



## Article

# Effect of Magnetic Field with Parabolic Motion on Fractional Second Grade Fluid

Nazish Iftikhar <sup>1</sup>, Muhammad Bilal Riaz <sup>2,3</sup>, Jan Awrejcewicz <sup>3,\*</sup> and Ali Akgül <sup>4</sup><sup>1</sup> Department of Science & Humanities, Lahore Campus, National University of Computer and Emerging Sciences, Lahore 54000, Pakistan; nazish.iftikhar289@gmail.com<sup>2</sup> Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan; Bilalsehole@gmail.com or muhammad.riaz@p.lodz.pl<sup>3</sup> Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowskiego St., 90-924 Lodz, Poland<sup>4</sup> Department of Mathematics, Art and Science Faculty, Siirt University, Siirt 56100, Turkey; aliakgul00727@gmail.com

\* Correspondence: jan.awrejcewicz@p.lodz.pl

**Abstract:** This paper is an analysis of the flow of magnetohydrodynamics (MHD) second grade fluid (SGF) under the influence of chemical reaction, heat generation/absorption, ramped temperature and concentration and thermodiffusion. The fluid was made to flow through a porous medium. It has been proven in many already-published articles that heat and mass transfer do not always follow the classical mechanics process that is known as memoryless process. Therefore, the model using classical differentiation based on the rate of change cannot really replicate such a dynamical process very accurately; thus, a different concept of differentiation is needed to capture such a process. Very recently, new classes of differential operators were introduced and have been recognized to be efficient in capturing processes following the power law, the decay law and the crossover behaviors. For the study of heat and mass transfer, we applied the newly introduced differential operators to model such flow. The equations for heat, mass and momentum are established in the terms of Caputo (C), Caputo–Fabrizio (CF) and Atangana–Baleanu in Caputo sense (ABC) fractional derivatives. The Laplace transform, inversion algorithm and convolution theorem were used to derive the exact and semi-analytical solutions for all cases. The obtained analytical solutions were plotted for different values of existing parameters. It is concluded that the fluid velocity shows increasing behavior for  $\kappa$ ,  $G_r$  and  $G_m$ , while velocity decreases for  $P_r$  and  $M$ . For  $K_r$ , both velocity and concentration curves show decreasing behavior. Fluid flow accelerates under the influence of  $S_r$  and  $R$ . Temperature and concentration profiles increase for  $S_r$  and  $R$ . Moreover, the ABC fractional operator presents a larger memory effect than C and CF fractional operators.

**Keywords:** second grade fluid; memory effect; magnetic field; inversion algorithm

**Citation:** Iftikhar, N.; Riaz, M.B.; Awrejcewicz, J.; Akgül, A. Effect of Magnetic Field with Parabolic Motion on Fractional Second Grade Fluid. *Fractal Fract.* **2021**, *5*, 163. <https://doi.org/10.3390/fractalfract5040163>

Academic Editors: Lanre Akinyemi, Mostafa M. A. Khater, Mehmet Senol and Hadi Rezazadeh

Received: 10 August 2021

Accepted: 24 September 2021

Published: 11 October 2021

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

Over the past thirty years, fractional derivatives have fascinated multiple investigators as compared to classical derivatives. Moreover, fractional derivatives are more credible in mathematical modeling of real-world problems [1–4].

In classical calculus, derivatives and integrals are uniquely computed. A similar situation exists in the case of fractional integrals. For example, Samko et al. [5], Podlubny [6], Oldham and Spanier [7], and Miller and Ross [8] used a similar definition to compute the fractional integrals. However, the circumstances are complicated in fractional order derivatives (FODs) because several different competing definitions exist in the literature. For instance, a few of those approaches include the Riemann–Liouville, the Caputo, the Hadamard, the Marchaud, the Granwald–Letnikov, the Erdelyi–Kober, the Riesz–Feller, the Caputo–Fabrizio and the Atangana–Baleanu approach. These definitions coincide only

with some particular cases. Among all these approaches to define fractional differentiation and fractional integration, the approach of Riemann–Liouville is significant. However, the approach of Riemann–Liouville does not properly address the physics of some fractional derivative initial-boundary value problems. Furthermore, this definition can exhibit the derivative of a constant function other than zero. To overcome this problem, Caputo proposed an alternate definition of FOD in 1967 [9], and it was used in fluid dynamics to explain the theory of viscoelasticity [10]. Recently, Caputo and Fabrizio [11] provided a modern definition of a non-integer order derivative including exponential function and the Atangana and Baleanu [12], based on Mittagag–Leffler function, which is a generalization of the exponential function.

The FOD inherits a nonlocal nature, so it is an excellent tool to obtain a better understanding of the hereditary properties of different processes and materials. Possibly, the first utilization of non-integer calculus in physical problems was noticed due to the work of Abel [13] while finding the solution of integer order equation, known as tautochrone problem. In this problem, the curve of an object (frictionless wire), lying in a vertical plane, was determined by using the operator  $D_0^{\frac{1}{2}}$  and assuming the dependence of the time position not on the starting point. Bagley [14] presented the first PhD thesis on the applications of FC in viscoelasticity models. Recently, the applications of FC have been observed in psychology to determine the time variation of the emotions of mankind [15,16]. The applications of FOD problems can be seen in dynamics and control systems [17], marine sciences and wave dynamics [18–21], diffusion processes [22–24], solid mechanics [25–27], medical sciences [28–31] and many more [32–35].

Convective flow is a self-sustained flow with a temperature gradient. Convective flow and magnetic effect combined play a very important role in real value problems. The SGF with warmth-transferring porous medium was discussed by Tan and Masuoka [36]. Mixed MHD convection on an upright plate with permeable space was analyzed by Aldose et al. [37]. Rashidi et al. [38] also investigated the difference between two normal kinds of liquid stream between clear liquid and permeable medium. The authors of [39] investigated the precarious MHD stream of turning SGF past a swaying plate. Khan et al. [40] discussed the precise answers for quickened flow behavior of a pivoting SGF in a permeable space. Bilal et al. [41] and Ali et al. [42] analyzed the SGF with a swaying plate under different conditions. The literature shows more interest developed in the numerical and approximate solutions on the convection flow of SGF [43–49]. Exact solutions for viscoelastic SGF fluid have been investigated by researchers [50–52]. In 2010, the SGF with Laplace transform methodology over a wavering plate was explored by Nazar et al. [53]. Ali et al. [54] explored the MHD fluid with permeable space. The procedure of heat transfer is expressed by using momentum, energy and continuity equations with stress and heat flux. The heat transfer phenomenon consists of the fins of the heat exchanger, the tabulator inside the tube or plates and the convective physical transport phenomenon [55]. MHD free convection radiative stream with different conditions has been considered by researchers [56–59]. Ali et al. [42] considered SGF with porous surface and determined the exact solutions. Some recent and useful work has also been done for SGF with fractional differential operators [60–65]. Heat and mass transfer phenomena in nanofluids with a porous medium have been investigated by [66–68]. Some significant work in the field of nanoparticles and carbon nanotubes via fractional derivatives have been done by researchers [69–71]. Recently, Rehman et al. [72] discussed heat and mass transfer of MHD unsteady SGF in the presence of ramped conditions. Song et al. [73] used the definition of ABC in order to study SGF with exponential heating and Darcy’s law. Moreover, Riaz et al. explored MHD SGF with ramped conditions via special functions [74].

In the present work, we propose the mathematical modeling of fractional SGF with the help of Laplace transform and fractional operators. Moreover, solutions are acquired for momentum, heat and mass profiles. An inversion algorithm is used for graphical interpretation of fractional models.

### 2. Mathematical Modeling

We begin with SGF on an upright and unbounded plate, with the impact of a magnetic field having strength  $B_0$ . The plate is perpendicular to  $\tilde{\eta}$ -axis and parallel to  $\tilde{x}$ -axis. At the start, the plate and fluid are not moving.  $\tilde{T}_\infty$  is fixed temperature and  $\tilde{C}_\infty$  is concentration at the surface. At the time  $\tilde{\tau} > 0$ , the temperature of the plate is either raised or lowered to  $\tilde{T}_\infty + (\tilde{T}_w - \tilde{T}_\infty) \frac{\tilde{\tau}}{\tau_0}$ , when  $\tilde{\tau} \leq \tau_0$ , and thereafter, for  $\tilde{\tau} > \tau_0$ , a constant temperature  $\tilde{T}_w$  is maintained and the level of mass transfer at the surface of the wall is either raised or lowered to  $\tilde{C}_\infty + (\tilde{C}_w - \tilde{C}_\infty) \frac{\tilde{\tau}}{\tau_0}$ , when  $\tilde{\tau} \leq \tau_0$ , and thereafter, for  $\tilde{\tau} > \tau_0$  is maintained at the constant surface concentration  $\tilde{C}_w$ , respectively. The physical model of the problem can be given as follows in Figure 1 [75]. The detailed method to solve the problem is shown in Figure 2.

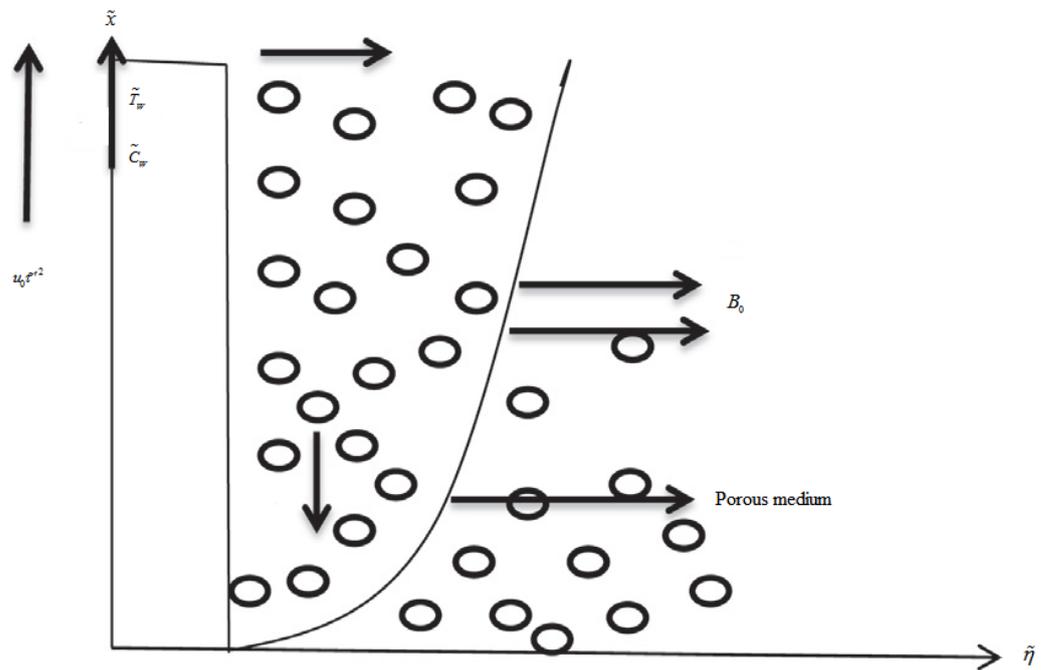


Figure 1. Geometrical presentation of the problem.

Governing equations for momentum, heat and mass are presented by [75]:

$$\frac{\partial \tilde{u}}{\partial \tilde{\tau}} = \left( \nu + \frac{\gamma}{\rho} \frac{\partial}{\partial \tilde{\tau}} \right) \frac{\partial^2 \tilde{u}}{\partial \tilde{\eta}^2} + g\beta_T \tilde{T} - g\beta_T \tilde{T}_\infty + g\beta_C \tilde{C} - g\beta_C \tilde{C}_\infty - \frac{\sigma B_0^2}{\rho} \tilde{u} - \frac{\Phi}{k_1} \left( \nu + \frac{\gamma}{\rho} \frac{\partial}{\partial \tilde{\tau}} \right) \tilde{u}, \tag{1}$$

$$\rho C_p \left( \frac{\partial \tilde{T}}{\partial \tilde{\tau}} \right) = k \left( \frac{\partial^2 \tilde{T}}{\partial \tilde{\eta}^2} \right) - \frac{\partial \tilde{q}_r}{\partial \tilde{\eta}} + Q_0 \tilde{T} - Q_0 \tilde{T}_\infty, \tag{2}$$

$$\left( \frac{\partial \tilde{C}}{\partial \tilde{\eta}} \right) = D_M \left( \frac{\partial^2 \tilde{C}}{\partial \tilde{\eta}^2} \right) + D_T \left( \frac{\partial^2 \tilde{T}}{\partial \tilde{\eta}^2} \right) - \tilde{k}_2 \tilde{C} - \tilde{k}_2 \tilde{C}_\infty, \tag{3}$$

and the imposed initial and boundary conditions are [75]:

$$\tilde{\tau} \leq 0, \quad \tilde{u}(\tilde{\eta}, 0) = 0, \quad \tilde{T}(\tilde{\eta}, 0) = \tilde{T}_\infty, \quad \tilde{C}(\tilde{\eta}, 0) = \tilde{C}_\infty, \quad \tilde{\eta} \geq 0, \tag{4}$$

$$\tilde{\tau} > 0, \quad \tilde{u}(\tilde{\eta}, \tilde{\tau}) = u_0 \tilde{\tau}^2, \quad \tilde{T}(0, \tilde{\tau}) = \begin{cases} \tilde{T}_\infty + (\tilde{T}_w - \tilde{T}_\infty) \frac{\tilde{\tau}}{\tau_0}, & 0 < \tilde{\tau} \leq \tau_0; \\ \tilde{T}_w, & \tilde{\tau} > \tau_0, \end{cases} \tag{5}$$

$$\tilde{C}(0, \tilde{\tau}) = \begin{cases} \tilde{C}_\infty + (\tilde{C}_w - \tilde{C}_\infty) \frac{\tilde{\tau}}{\tau_0}, & 0 < \tilde{\tau} \leq \tau_0; \\ \tilde{C}_w, & \tilde{\tau} > \tau_0, \end{cases} \quad \tilde{\eta} = 0, \quad (6)$$

$$\tilde{\tau} \geq 0, \quad \tilde{u}(\tilde{\eta}, \tilde{\tau}) \rightarrow 0, \quad \tilde{T}(\tilde{\eta}, \tilde{\tau}) \rightarrow \tilde{T}_\infty, \quad \tilde{C}(\tilde{\eta}, \tilde{\tau}) \rightarrow \tilde{C}_\infty, \quad \tilde{\eta} \rightarrow \infty. \quad (7)$$

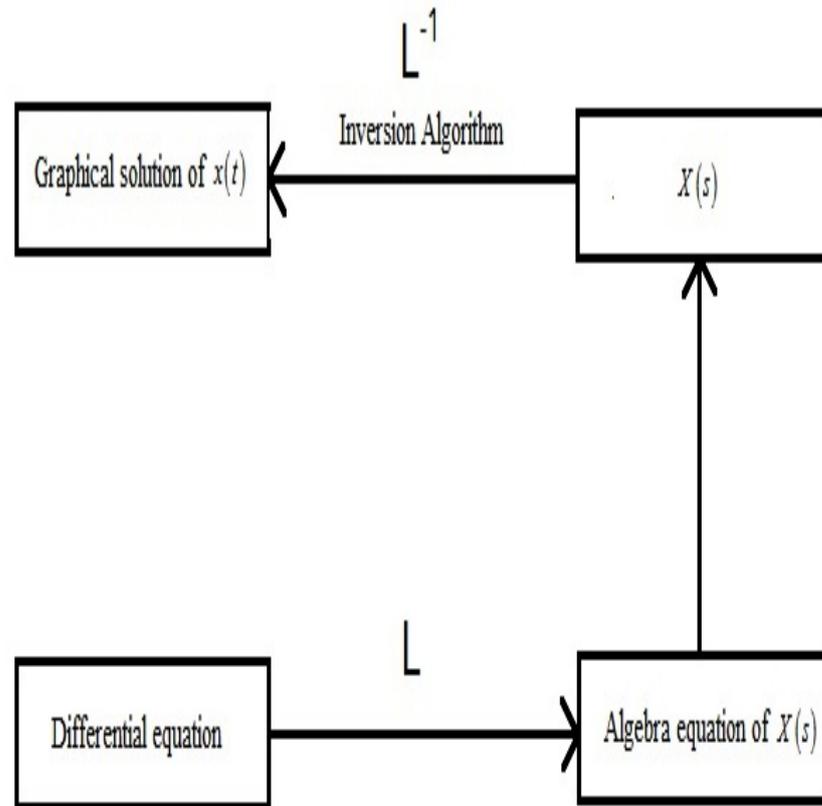


Figure 2. Flow chart of the method.

For the simplification of Equations (1)–(7), we introduce the dimensionless variables given below:

$$\zeta = \frac{u_0 \tilde{\tau}_0^2}{v} \tilde{\eta}, \quad w = \frac{\tilde{u}}{\tilde{\tau}_0^2 u_0}, \quad \theta = \frac{\tilde{T} - \tilde{T}_\infty}{\tilde{T}_w - \tilde{T}_\infty}, \quad C = \frac{\tilde{C} - \tilde{C}_\infty}{\tilde{C}_w - \tilde{C}_\infty}, \quad M^2 = \frac{\sigma B_0^2 \tau_0}{\rho}, \quad K_r = \tau_0 \tilde{k}_2, \quad (8)$$

$$S_c = \frac{v}{D_M}, \quad G_r = \frac{g \tilde{\beta}_T (\tilde{T}_w - \tilde{T}_\infty)}{\tilde{u}_0 \tau_0}, \quad G_m = \frac{g \tilde{\beta}_C (\tilde{C}_w - \tilde{C}_\infty)}{\tilde{u}_0 \tau_0}, \quad \tau_0 = \left( \frac{v}{\tilde{u}_0^2} \right)^{\frac{1}{5}}, \quad \gamma_1 = \frac{\gamma}{\rho v \tau_0}, \quad (9)$$

$$S_r = \frac{D_T (\tilde{T}_w - \tilde{T}_\infty)}{v (\tilde{C}_w - \tilde{C}_\infty)}, \quad P_r = \frac{\rho v C_p}{k}, \quad R = \frac{16 \sigma^* \tilde{T}_\infty^3}{3 k k^*}, \quad \frac{1}{k_1} = \frac{u_0 v \tau_0^2 \Phi}{\tilde{k}_1}, \quad H = \frac{Q_0 v \tau_0}{k}, \quad (10)$$

$$t = \frac{\tilde{\tau}}{\tau_0}, \quad h = \frac{1}{u_0 \tau_0 \tilde{k}_1}, \quad c = 1 + \gamma_1 h, \quad b = M^2 + h. \quad (11)$$

Therefore, the dimensionless momentum, energy and mass equations are [75]

$$c \left( \frac{\partial w(\zeta, t)}{\partial t} \right) = \frac{\partial^2 w}{\partial \zeta^2} + \gamma_1 \left( \frac{\partial^3 w(\zeta, t)}{\partial t \partial \zeta^2} \right) + G_r \theta(\zeta, t) + G_m C(\zeta, t) - b w(\zeta, t), \quad (12)$$

$$P_r \left( \frac{\partial \theta(\zeta, t)}{\partial t} \right) = (1 + R) \frac{\partial^2 \theta(\zeta, t)}{\partial \zeta^2} + H\theta(\zeta, t), \quad (13)$$

$$\frac{\partial C(\zeta, t)}{\partial t} = \frac{1}{S_c} \left( \frac{\partial^2 C(\zeta, t)}{\partial \zeta^2} \right) + S_r \left( \frac{\partial^2 \theta(\zeta, t)}{\partial \zeta^2} \right) - K_r C(\zeta, t), \quad (14)$$

with initial and boundary conditions

$$t \leq 0, \quad w(\zeta, 0) = 0, \quad \theta(\zeta, 0) = 0, \quad C(\zeta, 0) = 0, \quad \zeta \geq 0, \quad (15)$$

$$t > 0, \quad w(\zeta, t) = t^2, \quad \theta(0, t) = \begin{cases} t & 0 < t \leq 1; \\ 1 & t > 1, \end{cases}, \quad C(0, t) = \begin{cases} t & 0 < t \leq 1; \\ 1 & t > 1, \end{cases}, \quad \zeta = 0, \quad (16)$$

$$t > 0, \quad w(\zeta, t) \rightarrow 0, \quad \theta(\zeta, t) \rightarrow 0, \quad C(\zeta, t) \rightarrow 0, \quad \zeta \rightarrow \infty. \quad (17)$$

### 3. Solution for Caputo Fractional Operator

The Caputo (C) fractional time derivative and its Laplace transform [76] are given below:

$${}^C D_t^\kappa N(\zeta, \tau) = \frac{1}{\Gamma(n - \kappa)} \int_0^\tau \left( \frac{N^{(n)}(\xi)}{(\tau - \xi)^{\kappa+1-n}} \right) d\xi, \quad (18)$$

$$\mathcal{L} \left( {}^C D_t^\kappa N(\zeta, \tau) \right) = s^\kappa \mathcal{L} \left( N(\zeta, \tau) \right) - s^{\kappa-1} N(\zeta, 0). \quad (19)$$

#### 3.1. Temperature Field

The following Caputo derivative form of temperature Equation (13) is developed by using Equation (18):

$$P_r {}^C D_t^\kappa \theta(\zeta, t) = (1 + R) \frac{\partial^2 \theta}{\partial \zeta^2} + H\theta. \quad (20)$$

By implementing the Laplace transform on Equation (20), we obtain

$$\frac{\partial^2 \bar{\theta}_c(\zeta, s)}{\partial \zeta^2} - \left( \frac{P_r}{1 + R} \right) \left( s^\kappa - \frac{H}{P_r} \right) \bar{\theta}_c(\zeta, s) = 0, \quad (21)$$

where the homogenous solution of the above equation is

$$\bar{\theta}_c(\zeta, s) = c_1 e^{-\zeta \sqrt{\left(\frac{P_r}{1+R}\right)\left(s^\kappa - \frac{H}{P_r}\right)}} + c_2 e^{\zeta \sqrt{\left(\frac{P_r}{1+R}\right)\left(s^\kappa - \frac{H}{P_r}\right)}}. \quad (22)$$

$c_1$  and  $c_2$  can be determined by employing Equations (15)–(17), and the required solution is given below:

$$\bar{\theta}_c(\zeta, s) = \left( \frac{1 - e^{-s}}{s^2} \right) e^{-\zeta \sqrt{\left(\frac{P_r}{1+R}\right)\left(s^\kappa - \frac{H}{P_r}\right)}}. \quad (23)$$

#### 3.2. Concentration Field

The following Caputo derivative form of concentration Equation (14) is developed by using Equation (18):

$${}^C D_t^\kappa C(\zeta, t) = \frac{1}{S_c} \frac{\partial^2 C(\zeta, t)}{\partial \zeta^2} + S_r \frac{\partial^2 \theta(\zeta, t)}{\partial \zeta^2} - K_r C(\zeta, t). \quad (24)$$

By implementing the Laplace transform on Equation (24),

$$(s^\kappa + K_r)\bar{C}_c(\zeta, s) = \frac{1}{S_c} \frac{\partial^2 \bar{C}_c(\zeta, s)}{\partial \zeta^2} + S_r \frac{\partial^2 \bar{\theta}_c(\zeta, s)}{\partial \zeta^2}, \quad (25)$$

the homogenous solution of the above equation is

$$\bar{C}_h(\zeta, s) = c_1 e^{-\zeta \sqrt{S_c \left( \frac{s^\kappa}{(1-\kappa)s+\kappa} + K_r \right)}} + c_2 e^{\zeta \sqrt{S_c \left( \frac{s^\kappa}{(1-\kappa)s+\kappa} + K_r \right)}}, \quad (26)$$

the particular solution of Equation (25) is

$$\bar{C}_p(\zeta, s) = - \frac{S_c S_r (1 - e^{-s}) \left\{ (P_r - H + H\kappa)s^\kappa - H\kappa \right\}}{s^2 \left\{ \left( P_r - H(1 - \kappa) - S_c(1 + R) - S_c K_r(1 + R)(1 - \kappa) \right) s^\kappa - (H\kappa + S_c K_r \kappa(1 + R)) \right\}} e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s^\kappa}{(1-\kappa)s+\kappa} - \frac{H}{P_r} \right)}}, \quad (27)$$

and the required solution is given below:

$$\bar{C}_c(\zeta, s) = \left( \frac{1 - e^{-s}}{s^2} - A_1 \right) e^{-\zeta \sqrt{S_c(s^\kappa + K_r)}} + A_1 e^{-\zeta \sqrt{\left( \frac{P_r}{1+R} \right) (s^\kappa - \frac{H}{P_r})}}. \quad (28)$$

where

$$A_1 = \frac{S_c S_r \left( \frac{P_r}{1+R} \right) \left( s^\kappa - \frac{H}{P_r} \right) (1 - e^{-s})}{\left( \frac{H}{1+R} \right) + S_c K_r} \times \left( \frac{\left( \frac{P_r}{1+R} - S_c \right)}{s^\kappa \left( \frac{H}{1+R} \right) + S_c K_r} + \frac{1}{s^2} - \frac{1}{\left( \frac{H}{1+R} + S_c K_r \right) \left( s^\kappa \left( \frac{P_r}{1+R} - S_c \right) - \left( \frac{H}{1+R} + S_c K_r \right) \right)} \right). \quad (29)$$

### 3.3. Velocity Field

The following Caputo derivative form of velocity Equation (12) is developed by using Equation (18):

$$(1 + h\gamma_1)({}^C D_t^\kappa)w(\zeta, t) = \left( 1 + \gamma_1^C D_t^\kappa \right) \frac{\partial^2 w}{\partial \zeta^2} + G_r \theta + G_m C - (M^2 + h)w. \quad (30)$$

By implementing the Laplace transform on Equation (30),

$$(1 + \gamma_1 s^\kappa) \frac{\partial^2 \bar{w}_c(\zeta, s)}{\partial \zeta^2} - (cs^\kappa + b)\bar{w}_c(\zeta, s) = -G_r \bar{\theta}_c(\zeta, s) + G_m \bar{C}_c(\zeta, s), \quad (31)$$

the homogenous solution of above equation is

$$\bar{w}_h(\zeta, s) = c_1 e^{-\zeta \sqrt{\left( 1 + \frac{\gamma_1}{k_1} \right) \left( \frac{s^\kappa}{(1-\kappa)s+\kappa} \right) + (M^2 + k_1)}} + c_2 e^{\zeta \sqrt{\left( 1 + \frac{\gamma_1}{k_1} \right) \left( \frac{s^\kappa}{(1-\kappa)s+\kappa} \right) + (M^2 + k_1)}}, \quad (32)$$

the particular solution for Equation (31) is

$$\begin{aligned}
\bar{w}_p(\zeta, s) = & - \frac{G_r(1 - e^{-s})(s^\kappa + b_2)^2}{s^2 \left\{ (a_4s - a_5)(a_{12}s + b_2) - (a_{13}s + a_{14})(s + b_2) \right\}} e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s^\kappa}{(1-\kappa)s+\kappa} - \frac{H}{P_r} \right)}} \\
& - \frac{G_m(1 - e^{-s})(s^\kappa + b_2)^2}{s^2 \left\{ (a_7s^\kappa - a_8)(a_{12}s + b_2) - (a_{13}s^\kappa + a_{14})(s^\kappa + b_2) \right\}} \times \\
& \frac{S_c S_r \left\{ (P_r - H + H\kappa)s^\kappa - H\kappa \right\}}{s^2 \left\{ \left( P_r - H(1 - \kappa) - S_c(1 + R) - S_c K_r(1 + R)(1 - \kappa) \right) s^\kappa - (H\kappa + S_c K_r \kappa(1 + R)) \right\}} \\
& \left( \frac{1 - e^{-s}}{s^2} \right) e^{-\zeta \sqrt{S_c \left( \frac{s^\kappa}{(1-\kappa)s+\kappa} + K_r \right)}} - \frac{G_m(1 - e^{-s})(s^\kappa + b_2)^2}{s^2 \left\{ (a_7s^\kappa - a_8)(a_{12}s^\kappa + b_2) - (a_{13}s^\kappa + a_{14})(s^\kappa + b_2) \right\}} \\
& \times \left\{ e^{-\zeta \sqrt{S_c \left( \frac{s^\kappa}{(1-\kappa)s+\kappa} + K_r \right)}} - e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s^\kappa}{(1-\kappa)s+\kappa} - \frac{H}{P_r} \right)}} \right\} \quad (33)
\end{aligned}$$

and the required solution is given below:

$$\begin{aligned}
\bar{w}_c(\zeta, s) = & \frac{2}{s^3} e^{-\zeta \sqrt{\frac{cs^\kappa + b}{1 + \gamma_1 s^\kappa}}} + \frac{G_r(1 - e^{-s})}{s^2 \left\{ \left( \frac{P_r}{1+R} \right) (s^\kappa - \frac{H}{P_r}) (1 + \gamma_1 s^\kappa) - (cs^\kappa + b) \right\}} \times \\
& \left\{ e^{-\zeta \sqrt{\frac{cs^\kappa + b}{1 + \gamma_1 s^\kappa}}} - e^{-\zeta \sqrt{\left( \frac{P_r}{1+R} \right) (s^\kappa - \frac{H}{P_r})}} \right\} \\
& + \frac{G_m}{S_c(s^\kappa + K_r)(1 + \gamma_1 s^\kappa) - (cs^\kappa + b)} \times \\
& \left\{ \left( \frac{1 - e^{-s}}{s^2} \right) e^{-\zeta \sqrt{\frac{cs^\kappa + b}{1 + \gamma_1 s^\kappa}}} - \left( \frac{1 - e^{-s}}{s^2} \right) e^{-\zeta \sqrt{S_c(s^\kappa + K_r)}} + A_1 A_2 \right\}, \quad (34)
\end{aligned}$$

where

$$\begin{aligned}
A_2 = & e^{-\zeta \sqrt{S_c(s^\kappa + K_r)}} - e^{-\zeta \sqrt{\left( \frac{P_r}{1+R} \right) (s^\kappa - \frac{H}{P_r})}}, \quad b_1 = \frac{1}{1 - \kappa}, \quad b_2 = \kappa b_1, \\
a_1 = & \frac{1 + R}{P_r}, \quad a_2 = \frac{H}{P_r}, \quad a_3 = \frac{1}{a_1}, \quad a_4 = a_3 b_1 - a_3 a_2, \quad a_5 = a_3 a_2 b_2, \\
b_4 = & a_3 - S_c, \quad b_5 = a_3 a_2 + S_c K_r, \quad a_6 = \frac{b_5}{b_4}, \quad a_7 = S_c(b_1 + K_r), \\
a_8 = & S_c K_r b_2, \quad a_9 = a_4 - a_7, \quad a_{10} = a_5 + a_8, \quad a_{11} = \frac{a_{10}}{b_9}, \\
a_{12} = & \frac{\gamma}{1 - \kappa} + 1, \quad a_{13} = \frac{c}{1 - \kappa} + b, \quad a_{14} = \frac{b\kappa}{1 - \kappa} + b. \quad (35)
\end{aligned}$$

#### 4. Solution for Caputo–Fabrizio Fractional Operator

The CF derivative and its Laplace transform [77] are given below:

$${}^{CF}D_t^\kappa N(\zeta, \tau) = \frac{1}{1-\kappa} \int_0^\tau \exp\left(-\frac{\kappa(\tau-\xi)}{1-\kappa}\right) N'(\xi) d\xi, \quad 0 < \kappa < 1, \quad (36)$$

$$\mathcal{L}\left({}^{CF}D_t^\kappa N(\zeta, \tau)\right) = \frac{s\mathcal{L}\left(N(\zeta, \tau)\right) - N(\zeta, 0)}{(1-\kappa)s + \kappa}. \quad (37)$$

##### 4.1. Temperature Field

The following CF derivative form of temperature Equation (13) is developed by using Equation (36):

$$P_r {}^{CF}D_t^\kappa \theta(\zeta, t) = (1+R) \frac{\partial^2 \theta}{\partial \zeta^2} + H\theta. \quad (38)$$

By implementing the Laplace transform on Equation (38),

$$\frac{\partial^2 \bar{\theta}_{cf}(\zeta, s)}{\partial \zeta^2} - \frac{P_r}{1+R} \left( \frac{s}{(1-\kappa)s + \kappa} - \frac{H}{P_r} \right) \bar{\theta}_{cf}(\zeta, s) = 0, \quad (39)$$

the homogenous solution of above equation is

$$\bar{\theta}_{cf}(\zeta, s) = c_1 e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s}{(1-\kappa)s + \kappa} - \frac{H}{P_r} \right)}} + c_2 e^{\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s}{(1-\kappa)s + \kappa} - \frac{H}{P_r} \right)}}, \quad (40)$$

and  $c_1$  and  $c_2$  can be determined by employing Equations (15)–(17). The required solution is given below:

$$\bar{\theta}_{cf}(\zeta, s) = \left( \frac{1 - e^{-s}}{s^2} \right) e^{-\zeta \sqrt{B_2}}. \quad (41)$$

##### 4.2. Concentration Field

The following CF derivative form of concentration Equation (13) is developed by using Equation (36):

$${}^{CF}D_t^\kappa C(\zeta, t) = \frac{1}{S_c} \frac{\partial^2 C(\zeta, t)}{\partial \zeta^2} + S_r \frac{\partial^2 \theta(\zeta, t)}{\partial \zeta^2} - K_r C(\zeta, t). \quad (42)$$

By implementing the Laplace transform on Equation (42),

$$\left( \frac{s}{(1-\kappa)s + \kappa} + K_r \right) C_{cf}(\zeta, s) = \frac{1}{S_c} \frac{\partial^2 \bar{C}_{cf}(\zeta, s)}{\partial \zeta^2} + S_r \frac{\partial^2 \bar{\theta}_{cf}(\zeta, s)}{\partial \zeta^2}, \quad (43)$$

the homogenous solution of the above equation is

$$\bar{C}_h(\zeta, s) = c_1 e^{-\zeta \sqrt{S_c \left( \frac{s}{(1-\kappa)s + \kappa} + K_r \right)}} + c_2 e^{\zeta \sqrt{S_c \left( \frac{s}{(1-\kappa)s + \kappa} + K_r \right)}}, \quad (44)$$

the particular solution of Equation (44) is

$$\bar{C}_p(\zeta, s) = - \frac{S_c S_r (1 - e^{-s}) \left\{ (P_r - H + H\kappa)s - H\kappa \right\}}{s^2 \left\{ \left( P_r - H(1 - \kappa) - S_c(1 + R) - S_c K_r(1 + R)(1 - \kappa) \right) s - (H\kappa + S_c K_r \kappa(1 + R)) \right\}} e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s}{(1-\kappa)s+\kappa} - \frac{H}{P_r} \right)}}, \quad (45)$$

and the required solution is given below:

$$\bar{C}_{cf}(\zeta, s) = \left( \frac{1 - e^{-s}}{s^2} - A_3 \right) e^{-\zeta \sqrt{B_3}} + A_3 e^{-\zeta \sqrt{B_2}}, \quad (46)$$

where

$$A_3 = \frac{S_c S_r \left( \left( \frac{P_r}{1+R} \left( \frac{1}{1-\kappa} - \frac{H}{P_r} \right) \right) s - \left( \frac{H}{1+R} \right) \left( \frac{\kappa}{1-\kappa} \right) (1 - e^{-s}) \right)}{\left( \frac{H}{1+R} + S_c K_r \right) \left( \frac{\kappa}{1-\kappa} \right)} \times \left( \frac{\left( \frac{P_r}{1+R} \right) \left( \frac{1}{1-\kappa} - \frac{H}{P_r} \right) - S_c \left( \frac{1}{1-\kappa} + K_r \right)}{s \left( \frac{H}{1+R} + S_c K_r \right) \left( \frac{\kappa}{1-\kappa} \right)} + \frac{1}{s^2} - A_4 \right), \quad (47)$$

$$A_4 = \frac{1}{\left( \frac{H}{1+R} + S_c K_r \right) \left( \frac{\kappa}{1-\kappa} \right) \left( s \left( \frac{P_r}{1+R} \right) \left( \frac{1}{1-\kappa} - \frac{H}{P_r} \right) - \left( \frac{H}{1+R} + S_c K_r \right) \left( \frac{\kappa}{1-\kappa} \right) \right)}, \quad (48)$$

$$B_3 = \frac{S_c \left( \frac{1}{1-\kappa} + K_r \right) s + S_c K_r \left( \frac{\kappa}{1-\kappa} \right)}{s + \frac{\kappa}{1-\kappa}}, \quad (49)$$

$$B_2 = \frac{\left( \frac{P_r}{1+R} \right) \left( \frac{1}{1-\kappa} - \frac{H}{P_r} \right) s - \left( \frac{H}{1+R} \right) \left( \frac{\kappa}{1-\kappa} \right)}{s + \frac{\kappa}{1-\kappa}}. \quad (50)$$

#### 4.3. Velocity Field

The following CF derivative form of velocity Equation (12) is developed by using Equation (36):

$$(1 + h\gamma_1) ({}^{CF}D_t^\kappa) w(\zeta, t) = \left( 1 + \gamma_1 {}^{CF}D_t^\kappa \right) \frac{\partial^2 w}{\partial \zeta^2} + G_r \theta + G_m C - (M^2 + h) w. \quad (51)$$

By implementing the Laplace transform on Equation (51),

$$\left( \frac{\left( \frac{c}{1-\kappa} + b \right) s + \frac{b\kappa}{1-\kappa}}{s + \frac{\kappa}{1-\kappa}} \right) \bar{w}_{cf}(\zeta, s) = \left( \frac{\left( \frac{\gamma_1}{1-\kappa} + 1 \right) s + \frac{\kappa}{1-\kappa}}{s + \frac{\kappa}{1-\kappa}} \right) \frac{\partial^2 \bar{w}_{cf}(\zeta, s)}{\partial \zeta^2} + G_r \bar{\theta}_{cf}(\zeta, s) + G_m \bar{C}_{cf}(\zeta, s), \quad (52)$$

the homogenous solution of the above equation is

$$\bar{w}_h(\zeta, s) = c_1 e^{-\zeta \sqrt{\left( 1 + \frac{\gamma_1}{k_1} \right) \left( \frac{s}{(1-\kappa)s+\kappa} \right) + (M^2 + k_1)}} + c_2 e^{\zeta \sqrt{\left( 1 + \frac{\gamma_1}{k_1} \right) \left( \frac{s}{(1-\kappa)s+\kappa} \right) + (M^2 + k_1)}}. \quad (53)$$

The particular solution of Equation (52) is

$$\begin{aligned} \bar{w}_p(\zeta, s) = & - \frac{G_r(1 - e^{-s})(s + b_2)^2}{s^2 \left\{ (a_4s - a_5)(a_{12}s + b_2) - (a_{13}s + a_{14})(s + b_2) \right\}} e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s}{(1-\kappa)s+\kappa} - \frac{H}{P_r} \right)}} \\ & - \frac{G_m(1 - e^{-s})(s + b_2)^2}{s^2 \left\{ (a_7s - a_8)(a_{12}s + b_2) - (a_{13}s + a_{14})(s + b_2) \right\}} \times \\ & \frac{S_c S_r \left\{ (P_r - H + H\kappa)s - H\kappa \right\}}{s^2 \left\{ \left( P_r - H(1 - \kappa) - S_c(1 + R) - S_c K_r(1 + R)(1 - \kappa) \right) s - (H\kappa + S_c K_r \kappa(1 + R)) \right\}} \\ & \left( \frac{1 - e^{-s}}{s^2} \right) e^{-\zeta \sqrt{S_c \left( \frac{s}{(1-\kappa)s+\kappa} + K_r \right)}} - \frac{G_m(1 - e^{-s})(s + b_2)^2}{s^2 \left\{ (a_7s - a_8)(a_{12}s + b_2) - (a_{13}s + a_{14})(s + b_2) \right\}} \\ & \times \left\{ e^{-\zeta \sqrt{S_c \left( \frac{s}{(1-\kappa)s+\kappa} + K_r \right)}} - e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s}{(1-\kappa)s+\kappa} - \frac{H}{P_r} \right)}} \right\} \end{aligned} \quad (54)$$

and the required solution is given below:

$$\begin{aligned} \bar{w}_{cf}(\zeta, s) = & \frac{2}{s^3} e^{-\zeta \sqrt{B_1}} + \frac{G_r(1 - e^{-s})(s + \frac{\kappa}{1-\kappa})^2}{s^2 A_7} \left( e^{-\zeta \sqrt{B_1}} - e^{-\zeta \sqrt{B_2}} \right) \\ & + \frac{G_m(s + \frac{\kappa}{1-\kappa})^2}{s^2 A_8} \left( e^{-\zeta \sqrt{B_1}} - e^{-\zeta \sqrt{B_3}} \right) + A_5 \left( e^{-\zeta \sqrt{B_1}} - e^{-\zeta \sqrt{B_2}} \right) \end{aligned} \quad (55)$$

where

$$\begin{aligned} A_7 = & \left( \left( \frac{P_r}{1+R} \right) \left( \frac{1}{1-\kappa} - \frac{H}{P_r} \right) s - \left( \frac{H}{1+R} \right) \left( \frac{\kappa}{1-\kappa} \right) \right) \left( \left( \frac{\gamma_1}{1+\kappa} + 1 \right) s + \frac{\kappa}{1-\kappa} \right) \\ & - \left( \left( \frac{c}{1-\kappa} + b \right) s + \frac{b\kappa}{1-\kappa} \right) \left( s + \frac{\kappa}{1-\kappa} \right), \end{aligned} \quad (56)$$

$$\begin{aligned} A_8 = & \left( S_c \left( \frac{1}{1-\kappa} - K_r \right) s + S_c K_r \left( \frac{\kappa}{1-\kappa} \right) \right) \left( \left( \frac{\gamma_1}{1+\kappa} + 1 \right) s + \frac{\kappa}{1-\kappa} \right) \\ & - \left( \left( \frac{c}{1-\kappa} + b \right) s + \frac{b\kappa}{1-\kappa} \right) \left( s + \frac{\kappa}{1-\kappa} \right) \end{aligned} \quad (57)$$

$$B_1 = \frac{\left( \frac{c}{1-\kappa} + b \right) s + \frac{b\kappa}{1-\kappa}}{\left( \frac{\gamma_1}{1+\kappa} + 1 \right) s + \frac{\kappa}{1-\kappa}} \quad (58)$$

## 5. Solution for Atangana–Baleanu Fractional Operator

The ABC derivative and its Laplace transform [78] are given below:

$${}^{ABC}D_t^\alpha f(\xi, \tau) = \frac{1}{1-\alpha} \int_0^\tau \mathbf{E}_\alpha \left( -\frac{\alpha(t-\tau)^\alpha}{1-\alpha} \right) \frac{\partial f(\xi, \tau)}{\partial \tau} d\tau, \quad (59)$$

$$\mathcal{L} \left( {}^{ABC}D_t^\alpha f(\xi, \tau) \right) = \frac{s^\kappa \mathcal{L}(f(\xi, \tau)) - s^{\kappa-1} f(\xi, 0)}{(1-\kappa)s^\kappa + \kappa}. \quad (60)$$

### 5.1. Temperature Field

The following ABC derivative form of temperature Equation (13) is developed by using Equation (59):

$$P_r {}^{ABC}D_t^\kappa \theta(\zeta, t) = (1 + R) \frac{\partial^2 \theta}{\partial \zeta^2} + H\theta, \quad (61)$$

The Laplace transform of Equation (61) is

$$\frac{\partial^2 \bar{\theta}_{abc}(\zeta, s)}{\partial \zeta^2} - \frac{P_r}{1 + R} \left( \frac{s^\kappa}{(1 - \kappa)s^\kappa + \kappa} - \frac{H}{P_r} \right) \bar{\theta}_{abc}(\zeta, s) = 0, \quad (62)$$

the homogenous solution of above equation is

$$\bar{\theta}_{abc}(\zeta, s) = c_1 e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s^\kappa}{(1-\kappa)s^\kappa + \kappa} - \frac{H}{P_r} \right)}} + c_2 e^{\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s^\kappa}{(1-\kappa)s^\kappa + \kappa} - \frac{H}{P_r} \right)}}. \quad (63)$$

and  $c_1$  and  $c_2$  can be determined by employing Equations (15)–(17). The required solution is given below:

$$\bar{\theta}_{abc}(\zeta, s) = \left( \frac{1 - e^{-s}}{s^2} \right) e^{-\zeta \sqrt{B_2}}. \quad (64)$$

### 5.2. Concentration Field

The following ABC derivative form of concentration Equation (14) is developed by using Equation (59):

$${}^{ABC}D_t^\kappa C(\zeta, t) = \frac{1}{S_c} \frac{\partial^2 C(\zeta, t)}{\partial \zeta^2} + S_r \frac{\partial^2 \theta(\zeta, t)}{\partial \zeta^2} - K_r C(\zeta, t). \quad (65)$$

The Laplace transform of Equation (65) is

$$\left( \frac{s^\kappa}{(1 - \kappa)s^\kappa + \kappa} + K_r \right) \bar{C}_{abc}(\zeta, s) = \frac{1}{S_c} \frac{\partial^2 \bar{C}_{abc}(\zeta, s)}{\partial \zeta^2} + S_r \frac{\partial^2 \bar{\theta}_{abc}(\zeta, s)}{\partial \zeta^2}, \quad (66)$$

and the homogenous solution of the above equation is

$$\bar{C}_h(\zeta, s) = c_1 e^{-\zeta \sqrt{S_c \left( \frac{s^\kappa}{(1-\kappa)s^\kappa + \kappa} + K_r \right)}} + c_2 e^{\zeta \sqrt{S_c \left( \frac{s^\kappa}{(1-\kappa)s^\kappa + \kappa} + K_r \right)}}. \quad (67)$$

The particular solution of Equation (66) is

$$\bar{C}_p(\zeta, s) = - \frac{S_c S_r (1 - e^{-s}) \left\{ (P_r - H + H\kappa)s^\kappa - H\kappa \right\}}{s^2 \left\{ \left( P_r - H(1 - \kappa) - S_c(1 + R) - S_c K_r(1 + R)(1 - \kappa) \right) s^\kappa - (H\kappa + S_c K_r \kappa(1 + R)) \right\}} e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s^\kappa}{(1-\kappa)s^\kappa + \kappa} - \frac{H}{P_r} \right)}}, \quad (68)$$

and the required solution is given below:

$$\bar{C}_{abc}(\zeta, s) = \left( \frac{1 - e^{-s}}{s^2} - A_5 \right) e^{-\zeta \sqrt{B_6}} + A_5 e^{-\zeta \sqrt{B_5}}, \quad (69)$$

where

$$A_5 = \frac{S_c S_r \left( \left( \frac{P_r}{1+R} \left( \frac{1}{1-\kappa} - \frac{H}{P_r} \right) \right) s^\kappa - \left( \frac{H}{1+R} \right) \left( \frac{\kappa}{1-\kappa} \right) \right) (1 - e^{-s})}{\left( \frac{H}{1+R} + S_c K_r \right) \left( \frac{\kappa}{1-\kappa} \right)} \times \left( \frac{\left( \frac{P_r}{1+R} \right) \left( \frac{1}{1-\kappa} - \frac{H}{P_r} \right) - S_c \left( \frac{1}{1-\kappa} + K_r \right)}{s^\kappa \left( \frac{H}{1+R} + S_c K_r \right) \left( \frac{\kappa}{1-\kappa} \right)} + \frac{1}{s^2} - A_6 \right), \tag{70}$$

$$A_6 = \frac{1}{\left( \frac{H}{1+R} + S_c K_r \right) \left( \frac{\kappa}{1-\kappa} \right) \left( s^\kappa \left( \frac{P_r}{1+R} \right) \left( \frac{1}{1-\kappa} - \frac{H}{P_r} \right) - \left( \frac{H}{1+R} + S_c K_r \right) \left( \frac{\kappa}{1-\kappa} \right) \right)}, \tag{71}$$

$$B_6 = \frac{S_c \left( \frac{1}{1-\kappa} + K_r \right) s^\kappa + S_c K_r \left( \frac{\kappa}{1-\kappa} \right)}{s^\kappa + \frac{\kappa}{1-\kappa}}, \tag{72}$$

$$B_5 = \frac{\left( \frac{P_r}{1+R} \right) \left( \frac{1}{1-\kappa} - \frac{H}{P_r} \right) s^\kappa - \left( \frac{H}{1+R} \right) \left( \frac{\kappa}{1-\kappa} \right)}{s^\kappa + \frac{\kappa}{1-\kappa}}. \tag{73}$$

### 5.3. Velocity Field

The following ABC derivative form of velocity Equation (12) is developed by using Equation (59):

$$(1 + h\gamma_1)({}^{ABC}D_t^\kappa)w(\zeta, t) = \left( 1 + \gamma_1 {}^{ABC}D_t^\kappa \right) \frac{\partial^2 w}{\partial \zeta^2} + G_r \theta + G_m C - (M^2 + h)w. \tag{74}$$

The Laplace transform of Equation (74) is

$$\left( \frac{\left( \frac{c}{1-\kappa} + b \right) s^\kappa + \frac{b\kappa}{1-\kappa}}{s^\kappa + \frac{\kappa}{1-\kappa}} \right) \bar{w}_{abc}(\zeta, s) = \left( \frac{\left( \frac{\gamma_1}{1-\kappa} + 1 \right) s^\kappa + \frac{\kappa}{1-\kappa}}{s^\kappa + \frac{\kappa}{1-\kappa}} \right) \frac{\partial^2 \bar{w}_{abc}(\zeta, s)}{\partial \zeta^2} + G_r \bar{\theta}_{abc}(\zeta, s) + G_m \bar{C}_{abc}(\zeta, s), \tag{75}$$

the homogenous solution of the above equation is

$$\bar{w}_h(\zeta, s) = c_1 e^{-\zeta \sqrt{\left( 1 + \frac{\gamma_1}{k_1} \right) \left( \frac{s}{(1-\kappa)s + \kappa} \right) + (M^2 + k_1)}} + c_2 e^{\zeta \sqrt{\left( 1 + \frac{\gamma_1}{k_1} \right) \left( \frac{s}{(1-\kappa)s + \kappa} \right) + (M^2 + k_1)}}, \tag{76}$$

the particular solution of Equation (75) is

$$\begin{aligned} \bar{w}_p(\zeta, s) = & - \frac{G_r (1 - e^{-s}) (s + b_2)^2}{s^2 \left\{ (a_4 s - a_5) (a_{12} s + b_2) - (a_{13} s + a_{14}) (s + b_2) \right\}} e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s}{(1-\kappa)s + \kappa} - \frac{H}{P_r} \right)}} \\ & - \frac{G_m (1 - e^{-s}) (s + b_2)^2}{s^2 \left\{ (a_7 s - a_8) (a_{12} s + b_2) - (a_{13} s + a_{14}) (s + b_2) \right\}} \times \\ & \frac{S_c S_r \left\{ (P_r - H + H\kappa) s - H\kappa \right\}}{s^2 \left\{ \left( P_r - H(1 - \kappa) - S_c(1 + R) - S_c K_r(1 + R)(1 - \kappa) \right) s - (H\kappa + S_c K_r \kappa(1 + R)) \right\}} \\ & \left( \frac{1 - e^{-s}}{s^2} \right) e^{-\zeta \sqrt{S_c \left( \frac{s}{(1-\kappa)s + \kappa} + K_r \right)}} - \frac{G_m (1 - e^{-s}) (s + b_2)^2}{s^2 \left\{ (a_7 s - a_8) (a_{12} s + b_2) - (a_{13} s + a_{14}) (s + b_2) \right\}} \\ & \times \left\{ e^{-\zeta \sqrt{S_c \left( \frac{s}{(1-\kappa)s + \kappa} + K_r \right)}} - e^{-\zeta \sqrt{\frac{P_r}{1+R} \left( \frac{s}{(1-\kappa)s + \kappa} - \frac{H}{P_r} \right)}} \right\} \end{aligned} \tag{77}$$

and the required solution is given below

$$\begin{aligned} \bar{w}_{abc}(\zeta, s) = & \frac{2}{s^3} e^{-\zeta\sqrt{B_4}} + \frac{G_r(1 - e^{-s})(s^\kappa + \frac{\kappa}{1-\kappa})^2}{s^2 A_9} (e^{-\zeta\sqrt{B_4}} - e^{-\zeta\sqrt{B_5}}) \\ & + \frac{G_m(s^\kappa + \frac{\kappa}{1-\kappa})^2}{s^2 A_{10}} (e^{-\zeta\sqrt{B_4}} - e^{-\zeta\sqrt{B_6}}) + A_6 (e^{-\zeta\sqrt{B_4}} - e^{-\zeta\sqrt{B_5}}). \end{aligned} \quad (78)$$

where

$$\begin{aligned} A_9 = & \left( \left( \frac{P_r}{1+R} \right) \left( \frac{1}{1-\kappa} - \frac{H}{P_r} \right) s^\kappa - \left( \frac{H}{1+R} \right) \left( \frac{\kappa}{1-\kappa} \right) \right) \left( \left( \frac{\gamma_1}{1+\kappa} + 1 \right) s^\kappa + \frac{\kappa}{1-\kappa} \right) \\ & - \left( \left( \frac{c}{1-\kappa} + b \right) s^\kappa + \frac{b\kappa}{1-\kappa} \right) \left( s^\kappa + \frac{\kappa}{1-\kappa} \right), \end{aligned} \quad (79)$$

$$\begin{aligned} A_{10} = & \left( S_c \left( \frac{1}{1-\kappa} - K_r \right) s^\kappa + S_c K_r \left( \frac{\kappa}{1-\kappa} \right) \right) \left( \left( \frac{\gamma_1}{1+\kappa} + 1 \right) s^\kappa + \frac{\kappa}{1-\kappa} \right) \\ & - \left( \left( \frac{c}{1-\kappa} + b \right) s^\kappa + \frac{b\kappa}{1-\kappa} \right) \left( s^\kappa + \frac{\kappa}{1-\kappa} \right), \end{aligned} \quad (80)$$

$$B_4 = \frac{\left( \frac{c}{1-\kappa} + b \right) s^\kappa + \frac{b\kappa}{1-\kappa}}{\left( \frac{\gamma_1}{1+\kappa} + 1 \right) s^\kappa + \frac{\kappa}{1-\kappa}}. \quad (81)$$

As  $\kappa \rightarrow 1$  in Equations (23), (41) and (64) for temperature, Equations (28), (46) and (69) for concentration and Equations (34), (55) and (78) for velocity, we recuperate results for temperature, concentration and velocity profile for integer order shown in Kataria and Hari (Equations (13)–(15)) [75], respectively.

Stehfest's formula [79] is one of the simplest algorithms we use to sort out the inverse Laplace transform.

$$w(r, t) = \frac{e^{4.7}}{t} \left[ \frac{1}{2} \bar{w} \left( r, \frac{4.7}{t} \right) + \text{Re} \left\{ \sum_{k=1}^{N_1} (-1)^k \bar{w} \left( r, \frac{4.7 + k\pi i}{t} \right) \right\} \right],$$

where  $\text{Re}(\cdot)$  is the real part,  $i$  is the imaginary unit and  $N_1$  is a natural number.

## 6. Results and Discussion

This article shows the effect of heat and mass transfer in MHD SGF past a vertical plate. Three fractional models C, CF and ABC for flow, energy and mass equations are presented. Fractional derivatives and Laplace transform are applied to examine solutions for non-dimension fractional models. The limiting cases of fractional models are discussed. The impact of several parameters on momentum, heat and mass profiles are compared and studied by graphs.

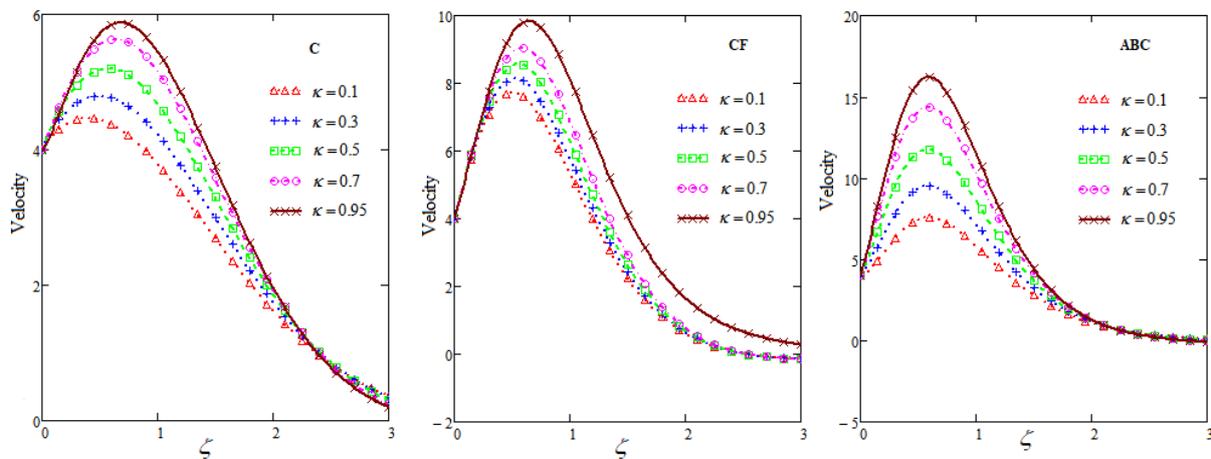
Figure 3 highlights the behavior of momentum curves for  $\kappa$ . We see the velocity accelerates by raising  $\kappa$ . The reason is by an increment in  $\kappa$ , the thickness of the boundary layer will enhance, so the velocity increases. Moreover, velocity is highest for the ABC.

Figure 4 reveals the deviation in velocity distribution under the MHD condition. As  $M$  increases, the frictional force rises and hence fluid velocity decreases. For different values of  $M$ , an increase in Lorentz force effectively decreases flow accelerating forces; as a result, velocity is decelerated. Fluid velocity is maximum, moderate and minimum for ABC, CF and C models, respectively.

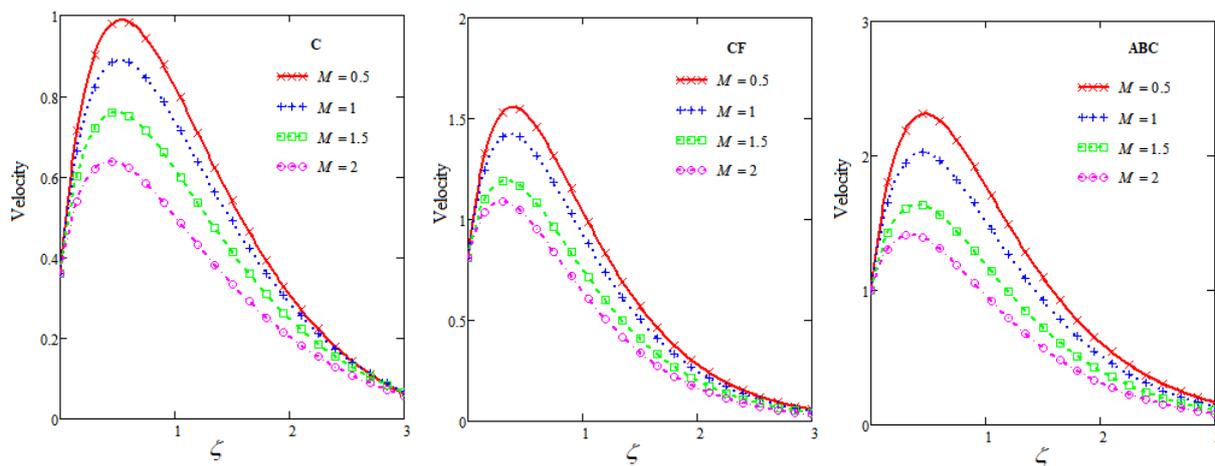
Figure 5 analyzes the influence of  $G_r$  on momentum profile. Physically, large values respond to significant buoyancy force as it is related to strong convection currents. As  $G_r$  increases, all buoyancy forces are dominant frictional forces and the hence momentum profile becomes amplified.

To highlight the velocity behavior for  $G_m$ , we present Figure 6. Physically, the increment in buoyancy forces reduces the viscous force that leads to augmenting the flow raise

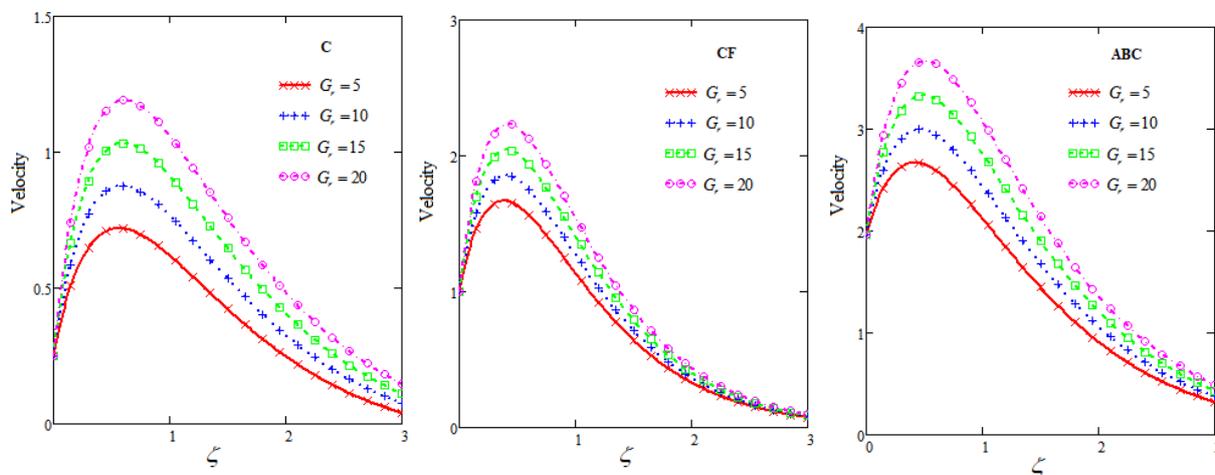
with higher values of  $G_m$ . The velocity curves show maximum behavior for the ABC model as compared to the other two models.



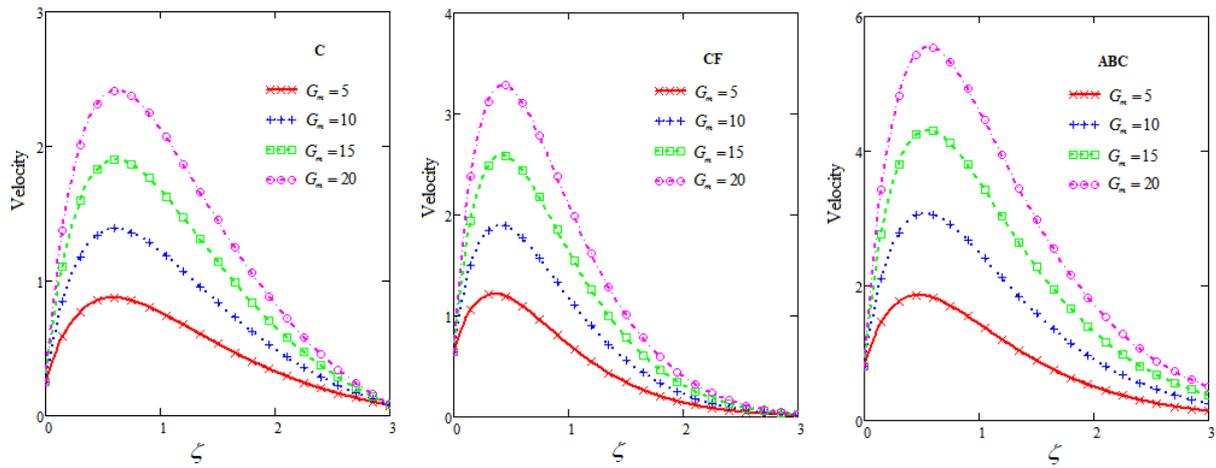
**Figure 3.** Velocity curves corresponding to C, CF and ABC with variable  $\kappa$  where  $P_r = 7, S_c = 0.66, G_r = 10, G_m = 5, H = 3, R = 5, S_r = 3, K_r = 2, M = 0.5, \gamma_1 = 0.1, h = 0.5, c = 1.05$  and  $b = 0.75$ .



**Figure 4.** Velocity curves corresponding to C, CF and ABC with variable  $M$  where  $P_r = 7, S_c = 0.66, G_r = 10, G_m = 5, H = 3, R = 5, S_r = 3, K_r = 2, \kappa = 0.5, \gamma_1 = 0.1, h = 0.5, c = 1.05$  and  $b = 0.75$ .

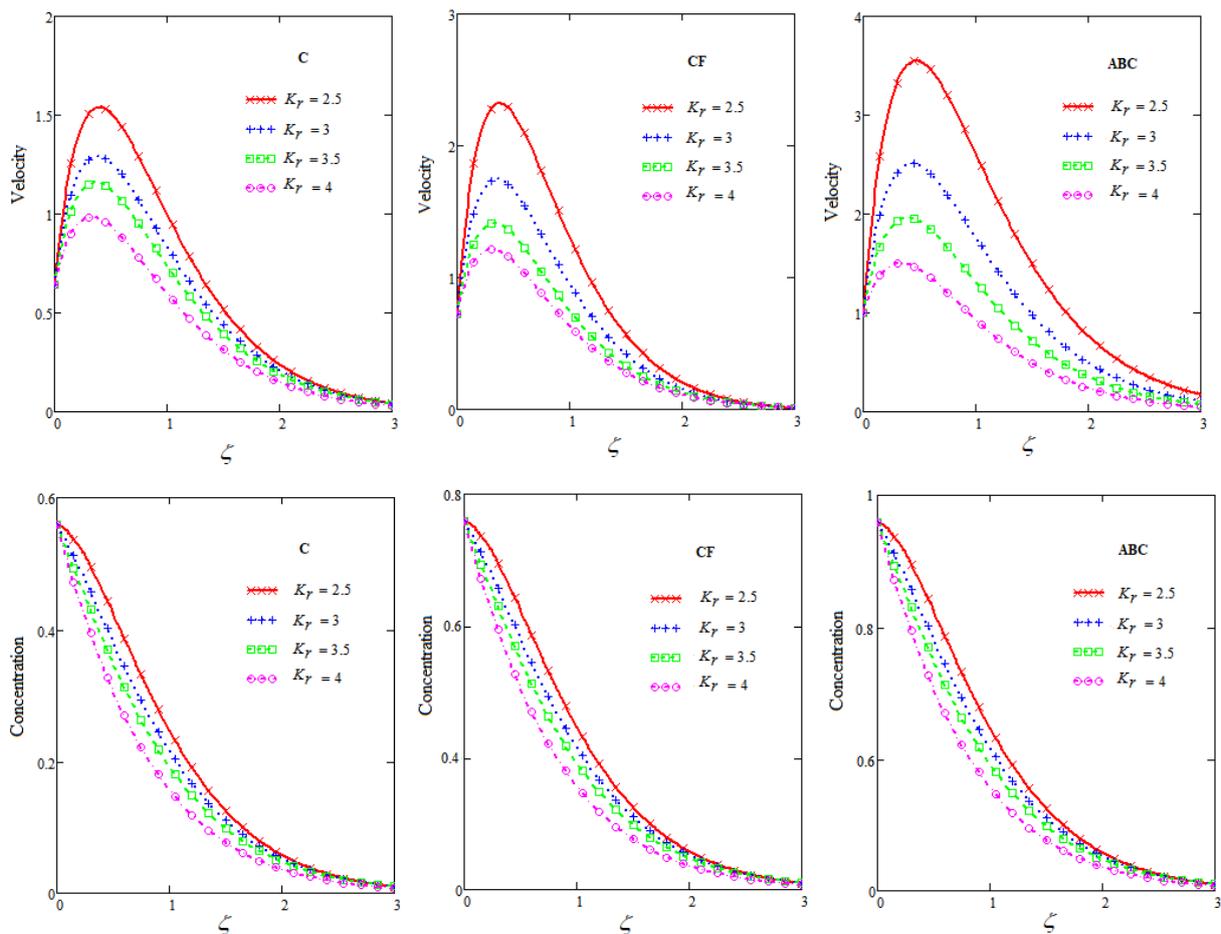


**Figure 5.** Velocity curves corresponding to C, CF and ABC with variable  $G_r$  where  $P_r = 7, S_c = 0.66, \kappa = 0.5, G_m = 5, H = 3, R = 5, S_r = 3, K_r = 2, M = 0.5, \gamma_1 = 0.1, h = 0.5, c = 1.05$  and  $b = 0.75$ .



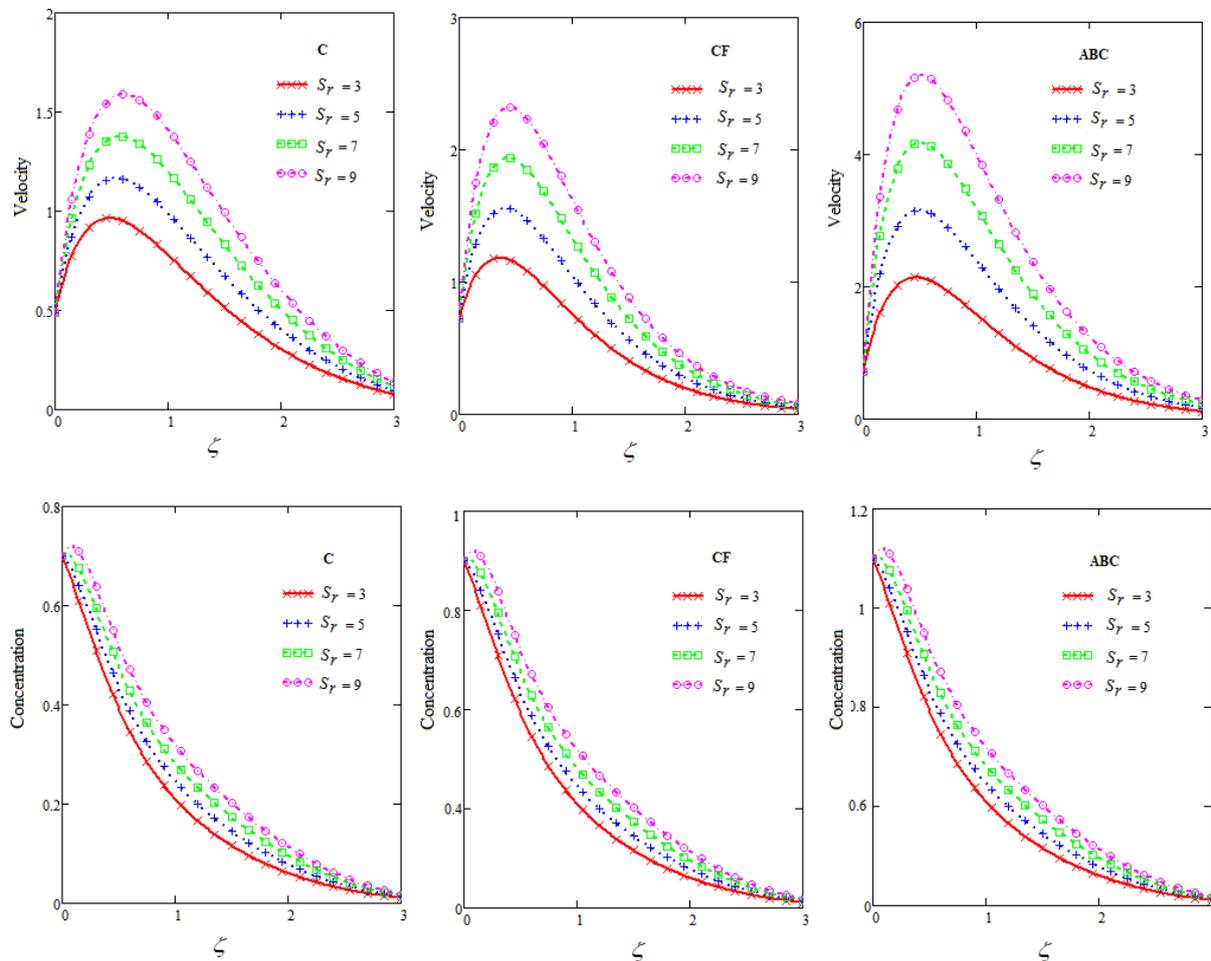
**Figure 6.** Velocity curves corresponding to C, CF and ABC with variable  $G_m$  where  $P_r = 7, S_c = 0.66, \kappa = 0.5, G_r = 10, H = 3, R = 5, S_r = 3, K_r = 2, M = 0.5, \gamma_1 = 0.1, h = 0.5, c = 1.05$  and  $b = 0.75$ .

The significant impact of  $K_r$  on momentum and mass profiles is shown in Figure 7. Both the fluid flow and the concentration decay with the rise in the  $K_r$ . The presence of chemical reaction reduces the buoyancy effects, which decreases and hence weakens the flow field. Clearly, the ABC model shows the highest velocity and concentration.



**Figure 7.** Velocity and concentration curves corresponding to C, CF and ABC with variable  $K_r$  where  $P_r = 7, S_c = 0.66, \kappa = 0.5, G_m = 5, H = 3, R = 5, S_r = 3, \kappa = 0.5, M = 0.5, \gamma_1 = 0.1, h = 0.5, c = 1.05$  and  $b = 0.75$ .

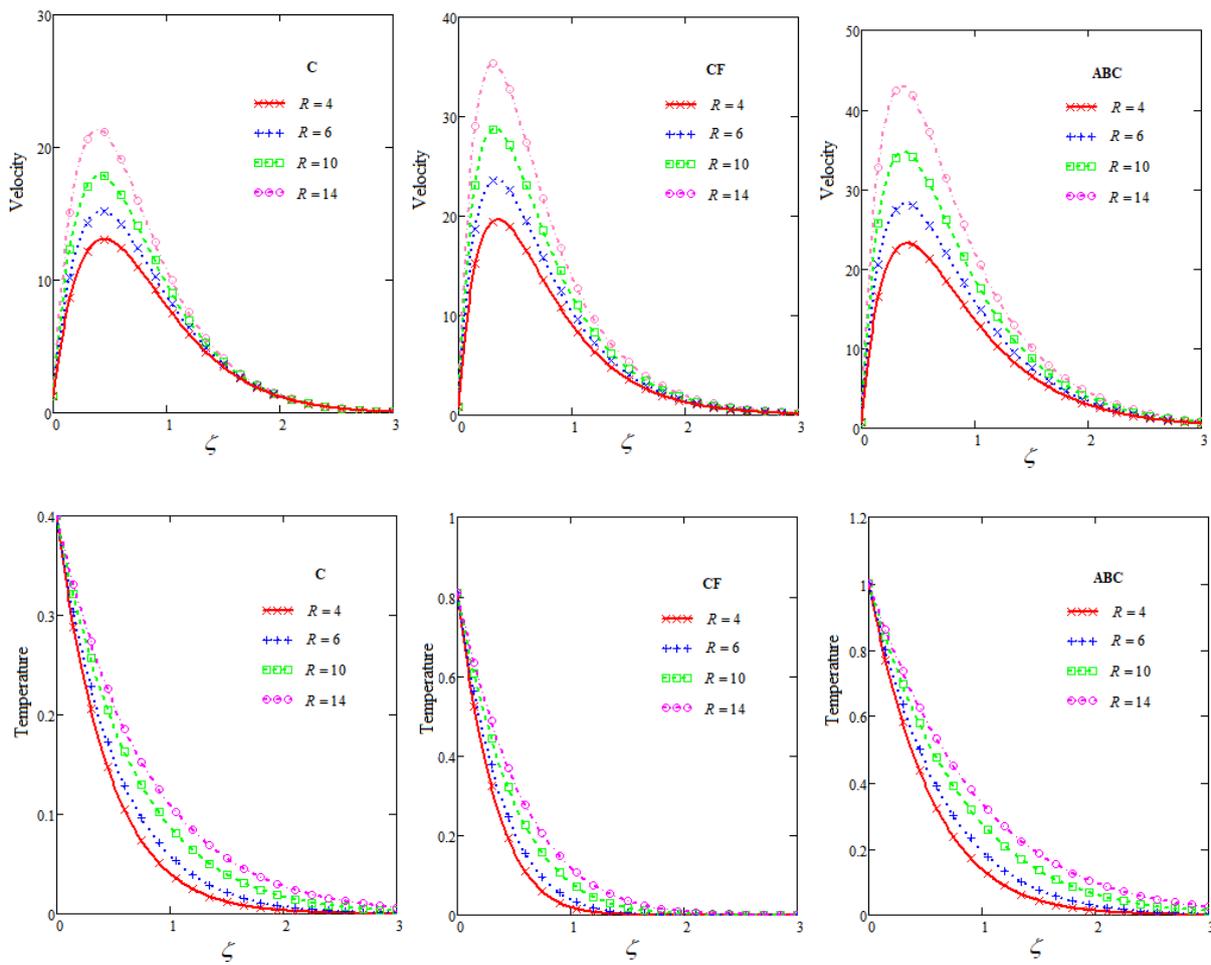
Figure 8 describes the behavior of momentum and mass profile with increasing values of  $S_r$ . As the velocity profile increases, the momentum boundary layer becomes thicker. Physically, large values of  $S_r$  respond with a significant increase in mass buoyancy force; as a result, momentum and mass profile are raised. The effect is greatest for the ABC fractional MHD SGF model.



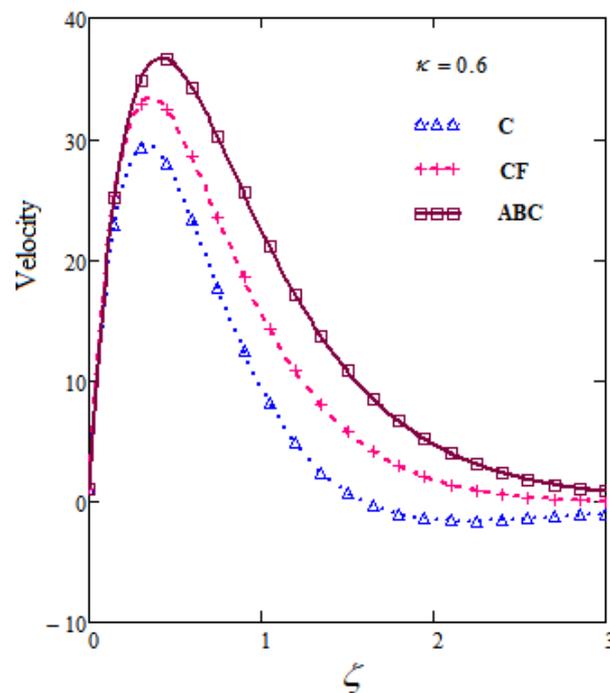
**Figure 8.** Velocity and concentration curves corresponding to C, CF and ABC with variable  $S_r$  where  $P_r = 7$ ,  $S_c = 0.66$ ,  $\kappa = 0.5$ ,  $G_m = 5$ ,  $H = 3$ ,  $R = 5$ ,  $G_r = 10$ ,  $K_r = 2$ ,  $M = 0.5$ ,  $\gamma_1 = 0.1$ ,  $h = 0.5$ ,  $c = 1.05$  and  $b = 0.75$ .

By raising thermal radiation parameter, momentum and heat profiles accelerate as shown in Figure 9. As the thermal radiation parameter increases, heat generation through flow increases, and as a result, bonds between fluid particles split which causes fluid to flow fast.

Figure 10 depicts velocity curves corresponding to C, CF and ABC for  $\kappa = 0.6$ . Clearly, the ABC model shows significant behavior as compared to the other two curves. The reason is that Atangana and Baleanu propounded an advanced fractional operator by utilizing the generalized Mittag–Leffler function as a non-local and non-singular kernel. Comparison of Nusselt number with ref. [72] at  $Pr = 0.71$  is given in Table 1.



**Figure 9.** Velocity and temperature curves corresponding to C, CF and ABC with variable  $R$  where  $P_r = 7, S_c = 0.66, \kappa = 0.5, G_m = 5, H = 3, S_r = 3, G_r = 10, K_r = 2, M = 0.5, \gamma_1 = 0.1, h = 0.5, c = 1.05$  and  $b = 0.75$ .



**Figure 10.** Velocity curves corresponding to fractional models with  $\kappa = 0.6$  and  $P_r = 7, S_c = 0.66, \kappa = 0.5, G_m = 5, H = 3, S_r = 3, G_r = 10, K_r = 2, M = 0.5, \gamma_1 = 0.1, h = 0.5, c = 1.05$  and  $b = 0.75$ .

**Table 1.** Comparison of Nusselt number with ref. [72] at  $P_r = 0.71$ .

$R$	$\phi$	$t$	$Nu$ (Ref. [75]) for Ramped Temp	$Nu$ (C) for Ramped Temp	$Nu$ (CF) for Ramped Temp	$Nu$ (ABC) for Ramped Temp
2	3	0.3	0.3837	0.384	0.385	0.386
2	3	0.5	0.5583	0.557	0.558	0.559
2	3	0.7	0.7289	0.727	0.728	0.729
2	3	0.5	0.4498	0.447	0.448	0.449
2	3	0.5	0.5583	0.557	0.558	0.559
2	5	0.5	0.6521	0.653	0.654	0.655
2	3	0.5	0.5583	0.557	0.558	0.559
4	3	0.5	0.4324	0.433	0.434	0.435
6	3	0.5	0.3655	0.366	0.367	0.368

## 7. Conclusions

This article is about the study of SGF with radiation and chemical reaction. Three fractional operators are applied to establish momentum, heat and mass profiles. Laplace transform and Stehfest's formula are utilized for the solutions of the mentioned equations. Several graphs are presented to illustrate the impact of incipient parameters for the solutions. Some main results are given below:

- Velocity curves are increasing for greater values of  $\kappa$ ,  $G_r$  and  $G_m$ .
- Fluid flow descends for  $P_r$  and  $M$ .
- Velocity and concentration curves show a decreasing behavior under the influence of  $K_r$ .
- Fluid velocity accelerates under the impact of  $S_r$  and  $R$ .
- Heat and mass profiles for  $S_r$  and  $R$  are show an increasing behavior.
- Curves show prominent behavior for ABC among C, CF and ABC.

Extending the work in this article as suggested below will be an interesting endeavor.

We have studied MHD SGF in the presence of ramped concentration and temperature with thermodiffusion. This study can be further carried out by considering more complex OBF models. Additionally, this study can be carried out by a rotational fluid model. Moreover, the present work can be extended by selecting different non-dimensional parameters and quantities.

**Author Contributions:** Conceptualization, N.I.; methodology, J.A.; software, A.A.; validation, M.B.R. and J.A.; formal analysis, A.A.; investigation, J.A.; data curation, M.B.R.; writing—original draft preparation, N.I. and J.A.; writing—review and editing, N.I. and J.A.; visualization, A.A.; supervision, A.A.; project administration, M.B.R.; Final checking, M.B.R. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work has been supported by the Polish National Science Centre under the grant OPUS 18 No. 2019/35/B/ST8/00980.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** My Manuscript has no associated data.

**Acknowledgments:** This work has been supported by the Polish National Science Centre under the grant OPUS 18 No. 2019/35/B/ST8/00980.

**Conflicts of Interest:** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Nomenclature

Symbol	Quantity
$w$	Velocity of the fluid
$\theta$	Temperature of the fluid
$C$	Concentration of the fluid
$g$	Acceleration due to gravity
$k$	Thermal conductivity of the fluid
$k_1$	Permeability parameter
$K_r$	Parameter of chemical reaction
$M$	Parameter of magnetic field
$Q_0$	Coefficient of heat absorption/generation
$P_r$	Prandtl number
$S_c$	Schmidt number
$G_r$	Thermal Grashof number
$G_m$	Mass Grashof number
$R$	Parameter of thermal radiation
$S_r$	Soret number
$D_M$	Coefficient of mass diffusion
$D_T$	Coefficient of thermal diffusion
$\theta_w$	Temperature of fluid at the plate
$\theta_\infty$	Temperature of fluid far away from the plate
$C_w$	Concentration level on the plate
$C_\infty$	Concentration of the fluid far away from the plate
$C_p$	Specific heat at constant temperature
$s$	Laplace transforms parameter
$\rho$	Fluid density
$\kappa$	Fractional parameter
$\gamma$	One of the material modules of second grade fluids
$\gamma_1$	Second grade parameter
$\mu$	Dynamic viscosity
$\nu$	Kinematic viscosity
$\beta_T$	Volumetric coefficient of thermal expansion
$\beta_C$	Volumetric coefficient of expansion for mass concentration
$\Phi$	Porosity

## References

- Bazhlekova, E.; Bazhlekov, I. Viscoelastic flows with fractional derivative models: computational approach by convolutional calculus of Dimovski. *Fract. Calc. Appl. Anal.* **2014**, *17*, 954–976. [[CrossRef](#)]
- Duan, J.S.; Qiu, X. The periodic solution of Stokes' second problem for viscoelastic fluids as characterized by a fractional constitutive equation. *J. Non-Newton. Fluid Mech.* **2014**, *205*, 11–15. [[CrossRef](#)]
- Friedrich, C.H.R. Relaxation and retardation functions of the Maxwell model with fractional derivatives. *Rheo Acta* **1991**, *30*, 151–158. [[CrossRef](#)]
- Meral, F.C.; Royston, T.J.; Magin, R. Fractional calculus in viscoelasticity: An experimental study. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 939–945. [[CrossRef](#)]
- Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach Science Publish: Langhorne, PA, USA, 1993.
- Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999; Volume 198.
- Oldham, K.; Spanier, J. *The Fractional Calculus*; Academic Press: New York, NY, USA; London, UK, 1974.
- Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Wiley and Sons Inc.: New York, NY, USA, 1993.
- Caputo, M. Linear model of dissipation whose Q is almost frequency independent—II. *Geophy. J. Int.* **1967**, *13*, 529–539. [[CrossRef](#)]
- Caputo, M. *Elasticità e Dissipazione*; Zanichelli: Bologna, Italy, 1969.
- Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Progr. Fract. Differ. Appl.* **2015**, *1*, 1–13.
- Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*. [[CrossRef](#)]
- Abel, N.H. *Solution de Quelques Problemes Al'aide D'integrales D'iniies, Oeuvres Completes*; Grondahl: Christiania, Norway, 1881; Volume 1, pp. 16–18.

14. Bagely, R.L. Applications of Generalized Derivatives to Viscoelasticity. Ph.D. Thesis, Air Force Institute of Technology, Kaduna, Nigeria, 1979.
15. Ahmad, W.M.; El-Khazalib, R. Fractional order dynamical models of love. *Chaos Solitons Fractals* **2007**, *33*, 1367–1375. [[CrossRef](#)]
16. Song, L.; Xu, S.; Yang, J. Dynamical models of happiness with fractional order. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 616–628. [[CrossRef](#)]
17. Aksikas, I.; Fuxman, A.; Forbes, J.F.; Winkin, J. LQ control design of a class of hyperbolic PDE systems: Application to xed-bed reactor. *Automatica* **2009**, *45*, 1542–1548. [[CrossRef](#)]
18. Arshad, M.S.; Mardan, S.A.; Riaz, M.B.; Altaf, S. Analysis of time-fractional semi-analytical solutions of strong interacting internal waves in rotating ocean. *Punjab Univ. J. Math.* **2020**, *52*, 99–111.
19. Jhangeer, A.; Munawar, M.; Riaz, M.B.; Baleanu, D. Construction of traveling waves patterns of (1+n)-dimensional modified Zakharov-Kuznetsov equation in plasma physics. *Results Phys.* **2020**, *19*, 103330. [[CrossRef](#)]
20. Lewis, P.A. A theory for the diffraction of the SH waves by randomly rough surfaces in two dimension. *Q. J. Mech. Appl. Math.* **1996**, *49*, 261–286. [[CrossRef](#)]
21. Okrasinski, W. On a nonlinear convolution equation occurring in the theory of water percolation. *Ann. Pol. Math.* **1980**, *37*, 223–229. [[CrossRef](#)]
22. Metzler, R.; Klafter, J. The random walks guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77. [[CrossRef](#)]
23. Owolabi, K.M.; Atangana, A.; Akgul, A. Modelling and analysis of fractal fractional partial differential equations: Application to reaction-diffusion model. *Alex. Eng. J.* **2020**, *59*, 2477–2490. [[CrossRef](#)]
24. Wyss, W. The fractional diffusion equation. *J. Math. Phys.* **1986**, *27*, 2782–2785. [[CrossRef](#)]
25. Heibig, A.; Palade, L.I. On the rest state stability of an objective fractional derivative viscoelastic uid model. *J. Math. Phys.* **2008**, *49*, 043101. [[CrossRef](#)]
26. Rosikin Y.; Shitikova, M. Application of fractional calculus for dynamic problem of solid mechanics, novel trends and recent results. *Appl. Mech. Rev.* **2010**, *63*, 1–52. [[CrossRef](#)]
27. Torvik, P.J.; Bagley, R.L. On the appearance of fractional derivatives in the behaviour of real materials. *J. Appl. Mech.* **1984**, *51*, 294–298. [[CrossRef](#)]
28. Atangana, A. Modelling the spread of COVID-19 with new fractal-fractional operators: Can the lockdown save mankind before vaccination? *Chaos Solitons Fractals* **2020**, *136*, 109860. [[CrossRef](#)] [[PubMed](#)]
29. Atangana, A.; Qureshi, S. Mathematical Modeling of an Autonomous Nonlinear Dynamical System for Malaria Transmission Using Caputo Derivative. *Fract. Order Anal. Theory Methods Appl.* **2020**, 225–252. [[CrossRef](#)]
30. Awrejcewicz, J.; Zafar, A.A.; Kudra, G.; Riaz, M.B. Theoretical study of the blood flow in arteries in the presence of magnetic particles and under periodic body acceleration. *Chaos Solitons Fractals* **2020**, *140*, 110204. [[CrossRef](#)]
31. Sweilam, N.H.; Al-Mekhla, S.M.; Assiri, T.; Atangana, A. Optimal control for cancer treatment mathematical model using Atangana Baleanu Caputo fractional derivative. *Adv. Diff. Equ.* **2020**, *2020*, 334. [[CrossRef](#)]
32. Abro, K.A.; Atangana, A. A comparative analysis of electromechanical model of piezoelectric actuator through Caputo Fabrizio and Atangana Baleanu fractional derivatives. *Math. Methods Appl. Sci.* **2020**, *43*, 9681–9691. [[CrossRef](#)]
33. Arshad, M.S.; Baleanu, D.; Riaz, M.B.; Abbas, M. A Novel 2-Stage Fractional RungeKutta Method for a Time-Fractional Logistic Growth Model. *Discret. Dyn. Nat. Soc.* **2020**, *2020*, 1020472. [[CrossRef](#)]
34. Asjad, M.I.; Aleem, M.; Riaz, M.B. Exact analysis of MHD Walters-B fluid flow with non-singular fractional derivatives of Caputo-Fabrizio in the presence of radiation and chemical reaction. *J. Polym. Sci. Eng.* **2018**, *1*, 599. [[CrossRef](#)]
35. Zafar, A.A.; Kudra, G.; Awrejcewicz, J.; Abdeljawad, T.; Riaz, M.B. A comparative study of the fractional oscillators. *Alex. Eng. J.* **2020**, *59*, 2649–2676. [[CrossRef](#)]
36. Tan, W.; Masuoka, T. Stoke's first problem for a second grade fluid in a porous half-space with heated boundary. *Int. J. Non-Linear Mech.* **2005**, *40*, 515–522. [[CrossRef](#)]
37. Aldoss, T.K.; Al-Nimr, M.A.; Jarrah, M.A.; Al-Shaer, B. Magnetohydrodynamics mixed convection from a vertical plate embedded in a porous medium. *Numer. Heat Transf. Appl.* **1995**, *28*, 635–645. [[CrossRef](#)]
38. Rashidi, S.; Nouri-Borujerdi, A.; Valipour, M.S.; Ellahi, R.; Pop, I. Stress-jump and continuity interface conditions for a cylinder embedded in porous medium. *Transp. Porous Media* **2015**, *107*, 171–186. [[CrossRef](#)]
39. Imran, M.A.; Imran, M.; Fetecau, C. MHD oscillating flows of rotating second grade fluid in a porous medium. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *2014*, 1–12. [[CrossRef](#)]
40. Khan, I.; Farhad, A.; Norzieha, M. Exact solutions for accelerated flows of a rotating second grade fluid in a porous medium. *World Appl. Sci. J.* **2010**, *9*, 55–68.
41. Riaz, M.B.; Zafar, A.A.; Vieru, D. On flows of generalized second grade fluids generated by an oscillating flat plate. In *Matematica, Mecanica, Teoretica, Fizica*; Buletinul Institutului Politehnic din Iași—Universitatea Tehnică: Iași, Romania, 2015; Volume 1.
42. Riaz, M.B.; Awrejcewicz, J.; Rehman, A.U.; Akgül, A. Thermophysical Investigation of Oldroyd-B Fluid with Functional Effects of Permeability: Memory Effect Study Using Non-Singular Kernel Derivative Approach. *Fractal Fract.* **2021**, *5*, 124. [[CrossRef](#)]
43. Hussain, M.; Hayat, T.; Asghar, S.; Fetecau, C. Oscillatory flows of second grade fluid in a porous space. *Nonlinear Anal. Real World Appl.* **2010**, *11*, 2403–2414. [[CrossRef](#)]

44. Erdogan, M.E.; Imrak, C.E. An exact solution of the governing equation of a fluid of second grade for three dimensional vortex flow. *Int. J. Eng. Sci.* **2005**, *43*, 721–729. [[CrossRef](#)]
45. Asghar, S.; Nadeem, S.; Hanif, K.; Hayat, T. Analytic solution of Stoke's second problem for second grade fluid. *Math Probl. Eng.* **2006**, *2006*, 072468. [[CrossRef](#)]
46. Tiwari, A.K.; Ravi, S.K. Analytical studies on transient rotating flow of a second grade fluid in a porous medium. *Adv. Theor. Appl. Mech.* **2009**, *2*, 33–41.
47. Fetecau, C.; Fetecau, C. Starting solutions for the motion of second grade fluid. *Int. J. Eng. Sci.* **2005**, *43*, 781–789. [[CrossRef](#)]
48. Khan, I.; Ellahi, R.; Fetecau, C. Some MHD flows of second grade fluid through the porous medium. *J. Porous Media* **2008**, *11*, 389–400.
49. Fetecau, C.; Fetecau, C. Starting solutions for the motion of a second grade fluid due to longitudinal and torsional oscillations of a circular cylinder. *Int. J. Eng. Sci.* **2006**, *44*, 788–796. [[CrossRef](#)]
50. Hsu, S.H.; Jamieson, A.M. Viscoelastic behavior at the thermal sol-gel transition of gelatin. *Polymer* **1993**, *34*, 2602–2608. [[CrossRef](#)]
51. Bandelli, R. Unsteady unidirectional flows of second grade fluids in domains with heated boundaries. *Int. J. Non-Linear Mech.* **1995**, *30*, 263–269. [[CrossRef](#)]
52. Damesh, R.A.; Shatnawi, A.S.; Chamkha, A.J.; Duwairi, H.M. Transient mixed convection flow of second grade viscoelastic fluid over a vertical surface. *Nonlinear Anal. Model. Control.* **2008**, *13*, 169–179. [[CrossRef](#)]
53. Nazar, M.; Fetecau, C.; Vieru, D.; Fetecau, C. New exact solutions corresponding to the second problem of stokes' for second grade fluids. *Nonlinear Anal. Real World Appl.* **2010**, *11*, 584–591. [[CrossRef](#)]
54. Ali, F.; Norzieha, M.; Sharidan, S.; Khan, I.; Hayat, T. New exact solutions of stokes' second problem for an MHD second grade fluid in a porous space. *Int. J. Nonlinear Mech.* **2012**, *47*, 521–525. [[CrossRef](#)]
55. Makinde, O.D.; Khan, W.A.; Culham, J.R. MHD variable viscosity reacting flow over a convectively heated plate in a porous medium with thermophoresis and radiative heat transfer. *Int. J. Heat Mass Transf.* **2016**, *93*, 595–604. [[CrossRef](#)]
56. Sheikholeslami, M.; Ganji, D.D.; Javed, M.Y.; Ellahi, R. Effect of thermal radiation on MHD nanofluid flow and heat transfer by means of two phase model. *J. Magn. Magn. Mater.* **2015**, *374*, 36–43. [[CrossRef](#)]
57. Zhang, C.; Zheng, L.; Zhang, X.; Chen, G. MHD flow and radiation heat transfer of nanofluid in porous media with variable surface heat flux and chemical reaction. *Appl. Math Model.* **2015**, *39*, 165–181. [[CrossRef](#)]
58. Rashidi, M.M.; Ali, M.; Freidoonimehr, N.; Rostami, B.; Hossain, M.A. Mixed convective heat transfer for MHD viscoelastic fluid flow over a porous wedge with thermal radiation. *Adv. Mech. Eng.* **2014**, *6*, 735939. [[CrossRef](#)]
59. Dehghan, M.; Rahmani, Y.; Ganji, D.D.; Saedodin, S.; Valipour, M.S.; Rashidi, S. Convection-radiation heat transfer in solar heat exchangers filled with a porous medium: homotopy perturbation method versus numerical analysis. *Renew. Energy* **2015**, *74*, 448–455. [[CrossRef](#)]
60. Imran, M.A.; Shah, N.A.; Aleem, M.; Khan, I. Heat transfer analysis of fractional second-grade fluid subject to Newtonian heating with Caputo and Caputo-Fabrizio fractional derivatives: A comparison. *Eur. Phys. J. Plus* **2017**, *132*, 340.
61. Tassaddiq, A. MHD flow of a fractional second grade fluid over an inclined heated plate. *Chaos Solitons Fractals* **2019**, *123*, 341–346. [[CrossRef](#)]
62. Sene, N. Second-grade fluid model with Caputo–Liouville generalized fractional derivative. *Chaos Solitons Fractals* **2020**, *133*, 109631. [[CrossRef](#)]
63. Haq, S.; Jan, S.; Jan, S.A.; Khan, I.; Singh, J. Heat and mass transfer of fractional second grade fluid with slippage and ramped wall temperature using Caputo-Fabrizio fractional derivative approach. *AIMS Math.* **2020**, *5*, 3056–3088. [[CrossRef](#)]
64. Fatecau, C.; Zafar, A.A.; Vieru, D.; Awrejcewicz, J. Hydromagnetic flow over a moving plate of second grade fluids with time fractional derivatives having non-singular kernel. *Chaos Solitons Fractals* **2020**, *130*, 109454. [[CrossRef](#)]
65. Siddique, I.; Tlili, I.; Bukhari, M.; Mahsud, Y. Heat transfer analysis in convective flows of fractional second grade fluids with Caputo–Fabrizio and Atangana–Baleanu derivative subject to Newtonian heating. *Mech.-Time-Depend. Mater.* **2019**, *25*, 291–311. [[CrossRef](#)]
66. Mehryan, S.A.; Ghalambaz, M.; Vaezi, M.; Zadeh, S.M.; Sedaghatzadeh, N.; Younis, O.; Chamkha, A.J.; Abulkhair, H. Non-Newtonian phase change study of nano-enhanced n-octadecane comprising mesoporous silica in a porous medium. *Appl. Math. Model.* **2021**, *97*, 463–482. [[CrossRef](#)]
67. Rana, B.M.; Arifuzzaman, S.M.; Islam, S.; Reza-E-Rabbi, S.; Al-Mamun, A.; Mazumder, M.; Roy, K.C.; Khan, M.S. Swimming of microbes in blood flow of nano-bioconvective Williamson fluid. *Ther. Sci. Eng. Prog.* **2021**, *25*, 101018. [[CrossRef](#)]
68. Al-Mamun, A.; Arifuzzaman, S.M.; Reza-E-Rabbi, S.; Alam, U.S.; Islam, S.; Khan, M.S. Numerical simulation of periodic MHD casson nanofluid flow through porous stretching sheet. *SN Appl. Sci.* **2021**, *3*, 1–14. [[CrossRef](#)]
69. Abro, K.A.; Gómez-Aguilar, J.F. Role of Fourier sine transform on the dynamical model of tensioned carbon nanotubes with fractional operator. *Math. Methods Appl. Sci.* **2020**. [[CrossRef](#)]
70. Abro, K.A.; Laghari, M.H.; Gómez-Aguilar, J.F. Application of Atangana-Baleanu Fractional Derivative to Carbon Nanotubes Based Non-Newtonian Nanofluid: Applications in Nanotechnology. *J. Appl. Comput. Mech.* **2020**, *6*, 1260–1269. [[CrossRef](#)]
71. Abro, K.A.; Khan, I.; Gomez-Aguilar, J.F. Heat transfer in magnetohydrodynamic free convection flow of generalized ferrofluid with magnetite nanoparticles. *J. Therm. Anal. Calorim.* **2021**, *143*, 3633–3642. [[CrossRef](#)]
72. Rehman, A.U.; Riaz, M.B.; Akgül, A.; Saeed, S.T.; Baleanu, D. Heat and mass transport impact on MHD second-grade fluid: A comparative analysis of fractional operators. *Heat Trans.* **2021**. [[CrossRef](#)]

73. Song, Y.Q.; Raza, A.; Al-Khaled, K.; Farid, S.; Khan, M.I.; Khan, S.U.; Shi, Q.H.; Malik, M.Y.; Khan, M.I. Significances of exponential heating and Darcy's law for second grade fluid flow over oscillating plate by using Atangana-Baleanu fractional derivatives. *Case Stud. Therm. Eng.* **2021**, *27*, 101266. [[CrossRef](#)]
74. Riaz, M.B.; Abro, K.A.; Abualnaja, K.M.; Akgül, A.; Rehman, A.U.; Abbas, M.; Hamed, Y.S. Exact solutions involving special functions for unsteady convective flow of magnetohydrodynamic second grade fluid with ramped conditions. *Adv. Diff. Equ.* **2021**, *2021*, 408. [[CrossRef](#)]
75. Kataria, H.R.; Patel, H.R. Effect of thermo-diffusion and parabolic motion on MHD Second grade fluid flow with ramped wall temperature and ramped surface concentration. *Alex. Eng. J.* **2018**, *57*, 173–185. [[CrossRef](#)]
76. Riaz, M.B.; Atangana, A.; Iftikhar, N. Heat and mass transfer in Maxwell fluid in view of local and non-local differential operators. *J. Therm. Anal. Calorim.* **2020**, *143*, 4313–4329. [[CrossRef](#)]
77. Iftikhar, N.; Baleanu, D.; Riaz, M.B.; Husnine, S.M. Heat and Mass Transfer of Natural Convective Flow with Slanted Magnetic Field via Fractional Operators. *J. Appl. Comput. Mech.* **2020**, *7*, 189–212. [[CrossRef](#)]
78. Riaz, M.B.; Iftikhar, N. A comparative study of heat transfer analysis of MHD Maxwell fluid in view of local and non-local differential operators. *Chaos Solitons Fractals* **2020**, *132*, 109556. [[CrossRef](#)]
79. Stehfest, H.A. Numerical inversion of Laplace transforms. In Proceedings of the Communications of the ACM, New York, NY, USA, 1 January 1970; Volume 13, pp. 9–47.