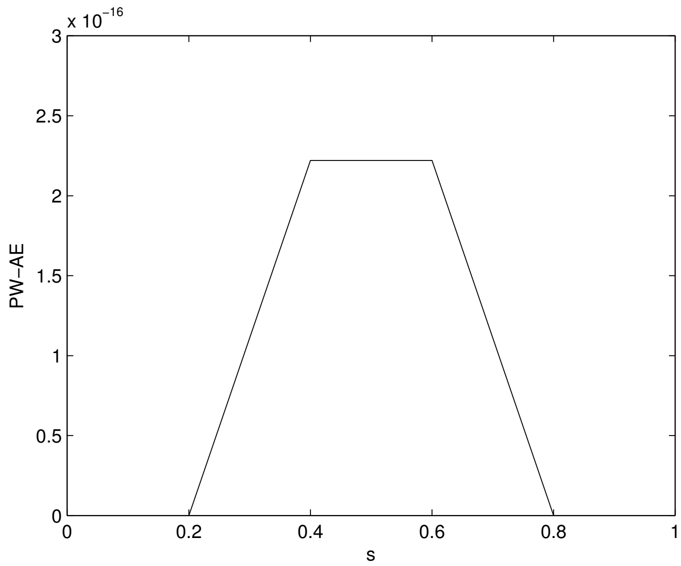

Figure 2. PW-AE for Example 3.

Example 4. *Consider 4^{th} order BVP [8]:*

$$32u^{(4)}(s) - 8u^{(2)}(s) - 2u(s) = (s-5)e^{\frac{s+1}{2}}; \qquad -1 \leq s \leq 1, \tag{42}$$

subject to $u(-1) = 1$, $u'(-1) = 0$, $u(1) = 0$, and exact solution $u(s) = (1-s)\,e^{(\frac{1+s}{2})}$. In Table 6, the PW-AE of Example 4 is listed. The values ensure both the accuracy and efficiency of the FDCHPs pseudo-Galerkin method.

Table 6. PW-AE for Example 4.

s	**FDCHPs Pseudo-Galerkin** $n = 12$	$n = 14$	**[8]** $n = 20$
-1	6.66×10^{-16}	2.22×10^{-16}	0
-0.6	3.10×10^{-14}	3.33×10^{-16}	2.17×10^{-14}
-0.2	3.84×10^{-14}	1.11×10^{-16}	3.46×10^{-14}
0.2	4.24×10^{-14}	1.11×10^{-16}	5.28×10^{-14}
0.6	3.00×10^{-14}	1.67×10^{-16}	1.23×10^{-14}
1	2.02×10^{-16}	4.67×10^{-17}	0

Example 5. *Consider 3^{rd} order non-linear BVP:*

$$u^{(3)}(s) + u(s)u^{(2)}(s) - \alpha\left(u^{(1)}(s)\right)^2 = 0, \qquad 0 < s < \infty, \tag{43}$$

subject to $u(0) = 0$, $u'(0) = 1$, and $u'(\infty) = 0$. This equation represents the study of the two-dimensional laminar flow due to the stretching wall [45–47] in the absence of the applied magnetic field.

Interestingly, the problem has exact solutions for some values of α. For the case of $\alpha = 1$, [48,49] $u(s) = 1 - e^{-s}$. By applying the method of FDCHPs pseudo-Galerkin, with shifting the given interval to $[-1, 1]$, we obtain 9.39×10^{-18} after four iterations as a MAE. On the other hand, the method in [50] obtained a MAE of almost 10^{-06} after $2^{9+1} = 1024$ iterations.

6. Conclusions

This paper has investigated a numerical technique, FDCHPs pseudo-Galerkin, based on Chebyshev polynomials' first derivative to treat some types of linear and nonlinear ODEs. Formulae and theorems for the novel basis functions have been introduced and proven. Based on this, we set up a new operational matrix for the differentiation of any integer order. Before proceeding to the computational steps, error analysis and approximation convergence have been discussed and investigated. Finally, the proposed technique has been applied to several types of BVPs, "Special applications were selected." The observed results reported from the tables and figures proved the efficiency and accuracy of our technique. These results were compatible and consistent with the discussed error analysis. As future work and open problems, we will extend the current relations to investigate the linearization relation. This will allow us to developed a Tau method for solving linear/nonlinear BVPs via Chebyshev polynomials' first derivative.

Author Contributions: Conceptualization, M.A.; methodology, M.A. and T.A.-E; software, M.A. and T.A.-E.; validation, M.A., T.A.-E. and M.G.A.; formal analysis, M.A. and T.A.-E; investigation, M.A., T.A.-E. and M.G.A.; data curation, D.B., M.G.A.; writing—original draft preparation, T.A.-E. and M.G.A.; writing—review and editing, M.A., T.A.-E., and D.B.; visualization, D.B., M.G.A., and M.E.-K.; supervision, M.A. and M.E.-K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors would like to thank the anonymous reviewers for carefully reading the article and also for their constructive and valuable comments, which have improved the paper in its present form. The authors are also sincerely grateful to the Helwan School of Numerical Analysis in Egypt (HSNAE) members for their valuable support during this research.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Danish, G.A.; Imran, M.; Tahir, M.; Waqas, H.; Asjad, M.I.; Akgül, A.; Baleanu, D. Effects of non-linear thermal radiation and chemical reaction on time dependent flow of Williamson nanofluid with combine electrical MHD and activation energy. *J. Appl. Comput. Mech.* **2021**, *7*, 546–558.
2. Farman, M.; Ahmad, A.; Akgül, A.; Saleem, M.U.; Naeem, M.; Baleanu, D. Epidemiological Analysis of the Coronavirus Disease Outbreak with Random Effects. *Comput. Mater. Contin.* **2021**, *67*, 3215–3227.
3. Nan, A.; Khan, U.; Ahmed, N.; Mohyud-Din, S.T.; Khan, I.; Baleanu, D.; Nisar, K.S. Al2O3 and gamma Al2O3 nanomaterials based nanofluid models with Surface Diffusion: Applications for thermal performance in multiple engineering systems and industries. *CMC* **2021**, *66*, 1563–1576.
4. Çelik, E.; Bayram, M. The numerical solution of physical problems modeled as a systems of differential algebraic equations (DAEs). *J. Frankl. Inst.* **2005**, *342*, 1–6.
5. Çelik, E. On the numerical solution of chemical differential-algebraic equations by Pade series. *Appl. Math. Comput.* **2004**, *153*, 13–17.
6. He, W.; Chen, N.; Dassios, I.; Shah, N.A.; Chung, J.D. Fractional system of Korteweg-De Vries equations via Elzaki transform. *Mathematics* **2021**, *9*, 973.
7. Kumar, S.; Shaw, P.K.; Abdel-Aty, A.H.; Mahmoud E.E. A numerical study on fractional differential equation with population growth model. *Numer. Methods Part. Differ. Eq.* **2020**, 1–2. doi: 10.1002/num.22684.
8. Youssri, Y.H.; Hafez, R.M. Chebyshev collocation treatment of Volterra–Fredholm integral equation with error analysis. *Arab. J. Math.* **2020**, *9*, 471–480.
9. Lu, Y.; Yin, Q.; Li, H. The LS-SVM algorithms for boundary value problems of high-order ordinary differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 195.
10. Akram, G.; Beck, C. Hierarchical cascade model leading to 7-th order initial value problem. *Appl. Numer. Math.* **2015**, *91*, 89–97.
11. Alsaedi, A.; Hayat, T.; Qayyum, S.; Yaqoob, R. Eyring-Powell nanofluid flow with nonlinear mixed convection: Entropy generation minimization. *Comput. Methods Programs Biomed.* **2020**, *186*, 105183.
12. Bhrawy, A.H. A new spectral algorithm for time-space fractional partial differential equations with subdiffusion and superdiffusion. *Proc. Rom. Acad. Ser. A* **2016**, *17*, 39–47.
13. Abdelkawy, M.A.; Ezz-Eldien, S.S.; Amin, A.Z.M. Jacobi spectral collocation scheme for solving Abel's integral equations. *Prog. Fract. Differ.* **2015**, *1*, 187–200.
14. Youssri, Y.H.; Abd-Elhameed, W.M.; Abdelhakem, M. A robust spectral treatment of a class of initial value problems using modified Chebyshev polynomials. *Math. Meth. Appl. Sci.* **2021**, *44*, 9224–9236.
15. Abdelhakem, M.; Doha, M.R.; Saadallah, A.F.; El-Kady, M. Monic Gegenbauer approximations for solving differential equations. *J. Sci. Eng. Res.* **2018**, *5*, 317–321.
16. Abdelhakem, M.; Ahmed, A.; El-kady, M. Spectral monic Chebyshev approximation for higher order differential equations. *Math. Sci. Lett.* **2019**, *8*, 11–17.

17. Sweilam, N.H.; Nagy, A.M.; Youssef, I.K.; Mokhtar M.M. New spectral second kind Chebyshev wavelets scheme for solving systems of integro-differential equations. *Int. J. Appl. Comput. Math.* **2017**, *3*, 333–345.
18. Abd-Elhameed, W.M.; Youssri, Y.H. Fifth-kind orthonormal Chebyshev Polynomial solutions for fractional differential equations. *Comp. Appl. Math.* **2018**, *37*, 2897—2921.
19. Abd-Elhameed, W.M.; Youssri, Y.H. Sixth-Kind Chebyshev spectral approach for solving fractional differential equations. *Int. J. Nonlin. Sci. Num.* **2019**, *20*, 191–203.
20. De Florio, M.; Schiassi, E.; D'Ambrosio, A.; Mortari, D.; Furfaro, R. Theory of Functional Connections Applied to Linear ODEs Subject to Integral Constraints and Linear Ordinary Integro-Differential Equations. *Math. Comput. Appl.* **2021**, *26*, 62, https://doi.org/10.3390/mca26030065.
21. Guan, X.; Zhong, R.; Qin, B.; Wang, Q.; Shao, W. Vibro-acoustic analysis of combined elliptical–cylindrical–elliptical shells immersed in acoustic medium. *J. Fluids Struct.* **2021**, *106*, 103391.
22. Shen, J. Efficient spectral-Galerkin Method I. direct solvers of second- and fourth-order equations. *SIAM J. Sci. Comput.* **1995**, *15*, 1489–1505.
23. Issa, K.; Adeniyi, R.B. Extension of generalized recursive Tau method to non-linear ordinary differential equations. *J. Niger. Math. Soc.* **2016**, *35*, 18–24.
24. Shahni, J.; Singh, R. Bernstein and Gegenbauer-wavelet collocation methods for Bratu-like equations arising in electrospinning process. *J. Math. Chem.* **2021**, 1–17, https://doi.org/10.1007/s10910-021-01290-y.
25. Izadi, M.; Yüzbaşı, Ş.; Baleanu, D. A Taylor–Chebyshev approximation technique to solve the 1D and 2D nonlinear Burgers equations. *Math. Sci.* **2021**, 1–13, https://doi.org/10.1007/s40096-021-00433-1
26. Abdelhakem, M.; Biomy, M.; Kandil, S, Baleanu, D.; El-kady, M. A numerical method based on Legendre differentiation matrices for higher order ODEs. *Inf. Sci. Lett.* **2020**, *9*, 175–180.
27. Napoli, A.; Abd-Elhameed, W.M. An innovative harmonic numbers operational matrix method for solving initial value problems. *Calcolo* **2017**, *54*, 57–76.
28. Youssri, Y.H. Two Fibonacci operational matrix pseudo-spectral schemes for nonlinear fractional Klein-Gordon equation. *Int. J. Mod. Phys. C* **2021**, doi:10.1142/S0129183122500498.
29. Abdelhakem, M.; Mahmoud, D.; Baleanu, D. Shifted ultraspherical pseudo-Galerkin method for approximating the solutions of some types of ordinary fractional problems. *Adv. Differ. Equ.* **2021**, *2021*, 110.
30. Elbarbary, E.M.E. Efficient Chebyshev–Petrov–Galerkin method for solving second-order equations. *J. Sci. Comput.* **2008**, *34*, 113-–126.
31. Abdelhakem, M.; Youssri,Y.H. Two Spectral Legendre's Derivative Algorithms for Lane-Emden, Bratu Equations, and Singular Perturbed Problems. *Appl. Numer. Math.* **2021**, *169*, 243–255. doi 10.1016/j.apnum.2021.07.006.
32. Mason, J.C.; Handscomb, D.C. *Chebyshev Polynomials*, 1st ed.; Chapman, Hall/CRC: New York, NY, USA, 2002; p. 360.
33. Shen, J.; Tang, T.; Wang, L. *Spectral Methods: Algorithms, Analyses and Applications*, 1st ed.; Springer: Berlin/Heidelberg, Germany, 2011; p. 472.
34. Darvishi, M.; Darvishi, N. SOR- Steffensen–Newton method to solve systems of nonlinear equations. *J. Appl. Math.* **2012**, *2*, 21–27.
35. Abd-Elhameed, W.; Youssri, Y.H. Numerical solutions for Volterra-Fredholm-Hammerstein integral equations via second kind Chebyshev quadrature collocation algorithm. *Adv. Math. Sci. Appl.* **2014**, *24*, 129–141.
36. Tomar, S. A Rapid-Converging Analytical Iterative Scheme for Solving Singular Initial Value Problems of Lane–Emden Type. *Int. J. Appl. Comput. Math.* **2021**, *7*, 1–17, https://doi.org/10.1007/s40819-021-01029-y
37. Hadian-Rasanan, A.H.; Rahmatic, D.; Gorgind, S.; Paranda, K. A single layer fractional orthogonal neural network for solving various types of Lane–Emden equation. *New Astron.* **2020**, *75*, 1384–1076.
38. Aydinlik, S. A high-order numerical method for solving nonlinear. *Astrophys. Space Sci.* **2018**, *363*, 264.
39. Zhu, H.; Niu, J.; Zhang R.; Lin Y. A new approach for solving nonlinear singular boundary value problems. *Math. Model. Anal.* **2018**, *23*, 33–43.
40. Mohsenyzadeh, M.; Maleknejad, K.; Ezzati, R. A numerical approach for the solution of a class of singular boundary value problems arising in physiology. *Adv. Differ. Equ.* **2015**, *2015*, 231.
41. Zhou, F.; Xu, X. Numerical solutions for the linear and nonlinear singular boundary value problems using Laguerre wavelets. *Adv. Diff. Equ.* **2016**, *2016*, 17.
42. Alam, M.P.; Begum, T.; Khan, A. A high-order numerical algorithm for solving Lane–Emden equations with various types of boundary conditions. *Comp. Appl. Math.* **2021**, *40*, https://doi.org/10.1007/s40314-021-01591-7.
43. Yüzbas, S. Improved Bessel collocation method for linear Volterra integro-differential equations with piecewise intervals and application of a Volterra population model. *Appl. Math. Model.* **2016**, *40*, 5349–5363.
44. Yüzbas, S.; Sezer, M.; Kemancı, B. Numerical solutions of integro-differential equations and application of a population model with an improved Legendre method. *Appl. Math. Model.* **2016**, *37*, 2086–2101.
45. Kudenatti, R.B. A new exact solution for boundary layer flow over a stretching plate. *Int. J. Non-Linear Mech.* **2012**, *47*, 727–733.
46. Kudenatti, R.B.; Awati, V.B.; Bujurke, N. Approximate analytical solutions of a class of boundary layer equations over nonlinear stretching surface. *Appl. Math. Comput.* **2011**, *218*, 2952–2959.
47. Sachdev, P.L.; Bujurke, N.M.; Awati, V.B. Boundary value problems for third-order nonlinear ordinary differential equations. *Stud. Appl. Math.* **2015**, *115*, 303–318.

48. Afzal, N. Momentum transfer on power law stretching plate with free stream pressure gradient. *Int. J. Eng. Sci.* **2003**, *41*, 1197–1207.
49. Crane, L.J. Flow past a stretching plate. *ZAMP* **1970**, *21*, 645–647.
50. Karkera, H.; Katagi, N.N.; Kudenatti, R.B. Analysis of general unified MHD boundary-layer flow of a viscous fluid—A novel numerical approach through wavelets. *Math. Comput. Simul.* **2020**, *168*, 135–154.