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# Some New Results on Hermite–Hadamard–Mercer-Type Inequalities Using a General Family of Fractional Integral Operators

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**Abstract:** The aim of this article is to obtain new Hermite–Hadamard–Mercer-type inequalities using Raina's fractional integral operators. We present some distinct and novel fractional Hermite–Hadamard–Mercer-type inequalities for the functions whose absolute value of derivatives are convex. Our main findings are generalizations and extensions of some results that existed in the literature.

**Keywords:** convex function; Hermite–Hadamard inequalities; Jensen–Mercer inequality; fractional integral operators

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## 1. Introduction

Convex functions have a very useful structure in terms of both definition and properties. This concept has an important role in the theory of inequality. This class of functions has many applications in the different branches of mathematics, and many important inequalities are obtained with the help of this class of functions. Hermite–Hadamard inequality, Jensen inequality, and Mercer inequality, which are well known in the literature, are some of them. Jensen inequality has been caught attention of many researchers, and many articles related to different versions of this inequality have been found in the literature. Jensen's famous inequality can be given as follows:

Let $0 < x_1 \leq x_2 \leq \ldots \leq x_n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be non-negative weights such that $\sum_{k=1}^{n} \mu_k = 1$. The famous Jensen inequality (see [1]) in the literature states that $f$ is convex function on the interval $[a, b]$; then

$$f\left(\sum_{k=1}^{n} \mu_k x_k\right) \leq \left(\sum_{k=1}^{n} \mu_k f(k)\right), \tag{1}$$

where $\forall x_k \in [a, b]$ and all $\mu_k \in [0, 1]$, $(k = \overline{1, n})$.

A new variant of Jensen inequality that has been established by Mercer can be presented as follows:

**Theorem 1.** *([2]) Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on $[a, b]$ then*

$$f\left(a + b - \sum_{k=1}^{n} \mu_k x_k\right) \leq f(a) + f(b) - \sum_{k=1}^{n} \mu_k f(x_k)$$

*for each $x_k \in [a, b]$ and $\mu_k \in [a, b]$ $(k = \overline{1, n})$ with $\sum_{k=1}^{n} \mu_k = 1$.*

Recently, many studies have been performed on the Jensen–Mercer inequality, see ([1,3–5]). For more recent and related results connected with Jensen–Mercer inequality, see ([3,6–11]).

Let us now recall another important inequality obtained by using convex functions. The Hermite–Hadamard inequality has been the focus of many researchers in the fields of inequality theory, numerical analysis, and applied mathematics for nearly a hundred years. A great number of generalizations, expansions, new variants, and improvements have been made regarding this inequality (see, e.g., [12]). The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

holds for convex functions and known as Hermite–Hadamard inequality. If $f$ is concave, both inequalities hold as a reverse direction.

Recently, a modern direction of research has been to investigate various likely ways to define fractional integrals and derivatives in fractional calculus. Fractional operators differ from each other with their kernel structures and further properties. Most of them have a general form of the previous operators. Motivated by this, several fractional operators are introduced that generalize ordinary integral operators.

Let us recall the fractional integral of Riemann–Liouville and its general form, which is called Raina's fractional integral operator.

**Definition 1.** *Let $f \in L_1[a,b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \ \ x > a \tag{2}$$

*and*

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \ \ x < b \tag{3}$$

*respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}u^{\alpha-1}du$. Here is $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.*

In addition, Raina [13] defined the following results related to the general class of fractional integral operators.

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k \quad (\rho,\lambda > 0; |x| < \mathbf{R}), \tag{4}$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N} = \mathbb{N} \cup \{0\})$ are a bounded sequence of positive real numbers and $\mathbf{R}$ is the set of real numbers. With the help of (4), Raina [13] and Agarwal et al. [14] defined the following left-sided and right-sided fractional integral operators, respectively, as follows:

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a^+;\omega}f\right)(x) = \int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(x-t)^{\rho}]f(t)dt \qquad (x > a > 0), \tag{5}$$

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;\omega}f\right)(x) = \int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-x)^{\rho}]f(t)dt \qquad (0 < x < b), \tag{6}$$

where $\lambda, \rho > 0$, $\omega \in \mathbb{R}$ and $f(t)$ is such that the integral on the right-hand side exits.

It is easy to verify that $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(x)$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(x)$ are bounded integral operators on $L(a,b)$ if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(a-b)^{\rho}] < \infty.$$

In fact, for $f \in L(a,b)$, we have

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f(x)||_1 \leq \mathfrak{M}(b-a)^{\lambda}||f||_1$$

and

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f(x)||_1 \leq \mathfrak{M}(b-a)^{\lambda}||f||_1$$

where

$$||f||_p := \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}.$$

Here, many useful fractional integral operators can be obtained by customizing the coefficient $\sigma(k)$.

**Remark 1.** *If we choose $\lambda = \alpha$, $\sigma(0) = 1$ and $\omega = 0$ in (5) and (6), we obtain the classical left- and right-RL fractional integrals (2) and (3), respectively.*

To provide more information about fractional integral operators and applications to the theory of inequality, we recommend the following papers to interested readers ([15–21]).

In this article, motivated by the Jensen–Mercer inequality and Raina's fractional integral operator, we establish new Hermite–Hadamard–Mercer-type integral inequalities for convex functions.

## 2. Hermite–Hadamard–Mercer Inequalities via the Raina's Fractional Integral Operator

In this section, we obtain some new Hermite–Hadamard–Mercer inequalities using the Jensen-Mercer inequality via Raina's fractional integral operator.

**Theorem 2.** *Suppose that $f : [a,b] \to \mathbb{R}$ is a convex function. Then*

$$\begin{aligned} & f\left(a+b-\tfrac{x+y}{2}\right) \\ \leq \;& f(a)+f(b) - \tfrac{1}{2(y-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;\omega}f)(y) + (\mathcal{J}^{\sigma}_{\rho,\lambda,y^-;\omega}f)(x)\right] \\ \leq \;& f(a)+f(b)-f\left(\tfrac{x+y}{2}\right) \end{aligned} \tag{7}$$

*and*

$$\begin{aligned} & f\left(a+b-\tfrac{x+y}{2}\right) \\ \leq \;& \tfrac{1}{2(y-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-y)^+;\omega}f)(a+b-x) + (\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-x)^-;\omega}f)(a+b-y)\right] \\ \leq \;& \tfrac{f(a+b-x)+f(a+b-y)}{2} \\ \leq \;& f(a)+f(b)-\tfrac{f(x)+f(y)}{2} \end{aligned} \tag{8}$$

*for all $x, y \in [a,b]$, $x < y$, and $\lambda, \rho, \omega > 0$.*

**Proof.** Using the Jensen-Mercer inequality, we can write

$$f\left(a+b-\frac{x_1+y_1}{2}\right) \leq f(a)+f(b)-\frac{f(x_1)+f(y_1)}{2} \tag{9}$$

for all $x_1, y_1 \in [a,b]$. By changing of the variables $x_1 = tx+(1-t)y$ and $y_1 = (1-t)x+ty$ for $x, y \in [a,b]$ and $t \in [0,1]$ in (9), we obtain

$$f\left(a+b-\frac{x+y}{2}\right) \leq f(a)+f(b)-\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}. \tag{10}$$

Multiplying both sides of (10) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]$ and then integrating the resulting inequality with respect to $t$ over $[0,1]$, we have

$$\begin{aligned}
&\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]f\left(a+b-\frac{x+y}{2}\right)\\
\leq\ & [f(a)+f(b)]\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]dt\\
&-\frac{1}{2}\left[\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]f(tx+(1-t)y)dt\right.\\
&\left.+\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]f((1-t)x+ty)dt\right]\\
=\ & [f(a)+f(b)]\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\\
&-\frac{1}{2}\left[\int_x^y\left(\frac{y-u}{y-x}\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(y-x)^{\rho}\left(\frac{y-u}{y-x}\right)^{\rho}\right]f(u)\frac{du}{y-x}\right.\\
&\left.+\int_x^y\left(\frac{u-x}{y-x}\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(y-x)^{\rho}\left(\frac{u-x}{y-x}\right)^{\rho}\right]f(u)\frac{du}{y-x}\right]\\
=\ & [f(a)+f(b)]\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\\
&-\frac{1}{2}\left[\frac{1}{(y-x)^{\lambda}}\int_x^y(y-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-u)^{\rho}]f(u)du\right.\\
&\left.+\frac{1}{(y-x)^{\lambda}}\int_x^y(u-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(u-x)^{\rho}]f(u)du\right]
\end{aligned}$$

namely

$$\begin{aligned}
&\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]f\left(a+b-\tfrac{x+y}{2}\right)\\
\leq\ & [f(a)+f(b)]\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]-\tfrac{1}{2(y-x)^{\lambda}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;\omega}f)(y)+(\mathcal{J}^{\sigma}_{\rho,\lambda,y^-;\omega}f)(x)\right]
\end{aligned} \tag{11}$$

and so the first inequality of (7) is proven. For the proof of the second inequality (7), we first note that if $f$ is a convex function, then, for $t\in[0,1]$, it yields

$$\begin{aligned}
f\left(\frac{x+y}{2}\right) &= f\left(\frac{tx+(1-t)y+(1-t)x+ty}{2}\right)\\
&\leq \frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}.
\end{aligned} \tag{12}$$

Multiplying both sides of (12) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]$ and then integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned}
&\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]f\left(\frac{x+y}{2}\right)\\
\leq\ & \frac{1}{2}\left[\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]f(tx+(1-t)y)dt\right.\\
&\left.+\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]f((1-t)x+ty)dt\right]\\
=\ & \frac{1}{2(y-x)^{\lambda}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;\omega}f)(y)+(\mathcal{J}^{\sigma}_{\rho,\lambda,y^-;\omega}f)(x)\right]
\end{aligned}$$

and so

$$-f\left(\frac{x+y}{2}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\geq\frac{1}{2(y-x)^{\lambda}}\left[(\mathcal{J}^{\sigma}_{\rho,\lambda,x^+;\omega}f)(y)+(\mathcal{J}^{\sigma}_{\rho,\lambda,y^-;\omega}f)(x)\right]. \tag{13}$$

Adding $f(a)+f(b)$ to both sides of (13), we find the second inequality of (7).

Now, we prove inequality (8). From the convexity of $f$, we have

$$\begin{aligned} f\left(a+b-\frac{x_1+y_1}{2}\right) &= f\left(\frac{a+b-x_1+a+b-y_1}{2}\right) \\ &\leq \frac{1}{2}[f(a+b-x_1)+f(a+b-y_1)] \end{aligned} \tag{14}$$

for all $x_1, y_1 \in [a,b]$. By changing the variables $a+b-x_1 = t(a+b-x)+(1-t)(a+b-y)$ and $a+b-y_1 = (1-t)(a+b-x)+t(a+b-y)$ for $x,y \in [a,b]$ and $t \in [0,1]$ in (14), we find that

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2}[f(t(a+b-x)+(1-t)(a+b-y))+f((1-t)(a+b-x)+t(a+b-y))]. \end{aligned} \tag{15}$$

Multiplying both sides of (15) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]$ and then integrating the resulting inequality with respect to $t$ over $[0,1]$, we have

$$\begin{aligned} & \mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2}\left[\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]f(t(a+b-x)+(1-t)(a+b-y))dt\right. \\ & \left.+\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]f((1-t)(a+b-x)+t(a+b-y))dt\right] \\ =\ & \frac{1}{2(y-x)^{\lambda}}\left[\int_{a+b-y}^{a+b-x}(u-(a+b-y))^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(u-(a+b-y))^{\rho}]f(u)du\right. \\ & \left.+\int_{a+b-y}^{a+b-x}((a+b-x)-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega((a+b-x)-u)^{\rho}]f(u)du\right] \\ =\ & \frac{1}{2(y-x)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-y)^{+};\omega}f\right)(a+b-x)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-x)^{-};\omega}f\right)(a+b-y)\right] \end{aligned}$$

and so

$$\begin{aligned} & \mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2(y-x)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-y)^{+};\omega}f\right)(a+b-x)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-x)^{-};\omega}f\right)(a+b-y)\right]. \end{aligned}$$

The proof of the first inequality of (8) is completed. On the other hand, using the convexity of $f$, we can write

$$f(t(a+b-x)+(1-t)(a+b-y)) \leq tf(a+b-x)+(1-t)f(a+b-y)$$

and

$$f((1-t)(a+b-x)+t(a+b-y)) \leq (1-t)f(a+b-x)+tf(a+b-y).$$

By adding these inequalities and using the Jensen–Mercer inequality, we have

$$\begin{aligned} & f(t(a+b-x)+(1-t)(a+b-y))+f((1-t)(a+b-x)+t(a+b-y)) \\ \leq\ & f(a+b-x)+f(a+b-y) \\ \leq\ & 2[f(a)+f(b)]-[f(x)+f(y)]. \end{aligned} \tag{16}$$

Multiplying both sides of (16) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]$ and then integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain the second and third inequalities of (8). □

**Theorem 3.** *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then*

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{2^{\lambda-1}}{(y-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]} \\ & \times\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{-};\omega}f\right)(a+b-y)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{+};\omega}f\right)(a+b-x)\right] \\ \leq\ & f(a)+f(b)-\frac{f(x)+f(y)}{2} \end{aligned} \tag{17}$$

*for all $x,y\in[a,b]$, $x<y$ and $\lambda,\rho,\omega>0$.*

**Proof.** To prove the first inequality of (17), by writing $x_1=\frac{t}{2}x+\frac{2-t}{2}y$ and $y_1=\frac{2-t}{2}x+\frac{t}{2}y$ for $x,y\in[a,b]$ and $t\in[0,1]$ in the inequality (14), we get

$$2f\left(a+b-\frac{x+y}{2}\right)\leq\left[f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}\right)y\right)+f\left(a+b-\left(\frac{2-t}{2}\right)x+\frac{t}{2}y\right)\right]. \tag{18}$$

Then, multiplying both sides of (18) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]$ and then integrating the resulting inequality with respect to $t$ over $[0,1]$, we have

$$\begin{aligned} & 2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)dt \\ & +\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)dt \\ =\ & \frac{2^{\lambda}}{(y-x)^{\lambda}}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}}(u-(a+b-y))^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(u-(a+b-y))^{\rho}]f(u)du\right. \\ & \left.+\int_{a+b-\frac{x+y}{2}}^{a+b-x}((a+b-x)-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega((a+b-x)-u)^{\rho}]f(u)du\right] \\ =\ & \frac{2^{\lambda}}{(y-x)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{-};\omega}f\right)(a+b-y)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{+};\omega}f\right)(a+b-x)\right] \end{aligned}$$

and so

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{2^{\lambda-1}}{(y-x)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]} \\ & \times\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{-};\omega}f\right)(a+b-y)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{+};\omega}f\right)(a+b-x)\right]. \end{aligned}$$

The first inequality of (17) is proven. For the proof of the second inequality of (17), by using Jensen–Mercer inequality, we obtain

$$f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\leq f(a)+f(b)-\left[\frac{t}{2}f(x)+\frac{2-t}{2}f(y)\right]$$

and
$$f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\leq f(a)+f(b)-\left[\frac{2-t}{2}f(x)+\frac{t}{2}f(y)\right].$$

By adding these inequalities, we have
$$\begin{aligned}&f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)+f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\\ \leq\;& 2[f(a)+f(b)]-(f(x)+f(y)).\end{aligned}\tag{19}$$

Multiplying both sides of (19) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]$ and then integrating the resulting inequality with respect to $t$ over $[0,1]$, we find the second inequality of (17). □

**Lemma 1.** *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integral holds:*
$$\begin{aligned}&\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\frac{f(a+b-x)+f(a+b-y)}{2}-\frac{1}{2(y-x)^{\lambda}}\\&\times\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-x)^{-};\omega}f\right)(a+b-y)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-y)^{+};\omega}f\right)(a+b-x)\right]\\=\;&\frac{y-x}{2}\left[\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]f'(a+b-(tx+(1-t)y))dt\right.\\&\left.-\int_0^1(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]f'(a+b-(tx+(1-t)y))dt\right]\end{aligned}\tag{20}$$

*for all $x,y\in[a,b]$, $x<y$, $\lambda,\rho,\omega>0$ and $t\in[a,b]$.*

**Proof.** It suffices to note that
$$I=\frac{y-x}{2}(I_1-I_2),\tag{21}$$

where
$$\begin{aligned}I_1&=\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]f'(a+b-(tx+(1-t)y))dt\\&=t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]\frac{f(a+b-(tx+(1-t)y))}{y-x}\Bigg|_0^1\\&\quad-\frac{1}{y-x}\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}t^{\rho}]f(a+b-(tx+(1-t)y))dt\\&=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]f(a+b-x)}{y-x}-\frac{1}{(y-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-x)^{-};\omega}f\right)(a+b-y)\end{aligned}\tag{22}$$

and
$$\begin{aligned}I_2&=\int_0^1(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]f'(a+b-(tx+(1-t)y))dt\\&=(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]\frac{f(a+b-(tx+(1-t)y))}{y-x}\Bigg|_0^1\\&\quad+\frac{1}{y-x}\int_0^1(1-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(y-x)^{\rho}(1-t)^{\rho}]f(a+b-(tx+(1-t)y))dt\\&=-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]f(a+b-y)}{y-x}+\frac{1}{(y-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-y)^{+};\omega}f\right)(a+b-x).\end{aligned}\tag{23}$$

By combining (22) and (23) with (21), we get (20). □

**Lemma 2.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a < b$. If $f' \in L[a,b]$, then*

$$\begin{aligned}
&\frac{2^{\lambda-1}}{(y-x)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{+};\omega}f\right)(a+b-x)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{-};\omega}f\right)(a+b-y)\right] \\
&-\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]f\left(a+b-\frac{x+y}{2}\right) \qquad (24)\\
=\ &\frac{y-x}{4}\left[\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]f'\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)dt\right. \\
&\left.-\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]f'\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)dt\right]
\end{aligned}$$

*for all $x, y \in [a,b]$, $x < y$, $\lambda, \rho, \omega > 0$, and $t \in [0,1]$.*

**Proof.** Let

$$\begin{aligned}
I\ =\ &\left[\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]f'\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)dt\right. \\
&\left.-\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]f'\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)dt\right] \\
=\ &I_2 - I_1.
\end{aligned}$$

Note that

$$\begin{aligned}
I_1\ =\ &\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]f'\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)dt \\
=\ &t^{\lambda}\frac{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]}{y-x}f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\Bigg|_0^1 \\
&-\frac{2}{y-x}\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)dt \\
=\ &\frac{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]}{y-x}f\left(a+b-\frac{x+y}{2}\right) \\
&-\frac{2}{y-x}\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]f\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)dt.
\end{aligned}$$

By substituting $u = a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)$ , we get, after some computations,

$$I_1 = \frac{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]}{y-x}f\left(a+b-\frac{x+y}{2}\right)-\frac{2^{\lambda+1}}{(y-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{-};\omega}f\right)(a+b-y). \qquad (25)$$

By proceeding with a similar process, we obtain

$$\begin{aligned}
I_2\ =\ &\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]f'\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)dt \qquad (26)\\
=\ &-\frac{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]}{y-x}f\left(a+b-\frac{x+y}{2}\right)+\frac{2^{\lambda+1}}{(y-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{+};\omega}f\right)(a+b-x).
\end{aligned}$$

By using (25) and (26), it follows that

$$
\begin{aligned}
I_2 - I_1 &= \frac{2^{\lambda+1}}{(y-x)^{\lambda+1}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^+;\omega}f\right)(a+b-x) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^-;\omega}f\right)(a+b-y)\right] \\
&\quad - f\left(a+b-\frac{x+y}{2}\right)\left(\frac{4\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]}{y-x}\right).
\end{aligned}
$$

Thus, by multiplying $\frac{y-x}{4}$ on both sides of the above equality, we get (24). □

## 3. Generalized Hermite–Hadamard–Mercer-Type Inequalities via Raina's Fractional Integral Operator

**Theorem 4.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds for fractional integral operators:*

$$
\begin{aligned}
&\Bigg|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\frac{f(a+b-x)+f(a+b-y)}{2} - \frac{1}{2(y-x)^{\lambda}} \\
&\times\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-x)^-;\omega}f\right)(a+b-y) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-y)^+;\omega}f\right)(a+b-x)\right]\Bigg| \qquad (27)\\
\leq\ & (y-x)\mathcal{F}^{\sigma_0}_{\rho,\lambda+2}[\omega(y-x)^{\rho}]\left[|f'(a)|+|f'(b)| - \frac{|f'(x)|+|f'(y)|}{2}\right]
\end{aligned}
$$

*where*

$$\sigma_0(k) = \sigma(k)\left(1 - \frac{1}{2^{\lambda+\rho k}}\right)$$

*for all $x, y \in [a,b]$, $x < y$ and $\lambda, \rho, \omega > 0$.*

**Proof.** By means of the Lemma 1 and the Jensen–Mercer inequality, we find that

$$
\begin{aligned}
&\Bigg|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\frac{f(a+b+-x)+f(a+b-y)}{2} - \frac{1}{2(y-x)^{\lambda}} \\
&\times\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-x)^-;\omega}f\right)(a+b-y) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-y)^+;\omega}f\right)(a+b-x)\right]\Bigg| \\
\leq\ & \frac{y-x}{2}\int_0^1 \left|t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}] - (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]\right| \\
&\times|f'(a+b-(tx+(1-t)y))|dt \\
\leq\ & \frac{y-x}{2}\int_0^1 \left|t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}] - (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]\right| \\
&\times[|f'(a)|+|f'(b)| - (t|f'(x)|+(1-t)|f'(y)|)]dt \\
=\ & \frac{y-x}{2}\Bigg\{\int_0^{\frac{1}{2}}\left[(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}] - t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]\right] \\
&\times[|f'(a)|+|f'(b)| - (t|f'(x)|+(1-t)|f'(y)|)]dt \\
&+\int_{\frac{1}{2}}^1\left[t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}] - (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]\right] \\
&\times[|f'(a)|+|f'(b)| - (t|f'(x)|+(1-t)|f'(y)|)]dt\Bigg\} \\
=\ & \frac{y-x}{2}(L_1+L_2).
\end{aligned}
$$

By calculating $L_1$ and $L_2$, we obtain

$$
\begin{aligned}
L_1 &= (|f'(a)|+|f'(b)|)\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]-t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]\right]dt\right)\\
&\quad -|f'(x)|\left(\int_0^{\frac{1}{2}}t(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]dt-\int_0^{\frac{1}{2}}t^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]dt\right)\\
&\quad +|f'(y)|\left(\int_0^{\frac{1}{2}}(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]dt-\int_0^{\frac{1}{2}}(1-t)t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]dt\right)\\
&= (|f'(a)|+|f'(b)|)\left(\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]-\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\right)\\
&\quad -|f'(x)|\left(\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]-\mathcal{F}^{\sigma_4}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\right)\\
&\quad +|f'(y)|\left(\mathcal{F}^{\sigma_5}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]-\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\right)\\
&= (|f'(a)|+|f'(b)|)\left(\frac{1}{\lambda+\rho k+1}-\frac{1}{2^{\lambda+\rho k}(\lambda+\rho k+1)}\right)\\
&\quad -\left\{|f'(x)|\left(\frac{1}{(\lambda+\rho k+1)(\lambda+\rho k+2)}-\frac{1}{2^{\lambda+\rho k+1}(\lambda+\rho k+1)}\right)\right.\\
&\quad \left.+|f'(y)|\left(\frac{1}{\lambda+\rho k+2}-\frac{1}{2^{\lambda+\rho k+1}(\lambda+\rho k+1)}\right)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
L_2 &= (|f'(a)|+|f'(b)|)\left(\int_{\frac{1}{2}}^{1}\left[t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]-(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]\right]dt\right)\\
&\quad -|f'(x)|\left(\int_{\frac{1}{2}}^{1}t^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]dt-\int_{\frac{1}{2}}^{1}t(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]dt\right)\\
&\quad +|f'(y)|\left(\int_{\frac{1}{2}}^{1}(1-t)t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}t^{\rho}]dt-\int_{\frac{1}{2}}^{1}(1-t)^{\lambda+1}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}(1-t)^{\rho}]dt\right)\\
&= (|f'(a)|+|f'(b)|)\left(\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]-\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\right)\\
&\quad -|f'(x)|\left(\mathcal{F}^{\sigma_5}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]-\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\right)\\
&\quad +|f'(y)|\left(\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]-\mathcal{F}^{\sigma_4}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\right)\\
&= (|f'(a)|+|f'(b)|)\left(\frac{1}{\lambda+\rho k+1}-\frac{1}{2^{\lambda+\rho k}(\lambda+\rho k+1)}\right)\\
&\quad -\left\{|f'(x)|\left(\frac{1}{\lambda+\rho k+2}-\frac{1}{2^{\lambda+\rho k+1}(\lambda+\rho k+1)}\right)\right.\\
&\quad \left.+|f'(y)|\left(\frac{1}{(\lambda+\rho k+1)(\lambda+\rho k+2)}-\frac{1}{2^{\lambda+\rho k+1}(\lambda+\rho k+1)}\right)\right\}
\end{aligned}
$$

where

$$\begin{aligned}
\sigma_1(k) &= \sigma(k)\left(\frac{1}{\lambda+\rho k+1} - \frac{1}{2^{\lambda+\rho k+1}(\lambda+\rho k+1)}\right),\\
\sigma_2(k) &= \sigma(k)\left(\frac{1}{2^{\lambda+\rho k+1}(\lambda+\rho k+1)}\right),\\
\sigma_3(k) &= \sigma(k)\left(-\frac{1}{\lambda+\rho k+2} + \frac{1}{2^{\lambda+\rho k+2}(\lambda+\rho k+2)} + \frac{1}{\lambda+\rho k+1} - \frac{1}{2^{\lambda+\rho k+1}(\lambda+\rho k+1)}\right),\\
\sigma_4(k) &= \sigma(k)\left(\frac{1}{2^{\lambda+\rho k+2}(\lambda+\rho k+2)}\right),\\
\sigma_5(k) &= \sigma(k)\left(\frac{1}{\lambda+\rho k+2} - \frac{1}{2^{\lambda+\rho k+2}(\lambda+\rho k+2)}\right),\\
\sigma_6(k) &= \sigma(k)\left(\frac{1}{2^{\lambda+\rho k+1}(\lambda+\rho k+1)} - \frac{1}{2^{\lambda+\rho k+2}(\lambda+\rho k+2)}\right).
\end{aligned}$$

By adding $L_1$ and $L_2$, we obtain the inequality (27). □

**Theorem 5.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds for fractional integral operators:*

$$\begin{aligned}
&\Bigg|\frac{2^{\lambda-1}}{(y-x)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{+};\omega}f\right)(a+b-x) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{-};\omega}f\right)(a+b-y)\right]\\
&-f\left(a+b-\frac{x+y}{2}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]\Bigg|\\
\leq\ &\frac{(y-x)\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]}{2}\left[|f'(a)|+|f'(b)| - \frac{|f'(x)|+|f'(y)|}{2}\right]
\end{aligned} \tag{28}$$

*for all $x, y \in [a,b]$, $x < y$, and $\lambda, \rho, \omega > 0$.*

**Proof.** Using the Lemma 2 and Jensen–Mercer inequality, we find

$$\begin{aligned}
&\Bigg|\frac{2^{\lambda-1}}{(y-x)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{+};\omega}f\right)(a+b-x) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{-};\omega}f\right)(a+b-y)\right]\\
&-f\left(a+b-\frac{x+y}{2}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]\Bigg|\\
\leq\ &\frac{y-x}{4}\left\{\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]\left|f'\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|dt\right.\\
&\left.+\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]\left|f'\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\right|dt\right\}\\
\leq\ &\frac{y-x}{4}\left\{\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]\left[|f'(a)|+|f'(b)| - \left(\frac{2-t}{2}|f'(x)|+\frac{t}{2}|f'(y)|\right)\right]dt\right.\\
&\left.+\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right]\left[|f'(a)|+|f'(b)| - \left(\frac{t}{2}|f'(x)|+\frac{2-t}{2}|f'(y)|\right)\right]\right\}\\
=\ &\frac{(y-x)\mathcal{F}^{\sigma}_{\rho,\lambda+2}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]}{2}\left[|f'(a)|+|f'(b)| - \frac{|f'(x)|+|f'(y)|}{2}\right].
\end{aligned}$$

□

**Theorem 6.** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a < b$. If $|f'|^{q}$ is convex on $[a,b]$, $q > 1$, then the following inequality holds for fractional integral operators:*

$$
\begin{aligned}
&\left|\frac{2^{\lambda-1}}{(y-x)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{+};\omega}f\right)(a+b-x)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{-};\omega}f\right)(a+b-y)\right]\right.\\
&\left.-f\left(a+b-\frac{x+y}{2}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]\right| \qquad (29)\\
\leq\ & \frac{y-x}{4}\mathcal{F}^{\sigma_7}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\\
&\times\left[\left(|f'(a)|^{q}+|f'(b)|^{q}-\frac{3|f'(x)|^{q}+|f'(y)|^{q}}{4}\right)^{\frac{1}{q}}+\left(|f'(a)|^{q}+|f'(b)|^{q}-\frac{|f'(x)|^{q}+3|f'(y)|^{q}}{4}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

*where*

$$\sigma_7=\sigma(k)\left(\frac{1}{\lambda p+\rho k p+1}\right)^{\frac{1}{p}}$$

*for all* $x,y\in[a,b]$, $x<y$, $\lambda,\rho,\omega>0$, *and* $\frac{1}{p}+\frac{1}{q}=1$.

**Proof.** From Lemma 2, using Hölder's inequality, we have

$$
\begin{aligned}
&\left|\frac{2^{\lambda-1}}{(y-x)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{+};\omega}f\right)(a+b-x)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{-};\omega}f\right)(a+b-y)\right]\right.\\
&\left.-f\left(a+b-\frac{x+y}{2}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{y-x}{4}\left(\int_0^1(t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\left(\frac{t}{2}\right)^{\rho}\right])^{p}dt\right)^{\frac{1}{p}}\\
&\times\left\{\left(\int_0^1\left|f'\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right\}\\
=\ & \frac{y-x}{4}\sum_{k=0}^{\infty}\frac{\sigma(k)[|\omega|^{k}(y-x)^{\rho k}\frac{1}{2^{\rho k}}]}{\Gamma(\lambda+\rho k+1)}\left(\int_0^1 t^{(\lambda+\rho k)p}\right)^{\frac{1}{p}}\\
&\times\left\{\left(\int_0^1\left|f'\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(a+b-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Using the Jensen–Mercer inequality and taking into account the convexity of $|f'|^{q}$, we have

$$
\begin{aligned}
&\left|\frac{2^{\lambda-1}}{(y-x)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{+};\omega}f\right)(a+b-x)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,(a+b-\frac{x+y}{2})^{-};\omega}f\right)(a+b-y)\right]\right.\\
&\left.-f\left(a+b-\frac{x+y}{2}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(y-x)^{\rho}\frac{1}{2^{\rho}}\right]\right|\\
\leq\ &\frac{y-x}{4}\sum_{k=0}^{\infty}\frac{\sigma(k)[|\omega|^{k}(y-x)^{\rho k}\frac{1}{2^{\rho k}}]}{\Gamma(\lambda+\rho k+1)}\left(\frac{1}{\lambda p+\rho kp+1}\right)^{\frac{1}{p}}\\
&\times\left\{\left(\int_0^1|f'(a)|^q+|f'(b)|^q-\left(\frac{2-t}{2}|f'(x)|^q+\frac{t}{2}|f'(y)|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1|f'(a)|^q+|f'(b)|^q-\left(\frac{t}{2}|f'(x)|^q+\frac{2-t}{2}|f'(y)|^q\right)dt\right)^{\frac{1}{q}}\right\}\\
=\ &\frac{y-x}{4}\sum_{k=0}^{\infty}\frac{\sigma(k)[|\omega|^{k}(y-x)^{\rho k}\frac{1}{2^{\rho k}}]}{\Gamma(\lambda+\rho k+1)}\left(\frac{1}{\lambda p+\rho kp+1}\right)^{\frac{1}{p}}\\
&\times\left[\left(|f'(a)|^q+|f'(b)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\right)^{\frac{1}{q}}+\left(|f'(a)|^q+|f'(b)|^q-\frac{|f'(x)|^q+3|f'(y)|^q}{4}\right)^{\frac{1}{q}}\right]\\
=\ &\frac{y-x}{4}\mathcal{F}^{\sigma_7}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]\\
&\times\left[\left(|f'(a)|^q+|f'(b)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\right)^{\frac{1}{q}}+\left(|f'(a)|^q+|f'(b)|^q-\frac{|f'(x)|^q+3|f'(y)|^q}{4}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

and so the proof is completed. □

## 4. Related Results

By using a similar arguments to the proof of the theorems that were obtained in the main results section, we will now use $E$ instead of $\mathcal{F}$, by which we will obtain the following new estimates for Prabhakar fractional integral operator.

Before giving the new results, let us remember the Prabhakar operator.

The function $E^{\gamma}_{\rho,\lambda}(x)$ is introduced by Prabhakar [22] in the following form

$$E^{\gamma}_{\rho,\lambda}(x)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\rho k+\lambda)}\frac{x^k}{k!}\quad(\rho,\lambda,\gamma\in\mathbb{C},\Re(\rho)>0)\tag{30}$$

where $(\gamma)_k$ is Pochhammer symbol ([23],Section 2.1.1)

$$(\gamma)_0=1,(\gamma)_k=\gamma(\gamma+1)\ldots(\gamma+k-1),\quad(k=1,2,\ldots).$$

This function is reduced to the Mittag–Leffler function for $\gamma=1$.

Prabhakar defined the following integral operator containing the function (30), in the kernel:

$$\left(\mathcal{E}^{\gamma}_{\rho,\lambda,a^{+};w}f\right)(x)=\int_a^x(x-t)^{\lambda-1}E^{\gamma}_{\rho,\lambda}[\omega(x-t)^{\rho}]f(t)dt$$

where $f\in[a,b]$, $0\leq a<t<b\leq\infty$ and $\rho,\lambda,\gamma,\omega\in\mathbb{C}$ with $\Re(\rho),\Re(\lambda)>0$. It is possible to define right-sided fractional integral operator in a natural way analogous to (6) as the following:

$$\left(\mathcal{E}^{\gamma}_{\rho,\lambda,b^{-};w}f\right)(x)=\int_x^b(t-x)^{\lambda-1}E^{\gamma}_{\rho,\lambda}[\omega(t-x)^{\rho}]f(t)dt.$$

**Theorem 7.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a convex function. Then*

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & f(a)+f(b)-\frac{1}{2(y-x)^{\lambda}E^{\gamma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]}\left[(\mathcal{E}^{\gamma}_{\rho,\lambda,x^{+};\omega}f)(y)+(\mathcal{E}^{\gamma}_{\rho,\lambda,y^{-};\omega}f)(x)\right] \\ \leq\ & f(a)+f(b)-f\left(\frac{x+y}{2}\right) \end{aligned}$$

*and*

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & \frac{1}{2(y-x)^{\lambda}E^{\gamma}_{\rho,\lambda+1}[\omega(y-x)^{\rho}]}\left[(\mathcal{E}^{\gamma}_{\rho,\lambda,(a+b-y)^{+};\omega}f)(a+b-x)+(\mathcal{E}^{\gamma}_{\rho,\lambda,(a+b-x)^{-};\omega}f)(a+b-y)\right] \\ \leq\ & \frac{f(a+b-x)+f(a+b-y)}{2} \\ \leq\ & f(a)+f(b)-\frac{f(x)+f(y)}{2} \end{aligned}$$

*for all $x, y \in [a, b]$, $x < y$, and $\lambda, \rho, \omega > 0$.*

**Proof.** The assertion follows from the definition of Prabhakar fractional integral operators in the proof of Theorem 2. □

Some similar results can be obtained for Theorem 3–6 Prabhakar fractional integral operators. We omit the details for the readers.

## 5. Conclusions

In this paper, we gave new Hermite–Hadamard–Mercer-type inequalities for convex functions. In order to prove these inequalities, we used the Raina's fractional integral operators and Jensen-Mercer inequality. Our results are the generalizations of the Hermite–Hadamard–Mercer-type inequalities that ones given via Riemann–Liouville fractional integrals in [24].

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## References


1. Matkovic, A.; Pecaric, J.; Peric, I. A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **2006**, *418*, 551–564. [CrossRef]
2. Mercer, A.M. A Variant of Jensen's Inequality. *J. Inequal. Pure Appl. Math.* **2003**, *4*, 73
3. Abramovich, S.; Baric, J.; Pecaric, J. A variant of Jessen's inequality of Mercer's type for superquadratic functions. *J. Inequal. Pure Appl. Math.* **2008**, *9*, 62.
4. Baric, J.; Matkovic, A. Bounds for the normalized Jensen Mercer functional. *J. Math. Inequal.* **2009**, 3, 529–541 [CrossRef]
5. Chu, H.H.; Rashid, S.; Hammouch, Z.; Chu, Y.M. New fractional estimates for Hermite-Hadamard-Mercer type inequalities. *Alex. Eng. J.* **2020**, *59*, 3079–3089 [CrossRef]
6. Ali, M.; Khan, A.R. Generalized integral Mercer's inequality and integral means. *J. Inequal. Spec. Funct.* **2019**, *10*, 60–76
7. Anjidani, E. Jensen-Mercer Operator Inequalities Involving Superquadratic Functions. *Mediterr. J. Math.* **2018**, *15*, 31. [CrossRef]

8. Anjidani, E.; Changalvaiy, M.R. Reverse Jensen-Mercer type operator inequalities. *Electron. J. Linear Algebra* **2016**, *31*, 87–99. [CrossRef]
9. Cho, Y.J.; Matic, M.; Pecaric, J. Two mappings in connection to Jensen inequality. *Pan-Am. Math. J.* **2002**, *12*, 43–50
10. Kian, M.; Moslehian, M.S. Refinements of the operator Jensen- Mercer inequality. *Electron. J. Linear Algebra* **2013**, *26*, 742–753 [CrossRef]
11. Moradi, H.R.; Furuichi, S. Improvement and Generalization of Some Jensen-Mercer-Type Inequalities. *J. Math. Inequal.* **2020**, *14*, 377–383. [CrossRef]
12. Srivastava, H.M.; Zhang, Z.H.; Wu, Y.D. Some further refinements and extensions of the Hermite-Hadamard and Jensen inequalities in several variables. *Math. Comput. Model.* **2011**, *54*, 2709–2717 [CrossRef]
13. Raina, R.K. On generalized Wright's hypergeometric functions and fractional calculus operators. *East Asian Math. J.* **2005**, *21*, 191–203.
14. Agarwal, R.P.; Luo, M.-J.; Raina, R.K. On Ostrowski type inequalities. *Fasc. Math.* **2016**, *204*, 5–27 [CrossRef]
15. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite-Hadamard Inequalities and Applications*; RGMIA Monographs; Victoria University: Footscray, Australia, 2000
16. Dragomir, S.S.; Agarwal, R.P. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95 [CrossRef]
17. Jarad, F.; Ugurlu, E.; Abdeljawad, T.; Baleanu, D. On a new class of fractional operators. *Adv. Differ. Equ.* **2017**, *2017*, 121–128. [CrossRef]
18. Mitrinovic, D.S.; Pecaric, J.E.; Fink, A.M. *Classical and New Inequalities in Analysis*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993
19. Srivastava, H.M.; Saxena, R.K. Operators of fractional integration and their applications. *Appl. Math. Comput.* **2001**, *118*, 1–52 [CrossRef]
20. Srivastava, H.M.; Harjule, P.; Jai, R. A general fractional differential equation associated with an integral operator with the H-function in the kernel. *Russ. J. Math. Phys.* **2015**, *22*, 112–126 [CrossRef]
21. Sarıkaya, M.Z.; Set, E.; Yaldız, H.; Başak, N. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407 [CrossRef]
22. Prabhakar, T.R. A singular integral equation with a generalized Mittag–Leffler function in the kernel. *Yokohama Math. J.* **1971**, *19*, 7–15.
23. Erdelyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill: NewYork, NY, USA; Toronto, ON, Canada; London, UK, 1953; Volume I
24. Öğülmüş, H.; Sarıkaya, M.Z. Hermite-Hadamard-Mercer Type Inequalities for Fractional Integrals. Available online: https://doi.org/10.13140/RG.2.2.30669.79844 (accessed on 12 April 2021).