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Qualitative Behavior of Unbounded Solutions of Neutral Differential Equations of Third-Order

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Abstract: New oscillatory properties for the oscillation of unbounded solutions to a class of third-order neutral differential equations with several deviating arguments are established. Several oscillation results are established by using generalized Riccati transformation and a integral average technique under the case of unbounded neutral coefficients. Examples are given to prove the significance of new theorems.

Keywords: neutral differential equation; oscillation; asymptotic behavior; deviating arguments

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1. Introduction

In this work, we investigate the oscillation properties of solutions to the third-order neutral differential equations with several deviating arguments

$$\left(r(\iota)(z''(\iota))^{\alpha}\right)' + \sum_{i=1}^{n} q_i(\iota)x^{\alpha}(\phi_i(\iota)) = 0, \quad \iota \geq \iota_0 > 0, \tag{1}$$

where $z(\iota) = x(\iota) + p(\iota)x(\varrho(\iota))$ and α is a quotient of odd positive integers.

The main results of this paper are obtained considering the following conditions:

$$\begin{cases} r \in C([\iota_0,\infty),(0,\infty)) \text{ and } \int_{\iota_0}^{\infty} r^{-1/\alpha}(s)ds = \infty; \\ q_i(\iota) \in C([\iota_0,\infty),[0,\infty)),\ \phi_i(\iota) \in C([\iota_0,\infty),\mathbb{R}) \text{ and } \lim_{\iota\to\infty}\phi_i(\iota) = \infty, \text{ where } i = 1,2,\cdots n; \\ \varrho \in C([\iota_0,\infty),\mathbb{R}) \text{ is strictly increasing, } \varrho(\iota) < \iota, \text{ and } \lim_{\iota\to\infty}\varrho(\iota) = \infty; \\ p(\iota) \in C([\iota_0,\infty),\mathbb{R}) \text{ with } p(\iota) \geq 1, \text{ and } p(\iota) \not\equiv 1, \text{ eventually.} \end{cases}$$

By a solution of (1), we mean a function $x : [\iota_x,\infty) \to \mathbb{R}$ such that $z(\iota) \in C^2([\iota_x,\infty),\mathbb{R})$ and $r(\iota)(z''(\iota))^{\alpha} \in C^1([\iota_x,\infty),\mathbb{R})$, and which satisfies Equation (1) on $[\iota_x,\infty)$. We only consider those solutions $x(\iota))$ of (1) defined on some ray $[\iota_x,\infty)$, for some $\iota_x \geq \iota_0$, which satisfy $\sup\{|x(\iota)| : \iota \geq T\} > 0$ for every $T \geq \iota_x$. We start with the assumption that Equation (1) does possess a proper solution. A proper solution of (1) is called oscillatory if it has a sequence of large zeros lending to ∞; otherwise we call nonoscillatory.

Because of the enormous advantage of neutral differential equations in describing several neutral phenomena, there is great scientific and academic value in studying neutral differential equations, both theoretically and practically; see [1]. Lately, there have been numerous articles investigating the oscillation of the solutions of third/higher order neutral differential equations with/without deviating arguments; see [2–16].

Baculíková et al. [17], Džurina et al. [18], and Li et al. [19] investigated third-order equations of the form:

$$\left[a(\iota)[x(\iota)+p(\iota)x(\delta(\iota))'']^{\gamma}\right]'+q(\iota)x^{\gamma}(\tau(\iota))=0,\quad \iota\geq\iota_0.$$

Jiang et al. [20] obtained several oscillation results for the third-order equation

$$\left[a(\iota)[x(\iota)+p(\iota)x(\delta(\iota))'']^{\alpha}\right]'+q(\iota)f(x(\tau(\iota)))=0,\quad \iota\geq\iota_0.$$

Tunç [21] investigated the third-order equation

$$\left(r(\iota)\left((x(\iota)+p(\iota)x(\tau(\iota)))''\right)^{\alpha}\right)'+\int_a^b q(\iota,\xi)x^{\alpha}(\phi(\iota,\xi))d\xi=0.$$

Soliman et al. [22] investigated a third-order delay differential equation

$$\left(a(\iota)\left((x(\iota)\pm\sum_{i=1}^{n}p_i(\iota)x(\sigma_i(\iota)))''\right)^{\alpha}\right)'+\sum_{j=1}^{m}f_j(\iota,x(\tau_j(\iota)))=0.$$

The articles listed above deal with the case when the neutral coefficient $p(\iota)$ is bounded, i.e., the cases where $0\leq p(\iota)\leq p_0<1$, $-1<p_0\leq p(\iota)\leq 0$, and $0\leq p(\iota)\leq p_0<\infty$ were considered, and so the results established in these papers cannot be applied to the case of $p(\iota)\to\infty$ as $\iota\to\infty$.

More precisely, the existing literature does not provide any criteria for the oscillation of third-order unbounded neutral differential equations with several deviating arguments in the case when $p(\iota)\to\infty$ as $\iota\to\infty$. With this motivation, we provide several criteria for oscillation of the differential Equation (1) under the assumptions of $\varrho(\iota)\geq\phi_i(\iota)$ and $\varrho(\iota)\leq\phi_i(\iota)$ for $i=1,2,\cdots,n$ when $p(\iota)\geq 1$. Furthermore, the results presented in this paper can be simply extended to more general third-order unbounded neutral differential equations with several deviating arguments in order to achieve more generalized oscillation results. As a result, it is envisaged that the present paper will make a significant contribution to the study of oscillations of solutions of (1).

2. Main Results

We start with the following lemmas, which are required to prove our main theorems. Through this paper, we will be using the following notations:

$$\zeta'_+(\iota):=\max\{0,\zeta'(\iota)\},$$
$$B_1(\iota,\iota_1):=\int_{\iota_1}^{\iota}\frac{ds}{r^{1/\alpha}(s)}\text{ for }\iota\geq\iota_1,$$
$$B_2(\iota,\iota_2):=\int_{\iota_2}^{\iota}B_1(s,\iota_1)ds\text{ for }\iota\geq\iota_2>\iota_1.$$

Furthermore, throughout this paper, we assume that

$$\psi_1(\iota):=\frac{1}{p(\varrho^{-1}(\iota))}\left[1-\frac{1}{p(\varrho^{-1}(\varrho^{-1}(\iota)))}\right]>0 \tag{2}$$

and

$$\psi_2(\iota) := \frac{1}{p(\varrho^{-1}(\iota))}\left[1 - \frac{1}{p(\varrho^{-1}(\varrho^{-1}(\iota)))}\frac{B_2(\varrho^{-1}(\varrho^{-1}(\iota)),\iota_2)}{B_2(\varrho^{-1}(\iota),\iota_2)}\right] > 0, \tag{3}$$

for all sufficiently large ι, where ϱ^{-1} is the inverse function of ϱ, and we consider

$$\Omega_1(\iota) := \sum_{i=1}^{n} q_i(\iota)(\psi_1(\phi_i(\iota)))^{\alpha}, \quad \Omega_2(\iota) := \sum_{i=1}^{n} q_i(\iota)(\psi_2(\phi_i(\iota)))^{\alpha}.$$

Lemma 1 ([23]). *If X and Y are nonnegative and $\lambda > 1$, then*

$$X^{\lambda} - \lambda X Y^{\lambda-1} + (\lambda-1)Y^{\lambda} \geq 0.$$

Lemma 2. *If $x(\iota)$ is an eventually positive solution of (1), then $z(\iota)$ satisfies either*
(C_I) $z(\iota) > 0, z'(\iota) > 0, z''(\iota) > 0$, and $(r(\iota)(z''(\iota))^{\alpha})' \leq 0$, or
(C_{II}) $z(\iota) > 0, z'(\iota) < 0, z''(\iota) > 0$, and $(r(\iota)(z''(\iota))^{\alpha})' \leq 0$.

The proof of the above lemma is standard and thus omitted.

Lemma 3. *Let (2) hold, and let $x(\iota)$ be an eventually positive solution of (1) with $z(\iota)$ satisfying (C_{II}) of Lemma 2. If*

$$\int_{\iota_0}^{\infty}\int_{v}^{\infty}\frac{1}{r^{1/\alpha}(u)}\left(\int_{u}^{\infty}\Omega_1(s)ds\right)^{1/\alpha} du\, dv = \infty, \tag{4}$$

then $\lim_{\iota\to\infty} x(\iota) = 0$.

Proof. Let $x(\iota)$ be an eventually positive solution of (1). Then, there exists $\iota_1 \in [\iota_0,\infty)$ such that, for $\iota \geq \iota_1$, $x(\iota) > 0$, $x(\varrho(\iota)) > 0$, $x(\phi_i(\iota)) > 0$ and $i = 1,2,\cdots,n$. From the definition of z, we have (see also [2] [(8.6)]):

$$\begin{aligned} x(\iota) &= \frac{1}{p(\varrho^{-1}(\iota))}(z(\varrho^{-1}(\iota)) - x(\varrho^{-1}(\iota))) \\ &= \frac{z(\varrho^{-1}(\iota))}{p(\varrho^{-1}(\iota))} - \frac{1}{p(\varrho^{-1}(\iota))p(\varrho^{-1}(\varrho^{-1}(\iota)))}\Big(z(\varrho^{-1}(\varrho^{-1}(\iota))) - x(\varrho^{-1}(\varrho^{-1}(\iota)))\Big) \\ &\geq \frac{z(\varrho^{-1}(\iota))}{p(\varrho^{-1}(\iota))} - \frac{1}{p(\varrho^{-1}(\iota))p(\varrho^{-1}(\varrho^{-1}(\iota)))}z(\varrho^{-1}(\varrho^{-1}(\iota))). \end{aligned} \tag{5}$$

From $\varrho(\iota) < \iota$, (iv) and the fact that $z(\iota)$ is decreasing, we have

$$z(\varrho^{-1}(\iota)) \geq z(\varrho^{-1}(\varrho^{-1}(\iota))),$$

using this in (5), we obtain

$$x(\iota) \geq \psi_1(\iota)z(\varrho^{-1}(\iota)),$$

so

$$x(\phi_i(\iota)) \geq \psi_1(\phi_i(\iota))z(\varrho^{-1}(\phi_i(\iota))), \quad i = 1,2,\cdots,n \tag{6}$$

for $\iota \geq \iota_2$. Using (6) in (1) gives

$$(r(\iota)(z''(\iota))^{\alpha})' + \sum_{i=1}^{n} q_i(\iota)(\psi_1(\phi_i(\iota)))^{\alpha} z^{\alpha}(\varrho^{-1}(\phi_i(\iota))) \leq 0, \tag{7}$$

for $\iota \geq \iota_2$. From (iv)–(v) and the fact that $z(\iota)$ is decreasing, (7) yields

$$(r(\iota)(z''(\iota))^\alpha)' + z^\alpha(\varrho^{-1}(\iota)) \sum_{i=1}^{n} q_i(\iota)(\psi_1(\phi_i(\iota)))^\alpha \leq 0 \text{ for } \iota \geq \iota_2. \tag{8}$$

Since $z(\iota) > 0$ and $z'(\iota) < 0$, there exists a constant κ such that

$$\lim_{\iota\to\infty} z(\iota) = \kappa < \infty,$$

where $\kappa \geq 0$. If $\kappa > 0$, then there exists $\iota_3 \geq \iota_2$ such that $\varrho^{-1}(\theta_1(\iota)) > \iota_2$ and

$$z(\iota) \geq \kappa \quad \text{for } \iota \geq \iota_3. \tag{9}$$

Integrating (8) from ι to ∞ two times we derive

$$-z'(\iota) \geq \kappa \int_\iota^\infty \frac{1}{r^{1/\alpha}(u)} \Big(\int_u^\infty \sum_{i=1}^{n} q_i(s)(\psi_1(\phi_i(s)))^\alpha \Big)^{1/\alpha} du.$$

Integrating the resulting inequality from ι_3 to ι, we obtain

$$z(\iota_3) \geq \kappa \int_{\iota_3}^{\iota} \int_v^\infty \frac{1}{r^{1/\alpha}(u)} \Big(\int_u^\infty \sum_{i=1}^{n} q_i(s)(\psi_1(\phi_i(s)))^\alpha \Big)^{1/\alpha} du\, dv,$$

which contradicts (4), and so we have $\kappa = 0$. Therefore, $\lim_{\iota\to\infty} z(\iota) = 0$. Since $0 < x(\iota) \leq z(\iota)$ on $[\iota_1, \infty)$, we obtain $\lim_{\iota\to\infty} x(\iota) = 0$. □

Theorem 1. *Assume that* (2)–(4) *hold and* $\varrho(\iota) \geq \phi_i(\iota)$ *for* $i = 1, 2, \cdots, n$. *If there exists a function* $\zeta \in C^1([\iota_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{\iota\to\infty} \int_T^\iota \left[\zeta(s) \sum_{i=1}^{n} q_i(s)(\psi_2(\phi_i(s)))^\alpha \Big(\frac{B_2(\varrho^{-1}(\phi_i(s)), \iota_2)}{B_1(s, \iota_1)} \Big)^\alpha - \frac{\zeta'_+(s)}{(B_1(s, \iota_1))^\alpha} \right] ds = \infty, \tag{10}$$

for all $\iota_1, \iota_2, T \in [\iota_0, \infty)$, *where* $T > \iota_2 > \iota_1$, *then any solution of* (1) *is either oscillatory or satisfies* $\lim_{\iota\to\infty} x(\iota) = 0$.

Proof. Assume that (1) has a nonoscillatory solution $x(\iota)$ on $[\iota_0, \infty)$, say there exists $\iota_1 \in [\iota_0, \infty)$ such that, for $\iota \geq \iota_1$, $x(\iota) > 0$, $x(\varrho(\iota)) > 0$, and $x(\phi_i(\iota)) > 0$, (2) and (3) hold, and $z(\iota)$ satisfies either (C_I) or (C_{II}) for $i = 1, 2, \cdots, n$. Assuming that (C_I) holds and proceeding as in the proof of Lemma 3, we obtain (5). Since $r(\iota)(z''(\iota))^\alpha$ is decreasing, we see that

$$z'(\iota) = z'(\iota_1) + \int_{\iota_1}^{\iota} \frac{(r(s)(z''(s))^\alpha)^{1/\alpha}}{r^{1/\alpha}(s)} ds \geq (r(\iota)(z''(\iota))^\alpha)^{1/\alpha} B_1(\iota, \iota_1) \quad \text{for } \iota \geq \iota_1. \tag{11}$$

From (11), we have for all $\iota \geq \iota_2 := \iota_1 + 1$ that

$$\Big(\frac{z'(\iota)}{B_1(\iota, \iota_1)} \Big)' = \frac{r^{-1/\alpha}(\iota)[r^{1/\alpha}(\iota) z''(\iota) B_1(\iota, \iota_1) - z'(\iota)]}{(B_1(\iota, \iota_1))^2} \leq 0,$$

so $z'(\iota)/B_1(\iota,\iota_1)$ is decreasing for $\iota \geq \iota_2$. Next, using the fact that $z'(\iota)/B_1(\iota,\iota_1)$ is decreasing for $\iota \geq \iota_2$, we obtain

$$\begin{aligned} z(\iota) &= z(\iota_2) + \int_{\iota_2}^{\iota} \frac{z'(s)}{B_1(s,\iota_1)} B_1(s,\iota_1) ds \\ &\geq \frac{z'(\iota)}{B_1(\iota,\iota_1)} \int_{\iota_2}^{\iota} B_1(s,\iota_1) ds \\ &= \frac{B_2(\iota,\iota_2)}{B_1(\iota,\iota_1)} z'(\iota) \quad \text{for } \iota \geq \iota_2. \end{aligned} \tag{12}$$

From (12), for all $\iota \geq \iota_3 := \iota_2 + 1$ we have that

$$\left(\frac{z(\iota)}{B_2(\iota,\iota_2)}\right)' = \frac{z'(\iota)B_2(\iota,\iota_2) - z(\iota)B_1(\iota,\iota_1)}{(B_2(\iota,\iota_2))^2} \leq 0,$$

so $z(\iota)/B_2(\iota,\iota_2)$ is decreasing for $\iota \geq \iota_3$. Next, in view of the fact that $z(\iota)/B_2(\iota,\iota_2)$ is decreasing for $\iota \geq \iota_3$ and $\varrho(\iota) < \iota$ or $\varrho^{-1}(\iota) \leq \varrho^{-1}(\varrho^{-1}(\iota))$, we obtain

$$\frac{B_2(\varrho^{-1}(\varrho^{-1}(\iota)),\iota_2) z(\varrho^{-1}(\iota))}{B_2(\varrho^{-1}(\iota),\iota_2)} \geq z(\varrho^{-1}(\varrho^{-1}(\iota))). \tag{13}$$

Using (13) in (5) yields

$$x(\iota) \geq \frac{1}{p(\varrho^{-1}(\iota))}\left[1 - \frac{1}{p(\varrho^{-1}(\varrho^{-1}(\iota)))} \frac{B_2(\varrho^{-1}(\varrho^{-1}(\iota)),\iota_2)}{B_2(\varrho^{-1}(\iota),\iota_2)}\right] z(\varrho^{-1}(\iota)) = \psi_2(\iota) z(\varrho^{-1}(\iota)),$$

so

$$x(\phi_i(\iota)) \geq \psi_2(\phi_i(\iota)) z(\varrho^{-1}(\phi_i(\iota))), \quad i = 1,2,\cdots,n \tag{14}$$

for $\iota \geq \iota_3$. Using (14) in (1) gives

$$(r(\iota)(z''(\iota))^\alpha)' + \sum_{i=1}^{n} q_i(\iota)(\psi_2(\phi_i(\iota)))^\alpha z^\alpha(\varrho^{-1}(\phi_i(\iota))) \leq 0. \tag{15}$$

Next, we define

$$w(\iota) = \zeta(\iota) \frac{r(\iota)(z''(\iota))^\alpha}{(z'(\iota))^\alpha} \quad \text{for } \iota \geq \iota_1. \tag{16}$$

Then $w(\iota) > 0$, and from (15), we see that

$$\begin{aligned} w'(\iota) &= \frac{\zeta(\iota)}{(z'(\iota))^\alpha}\left[r(\iota)(z''(\iota))^\alpha\right]' + \left[\frac{\zeta(\iota)}{(z'(\iota))^\alpha}\right]' r(\iota)(z''(\iota))^\alpha \\ &= \zeta'(\iota)\frac{r(\iota)(z''(\iota))^\alpha}{(z'(\iota))^\alpha} + \zeta(\iota)\left[\frac{(r(\iota)(z''(\iota))^\alpha)'}{(z'(\iota))^\alpha} - \frac{r(\iota)(z''(\iota))^\alpha((z'(\iota))^\alpha)'}{(z'(\iota))^{2\alpha}}\right] \\ &\leq \zeta'_+(\iota)\frac{r(\iota)(z''(\iota))^\alpha}{(z'(\iota))^\alpha} - \zeta(\iota)\left[\sum_{i=1}^{n} q_i(s)(\psi_2(\phi_i(s)))^\alpha \frac{z^\alpha(\varrho^{-1}(\phi_i(\iota)))}{(z'(\iota))^\alpha}\right] \\ &\quad -\alpha\zeta(\iota)r(\iota)\frac{(z''(\iota))^{\alpha+1}}{(z'(\iota))^{\alpha+1}} \end{aligned} \tag{17}$$

for $\iota \geq \iota_3$ with $\iota_3 \in (\iota_2,\infty)$ and $\iota_2 \in (\iota_1,\infty)$. From (11), $z'(\iota) > 0$ and $z''(\iota) > 0$, (17) yields

$$w'(\iota) \leq \frac{\zeta'_+(\iota)}{(B_1(\iota,\iota_1))^\alpha} - \zeta(\iota)\left[\sum_{i=1}^{n} q_i(s)(\psi_2(\phi_i(s)))^\alpha \frac{z^\alpha(\varrho^{-1}(\phi_i(\iota)))}{(z'(\iota))^\alpha}\right] \frac{z^\alpha(\iota)}{(z'(\iota))^\alpha} \quad \text{for } \iota \geq \iota_3. \tag{18}$$

Using the fact that $z(\iota)/B_2(\iota,\iota_2)$ is nonincreasing for $\iota \geq \iota_3$, and noting that $\varrho(\iota) \geq \phi_i(\iota)$ implies $\varrho^{-1}(\phi_i(\iota)) \leq \iota$, we obtain

$$\frac{z(\varrho^{-1}(\phi_i(\iota)))}{z(\iota)} \geq \frac{B_2(\varrho^{-1}(\phi_i(\iota)),\iota_2)}{B_2(\iota,\iota_2)}, \quad i = 1,2,\cdots,n \tag{19}$$

for $\iota \geq \iota_3$. Substituting (19) and (12) into (18), we obtain

$$w'(\iota) \leq \frac{\zeta'_+(\iota)}{(B_1(\iota,\iota_1))^\alpha} - \zeta(\iota)\sum_{i=1}^{n} q_i(\iota)(\psi_2(\phi_i(\iota)))^\alpha \left(\frac{B_2(\varrho^{-1}(\phi_i(\iota)),\iota_2)}{B_1(\iota,\iota_1)}\right)^\alpha \quad \text{for } \iota \geq \iota_3. \tag{20}$$

An integration of (20) from ι_3 to ι yields

$$\int_{\iota_3}^{\iota}\left[\zeta(s)\sum_{i=1}^{n} q_i(s)(\psi_2(\phi_i(s)))^\alpha \left(\frac{B_2(\varrho^{-1}(\phi_i(s)),\iota_2)}{B_1(s,\iota_1)}\right)^\alpha - \frac{\zeta'_+(s)}{(B_1(s,\iota_1))^\alpha}\right]ds \leq w(\iota_3),$$

which contradicts (10).

This implies that (C_{II}) holds, and so from Lemma 3, we have $\lim_{\iota\to\infty} x(\iota) = 0$. This completes the proof. □

Theorem 2. *Assume that* (2)–(4) *hold and* $\varrho(\iota) \geq \phi_i(\iota)$ *for* $i = 1,2,\cdots,n$. *If there exists a function* $\zeta \in C^1([\iota_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{\iota\to\infty}\int_T^\iota\left[\zeta(s)\sum_{i=1}^{n} q_i(s)(\psi_2(\phi_i(s)))^\alpha \left(\frac{B_2(\varrho^{-1}(\phi_i(s)),\iota_2)}{B_1(s,\iota_1)}\right)^\alpha - \frac{r(s)(\zeta'_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\zeta^\alpha(s)}\right]ds = \infty, \tag{21}$$

for all $\iota_1,\iota_2,T \in [\iota_0,\infty)$, *where* $T > \iota_2 > \iota_1$, *then any solution of* (1) *is either oscillatory or satisfies* $\lim_{\iota\to\infty} x(\iota) = 0$.

Proof. Assume that (1) has a nonoscillatory solution $x(\iota)$ on $[\iota_0,\infty)$, say there exists $\iota_1 \in [\iota_0,\infty)$ such that, for $\iota \geq \iota_1$, $x(\iota) > 0$, $x(\varrho(\iota)) > 0$, and $x(\phi_i(\iota)) > 0$, (2) and (3) hold, for $z(\iota)$ satisfies either (C_I) or (C_{II}) and $i = 1,2,\cdots,n$. Assume that (C_I) holds. We use the same type of argument as in the proof of the Theorem 1, and arrive at (17). In view of (16), inequality (17) takes the form

$$w'(\iota) \leq \frac{\zeta'_+(\iota)}{\zeta(\iota)}w(\iota) - \zeta(\iota)\left[\sum_{i=1}^{n} q_i(\iota)(\psi_2(\phi_i(\iota)))^\alpha \frac{z^\alpha(\varrho^{-1}(\phi_i(\iota)))}{(z'(\iota))^\alpha}\right]\frac{z^\alpha(\iota)}{(z'(\iota))^\alpha} - \frac{\alpha w^{(\alpha+1)/\alpha}(\iota)}{(\zeta(\iota)r(\iota))^{1/\alpha}}. \tag{22}$$

Using (12) and (19) in (22), for $\iota \geq \iota_3$, we obtain

$$w'(\iota) \leq \frac{\zeta'_+(\iota)}{\zeta(\iota)}w(\iota) - \frac{\alpha w^{(\alpha+1)/\alpha}(\iota)}{(\zeta(\iota)r(\iota))^{1/\alpha}} - \zeta(\iota)\sum_{i=1}^{n} q_i(\iota)(\psi_2(\phi_i(\iota)))^\alpha \left(\frac{B_2(\varrho^{-1}(\phi_i(\iota)),\iota_2)}{B_1(\iota,\iota_1)}\right)^\alpha. \tag{23}$$

If we apply Lemma 1 with $X = \frac{\alpha^{1/\lambda}}{[(\zeta(\iota)r(\iota))^{1/\alpha}]^{1/\lambda}}w(\iota)$, $Y = \left[\frac{\alpha}{\alpha+1}\frac{[(\zeta(\iota)r(\iota))^{1/\alpha}]^{1/\lambda}}{\alpha^{1/\lambda}}\frac{\zeta'_+(\iota)}{\zeta(\iota)}\right]^\alpha$ and $\lambda = \frac{\alpha+1}{\alpha}$, we see that

$$\frac{\zeta'_+(\iota)}{\zeta(\iota)}w(\iota) - \frac{\alpha}{(\zeta(\iota)r(\iota))^{1/\alpha}}w^{(\alpha+1)/\alpha}(\iota) \leq \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(\iota)(\zeta'_+(\iota))^{\alpha+1}}{\zeta^\alpha(\iota)}.$$

Using this in (23) gives

$$w'(\iota) \leq \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(\iota)(\zeta'_+(\iota))^{\alpha+1}}{\zeta^\alpha(\iota)} - \zeta(\iota)\sum_{i=1}^{n} q_i(\iota)(\psi_2(\phi_i(\iota)))^\alpha \left(\frac{B_2(\varrho^{-1}(\phi_i(\iota)),\iota_2)}{B_1(\iota,\iota_1)}\right)^\alpha.$$

Integrating the latter inequality from ι_3 to ι yields

$$\limsup_{\iota\to\infty}\int_T^\iota\left[\zeta(s)\sum_{i=1}^n q_i(s)(\psi_2(\phi_i(s)))^\alpha\left(\frac{B_2(\varrho^{-1}(\phi_i(s)),\iota_2)}{B_1(s,\iota_1)}\right)^\alpha-\frac{r(s)(\zeta'_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\zeta^\alpha(s)}\right]ds\le w(\iota_3),$$

which contradicts (21). Therefore (C_{II}) holds, and so $\lim_{\iota\to\infty}x(\iota)=0$ by Lemma 3. This completes the proof. $\square$

Next, we examine the oscillation results of solutions of (1) by Philos-type [3]. Let $\mathbb{S}_0=\{(\iota,s):a\le s<\iota<+\infty\}$, $\mathbb{S}=\{(\iota,s):a\le s\le\iota<+\infty\}$; the continuous function $E(\iota,s)$, $E:\mathbb{S}\to\mathbb{R}$ belongs to the class function $\Re$

(C_I) $E(\iota,\iota)=0$ for $\iota\ge\iota_0$ and $E(\iota,s)>0$ for $(\iota,s)\in\mathbb{S}_0$,

(C_{II}) $\frac{\partial E(\iota,s)}{\partial s}\le 0$, $(\iota,s)\in\mathbb{S}_0$ and some locally integrable function $e(\iota,s)$ such that

$$\frac{\partial E(\iota,s)}{\partial s}+E(\iota,s)\frac{\zeta'(\iota)}{\zeta(\iota)}=\frac{e_+(\iota,s)}{\zeta(\iota)}(E(\iota,s))^{\frac{1}{\alpha}}\quad\text{for all }(\iota,s)\in\mathbb{S}_0.$$

Theorem 3. *Assume that* (2)–(4) *hold and* $\varrho(\iota)\ge\phi_i(\iota)$ *for* $i=1,2,\cdots,n$. *If there exists a function* $\zeta\in C^1([\iota_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{\iota\to\infty}\frac{1}{E(\iota,\iota_*)}\int_{\iota_*}^\iota\left[E(\iota,s)\zeta(s)\sum_{i=1}^n q_i(s)(\psi_2(\phi_i(s)))^\alpha\left(\frac{B_2(\varrho^{-1}(\phi_i(s)),\iota_2)}{B_1(s,\iota_1)}\right)^\alpha-\frac{r(s)(\zeta'_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\zeta^\alpha(s)}\right]ds=\infty, \tag{24}$$

for all $\iota_1,\iota_2,\iota_*\in[\iota_0,\infty)$, *where* $\iota_*>\iota_2>\iota_1$, *then any solution of* (1) *is either oscillatory or satisfies* $\lim_{\iota\to\infty}x(\iota)=0$.

Proof. Assume that (1) has a nonoscillatory solution $x(\iota)$ on $[\iota_0,\infty)$, say there exists $\iota_1\in[\iota_0,\infty)$ such that, for $\iota\ge\iota_1$, $x(\iota)>0$, $x(\varrho(\iota))>0$, and $x(\phi_i(\iota))>0$, (2) and (3) hold, for $z(\iota)$ satisfies either (C_I) or (C_{II}) and $i=1,2,\cdots,n$. Assume that (C_I) holds. Following the same arguments as in the proof of the Theorem 1, we arrive at (17). In view of (16), inequality (17) takes the form

$$\zeta(\iota)\textstyle\sum_{i=1}^n q_i(\iota)(\psi_2(\phi_i(\iota)))^\alpha\left(\frac{B_2(\varrho^{-1}(\phi_i(\iota)),\iota_2)}{B_1(\iota,\iota_1)}\right)^\alpha\le -w'(\iota)+\frac{\zeta'_+(\iota)}{\zeta(\iota)}w(\iota)-\frac{\alpha w^{(\alpha+1)/\alpha}(\iota)}{(\zeta(\iota)r(\iota))^{1/\alpha}}. \tag{25}$$

Multiplying by $E(\iota,s)$ and integrating (25) from ι_3 to ι, one can obtain that

$$\begin{aligned}
&\int_{\iota_3}^\iota E(\iota,s)\zeta(s)\sum_{i=1}^n q_i(s)(\psi_2(\phi_i(s)))^\alpha\left(\frac{B_2(\varrho^{-1}(\phi_i(s)),\iota_2)}{B_1(s,\iota_1)}\right)^\alpha ds\\
&\le-\int_{\iota_3}^\iota E(\iota,s)w'(s)ds+\int_{\iota_3}^\iota E(\iota,s)\frac{\zeta'_+(s)}{\zeta(s)}w(s)ds-\int_{\iota_3}^\iota E(\iota,s)\frac{\alpha w^{(\alpha+1)/\alpha}(s)}{(\zeta(s)r(s))^{1/\alpha}}ds\\
&\le E(\iota,\iota_3)w(\iota_3)+\int_{\iota_3}^\iota\left\{\frac{\partial E(\iota,s)}{\partial s}+E(\iota,s)\frac{\zeta'_+(s)}{\zeta(s)}\right\}w(s)ds-\int_{\iota_3}^\iota E(\iota,s)\frac{\alpha w^{(\alpha+1)/\alpha}(s)}{(\zeta(s)r(s))^{1/\alpha}}ds\\
&\le E(\iota,\iota_3)w(\iota_3)+\int_{\iota_3}^\iota(E(\iota,s))^{\frac{1}{\alpha}}\frac{e_+(\iota,s)}{\zeta(s)}w(s)ds-\int_{\iota_3}^\iota E(\iota,s)\frac{\alpha w^{(\alpha+1)/\alpha}(s)}{(\zeta(s)r(s))^{1/\alpha}}ds.
\end{aligned} \tag{26}$$

Now, using the Lemma 1, set

$$X=\left[\frac{\alpha E(\iota,s)}{(\zeta(s)r(s))^{1/\alpha}}\right]^{1/\lambda}w(s)$$

and

$$\Upsilon = \left[\frac{\alpha}{1+\alpha}\left[\frac{\left((\zeta(s)r(s))^{1/\alpha}\right)^{1/\lambda}}{\alpha^{1/\lambda}}\frac{e_+(\iota,s)}{\zeta(s)}\right]^{\alpha}\right]$$

we obtain that

$$\int_{\iota_3}^{\iota}\left[E(\iota,s)\zeta(s)\sum_{i=1}^{n}q_i(s)(\psi_2(\phi_i(s)))^{\alpha}\left(\frac{B_2(\varrho^{-1}(\phi_i(s)),\iota_2)}{B_1(s,\iota_1)}\right)^{\alpha}-\frac{r(s)(\zeta'_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\zeta^{\alpha}(s)}\right]ds \le E(\iota,\iota_3)w(\iota_3),$$

which contradicts (24). Therefore (C_{II}) holds, and so $\lim_{\iota\to\infty}x(\iota)=0$ by Lemma 3. This completes the proof. □

Corollary 1. *Suppose that all conditions of Theorem 3 are satisfied with (24) replaced by*

$$\limsup_{\iota\to\infty}\frac{1}{E(\iota,\iota_*)}\int_{\iota_*}^{\iota}E(\iota,s)\zeta(s)\sum_{i=1}^{n}q_i(s)(\psi_2(\phi_i(s)))^{\alpha}\left(\frac{B_2(\varrho^{-1}(\phi_i(s)),\iota_2)}{B_1(s,\iota_1)}\right)^{\alpha}ds=\infty$$

and

$$\limsup_{\iota\to\infty}\frac{1}{E(\iota,\iota_*)}\int_{\iota_*}^{\iota}\frac{r(s)(\zeta'_+(s))^{\alpha+1}}{\zeta^{\alpha}(s)}ds<\infty,$$

then any solution of (1) is either oscillatory or satisfies $\lim_{\iota\to\infty}x(\iota)=0$.

Theorem 4. *Let $\alpha\ge 1$. Assume that (2)–(4) hold and $\varrho(\iota)\ge\phi_i(\iota)$ for $i=1,2,\cdots,n$. If there exists a function $\zeta\in C^1([\iota_0,\infty),\mathbb{R})$ such that*

$$\limsup_{\iota\to\infty}\int_{T}^{\iota}\left[\zeta(s)\sum_{i=1}^{n}q_i(s)(\psi_2(\phi_i(s)))^{\alpha}\left(\frac{B_2(\varrho^{-1}(\phi_i(s)),\iota_2)}{B_1(s,\iota_1)}\right)^{\alpha}-\frac{r^{1/\alpha}(s)(\zeta'_+(s))^2}{4\alpha\zeta(s)[B_1(s,\iota_1)]^{\alpha-1}}\right]ds=\infty, \tag{27}$$

for all $\iota_1,\iota_2,T\in[\iota_0,\infty)$, where $T>\iota_2>\iota_1$, then any solution of (1) is either oscillatory or satisfies $\lim_{\iota\to\infty}x(\iota)=0$.

Proof. Let (1) have a nonoscillatory solution $x(\iota)$ on $[\iota_0,\infty)$, and say there exists $\iota_1\in[\iota_0,\infty)$ such that, for $\iota\ge\iota_1$, $x(\iota)>0$, $x(\varrho(\iota))>0$, and $x(\phi_i(\iota))>0$, (2) and (3) hold, and $z(\iota)$ satisfies either (C_I) or (C_{II}) for $i=1,2,\cdots,n$. Assume (C_I) holds. Following the same arguments as in the proof of the Theorem 2, we arrive at (23), which can be rewritten as

$$w'(\iota)\le\frac{\zeta'_+(\iota)}{\zeta(\iota)}w(\iota)-\zeta(\iota)\Omega_2(\iota)\left(\frac{B_2(\varrho^{-1}((\iota)),\iota_2)}{B_1(\iota,\iota_1)}\right)^{\alpha}-\frac{\alpha w^2(\iota)w^{\frac{1}{\alpha}-1}(\iota)}{(\zeta(\iota)r(\iota))^{1/\alpha}}. \tag{28}$$

From (11) and (16) , we see that

$$\begin{aligned}
w^{\frac{1}{\alpha}-1}(\iota)&=(\zeta(\iota)r(\iota))^{\frac{1}{\alpha}-1}\frac{(z''(\iota))^{1-\alpha}}{(z'(\iota))^{1-\alpha}}\\
&=(\zeta(\iota)r(\iota))^{\frac{1}{\alpha}-1}\left(\frac{z'(\iota)}{z''(\iota)}\right)^{\alpha-1}\\
&\ge(\zeta(\iota)r(\iota))^{\frac{1}{\alpha}-1}\left[r^{1/\alpha}(\iota)B_1(\iota,\iota_1)\right]^{\alpha-1}\\
&=\zeta^{\frac{1}{\alpha}-1}(\iota)[B_1(\iota,\iota_1)]^{\alpha-1}.
\end{aligned}\tag{29}$$

Using (29) in (28), for $\iota\ge\iota_3$, we obtain

$$w'(\iota)\le-\zeta(\iota)\sum_{i=1}^{n}q_i(\iota)(\psi_2(\phi_i(\iota)))^{\alpha}\left(\frac{B_2(\varrho^{-1}(\phi_i(\iota)),\iota_2)}{B_1(\iota,\iota_1)}\right)^{\alpha}+\frac{\zeta'_+(\iota)}{\zeta(\iota)}w(\iota)-\frac{\alpha[B_1(\iota,\iota_1)]^{\alpha-1}}{\zeta(\iota)r^{1/\alpha}(\iota)}w^2(\iota). \tag{30}$$

Bringing the square to a close with respect to w, from (30) it follows that

$$w'(\iota) \leq -\zeta(\iota)\sum_{i=1}^{n} q_i(\iota)(\psi_2(\phi_i(\iota)))^\alpha \left(\frac{B_2(\varrho^{-1}(\phi_i(\iota)),\iota_2)}{B_1(\iota,\iota_1)}\right)^\alpha + \frac{r^{1/\alpha}(\iota)}{4\alpha[B_1(\iota,\iota_1)]^{\alpha-1}}\frac{(\zeta'_+(\iota))^2}{\zeta(\iota)}.$$

Integrating this inequality from ι_3 to ι gives

$$\int_T^\iota \left[\zeta(s)\sum_{i=1}^{n} q_i(s)(\psi_2(\phi_i(s)))^\alpha \left(\frac{B_2(\varrho^{-1}(\theta_2(s)),\iota_2)}{B_1(s,\iota_1)}\right)^\alpha - \frac{r^{1/\alpha}(s)(\zeta'_+(s))^2}{4\alpha\zeta(s)[B_1(s,\iota_1)]^{\alpha-1}}\right]ds \leq w(\iota_3),$$

which contradicts (27).

If (C_{II}) holds, then again from Lemma 3, we have $\lim_{\iota\to\infty} x(\iota) = 0$. The proof is complete. □

Next, we give oscillation results in the case when $\varrho(\iota) \leq \phi_i(\iota)$ for $i = 1, 2, \cdots, n$ holds.

Theorem 5. *Assume that (2)–(4) hold and $\varrho(\iota) \leq \phi_i(\iota)$ for $i = 1, 2, \cdots, n$. If there exists a function $\zeta \in C^1([\iota_0,\infty),\mathbb{R})$ such that*

$$\limsup_{\iota\to\infty} \int_T^\iota \left[\zeta(s)\Omega_2(s)\left(\frac{B_2(s,\iota_2)}{B_1(s,\iota_1)}\right)^\alpha - \frac{\zeta'_+(s)}{(B_1(s,\iota_1))^\alpha}\right]ds = \infty, \tag{31}$$

for all $\iota_1, \iota_2, T \in [\iota_0,\infty)$, where $T > \iota_2 > \iota_1$, then any solution of (1) is either oscillatory or satisfies $\lim_{\iota\to\infty} x(\iota) = 0$.

Proof. Let (1) has a nonoscillatory solution $x(\iota)$ on $[\iota_0,\infty)$, say there exists $\iota_1 \in [\iota_0,\infty)$ such that, for $\iota \geq \iota_1$, $x(\iota) > 0$, $x(\varrho(\iota)) > 0$, and $x(\phi_i(\iota)) > 0$, (2) and (3) hold, for $z(\iota)$ satisfies either (C_I) or (C_{II}) and $i = 1, 2, \cdots, n$. Assume that (C_I) holds. Following the same arguments as in the proof of the Theorem 1, we arrive at (18). Using the fact that $\varrho(\iota)$ is strictly increasing and $\varrho(\iota) \leq \phi_i(\iota)$, we have

$$\iota \leq \varrho^{-1}(\phi_i(\iota)), \quad i = 1, 2, \cdots, n.$$

Thus, in view of the fact that $z(\iota)$ is increasing, we obtain

$$\frac{z(\varrho^{-1}(\phi_i(\iota)))}{z(\iota)} \geq 1, \quad i = 1, 2, \cdots, n. \tag{32}$$

Using (32) in (18), we obtain that

$$w'(\iota) \leq \frac{\zeta'_+(\iota)}{(B_1(\iota,\iota_1))^\alpha} - \zeta(\iota)\frac{z^\alpha(\iota)}{(z'(\iota))^\alpha}\sum_{i=1}^{n} q_i(s)(\psi_2(\phi_i(s)))^\alpha \quad \text{for } \iota \geq \iota_3. \tag{33}$$

In view of (12), (33) takes the form

$$w'(\iota) \leq \frac{\zeta'_+(\iota)}{(B_1(\iota,\iota_1))^\alpha} - \zeta(\iota)\Omega_2(\iota)\left(\frac{B_2(\iota,\iota_2)}{B_1(\iota,\iota_1)}\right)^\alpha \quad \text{for } \iota \geq \iota_3. \tag{34}$$

The remainder of the proof is similar to that of Theorem 1 and so we omit it. □

Theorem 6. *Assume that (2)–(4) hold and $\varrho(\iota) \leq \phi_i(\iota)$ for $i = 1, 2, \cdots, n$. If there exists a function $\zeta \in C^1([\iota_0,\infty),\mathbb{R})$ such that*

$$\limsup_{\iota\to\infty} \int_T^\iota \left[\zeta(s)\Omega_2(s)\left(\frac{B_2(s,\iota_2)}{B_1(s,\iota_1)}\right)^\alpha - \frac{r(s)(\zeta'_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\zeta^\alpha(s)}\right]ds = \infty, \tag{35}$$

for all $\iota_1, \iota_2, T \in [\iota_0, \infty)$, *where* $T > \iota_2 > \iota_1$, *then any solution of* (1) *is either oscillatory or satisfies* $\lim_{\iota\to\infty} x(\iota) = 0$.

Theorem 7. *Assume that* (2)–(4) *hold and* $\varrho(\iota) \le \phi_i(\iota)$ *for* $i = 1, 2, \cdots, n$. *If there exists a function* $\zeta \in C^1([\iota_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{\iota\to\infty} \frac{1}{E(\iota,\iota_*)} \int_{\iota_*}^{\iota} \Big[E(\iota,s)\zeta(s)\Omega_2(s) \Big(\frac{B_2(s,\iota_2)}{B_1(s,\iota_1)} \Big)^{\alpha} - \frac{r(s)(\zeta'_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\zeta^{\alpha}(s)} \Big] ds = \infty, \tag{36}$$

for all $\iota_1, \iota_2, \iota_* \in [\iota_0, \infty)$, *where* $\iota_* > \iota_2 > \iota_1$, *then any solution of* (1) *is either oscillatory or satisfies* $\lim_{\iota\to\infty} x(\iota) = 0$.

Corollary 2. *Suppose that all conditions of Theorem 7 are satisfied with* (36) *replaced by*

$$\limsup_{\iota\to\infty} \frac{1}{E(\iota,\iota_*)} \int_{\iota_*}^{\iota} E(\iota,s)\zeta(s)\Omega_2(s) \Big(\frac{B_2(s,\iota_2)}{B_1(s,\iota_1)} \Big)^{\alpha} ds = \infty$$

and

$$\limsup_{\iota\to\infty} \frac{1}{E(\iota,\iota_*)} \int_{\iota_*}^{\iota} \frac{r(s)(\zeta'_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\zeta^{\alpha}(s)} ds < \infty,$$

then any solution of (1) *is either oscillatory or satisfies* $\lim_{\iota\to\infty} x(\iota) = 0$.

Theorem 8. *Let* $\alpha \ge 1$. *Assume that* (2)–(4) *hold and* $\varrho(\iota) \le \phi_i(\iota)$ *for* $i = 1, 2, \cdots, n$. *If there exists a function* $\zeta \in C^1([\iota_0, \infty), \mathbb{R})$ *such that*

$$\limsup_{\iota\to\infty} \int_T^{\iota} \Big[\zeta(s)\Omega_2(s) \Big(\frac{B_2(s,\iota_2)}{B_1(s,\iota_1)} \Big)^{\alpha} - \frac{r^{1/\alpha}(s)}{4\alpha[B_1(s,\iota_1)]^{\alpha-1}} \frac{(\zeta'_+(s))^2}{\zeta(s)} \Big] ds = \infty, \tag{37}$$

for all $\iota_1, \iota_2, T \in [\iota_0, \infty)$, *where* $T > \iota_2 > \iota_1$, *then any solution of* (1) *is either oscillatory or satisfies* $\lim_{\iota\to\infty} x(\iota) = 0$.

Example 1. *Consider the differential equation*

$$\Big(\big((x(\iota) + 8x(\iota/2))'' \big)^3 \Big)' + \iota^2 x^3(\iota/4) + \iota^3 x^3(\iota/8) = 0, \quad \iota \ge 1 \tag{38}$$

where $\alpha = 3$, $r(\iota) = 1$, $p(\iota) = 8$, $\varrho(\iota) = \iota/2$, $q_1(\iota) = \iota^2$, $q_2(\iota) = \iota^3$, $\phi_1(\iota) = \iota/4$ *and* $\phi_2(\iota) = \iota/8$. *Then, we obtain*

$$\begin{aligned}
B_1(\iota,\iota_1) &= B_1(\iota,1) = \iota - 1, \\
B_2(\iota,\iota_2) &= B_2(\iota,2) = (\iota^2 - 2\iota)/2, \\
B_2(\varrho^{-1}(\iota),\iota_2) &= B_2(2\iota,2) = 2\iota^2 - 2\iota, \\
B_2(\varrho^{-1}(\varrho^{-1}(\iota)),\iota_2) &= B_2(4\iota,2) = 8\iota^2 - 4\iota \\
B_2(\varrho^{-1}(\phi_1(\iota)),\iota_2) &= B_2(\iota/2,2) = \frac{4\iota^2 - \iota}{8} \\
B_2(\varrho^{-1}(\phi_2(\iota)),\iota_2) &= B_2(\iota/4,2) = \frac{8\iota^2 - \iota}{32},
\end{aligned}$$

and

$$\begin{aligned}
\psi_1(\iota) &= \frac{1}{8}\Big(1-\frac{1}{8}\Big) = 7/64 > 0,\\
\psi_2(\iota) &= \frac{1}{8}\Big(1-\frac{1}{8}\frac{8\iota^2-4\iota}{2\iota^2-2\iota}\Big) = \frac{1}{8}\Big(\frac{2\iota-1}{\iota-1}\Big) \geq \frac{1}{32} > 0, \quad \textit{for } \iota \geq \iota_2 = 2,\\
\Omega_1(\iota) &= \sum_{i=1}^{2} q_i(\iota)(\psi_1(\phi_i(\iota)))^{\alpha} = \iota^2\Big(\frac{7}{64}\Big)^3 + \iota^3\Big(\frac{7}{64}\Big)^3 = \iota^2\Big(\frac{7}{64}\Big)^3(1+\iota).
\end{aligned}$$

It is easy to verify that

$$\int_{\iota_0}^{\infty}\int_{v}^{\infty}\frac{1}{r^{1/\alpha}(u)}\Big(\int_{u}^{\infty}\Omega_1(s)ds\Big)^{1/\alpha} du\,dv = \int_{1}^{\infty}\int_{v}^{\infty}\int_{u}^{\infty}\Big(\frac{7}{64}\Big)(s^2(1+s))^{\frac{1}{3}} ds\,du\,dv = \infty,$$

and picking $\zeta(\iota) = \iota$, we see that

$$\begin{aligned}
&\int_{T}^{\infty}\left[s\sum_{i=1}^{2} q_i(s)(\psi_2(\phi_i(s)))^3\Big(\frac{B_2(\varrho^{-1}(\phi_i(s)),\iota_2)}{B_1(s,\iota_1)}\Big)^3 - \frac{1}{(B_1(s,\iota_1))^3}\right] ds\\
= &\int_{2}^{\infty}\left\{s^3\Big(\frac{7}{64}\Big)^3\Big(\frac{4s^2-s}{8(s-1)}\Big)^3 + s^4(\frac{1}{32})^3\Big(\frac{8s^2-s}{32(s-1)}\Big)^3 - \frac{1}{(s-1)^3}\right\} ds = \infty.
\end{aligned}$$

Hence, any solution of (38) *is either oscillatory or satisfies* $\lim_{\iota\to\infty} x(\iota) = 0$ *by Theorem 1.*

Example 2. *Consider the differential equation*

$$\Big(\Big(\big(x(\iota)+\tfrac{7\iota+8}{\iota+1}x(\iota-2)\big)''\Big)^{1/5}\Big)' + (\iota^2+\iota)x^{1/5}(\iota-\tfrac{3}{2}) + (\iota^3+\iota)x^{1/5}(\iota-\tfrac{1}{2}) = 0, \quad \iota \geq 2 \tag{39}$$

where $\alpha = 1/5$, $r(\iota) = 1$, $p(\iota) = \frac{7\iota+8}{\iota+1}$, $\varrho(\iota) = \iota-2$, $q_1(\iota) = \iota^2+\iota$, $q_2(\iota) = \iota^3+\iota$, $\phi_1(\iota) = \iota-3/2$ *and* $\phi_2(\iota) = \iota-1/2$. *Then, we obtain*

$$\begin{aligned}
&7 \leq p(\iota) < 8,\\
&B_1(\iota,\iota_1) = B_1(\iota,2) = \iota-2,\\
&B_2(\iota,\iota_2) = B_2(\iota,3) = (\iota^2-4\iota+3)/2,\\
&B_2(\varrho^{-1}(\iota),\iota_2) = B_2(\iota+2,3) = (\iota^2-1)/2,\\
&B_2(\varrho^{-1}(\varrho^{-1}(\iota)),\iota_2) = B_2(\iota+4,3) = (\iota^2+4\iota+3)/2\iota,
\end{aligned}$$

and

$$\begin{aligned}
\psi_1(\iota) &\geq \frac{1}{8}\Big(1-\frac{1}{7}\Big) = 3/28 > 0,\\
\psi_2(\iota) &= \frac{1}{8}\Big(1-\frac{1}{7}\frac{\iota^2+4\iota+3}{\iota^2-1}\Big) \geq \frac{1}{14} > 0, \quad \textit{for } \iota \geq \iota_2 = 3,\\
\Omega_1(\iota) &= \sum_{i=1}^{2} q_i(\iota)(\psi_1(\phi_i(\iota)))^{\frac{1}{5}} = \Big(\frac{3}{28}\Big)^{\frac{1}{5}}(\iota^3+\iota^2+2\iota),\\
\Omega_2(\iota) &= \sum_{i=1}^{2} q_i(\iota)(\psi_2(\phi_i(\iota)))^{\alpha} \geq \Big(\frac{1}{14}\Big)^{1/5}(\iota^3+\iota^2+2\iota), \quad \textit{for } \iota \geq \iota_2 = 3.
\end{aligned}$$

It is easy to verify that

$$\int_{\iota_0}^{\infty}\int_{v}^{\infty}\frac{1}{r^{1/\alpha}(u)}\Big(\int_{u}^{\infty}\Omega_1(s)ds\Big)^{1/\alpha}du\,dv =$$
$$(\frac{3}{28})^{\frac{1}{5}}\int_{2}^{\infty}\int_{v}^{\infty}\int_{u}^{\infty}(s^3+s^2+2s)^{1/5}ds\,du\,dv=\infty,$$

and picking $\zeta(\iota)=1$*, we see that*

$$\int_{T}^{\infty}\Big[\zeta(s)\Omega_2(s)\Big(\frac{B_2(s,\iota_2)}{B_1(s,\iota_1)}\Big)^{1/5}-\frac{\zeta'_+(s)}{(B_1(s,\iota_1))^{1/5}}\Big]ds$$
$$=\int_{3}^{\infty}\left\{\Big(\frac{1}{14}\Big)^{1/5}(s^3+s^2+2s)\Big(\frac{s^2-4s+3}{2s-4}\Big)^{1/5}\right\}ds=\infty.$$

Hence, any solution of (39) *is either oscillatory or satisfies* $\lim_{\iota\to\infty}x(\iota)=0$*, by Theorem 5.*

3. Conclusions

We established several oscillation theorems for (1) under the assumptions of $\varrho(\iota)\geq\phi_i(\iota)$ and $\varrho(\iota)\leq\phi_i(\iota)$ for $i=1,2,\cdots,n$, when $p(\iota)\geq 1$. The main outcomes were proven via the means of a generalized Riccati technique, integral averaging conditions under the assumptions of $\int_{\iota_0}^{\infty}r^{-1/\alpha}(s)ds=\infty$. Two examples were given to prove the significance of new theorems. The primary conclusions given in this work are basically new and have a high degree of generality. For future consideration, it will be of great importance to study the oscillation of (1) when $\int_{\iota_0}^{\infty}r^{-1/\alpha}(s)ds<\infty$.

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