Article

Weak Solutions for Time-Fractional Evolution Equations in Hilbert Spaces

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Abstract: Our purpose is to introduce a notion of weak solution for a class of abstract fractional differential equations. We point out that the time fractional derivative occurring in the equations is in the sense of the Caputo derivative. We prove existence results for weak and strong solutions. To justify the abstract theory we develop, we apply two examples of concrete equations: time-fractional wave equations and time-fractional Petrovsky systems. Both these concrete examples are of great interest in the theory of fractional partial differential equations.

Keywords: abstract evolution equations; fractional analysis; strong and weak solutions

1. Introduction

Our aim is to establish the existence and regularity results regarding weak solutions of the abstract fractional equation

$$\partial_t^\alpha u + Au = 0 \quad \text{in } (0, T), \quad (1)$$

where u takes its values in a Hilbert space H , A is a densely defined, linear self-adjoint positive operator on H and the Caputo derivative $\partial_t^\alpha u$ of order $\alpha \in (1, 2)$ is defined as

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \frac{d^2 u}{d\tau^2}(\tau) d\tau.$$

In order to carry out our task, we need to give an appropriate notion of weak solutions that constitute an agreement between smoothness requirement and applicability to concrete models.

Our study is inspired from the literature concerning partial differential equations of the type (1) in the case the operator A is a concrete differential operator. Typically, the operator A is the Laplacian or, more general, a uniformly elliptic operator. To our knowledge, the biharmonic operator has not been studied yet in the framework of the time-fractional differential equations.

We mention [1], where the author gives a first analysis about evolution equations, related to fractional differential ones, that interpolate the heat equation and the wave equation in the domain $(0, +\infty) \times \mathbb{R}$. In the survey [2], the interested reader can find a historical note on the origin of the fractional derivative in the Caputo sense. Moreover, the authors provide some arguments showing the usefulness of the Caputo derivatives in the theory of viscoelasticity.

To begin with, we introduce strong solutions of (1) as functions u belonging to $C([0, T]; D(A)) \cap C^1([0, T]; H)$, $\partial_t^\alpha u \in C([0, T]; H)$ and u satisfies Equation (1) for any $t \in [0, T]$.

Citation: Loreti, P.; Sforza, D. Weak Solutions for Time-Fractional Evolution Equations in Hilbert Spaces. *Fractal Fract.* **2021**, *5*, 138. <https://doi.org/10.3390/fractalfract5040138>

Academic Editor: Mahamadi Warma

Received: 5 July 2021

Accepted: 16 September 2021

Published: 25 September 2021

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The Definition of weak solutions that we propose is suggested by the formula

$$\partial_t^\alpha u(t) = \frac{d}{dt} I^{2-\alpha} (u' - u'(0))(t), \quad (2)$$

where $I^{2-\alpha}$ is the Riemann–Liouville operator of order $2 - \alpha$, see Formula (12) in Section 2.2.

The novelty of the paper consists of introducing the following notion: we define a weak solution of the fractional differential Equation (1) as a function u belonging to $C([0, T]; D(\sqrt{A}))$, $u' \in L^2(0, T; H) \cap C([0, T]; D(A^{-\theta}))$, for some $\theta \in (0, 1)$, and for any $v \in D(\sqrt{A})$ one has $\langle I^{2-\alpha} (u' - u'(0))(t), v \rangle \in C^1([0, T])$ and

$$\frac{d}{dt} \langle I^{2-\alpha} (u' - u'(0))(t), v \rangle + \langle \sqrt{A} u(t), \sqrt{A} v \rangle = 0 \quad t \in [0, T]. \quad (3)$$

Thanks to the above Definition we are able to prove in Section 3 the following Theorem.

Theorem 1. *If $u_0 \in D(\sqrt{A})$ and $u_1 \in H$, then the function*

$$u(t) = \sum_{n=1}^{\infty} [\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)] e_n \quad (4)$$

is the unique weak solution of (1) satisfying the initial conditions

$$u(0) = u_0, \quad u'(0) = u_1.$$

Formula (4) gives an explicit representation through expansions involving the Mittag–Leffler functions, the eigenvalues λ_n and the eigenfunctions e_n of the spatial operator. This representation formula is suggested by the spectral approach given in [3,4], where the authors deal with uniformly elliptic operators in a different setting. It is worthwhile to mention the paper [5] (see also references therein), where the authors give the representation of classical solutions by means of the α -resolvent family. A further study in the viewpoint of [3] can be found in [6,7]. Our efforts are to introduce a weaker notion of solutions to guarantee existence results adopting weak assumptions on the initial data.

Our abstract results can be applied to various partial differential equations to obtain the existence of weak solutions. The last section is devoted to the discussion of two examples: time-fractional wave equations and time-fractional Petrovsky systems.

2. Preliminaries

In this section, we collect some notations, Definitions and known results that we use to prove our main results.

2.1. Abstract Operators

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A is a linear self-adjoint positive operator on H with dense domain $D(A)$, satisfying

$$\langle Ax, x \rangle \geq a \|x\|^2 \quad \forall x \in D(A) \quad (5)$$

for some $a > 0$. We assume that the spectrum of A consists of a sequence of positive eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ such that $\lambda_n \xrightarrow{n} \infty$. Moreover, λ_n are all distinct numbers, whence the eigenspace generated by every single λ_n has dimension one.

Moreover, the eigenfunctions e_n of A ($Ae_n = \lambda_n e_n$) constitute an orthonormal basis of H .

The fractional powers A^θ are defined for $\theta > 0$, see, e.g., [8,9]. The domain $D(A^\theta)$ of A^θ consists of $u \in H$ such that

$$\sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n \rangle|^2 < +\infty$$

and

$$A^\theta u = \sum_{n=1}^{\infty} \lambda_n^\theta \langle u, e_n \rangle e_n, \quad u \in D(A^\theta).$$

Moreover, $D(A^\theta)$ is a Hilbert space with the norm given by

$$\|u\|_{D(A^\theta)} = \|A^\theta u\| = \left(\sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n \rangle|^2 \right)^{1/2}, \quad u \in D(A^\theta), \quad (6)$$

and for any $0 < \theta_1 < \theta_2$ we have $D(A^{\theta_2}) \subset D(A^{\theta_1})$.

In particular, the norm of the space $D(\sqrt{A})$ is given by

$$\|u\|_{D(\sqrt{A})} = \|\sqrt{A}u\| = \left(\sum_{n=1}^{\infty} \lambda_n |\langle u, e_n \rangle|^2 \right)^{1/2}, \quad u \in D(\sqrt{A}). \quad (7)$$

If we identify the dual H' with H itself, then we have $D(A^\theta) \subset H \subset (D(A^\theta))'$. From now on we set

$$D(A^{-\theta}) := (D(A^\theta))', \quad (8)$$

whose elements are bounded linear functionals on $D(A^\theta)$. If $u \in D(A^{-\theta})$ and $\varphi \in D(A^\theta)$ the value $u(\varphi)$ is denoted by

$$\langle u, \varphi \rangle_{-\theta, \theta} := u(\varphi). \quad (9)$$

This notation is analogous to that used in [3]. In addition, $D(A^{-\theta})$ is a Hilbert space with the norm given by

$$\|u\|_{D(A^{-\theta})} = \left(\sum_{n=1}^{\infty} \lambda_n^{-2\theta} |\langle u, e_n \rangle_{-\theta, \theta}|^2 \right)^{1/2}, \quad u \in D(A^{-\theta}), \quad (10)$$

and for any $0 < \theta_1 < \theta_2$ we have $D(A^{-\theta_1}) \subset D(A^{-\theta_2})$. We also recall that

$$\langle u, \varphi \rangle_{-\theta, \theta} = \langle u, \varphi \rangle \quad \text{for } u \in H, \varphi \in D(A^\theta), \quad (11)$$

see, e.g., [10] (Chapitre V).

2.2. Fractional Derivatives

Definition 1. For any $\beta > 0$ we denote the Riemann–Liouville fractional integral operator of order β by

$$I^\beta(f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} f(\tau) d\tau, \quad f \in L^1(0, T), \text{ a.e. } t \in (0, T), \quad (12)$$

where $T > 0$ and $\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt$ is the Euler Gamma function.

The Caputo fractional derivative of order $\alpha \in (1, 2)$ is given by

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t - \tau)^{1-\alpha} \frac{d^2 f}{d\tau^2}(\tau) d\tau.$$

By means of the Riemann–Liouville integral operator $I^{2-\alpha}$ we can write

$$\partial_t^\alpha f(t) = I^{2-\alpha} \left(\frac{d^2 f}{d\tau^2} \right)(t).$$

We also note that if f' is absolutely continuous, then

$$\partial_t^\alpha f(t) = \frac{d}{dt} I^{2-\alpha} (f' - f'(0))(t). \quad (13)$$

For arbitrary constants $\alpha, \beta > 0$, we denote the Mittag-Leffler functions by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}.$$

By the power series, one can note that $E_{\alpha,\beta}(z)$ is an entire function of $z \in \mathbb{C}$. We note that $E_{\alpha,1}(0) = 1$.

The proof of the following result can be found in [11] (p. 35), see also [3] (Lemma 3.1). In the following we denote the Laplace transform of a function $f(t)$ by the symbol

$$\mathcal{L}[f(t)](z) := \int_0^\infty e^{-zt} f(t) dt, \quad z \in \mathbb{C}.$$

Lemma 1.

1. Let $\alpha \in (1, 2)$ and $\beta > 0$ be. Then for any $\mu \in \mathbb{R}$ such that $\pi\alpha/2 < \mu < \pi$ there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|}, \quad z \in \mathbb{C}, \quad \mu \leq |\arg(z)| \leq \pi. \quad (14)$$

2. For $\alpha, \beta, \lambda > 0$ one has

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)](z) = \frac{z^{\alpha-\beta}}{z^\alpha + \lambda}, \quad \Re z > \lambda^{\frac{1}{\alpha}}. \quad (15)$$

3. If $\alpha, \lambda > 0$, then we have

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0, \quad (16)$$

$$\frac{d}{dt} (t^k E_{\alpha,k+1}(-\lambda t^\alpha)) = t^{k-1} E_{\alpha,k}(-\lambda t^\alpha), \quad k \in \mathbb{N}, t \geq 0, \quad (17)$$

$$\frac{d}{dt} (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha), \quad t \geq 0. \quad (18)$$

We also exhibit an elementary result that will be useful in the estimates.

Lemma 2. For any $0 < \beta < 1$ the function $x \rightarrow \frac{x^\beta}{1+x}$ gains its maximum on $[0, +\infty]$ at point $\frac{\beta}{1-\beta}$ and the maximum value is given by

$$\max_{x \geq 0} \frac{x^\beta}{1+x} = \beta^\beta (1-\beta)^{1-\beta}, \quad \beta \in (0, 1). \quad (19)$$

Now we recall the Definition of fractional vector-valued Sobolev spaces. For $\beta \in (0, 1)$, $T > 0$ and a Hilbert space H , endowed with the norm $\|\cdot\|_H$, $H^\beta(0, T; H)$ is the space of all $u \in L^2(0, T; H)$ such that

$$[u]_{H^\beta(0,T;H)} := \left(\int_0^T \int_0^T \frac{\|u(t) - u(\tau)\|_H^2}{|t - \tau|^{1+2\beta}} dt d\tau \right)^{1/2} < +\infty, \quad (20)$$

that is $[u]_{H^\beta(0,T;H)}$ is the so-called Gagliardo semi-norm of u . $H^\beta(0,T;H)$ is endowed with the norm

$$\|\cdot\|_{H^\beta(0,T;H)} := \|\cdot\|_{L^2(0,T;H)} + [\cdot]_{H^\beta(0,T;H)}. \quad (21)$$

The following result is a generalization to the case of vector valued functions of [12] (Theorem 2.1). We will use the symbol $\sim$ to indicate equivalent norms.

Theorem 2. Let H be a separable Hilbert space.

(i) The Riemann–Liouville operator $I^\beta : L^2(0,T;H) \rightarrow L^2(0,T;H)$, $0 < \beta \leq 1$, is injective and the range $\mathcal{R}(I^\beta)$ of I^β is given by

$$\mathcal{R}(I^\beta) = \begin{cases} H^\beta(0,T;H), & 0 < \beta < \frac{1}{2}, \\ \left\{ v \in H^{\frac{1}{2}}(0,T;H) : \int_0^T t^{-1}|v(t)|^2 dt < \infty \right\}, & \beta = \frac{1}{2}, \\ {}_0H^\beta(0,T;H), & \frac{1}{2} < \beta \leq 1, \end{cases}$$

where ${}_0H^\beta(0,T) = \{u \in H^\beta(0,T) : u(0) = 0\}$.

(ii) For the Riemann–Liouville operator I^β and its inverse operator $I^{-\beta}$ the norm equivalences

$$\begin{aligned} \|I^\beta(u)\|_{H^\beta(0,T;H)} &\sim \|u\|_{L^2(0,T;H)}, & u \in L^2(0,T;H), \\ \|I^{-\beta}(v)\|_{L^2(0,T;H)} &\sim \|v\|_{H^\beta(0,T;H)}, & v \in \mathcal{R}(I^\beta), \end{aligned}$$

hold true.

3. Existence and Regularity of Solutions

First, we introduce the notions of weak and strong solutions. Let $\alpha \in (1,2)$ and $T > 0$.

Definition 2.

1. A function u is called a weak solution of the abstract time-fractional equation

$$\partial_t^\alpha u + Au = 0 \quad (22)$$

if $u \in C([0,T];D(\sqrt{A}))$, $u' \in L^2(0,T;H) \cap C([0,T];D(A^{-\theta}))$, for some $\theta \in (0,1)$, and for any $v \in D(\sqrt{A})$ one has $\langle I^{2-\alpha}(u' - u'(0))(t), v \rangle \in C^1([0,T])$ and

$$\frac{d}{dt} \langle I^{2-\alpha}(u' - u'(0))(t), v \rangle + \langle \sqrt{A}u(t), \sqrt{A}v \rangle = 0, \quad t \in [0,T]. \quad (23)$$

2. A function u is called a strong solution of (22) if $u \in C([0,T];D(A)) \cap C^1([0,T];H)$, $\partial_t^\alpha u \in C([0,T];H)$ and satisfies Equation (22) for any $t \in [0,T]$.

Remark 1. We observe that a strong solution is also a weak solution.

Furthermore, we note that for a weak solution u of (22) we have $\partial_t^\beta u \in H^{1-\beta}(0,T;H)$, $\beta \in (0,1)$, where the Caputo fractional derivative of order $\beta \in (0,1)$ is defined as

$$\partial_t^\beta u(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} u'(\tau) d\tau = I^{1-\beta}(u')(t). \quad (24)$$

Indeed, since $u' \in L^2(0,T;H)$ we can apply Theorem 2 to obtain $I^{1-\beta}(u') \in H^{1-\beta}(0,T;H)$, that is $\partial_t^\beta u \in H^{1-\beta}(0,T;H)$.

In particular, for $\beta = \alpha/2$ and $\beta = 1 - \alpha/2$ we have $\partial_t^{\alpha/2} u \in H^{1-\alpha/2}(0,T;H)$ and $\partial_t^{1-\alpha/2} u \in H^{\alpha/2}(0,T;H)$, respectively.

A classical approach to solve scalar fractional differential equations is using Laplace transform methods.

Lemma 3. For any $\lambda > 0$ and $x, y \in \mathbb{R}$ the solution of problem

$$\begin{cases} \partial_t^\alpha u(t) + \lambda u(t) = 0, & t \geq 0, \\ u(0) = x, \quad u'(0) = y, \end{cases}$$

is given by

$$u(t) = x E_{\alpha,1}(-\lambda t^\alpha) + y t E_{\alpha,2}(-\lambda t^\alpha), \quad t \geq 0.$$

We omit the proof of the Lemma because it is well known, see, e.g., [13].

Theorem 3.

(i) If $u_0 \in D(\sqrt{A})$ and $u_1 \in H$, then the function

$$u(t) = \sum_{n=1}^{\infty} [\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)] e_n \quad (25)$$

is the unique weak solution of (22) satisfying the initial conditions

$$u(0) = u_0, \quad u'(0) = u_1. \quad (26)$$

In addition

$$u'(t) = \sum_{n=1}^{\infty} [-\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha)] e_n, \quad (27)$$

and $u' \in C([0, T]; D(A^{-\theta}))$ for $\theta \in (\frac{2-\alpha}{2\alpha}, \frac{1}{2})$.

(ii) For $u_0 \in D(A)$ and $u_1 \in D(\sqrt{A})$ the weak solution given by (25) is a strong one and

$$\partial_t^\alpha u(t) = - \sum_{n=1}^{\infty} [\lambda_n \langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \lambda_n \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)] e_n. \quad (28)$$

Proof. To begin with, we intend to justify the expression of the solution given by the series (25). To this end we search the solution in the form

$$u(t) = \sum_{n=1}^{\infty} u_n(t) e_n$$

where the functions $u_n(t) = \langle u(t), e_n \rangle$ are unknown. First, we assume that there exists a strong solution u , see Definition 2-(2). By means of the scalar product in H we multiply the equation $\partial_t^\alpha u + Au = 0$ by e_n and take into account that the operator A is self-adjoint, so we have

$$0 = \langle \partial_t^\alpha u, e_n \rangle + \langle Au, e_n \rangle = \partial_t^\alpha u_n + \langle u, Ae_n \rangle = \partial_t^\alpha u_n + \lambda_n \langle u, e_n \rangle = \partial_t^\alpha u_n + \lambda_n u_n.$$

Therefore, in virtue of the initial conditions (26) $u_n(t)$ is the solution of problem

$$\begin{cases} \partial_t^\alpha u_n(t) + \lambda_n u_n(t) = 0, & t \geq 0, \\ u_n(0) = \langle u_0, e_n \rangle, \quad u'_n(0) = \langle u_1, e_n \rangle, \end{cases} \quad (29)$$

and hence by Lemma 3 we get

$$u_n(t) = \langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha), \quad t \geq 0. \quad (30)$$

(i) Now, we take $u_0 \in D(\sqrt{A})$, $u_1 \in H$ and show that

$$u(t) = \sum_{n=1}^{\infty} u_n(t) e_n,$$

with $u_n(t)$ given by (30), is a weak solution of (22) satisfying the initial conditions (26). First, we note that for any $t \in [0, T]$ we have $u(t) \in D(\sqrt{A})$. Indeed, since

$$\|\sqrt{A}u(t)\|^2 = \sum_{n=1}^{\infty} \lambda_n |u_n(t)|^2 \leq 2 \sum_{n=1}^{\infty} \lambda_n |\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha)|^2 + 2 \sum_{n=1}^{\infty} \lambda_n |\langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)|^2,$$

thanks to (14) we have

$$\begin{aligned} \lambda_n |\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha)|^2 &\leq C \lambda_n |\langle u_0, e_n \rangle|^2, \\ \lambda_n |\langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)|^2 &\leq C t^{2-\alpha} |\langle u_1, e_n \rangle|^2 \frac{\lambda_n t^\alpha}{(1 + \lambda_n t^\alpha)^2} \leq C t^{2-\alpha} |\langle u_1, e_n \rangle|^2, \end{aligned}$$

and hence, being $\alpha < 2$, we get

$$\|\sqrt{A}u(t)\|^2 \leq C \|\sqrt{A}u_0\|^2 + C T^{2-\alpha} \|u_1\|^2. \quad (31)$$

Following the same reasoning pursued to get (31), we obtain for any $n \in \mathbb{N}$

$$\left\| \sqrt{A} \sum_{k=n}^{\infty} u_k(t) e_k \right\|^2 \leq C \sum_{k=n}^{\infty} \lambda_k |\langle u_0, e_k \rangle|^2 + C T^{2-\alpha} \sum_{k=n}^{\infty} |\langle u_1, e_k \rangle|^2,$$

and hence

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left\| \sqrt{A} \sum_{k=n}^{\infty} u_k(t) e_k \right\| = 0.$$

As a consequence, the series $\sum_{n=1}^{\infty} u_n(t) e_n$ is convergent in $D(\sqrt{A})$ uniformly in $t \in [0, T]$, so $u \in C([0, T]; D(\sqrt{A}))$. Moreover, $u(0) = \sum_{n=1}^{\infty} \langle u_0, e_n \rangle e_n = u_0$.

Concerning Formula (27) for u' , thanks to (16) and (17) with $k = 1$ we note that for $u_n(t)$ given by (30) we have

$$u'_n(t) = -\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha). \quad (32)$$

We prove that u' is given by (27) and belongs to $C([0, T]; D(A^{-\theta}))$ for $\theta \in (\frac{2-\alpha}{2\alpha}, \frac{1}{2})$. Indeed, if $u'_n(t)$ is given by (32), then we have

$$\left\| \sum_{n=1}^{\infty} u'_n(t) e_n \right\|_{D(A^{-\theta})}^2 = \sum_{n=1}^{\infty} \lambda_n^{-2\theta} |u'_n(t)|^2.$$

Since $0 < \theta < \frac{1}{2}$, thanks to (19) we have

$$\begin{aligned} \lambda_n^{-2\theta} | -\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) |^2 \\ \leq C t^{2\alpha\theta+\alpha-2} \lambda_n |\langle u_0, e_n \rangle|^2 \left(\frac{(\lambda_n t^\alpha)^{\frac{1}{2}-\theta}}{1 + \lambda_n t^\alpha} \right)^2 \leq C t^{2\alpha\theta+\alpha-2} \lambda_n |\langle u_0, e_n \rangle|^2. \end{aligned}$$

Therefore, taking into account that $\theta > \frac{2-\alpha}{2\alpha}$, for any $t \in [0, T]$ we get

$$\left\| \sum_{n=1}^{\infty} u'_n(t) e_n \right\|_{D(A^{-\theta})}^2 \leq C T^{2\alpha\theta+\alpha-2} \|\sqrt{A}u_0\|^2 + C \|u_1\|^2.$$

Arguing as above, we can show that the series $\sum_{n=1}^{\infty} u'_n(t)e_n$ is convergent in $D(A^{-\theta})$ uniformly in $t \in [0, T]$. For that reason, the function u is differentiable and

$$u'(t) = \sum_{n=1}^{\infty} u'_n(t)e_n,$$

that is, Formula (27) holds for u' . Moreover, $u' \in C([0, T]; D(A^{-\theta}))$ and

$$u'(0) = \sum_{n=1}^{\infty} \langle u_1, e_n \rangle e_n = u_1.$$

In addition,

$$\begin{aligned} \|u'(t)\|^2 &= \sum_{n=1}^{\infty} |u'_n(t)|^2 \\ &\leq 2 \sum_{n=1}^{\infty} |-\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha})|^2 + 2 \sum_{n=1}^{\infty} |\langle u_1, e_n \rangle E_{\alpha, 1}(-\lambda_n t^{\alpha})|^2. \end{aligned} \quad (33)$$

Using (14) and (19) we obtain

$$|-\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha})|^2 \leq C t^{\alpha-2} \lambda_n |\langle u_0, e_n \rangle|^2 \left(\frac{(\lambda_n t^{\alpha})^{\frac{1}{2}}}{1 + \lambda_n t^{\alpha}} \right)^2 \leq C t^{\alpha-2} \lambda_n |\langle u_0, e_n \rangle|^2.$$

Putting the above estimate into (33) we get

$$\int_0^T \|u'(t)\|^2 dt \leq C \frac{T^{\alpha-1}}{\alpha-1} \|\sqrt{A}u_0\|^2 + C \|u_1\|^2,$$

so $u' \in L^2(0, T; H)$.

To evaluate $I^{2-\alpha}(u' - u_1)(t)$, where

$$u'(t) - u_1 = \sum_{n=1}^{\infty} [-\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_1, e_n \rangle (E_{\alpha, 1}(-\lambda_n t^{\alpha}) - 1)] e_n,$$

we observe that

$$\begin{aligned} \mathcal{L}[I^{2-\alpha}(t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}))](z) &= \frac{z^{\alpha-2}}{z^{\alpha} + \lambda_n} = \mathcal{L}[t E_{\alpha, 2}(-\lambda_n t^{\alpha})](z), \\ \mathcal{L}[I^{2-\alpha}(E_{\alpha, 1}(-\lambda_n t^{\alpha}) - 1)](z) &= z^{\alpha-2} \left(\frac{z^{\alpha-1}}{z^{\alpha} + \lambda_n} - \frac{1}{z} \right) \\ &= -\lambda_n \frac{z^{\alpha-3}}{z^{\alpha} + \lambda_n} = -\lambda_n \mathcal{L}[t^2 E_{\alpha, 3}(-\lambda_n t^{\alpha})](z). \end{aligned}$$

Therefore, by Lerch's Theorem that assures the uniqueness of inverse Laplace transforms and repeating the previous argumentations one proves

$$I^{2-\alpha}(u' - u_1)(t) = - \sum_{n=1}^{\infty} \lambda_n [\langle u_0, e_n \rangle t E_{\alpha, 2}(-\lambda_n t^{\alpha}) + \langle u_1, e_n \rangle t^2 E_{\alpha, 3}(-\lambda_n t^{\alpha})] e_n, \quad (34)$$

and $I^{2-\alpha}(u' - u_1)(t) \in C([0, T]; H)$.

Next, if $v = \sum_{n=1}^{\infty} \langle v, e_n \rangle e_n$ belongs to $D(\sqrt{A})$, then we have

$$\langle I^{2-\alpha}(u' - u_1)(t), v \rangle = - \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} [\langle u_0, e_n \rangle t E_{\alpha, 2}(-\lambda_n t^{\alpha}) + \langle u_1, e_n \rangle t^2 E_{\alpha, 3}(-\lambda_n t^{\alpha})] \lambda_n^{\frac{1}{2}} \langle v, e_n \rangle.$$

We observe that by (17) for $k = 1$ and $k = 2$ we have

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \frac{d}{dt} \left[\langle u_0, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t^2 E_{\alpha,3}(-\lambda_n t^\alpha) \right] \lambda_n^{\frac{1}{2}} \langle v, e_n \rangle \\ = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \left[\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] \lambda_n^{\frac{1}{2}} \langle v, e_n \rangle. \end{aligned}$$

Thanks to (14) and (19) we get

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n \left| \langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right|^2 \\ \leq C \sum_{n=1}^{\infty} \lambda_n \left| \langle u_0, e_n \rangle \right|^2 \frac{1}{(1 + \lambda_n t^\alpha)^2} + C t^{2-\alpha} \sum_{n=1}^{\infty} \left| \langle u_1, e_n \rangle \right|^2 \left(\frac{\lambda_n^{\frac{1}{2}} t^{\frac{\alpha}{2}}}{1 + \lambda_n t^\alpha} \right)^2 \\ \leq C (\|\sqrt{A}u_0\|^2 + t^{2-\alpha} \|u_1\|^2). \end{aligned}$$

Therefore, since $\alpha < 2$ for any $t \in [0, T]$ we have

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \lambda_n^{\frac{1}{2}} \frac{d}{dt} \left[\langle u_0, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t^2 E_{\alpha,3}(-\lambda_n t^\alpha) \right] \lambda_n^{\frac{1}{2}} \langle v, e_n \rangle \right| \\ \leq C (\|\sqrt{A}u_0\|^2 + \|u_1\|^2 + \|\sqrt{A}v\|^2), \end{aligned}$$

whence

$$\begin{aligned} \frac{d}{dt} \langle I^{2-\alpha}(u' - u_1)(t), v \rangle \\ = - \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \frac{d}{dt} \left[\langle u_0, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t^2 E_{\alpha,3}(-\lambda_n t^\alpha) \right] \lambda_n^{\frac{1}{2}} \langle v, e_n \rangle \\ = - \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \left[\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] \lambda_n^{\frac{1}{2}} \langle v, e_n \rangle. \end{aligned}$$

On the other hand

$$\langle \sqrt{A}u(t), \sqrt{A}v \rangle = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \left[\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] \lambda_n^{\frac{1}{2}} \langle v, e_n \rangle,$$

and hence for any $t \in [0, T]$ we have

$$\frac{d}{dt} \langle I^{2-\alpha}(u' - u_1)(t), v \rangle + \langle \sqrt{A}u(t), \sqrt{A}v \rangle = 0,$$

that is (23) holds.

In conclusion, u given by (25) is the weak solution of (22) satisfying the initial conditions (26).

(ii) We assume that $u_0 \in D(A)$ and $u_1 \in D(\sqrt{A})$. Repeating similar argumentations to those done before, involving again (14) and (19), we have that the function u given by (25) belongs to $C([0, T]; D(A)) \cap C^1([0, T]; H)$.

From (34) it follows that

$$\frac{d}{dt} I^{2-\alpha}(u' - u_1)(t) = - \sum_{n=1}^{\infty} \lambda_n \left[\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n \quad (35)$$

and $\frac{d}{dt} I^{2-\alpha} (u' - u_1)(t) \in C([0, T]; H)$. Indeed, for any $t \in [0, T]$

$$\lambda_n^2 |\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)|^2 \leq C(\lambda_n^2 |\langle u_0, e_n \rangle|^2 + T^{2-\alpha} \lambda_n |\langle u_1, e_n \rangle|^2),$$

whence

$$\sum_{n=1}^{\infty} |\lambda_n \langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \lambda_n \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)|^2 \leq C(\|Au_0\|^2 + T^{2-\alpha} \|\sqrt{A}u_1\|^2).$$

From (32), taking into account (18) and (16), we have

$$u_n''(t) = -\lambda_n [\langle u_0, e_n \rangle t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)].$$

Moreover,

$$\left\| \sum_{n=1}^{\infty} u_n''(t) e_n \right\|^2 \leq C(t^{2\alpha-4} \|Au_0\|^2 + t^{\alpha-2} \|\sqrt{A}u_1\|^2),$$

and hence

$$\left\| \sum_{n=1}^{\infty} u_n''(t) e_n \right\| \leq C(t^{\alpha-2} \|Au_0\| + t^{\frac{\alpha}{2}-1} \|\sqrt{A}u_1\|).$$

Therefore,

$$\int_0^T \left\| \sum_{n=1}^{\infty} u_n''(t) e_n \right\| dt \leq C(T^{\alpha-1} \|Au_0\| + T^{\frac{\alpha}{2}} \|\sqrt{A}u_1\|),$$

that is $\sum_{n=1}^{\infty} u_n''(t) e_n$ belongs to $L^1(0, T; H)$. Therefore, for any $t \in [0, T]$ we have

$$\int_0^t \sum_{n=1}^{\infty} u_n''(s) e_n ds = \sum_{n=1}^{\infty} \int_0^t u_n''(s) ds e_n = \sum_{n=1}^{\infty} (u_n'(t) - \langle u_1, e_n \rangle) e_n = u'(t) - u_1.$$

Since u' is absolutely continuous, taking into account (13), we have

$$\frac{d}{dt} I^{2-\alpha} (u' - u_1)(t) = \partial_t^\alpha u(t). \quad (36)$$

In particular, $\partial_t^\alpha u \in C([0, T]; H)$ and, thanks to (35), Formula (28) holds.

Finally, from (23) and (36) we have for any $v \in D(\sqrt{A})$ and $t \in (0, T)$

$$0 = \left\langle \frac{d}{dt} I^{2-\alpha} (u' - u_1)(t), v \right\rangle + \langle \sqrt{A}u(t), \sqrt{A}vs. \rangle = \langle \partial_t^\alpha u(t), v \rangle + \langle Au(t), vs. \rangle = \langle \partial_t^\alpha u(t) + Au(t), v \rangle,$$

that is $\partial_t^\alpha u + Au = 0$ in $(0, T)$.

In conclusion, u satisfies the conditions of Definition 2-2, that is, u is a strong solution of (22). $\square$

4. Examples

In this section, we apply our well-posedness results by discussing two examples of concrete models involving well-known partial differential operators. Throughout the section, we denote by Ω a bounded open domain in $\mathbb{R}^N$, $N \geq 1$, with sufficiently smooth boundary $\partial\Omega$. In both examples we consider the Hilbert space $H = L^2(\Omega)$, endowed with the inner product and norm defined by

$$\langle u, v \rangle = \int_{\Omega} u(x)v(x) dx, \quad \|u\| = \left(\int_{\Omega} |u(x)|^2 dx \right)^{1/2} \quad u, v \in L^2(\Omega).$$

4.1. Time-Fractional Wave Equations

We analyze the fractional boundary value problem

$$\begin{cases} \partial_t^\alpha u = \Delta u & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega. \end{cases} \quad (37)$$

We rewrite (37) as an abstract equation of the type (1) by introducing the operator A as follows:

$$\begin{aligned} D(A) &= H^2(\Omega) \cap H_0^1(\Omega) \\ (Au)(x) &= -\Delta u(x), \quad u \in D(A), \quad x \in \Omega. \end{aligned}$$

It is well known that A is a linear self-adjoint positive operator on $L^2(\Omega)$ with dense domain. Moreover, the fractional power $\sqrt{A}$ of A is well defined and $D(\sqrt{A}) = H_0^1(\Omega)$.

We assume that the eigenvalues λ_n , $n \in \mathbb{N}$, of the operator A are all distinct numbers, whence the eigenspace generated by λ_n has dimension one.

We are in conditions to apply Theorem 3 to get a well-posedness result for problem (37).

Theorem 4.

(i) If $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then the function

$$u(t, x) = \sum_{n=1}^{\infty} [\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)] e_n(x) \quad (38)$$

is the unique weak solution of (37) satisfying the initial conditions

$$u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1.$$

In addition

$$u_t(t, x) = \sum_{n=1}^{\infty} [-\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha)] e_n(x),$$

and $u_t \in C([0, T]; D(A^{-\theta}))$ for $\theta \in (\frac{2-\alpha}{2\alpha}, \frac{1}{2})$.

(ii) For $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$ the weak solution given by (38) is a strong one and

$$\partial_t^\alpha u(t, x) = - \sum_{n=1}^{\infty} [\lambda_n \langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \lambda_n \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)] e_n(x).$$

4.2. Time-Fractional Petrovsky Systems

We consider the system

$$\begin{cases} \partial_t^\alpha u + \Delta^2 u = 0 & \text{in } (0, T) \times \Omega, \\ u = \Delta u = 0 & \text{on } (0, T) \times \partial\Omega. \end{cases} \quad (39)$$

We can recast (39) as an abstract problem by defining the operator A in this way:

$$\begin{aligned} D(A) &= H^4(\Omega) \cap \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\} \\ (Au)(x) &= \Delta^2 u(x), \quad u \in D(A), \quad x \in \Omega. \end{aligned} \quad (40)$$

We assume that the eigenvalues λ'_n , $n \in \mathbb{N}$, of the operator $-\Delta$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ are all distinct numbers, and hence the eigenspace generated by λ'_n has dimension one.

The biharmonic operator given by (40) satisfies:

- A is self-adjoint, because, integrating by parts and taking into account the boundary conditions satisfied by the elements of $D(A)$, we have

$$\langle Au, v \rangle = \int_{\Omega} \Delta u(x) \Delta v(x) dx = \langle u, Av \rangle \quad u, v \in D(A);$$

- the fractional power $\sqrt{A}$ of the operator A is $-\Delta$ with domain $D(\sqrt{A}) = H^2(\Omega) \cap H_0^1(\Omega)$;
- A is positive, since (5) is satisfied. Indeed,

$$\|\Delta u\| \cdot \|u\| \geq \langle -\Delta u, u \rangle = \int_{\Omega} |\nabla u(x)|^2 dx \geq C \int_{\Omega} |u(x)|^2 dx$$

and hence

$$\langle Au, u \rangle = \int_{\Omega} |\Delta u(x)|^2 dx = \|\Delta u\|^2 \geq C \|u\|^2 \quad u \in D(A);$$

- the domain $D(A)$ is dense in $L^2(\Omega)$ by the density of $C_c^\infty(\Omega)$ in $L^2(\Omega)$. (As usual the symbol $C_c^\infty(\Omega)$ denotes the space of the functions of class C^∞ with compact support in Ω .)

Lemma 4. The spectrum of the operator A consists of the sequence tending to $+\infty$ of eigenvalues $\lambda_n = (\lambda'_n)^2$, where λ'_n are the eigenvalues of the operator $-\Delta$ in $H^2(\Omega) \cap H_0^1(\Omega)$.

The eigenfunctions e_n of $-\Delta$ ($-\Delta e_n = \lambda'_n e_n$), which constitutes an orthonormal basis of $L^2(\Omega)$, are also eigenfunctions of Δ^2 ($\Delta^2 e_n = \lambda_n e_n$).

Proof. It is enough to note that

$$\Delta^2 e_n = -\Delta(-\Delta e_n) = -\lambda'_n \Delta(e_n) = (\lambda'_n)^2 e_n.$$

The operator Δ^2 cannot have other eigenfunctions, because the sequence $\{e_n\}$ of the eigenfunctions of $-\Delta$ constitutes an orthonormal basis of $L^2(\Omega)$. $\square$

In the case of the biharmonic operator a weak solution can be called a H^2 -solution. This terminology is suggested by the analysis of the stationary case given in [14].

Finally, by Theorem 3 a well-posedness result for system (39) follows.

Theorem 5.

- (i) If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then the function

$$u(t, x) = \sum_{n=1}^{\infty} [\langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)] e_n(x) \quad (41)$$

is the unique H^2 -solution of (39) satisfying the initial conditions

$$u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1.$$

In addition

$$u_t(t, x) = \sum_{n=1}^{\infty} [-\lambda_n \langle u_0, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_1, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha)] e_n(x),$$

and $u_t \in C([0, T]; D(A^{-\theta}))$ for $\theta \in (\frac{2-\alpha}{2\alpha}, \frac{1}{2})$.

- (ii) For $u_0 \in H^4(\Omega) \cap \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$ and $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ the H^2 -solution given by (41) is a strong one and

$$\partial_t^\alpha u(t, x) = - \sum_{n=1}^{\infty} [\lambda_n \langle u_0, e_n \rangle E_{\alpha,1}(-\lambda_n t^\alpha) + \lambda_n \langle u_1, e_n \rangle t E_{\alpha,2}(-\lambda_n t^\alpha)] e_n(x).$$

5. Conclusions

In this paper, we consider in a general framework the question of existence and uniqueness of the solution of a class of PDEs with time fractional derivative (in the Caputo sense). Our aim is to give a notion of weak solutions in a wide class of functions: this requires a good compromise between smoothness requirement and applicability to concrete models. For a general discussion in the fractional setting see [15].

We show that the notion of weak solutions we propose may be strengthened by the spectral approach followed in [3,4]. Indeed, we prove that the weak solutions may be written by means of an explicit representation formula through expansions involving the Mittag-Leffler functions and the eigenvalues of the spatial operator.

We also give two examples of applications involving well-known partial differential operators.

The Definition of weak solutions we propose is a preliminary step to other research problems concerning further regularity properties of the solutions. For example, the problem regarding the hidden regularity for time-fractional wave equations has been studied in [16]. A recent paper for the trace regularity for time-fractional Petrovsky systems which involves the notion of weak solution is given in [17].

Further studies can be performed for more general classes of operators.

Author Contributions: Conceptualization; methodology; validation; formal analysis; investigation; writing—original draft preparation; writing—review and editing: P.L. and D.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially supported by Progetto Ateneo Sapienza Università di Roma “Mathematical models for interacting systems”, grant number RP11715C3FC26FE6.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Informed consent was obtained from all subjects involved in the study.

Data Availability Statement: Exclude this statement because the study did not report any data.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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