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Article

Existence, Uniqueness, and E_q–Ulam-Type Stability of Fuzzy Fractional Differential Equation

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Abstract: This paper concerns with the existence and uniqueness of the Cauchy problem for a system of fuzzy fractional differential equation with Caputo derivative of order $q \in (1,2]$, ${}_0^C D_{0^+}^q u(t) = \lambda u(t) \oplus f(t, u(t)) \oplus B(t)C(t)$, $t \in [0,T]$ with initial conditions $u(0) = u_0$, $u'(0) = u_1$. Moreover, by using direct analytic methods, the E_q–Ulam-type results are also presented. In addition, several examples are given which show the applicability of fuzzy fractional differential equations.

Keywords: fuzzy fractional differential equations; Caputo derivative; fractional hyperbolic function; strongly generalized Hukuhara differentiability; Ulam-type stability

MSC: 34K37; 34B15

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1. Introduction

In real-life phenomena, numerous physical processes are used to present fractional-order sets that may change with space and time. The operations of differentiation and integration of fractional order are authorized by fractional calculus. The fractional order may be taken on imaginary and real values [1–3]. The theory of fuzzy sets is continuously drawing the attention of researchers. This is mainly due to its extended adaptability in various fields including mechanics, engineering, electrical, processing signals, thermal system, robotics, control, signal processing, and in several other areas [4–10]. Therefore, it has been a topic of increasing concern for researchers during the past few years.

Fuzzy fractional differential equations appeared for the first time in 2010 when an idea of the solution was initially proposed by Agarwal et al. [11]. However, the Riemann–Liouville H derivative based on the strongly generalizing Hukuhara differentiability [12,13] was defined by Allahviranloo and Salahshour [14,15]. They worked on solutions to Cauchy problems under this kind of derivative.

$$\begin{array}{l} {}_0^{RL}D_{a^+}^q u(t) = \lambda u(t) + f(t),\ t \in [a,b], \\ {}_0^{RL}D_{a^+}^{q-1} u(t) = u_0 \in \mathbb{E}^1 \end{array}.$$

In the above, $q \in (0,1]$, through using Laplace transforms [13] and Mittag–Leffler functions [12]. By using fractional hyperbolic functions and the properties of these functions, Chehlabi et al. obtain some new results [16]. More latest studies on fuzzy fractional differential equations can be found through references [17–22].

In 1940, Ulam promoted the Ulam stability. Lately, Hyers and Rassias used this concept of stability. Since then, in mathematical analysis and differential equations, the Ulam-type stability has had great significance. In fractional differential equations, E_α–Ulam-type stabilities were promoted by Wang in 2014 [23].

$$\begin{array}{l} {}_0^cD_t^\alpha u(t) + \lambda u(t) = f(t, u(t)),\ t \in [0, T] \\ u(0) = u_0 \in \mathbb{R} \end{array}.$$

In the above equation, $\alpha \in (0,1]$ and $\lambda > 0$. Shen studied the Ulam stability under the generalization of Hukuhara differentiability of a first-order linear fuzzy differential equation in 2015 [24]. Later, Shen et al. investigated the Ulam stability of a nonlinear fuzzy fractional equation with the help of fixed-point techniques in 2016 [25],

$${}_0^{RL}D_{0+}^q u(t) = \lambda u(t) \oplus f(t, u(t)),\ t \in [0, T],$$

by focusing on the initial condition

$${}_0^{RL}D_{0+}^{q-1} u(0) = u_0 \in \mathbb{E}^1,$$

where ${}_0^{RL}D_{0+}^q$ denoted Riemann–Liouville H derivative with respect to order $q \in (0,1]$, $f : (0,T] \times \mathbb{E}^1 \to \mathbb{E}^1$, $T \in \mathbb{R}_+$, and $\lambda \in \mathbb{R}$.

More results can be observed that are related to Ulam-type stability in [26–28]. Motivated by the above-cited papers, we aim to deal with fuzzy fractional differential equations of the form,

$${}_0^cD_{0+}^q u(t) = \lambda u(t) \oplus f(t, u(t)) \oplus B(t)C(t),\ t \in [0, T], \tag{1}$$

with initial conditions

$$u(0) = u_0, u'(0) = u_1, \tag{2}$$

Here, ${}_0^cD_{0+}^q$ denotes the Caputo derivative of order $q \in (1,2]$, $f : (0,T] \times \mathbb{E}^1 \to \mathbb{E}^1$, $T \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$.

This paper focuses on facilitating, with as few conditions as possible, to assure the uniqueness and existence of a solution to Cauchy problems (1) and (2). It establishes a link between fuzzy fractional differential equations and the Ulam-type stability, which enhances and generalizes some familiar outputs in the existing literature.

2. Basic Concepts

Assume that $P_k(\mathbb{R})$ denotes the collection of all nonempty convex and compact subsets of $\mathbb{R}$ and define sums and scalar products in $P_k(\mathbb{R})$ in the usual manner. Let A and B be two nonempty bounded subsets in $\mathbb{R}$. The distance between A and B is defined through the Hausdorff metric,

$$D(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} ||a - b||, \sup_{b \in B} \inf_{a \in A} ||a - b||\}.$$

In the above equality, $||x||$ stands for the usual Euclidean norm in $\mathbb{R}$. Now it is well known that the metric D turns the space $(P_k(\mathbb{R}), D)$ into a complete and separable metric space [26].

Denote

$$\mathbb{E}^1 = \{u : \mathbb{R} \to [0, 1] \mid u\ satisfies\ (1)\text{–}(4)\}$$

where (1)–(4) stands for the following properties of the function u:

(1) u is normal in the sense that there exists an $s_0 \in \mathbb{R}$ such that $u(s_0) = 2$;
(2) u is fuzzy convex, that is $u(qs + (1-q)y) \geqslant min\{u(s), u(y)\}$ for any $s, y \in \mathbb{R}$ and $q \in (1,2]$;
(3) u is an upper semicontinuous function on $\mathbb{R}$;
(4) The set $[u]^1$ defined by $[u]^1 = \overline{\{t \in \mathbb{R} | u(t) > 1\}}$ is compact.

For $1 < q \leqslant 2$, denote $[u]^q = \overline{\{t \in \mathbb{R} | u(t) \geqslant q\}}$. Now, from (1)–(4), it follows that the q-level set $[u]^q \in P_k(\mathbb{R})\ \forall\, 1 \leqslant q \leqslant 2$.

Define $\underline{u}$ as the lower branch and $\bar{u}$ as the upper branch of the fuzzy number $u \in \mathbb{E}^1$. The set $[u]^q = \{t \in \mathbb{R} | u(t) \geq q\} := [\underline{u}^q, \overline{u}^q]$ is known as the q-level set of fuzzy number u, where $q \in (1, 2]$. The length of q-level set is calculated as $\operatorname{diam}[u]^q = \overline{u}^q - \underline{u}^q$.

Lemma 1 ([29,30])**.** *If $u, v, s, y \in \mathbb{E}^1$, then*

(i) $(\mathbb{E}^1, D)$ *is a complete metric space;*
(ii) $D(u \oplus s, v \oplus s) = D(u, v)$;
(iii) $D(\lambda u, \lambda v) = |\lambda| D(u, v) \ \ \lambda \in \mathbb{R}$;
(iv) $D(u \oplus s, v \oplus y) \leqslant D(u, v) + D(s, y)$;
(v) $D(\lambda u, \mu u) = |\lambda - \mu| D(u, \hat{0}), \lambda, \mu \geqslant 0$.

Let $C^{\mathbb{E}}[a, b]$ and $L^{\mathbb{E}}[a, b]$ be spaces for all continuous and Lebesgue integrable fuzzy-valued functions on $[a, b]$, respectively. Moreover, $(C^{\mathbb{E}}[a, b], D)$ stands for the complete metric space, where

$$D(u, v) = \sup_{t \in [a,b]} d(u(t), v(t)).$$

Remark 1. *On $\mathbb{E}^1$, we can define the subtraction $\ominus$, called the H difference as follows: $u \ominus v$ makes sense if there exists $\omega \in \mathbb{E}^1$ such that $u = v \oplus \omega$. Then, by definition, $\omega = u \oplus v$.*

Let $u, v \in \mathbb{E}^1$ be such that $u \ominus v$ is well defined. Then, its q-level is determined by

$$[u \ominus v]^q = [\underline{u}^q - \underline{v}^q, \overline{u}^q - \overline{v}^q].$$

Through a generalization of the Hausdorff–Pompeiu metric on convex and compact sets, the metric D on $\mathbb{E}^1$ can be defined by

$$D(u, v) = \sup_{1 \leqslant q \leqslant 2} \max\{|\underline{u}^q - \underline{v}^q|, |\overline{u}^q - \overline{u}^q|\}.$$

Definition 1 ([13])**.** *Assume that $F \in C^{\mathbb{E}}(a, b] \bigcap L^{\mathbb{E}}(a, b]$. The fuzzy Riemann–Liouville integral for a fuzzy-valued function F is defined by*

$$\mathfrak{T}^q F(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{F(x)}{(t - x)^{1-q}} dx, \ t \in (a, b),$$

$q \in (1, 2]$. For $q = 1$ we obtain $\mathfrak{T}^1 F(t) = \int_a^t F(x) ds$, which is the classical fuzzy integral operator.

Definition 2 ([13])**.** *Assume that $F \in C^{\mathbb{E}}(a, b] \bigcap L^{\mathbb{E}}(a, b]$, $\ \ t_1 \in (a, b)$ and*

$$\phi(t) = \frac{1}{\Gamma(1 - q)} \int_a^t (t - x)^q F(x) dx.$$

It is said that F is Caputo H-differentiable of order $1 < q \leqslant 2$ at t_1, if there exists an element ${}_0^c D_t^q F(t_1) \in \mathbb{E}^1$ such that the following fuzzy equalities are valid:

(i) ${}_0^c D_t^q F(t_1) = \lim_{h \to 0^+} \frac{\phi(t_1 + h) \ominus \phi(t_1)}{h}$
(ii) ${}_0^c D_t^q F(t_1) = \lim_{h \to 0^+} \frac{\phi(t_1) \ominus \phi(t_1 - h)}{h}$
(iii) ${}_0^c D_t^q F(t_1) = \lim_{h \to 0^+} \frac{\phi(t_1) \ominus \phi(t_1 + h)}{-h}$
(iv) ${}_0^c D_t^q F(t_1) = \lim_{h \to 0^+} \frac{\phi(t_1 - h) \ominus \phi(t_1)}{-h}$

Here, we use only the first two cases [23]. These derivatives are trivial because they reduce to crisp elements. Regarding other fuzzy cases, the reader is referred to [23]. Furthermore, regarding this simplicity, a fuzzy-valued function F is called ^{c}[(i)-GH]-differentiable or ^{c}[(ii)-GH]-differentiable if it is differentiable according to concept (i) or to (ii) of Definition 2, respectively.

The Mittag–Leffler and fractional hyperbolic functions frequently occur in solutions to fractional systems; see, e.g., [16,23]. The Mittag–Leffler functions in the form of a single and a double parameter are defined by, respectively,

$$\begin{aligned} E_{\alpha}(x) &= \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)} \\ E_{\alpha,\beta}(x) &= \sum_{k=1}^{\infty} \frac{x^k}{\alpha k+\beta}. \end{aligned}$$

Some properties of these functions can be found in [31–33].

Lemma 2. *Let $\delta > 0$. Some properties of the functions $E_{\alpha}(.)$ and $E_{\alpha,\beta}(.)$ are listed below:*

(i) *Let $1 < \alpha < 2$. Then $E_{\alpha}(-\delta t^{\alpha}) \leqslant 2$ and $E_{\alpha,\alpha}(-\delta t^{\alpha}) \leqslant \frac{1}{\Gamma(\alpha)}$;*

(ii) *Let $1 < \alpha \leqslant 2$ and $\beta < \alpha + 1$. Then $E_{\alpha}(.)$ and $E_{\alpha,\beta}(.)$ are positive. If, moreover, $0 \leqslant t_2 \leqslant t_3$, then $E_{\alpha}(\delta t_2^{\alpha}) \leqslant E_{\alpha}(\delta t_3^{\alpha})$ and $E_{\alpha,\beta}(\delta t_2^{\alpha}) \leqslant E_{\alpha,\beta}(\delta t_3^{\alpha})$;*

(iii) $\int_0^z E_{\alpha,\beta}(t^{\alpha}) t^{\beta-1} dt = z^{\beta} E_{\alpha,\beta+1}(z^{\alpha}),\ \alpha > 1.$

Remark 2. *According to the lemma given above, it can be observed that $E_{\alpha,\alpha}(-s) \leqslant \frac{1}{\Gamma(\alpha)} \leqslant E_{\alpha,\alpha}(s)$ for $\alpha \in (1,2]$ and $s \in \mathbb{R}_+$. Fractional hyperbolic functions that are generalizations of standard hyperbolic functions can be defined through Mittag–Leffler functions (see, e.g., [16]) as follows:*

$$cosh_{\alpha,\beta}(s) = \sum_{k=0}^{\infty} \frac{s^{2k}}{\Gamma(2\alpha k+\beta)} = E_{2\alpha,\beta}(s^2),$$

$$sinh_{\alpha,\beta}(s) = \sum_{k=0}^{\infty} \frac{s^{2k+1}}{\Gamma(2\alpha k+\alpha+\beta)} = sE_{2\alpha,\alpha+\beta}(s^2),$$

for $\alpha, \beta > 1$. It is noticed that $cosh_{\alpha,\beta}(s)$ is an even function and that $sinh_{\alpha,\beta}(s)$, $s \in \mathbb{R}$, is an odd function. For $\alpha = \beta$, we write $Ch_{\alpha}(s)$ and $Sh_{\alpha}(s)$ instead of $cosh_{\alpha,\alpha}(s)$ and $sinh_{\alpha,\alpha}(s)$ respectively. It is not difficult to observe that $Ch_{\alpha}(s) + Sh_{\alpha}(s) = E_{\alpha,\alpha}(s)$ and $Ch_{\alpha}(s) - Sh_{\alpha}(s) = E_{\alpha,\alpha}(-s), s \in \mathbb{R}$ (see, e.g., [16]).

Remark 3. *According to the above arguments and Remark 2, we have $|Ch_{\alpha}(s) \pm Sh_{\beta}(s)| \leqslant E_{\alpha,\alpha}(|s|)$ for any $s \in \mathbb{R}$.*

Lemma 3. *(Gronwall lemma) [34] Let μ , $v \in C([0,1], \mathbb{R}_+)$. Suppose μ is increasing. If $s \in C([0,1], \mathbb{R}_+)$ obeys the inequality*

$$s(t) \leqslant \mu(t) + \int_0^t v(x)s(x)dx,\ t \in [0,1],$$

then

$$s(t) \leqslant \mu(t) exp\left(\int_0^t v(x)s(x)dx \right),\ t \in [0,1].$$

3. Existence and Uniqueness Results

In this part, existence and uniqueness of solutions to the Cauchy problem in (1) and (2) are discussed. We can start with the lemma given below.

Lemma 4 ([16])**.** *When $\lambda > 1$, the ^{c}[(i)-GH]-differentiable solution to problem (1) is given by*

$$u(t) = E_{q,1}(-\lambda t^q)u_0 \oplus t E_{q,2}(-\lambda t^q)u_1 \oplus \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)f(x)}{(t-x)^{1-q}} dx;$$

when $\lambda < 1$, the ^{c}[(ii)-GH]-differentiable solution to problem (1) is given by

$$u(t) = E_{q,1}(-\lambda t^q)u_0 \ominus (-1)tE_{q,2}(-\lambda t^q)u_1 \ominus (-1)\int_0^t \frac{E_{q,q}(\lambda(t-x)^q)f(x)}{(t-x)^{1-q}}dx;$$

when $\lambda < 1$, the ^{c}[(i)-GH]-differentiable solution to problem (1) is given by

$$u(t) = [Ch_{q,1}(-\lambda t^q)u_0 \oplus Sh_{q,1}(-\lambda t^q)u_0] \oplus [tCh_{q,2}(-\lambda t^q)u_1 \oplus tSh_{q,2}(-\lambda t^q)u_1] \oplus \int_0^t \frac{Ch_q\lambda f(x) \oplus Sh_q\lambda f(x)}{(t-x)^{1-q}}dx;$$

when $\lambda > 1$, the ^{c}[(ii)-GH]-differentiable solution to problem (1) is given by

$$\begin{aligned} u(t) \;=\; & [Ch_{q,1}(-\lambda t^q)u_0 \ominus (-1)Sh_{q,1}(-\lambda t^q)u_0] \ominus (-1)[tCh_{q,2}(-\lambda t^q)u_1 \ominus (-1)tSh_{q,2}(-\lambda t^q)u_1] \ominus (-1) \\ & \int_0^t \frac{Ch_q\lambda f(x) \ominus (-1)Sh_q\lambda f(x)}{(t-x)^{1-q}}dx; \end{aligned}$$

when $\lambda = 1$, the ^{c}[(i)-GH]-differentiable solution to problem (1) is given by

$$u(t) = u_0 \oplus tu_1 \oplus \int_0^t \frac{f(x)}{(t-x)^{1-q}}dx;$$

when $\lambda = 1$, the ^{c}[(ii)-GH]-differentiable solution to problem (1) is given by

$$u(t) = u_0 \ominus (-1)tu_1 \ominus (-1)\int_0^t \frac{f(x)}{(t-x)^{1-q}}dx;$$

Remark 4. *If $\lambda = 0$, then problem (1) reduces to*

$$\begin{aligned} {}_0^cD_t^q u(t) &= f(t), \\ {}_0^cD_{0^+}^{q-1} u(0) &= u_0 \in \mathbb{E}^1, \\ {}_0^cD_{0^+}^{q-1} u'(0) &= u_1 \end{aligned}$$

By applying Lemma 4 and Remark 4 with $f(t,u(t)) \oplus B(t)C(t)$ instead of $f(t)$, it follows that the Cauchy problem in (1) and (2) possesses an integral version. In case $\lambda \geqslant 1$ and the function $t \mapsto u(t), t \in [0,T]$ is assumed to be ^{c}[(i)-GH]-differentiable, then the function u satisfies

$$u(t) = E_{q,1}(-\lambda t^q)u_0 \oplus tE_{q,2}(-\lambda t^q)u_1 \oplus (-1)\int_0^t \frac{E_{q,q}(\lambda(t-x)^q)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx.$$

In case $\lambda \leqslant 1$ and the function $t \mapsto u(t)$ is supposed to be ^{c}[(ii)-GH]-differentiable, then the function u satisfies

$$u(t) = E_{q,1}(-\lambda t^q)u_0 \ominus (-1)tE_{q,2}(-\lambda t^q)u_1 \ominus (-1)\int_0^t \frac{E_{q,q}(\lambda(t-x)^q)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx.$$

In case $\lambda < 1$ and the function $t \mapsto u(t)$ is ^{c}[(i)-GH]-differentiable, then the function u satisfies

$$\begin{aligned} u(t) \;=\; & [Ch_{q,1}(-\lambda t^q)u_0 \oplus Sh_{q,1}(-\lambda t^q)u_0] \oplus [tCh_{q,2}(-\lambda t^q)u_1 \oplus tSh_{q,2}(-\lambda t^q)u_1] \\ & \oplus \int_0^t \frac{Ch_q\lambda[f(x,u(x)) + B(x)C(x)] \oplus Sh_q\lambda[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx. \end{aligned}$$

In case $\lambda > 1$ and the function $t \mapsto u(t)$ is ^{c}[(ii)-GH]-differentiable, then the function u satisfies

$$\begin{aligned} u(t) &= [Ch_{q,1}(-\lambda t^q)u_0 \ominus (-1)Sh_{q,1}(-\lambda t^q)u_0] \ominus (-1)[tCh_{q,2}(-\lambda t^q)u_1 \ominus (-1)tSh_{q,2}(-\lambda t^q)u_1] \\ &\oplus \int_0^t \frac{Ch_q\lambda[f(x,u(x)) + B(x)C(x)] \ominus (-1)Sh_q\lambda[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx. \end{aligned}$$

We should formulate the basic assumptions before initiating our main work:

(H_1) The function $f : [0,T] \times \mathbb{E}^1 \to \mathbb{E}^1$ is continuous;

(H_2) There exists a finite constant $L > 0$ such that for all $t \in [0,T]$ and for all $u, v \in \mathbb{E}^1$ the inequality

$$D(f(t,u), f(t,v)) \leqslant LD(u,v)$$

is valid and such that $\lambda \in \mathbb{R}$ is satisfied;

(H_3) $LT^qE_{(q,q+1)}(|\lambda|T^q) < 1$.

Theorem 1. *Let $\lambda \geqslant 1$ and suppose that the conditions (H_1)–(H_3) are satisfied. Then, the Cauchy problem (1) and (2) has a unique ^{c}[(i)-GH]-differentiable solution u in $C^{\mathbb{E}}[0,T]$.*

Proof. Let the operator $P_1 : C^{\mathbb{E}}[0,T] \to C^{\mathbb{E}}[0,T]$ be defined as

$$P_1u(t) = E_{q,1}(-\lambda t^q)u_0 \oplus tE_{q,2}(-\lambda t^q)u_1 \oplus \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx$$

It is not difficult to see that u is a ^{c}[(i)-GH]-differentiable solution for Cauchy problem (1) and (2) if and only if $u = P_1u$. Let u and v belong to $\mathbb{E}^1$. From the above Lemmas 1 and 2 we infer

$$\begin{aligned} D(P_1u(t), P_1v(t)) &= D\Bigg[\int_0^t \frac{E_{q,q}(\lambda(t-x)^q)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx, \\ &\quad \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)[f(x,v(x)) + B(x)C(x)]}{(t-x)^{1-q}}ds\Bigg] \\ &\leqslant LD(u,v) \oplus \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)[f(x,u(x)) + B(x)C(x), f(x,v(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx \\ &\leqslant LD(u,v) \oplus \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)[f(x,u(x)), f(x,v(x))]}{(t-x)^{1-q}}dx \\ &\leqslant LD(u,v) \oplus L\int_0^t \frac{E_{q,q}(\lambda(t-x)^q)f(u(x),v(x))}{(t-x)^{1-q}}dx \\ &\leqslant LD(u,v) \oplus LD(u,v)\int_0^t \frac{E_{q,q}(\lambda(t-x)^q)}{(t-x)^{1-q}}dx \\ &= LD(u,v) \oplus t^qE_{q,q+1}(\lambda t^q)LD(u,v) \end{aligned}$$

for $u, v \in \mathbb{E}^1$, and for all $t \in [0,T]$, which means that

$$D(P_1u, P_1v) \leqslant L[1 \oplus t^qE_{q,q+1}(|\lambda|T^q)]D(u,v)$$

Thus, the Banach contraction mapping (BCM) principle shows the operator P_1 has a unique fixed point $u^* \in C^{\mathbb{E}}[0,T]$. It represents the unique ^{c}[(i)-GH]-differentiable solution to the Cauchy problem (1) and (2). □

Theorem 2. *Let $\lambda \leqslant 1$ and suppose the conditions (H_1)–(H_3) are satisfied. Assume that (H_4) for any $t \in (0,T]$,*

$$E_{q,1}(\lambda t^q)\underline{u_0^\alpha} + tE_{q,2}(-\lambda t^\alpha)\underline{u_1^\alpha} + \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)\overline{[f(x,u(x)) + B(x)C(x)]^\alpha}}{(t-x)^{1-q}}dx$$

is non-decreasing in α,

$$E_{q,1}(\lambda t^q)\overline{u_0^\alpha} + tE_{q,2}(-\lambda t^q)\overline{u_1^\alpha} + \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)\overline{[f(x,u(x)) + B(x)C(x)]^\alpha}}{(t-x)^{1-q}}dx$$

is non-increasing in α, and for any $q \in [1,2]$ and $t \in (0,T]$

$$\begin{aligned} tE_q(-\lambda t^q)u_1^\alpha + \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)diam[f(x,u(x)) + B(x)C(x)]^\alpha}{(t-x)^{1-q}}dx \\ \leqslant t^q E_{q,2}(-\lambda t^q)diam[u_1]^\alpha + t^{q-1}E_{q,1}(\lambda t^q)diam[u_0]^\alpha. \end{aligned}$$

Then, the Cauchy problem (1) and (2) has a unique ^{c}[(ii)-GH]-differentiable solution in $C^{\mathbb{E}}[0,T]$.

Proof. Let the operator $P_2 : C^{\mathbb{E}}[0,T] \to C^{\mathbb{E}}[0,T]$ be defined by

$$P_2u(t) = E_{q,1}(\lambda t^q)u_0 \ominus E_{q,2}(\lambda t^q)u_1 \ominus (-1)\int_0^t \frac{E_{q,q}(\lambda(t-x)^q)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx$$

For condition (H_4) and [35], we know that P_2 is well defined on $C^{\mathbb{E}}[0,T]$. Moreover, it is not difficult to see that u is a ^{c}[(ii)-GH]-differentiable solution for Cauchy problem (1) and (2) if and only if $u = P_2u$. Let u and v belong to $C^{\mathbb{E}}[0,T]$. From the above Lemmas 1 and 2 and Remark 2 we infer

$$\begin{aligned} D(P_2u(t), P_2v(t)) &\leqslant LD(u,v) \ominus (-1)LD(u,v)\int_0^t \frac{E_{q,q}(\lambda(t-x)^q)}{(t-x)^{1-q}}dx \\ &\leqslant LD(u,v) \ominus (-1)LD(u,v)\int_0^t \frac{E_{q,q}(|\lambda|(t-x)^q)}{(t-x)^{1-q}}dx \\ &= LD(u,v) \ominus (-1)t^q E_{q,q+1}(|\lambda|t^q)LD(u,v) \end{aligned}$$

for $u, v \in \mathbb{E}^1$ and for all $t \in [0,T]$, which means that

$$D(P_2u(t), P_2v(t)) \leqslant LD(u,v) \ominus (-1)Lt^q E_{q,q+1}(|\lambda|t^q)D(u,v)$$

Thus, the Banach contraction mapping (BCM) principle shows the operator P_2 has a unique fixed point $u^* \in C^{\mathbb{E}}[0,T]$. It represents the unique ^{c}[(ii)-GH]-differentiable solution to the Cauchy problem (1) and (2). Now, the proof is completed. □

Theorem 3. *Let $\lambda < 1$, and suppose that the conditions (H_1)–(H_3) are satisfied. Then, the Cauchy problem (1) and (2) has a ^{c}[(i)-GH]-differentiable solution u in $C^{\mathbb{E}}[0,T]$.*

Proof. Let the operator $P_3 : C^{\mathbb{E}}[0,T] \to C^{\mathbb{E}}[0,T]$ be defined as

$$\begin{aligned} P_3u(t) = & [Ch_{q,1}(\lambda t^q)u_0 \oplus Sh_{q,1}(\lambda t^q)u_0] \oplus t[Ch_{q,2}(\lambda t^q)u_1 \oplus Sh_{q,2}(\lambda t^q)u_1] \\ & \oplus \int_0^t \frac{C_q^\lambda(t,x)[f(x,u(x)) + B(x)C(x)] \oplus S_q^\lambda(t,x)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx, \end{aligned}$$

$t \in [0,T]$. It is not difficult to see that u is a ^{c}[(i)-GH]-differentiable solution for Cauchy problem (1) and (2) if and only if $u = P_3 u$. Let u and v belong to $C^{\mathbb{E}}[0,T]$. From the above Lemmas 1 and 2 and Remarks 2 and 3 we deduce

$$\begin{aligned} D(P_3u(t), P_3v(t)) &\leqslant LD(u,v) \oplus LD(u,v) \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)}{(t-x)^{1-q}} dx \\ &\leqslant LD(u,v) \oplus LD(u,v) \int_0^t \frac{E_{q,q}(|\lambda|(t-x)^q)}{(t-x)^{1-q}} dx \\ &= LD(u,v) \oplus t^q E_{q,q+1}(|\lambda| t^q) LD(u,v) \end{aligned}$$

For $u, v \in E^1$ and for all $t \in [0,T]$, which signifies as

$$D(P_3u(t), P_3v(t)) \leqslant LD(u,v) \oplus Lt^q E_{q,q+1}(|\lambda| t^q) D(u,v)$$

Thus, the Banach contraction mapping (BCM) principle shows the operator P_3 has a unique fixed point $u^* \in C^{\mathbb{E}}[0,T]$. It represents the unique ^{c}[(i)-GH]-differentiable solution to the Cauchy problem (1) and (2). Now the proof is done. □

Theorem 4. *Let $\lambda > 0$ and suppose that the conditions (H_1)–(H_3) are satisfied. Assume that (H_5) for all $t \in (0,T]$ the functions*

$$\begin{aligned} \xi_1(t,\alpha) &= Ch_{q,1}(\lambda t^q)\underline{u_0}^{\alpha} + Sh_{q,1}(\lambda t^q)\overline{u_0}^{\alpha} \\ \mu_1(t,\alpha) &= tCh_{q,2}(\lambda t^q)\underline{u_1}^{\alpha} + tSh_{q,2}(\lambda t^q)\overline{u_1}^{\alpha} \end{aligned}$$

is non-decreasing in α. in addition, the function

$$\begin{aligned} \xi_2(t,\alpha) &= Ch_{q,1}(\lambda t^q)\overline{u_0}^{q} + Sh_{q,1}(\lambda t^q)\underline{u_0}^{\alpha} \\ \mu_2(t,\alpha) &= tCh_{q,2}(\lambda t^q)\overline{u_1}^{q} + tSh_{q,2}(\lambda t^q)\underline{u_1}^{\alpha} \end{aligned}$$

are non-increasing in α. Furthermore, assume (H_6) for all $t \in (0,T]$, the function

$$\psi_1(t,x,\alpha) = Ch_q(\lambda(t-x)^q)\underline{[f(x,u(x)) + B(x)C(x)]^{\alpha}} + Sh_q(\lambda(t-x)^q)\overline{[f(x,u(x)) + B(x)C(x)]^{\alpha}}$$

is non-decreasing in α. In addition, the function

$$\psi_2(t,x,\alpha) = Ch_q(\lambda(t-x)^q)\overline{[f(x,u(x)) + B(x)C(x)]^{\alpha}} + Sh_q(\lambda(t-x)^q)\underline{[f(x,u(x)) + B(x)C(x)]^{\alpha}}$$

is non-increasing in α. In addition, the function (H_7) for all $t \in (0,T]$

$$\xi_1(t,\alpha) + \mu_1(t,\alpha) + \int_0^t \frac{\psi_1(t,x,\alpha)}{(t-x)^{1-q}} dx$$

is non-decreasing in α, the expression

$$\xi_2(t,\alpha) + \mu_2(t,\alpha) + \int_0^t \frac{\psi_2(t,x,\alpha)}{(t-x)^{1-q}} dx$$

is non-increasing in α, and for all $q \in (1,2]$ and $t \in (0,T]$,

$$\int_0^t \frac{E_{q,q}(-\lambda(t-x)^q) diam[f(x,u(x)) + B(x)C(x)]^{\alpha}}{(t-x)^{1-q}} dx \leqslant t^{q-1} E_{q,q}(-\lambda t^q) diam[u_0]^q$$

Then, the Cauchy problem (1) and (2) has a unique ^{c}[(ii)-GH]-differentiable solution for $C^{\mathbb{E}}[0,T]$.

Proof. Let the operator $P_4 : C^{\mathbb{E}}[0,T] \to C^{\mathbb{E}}[0,T]$ be defined as

$$\begin{aligned} P_4u(t) &= [Ch_{q,1}(\lambda t^q)u_0 \ominus (-1)Sh_{q,1}(\lambda t^q)u_0] \ominus (-1)t[Ch_{q,2}(\lambda t^q)u_1 \ominus (-1)Sh_{q,2}(\lambda t^q)u_1] \\ &\ominus(-1)\int_0^t \frac{C_q^\lambda(t,x)[f(x,u(x)) + B(x)C(x)] \ominus (-1)S_q^\lambda(t,x)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx \end{aligned}$$

According to conditions (H_5)–(H_7) and [21], it is known that P_4 is well illustrated on $C^{\mathbb{E}}[0,T]$. From the above Lemmas 1, 2, and Remark 2,

$$\begin{aligned} D(P_4u(t), P_4v(t)) &\leqslant LD(u,v)\int_0^t \frac{E_{q,q}(\lambda(t-x)^q)}{(t-x)^{1-q}}dx \\ D(P_4u(t), P_4v(t)) &= t^q E_{q,q+1}(\lambda t^q)LD(u,v) \end{aligned}$$

for $u, v \in \mathbb{E}^1$ and for all $t \in [0,T]$, which means that

$$D(P_4u(t), P_4v(t)) \leqslant Lt^q E_{q,q+1}(|\lambda|t^q)D(u,v)$$

Thus, the Banach contraction mapping (BCM) principle shows the operator P_4 has a unique fixed point $u^* \in C^{\mathbb{E}}[0,T]$. It represents the unique ^{c}[(ii)-GH]-differentiable solution to the Cauchy problem (1) and (2). □

4. Stability Results

In various studies, E_α–Ulam-type stability approaches regarding fractional differential equations [23] and Ulam-type stability approaches regarding fuzzy differential equations [24,25] were established. Afterward, Yupin Wang and Shurong Sun worked on E_q–Ulam-type stability concepts regarding fuzzy fractional differential equation where $q \in (0,1]$. We offer some new E_q–Ulam-type stability concepts regarding fuzzy fractional differential equation where $q \in (1,2]$.

Assume that $\gamma > 0$ is a constant and that $t \mapsto \zeta(t),\ t \in [0,T]$ is a positive continuous function. In addition, suppose that $t \mapsto u(t), t \in [0,T]$ is a continuous function that solves the equation in (1) and consider the following related inequalities:

$$D({}_0^cD_t^q u(t), \lambda u(t) \oplus (f(t,u(t)) \oplus B(t)C(t)) \leqslant \gamma, \tag{3}$$

$$D({}_0^cD_t^q u(t), \lambda u(t) \oplus (f(t,u(t)) \oplus B(t)C(t)) \leqslant \zeta(t), \tag{4}$$

$$D({}_0^cD_t^q u(t), \lambda u(t) \oplus (f(t,u(t)) \oplus B(t)C(t)) \leqslant \gamma\zeta(t), \tag{5}$$

where $t \in [0,T]$.

Definition 3. *Equation (1) is called E_q–Ulam–Hyers stable in case there exist a finite constant $c > 1$ and a function $v \in C^{\mathbb{E}}[0,T]$ that satisfies the equation in (1) such that for all $\gamma > 1$ and for all solutions $u \in C^{\mathbb{E}}[0,T]$ of Equation (1) that satisfy the inequality in (3), the following inequality is valid:*

$$D(u(t), v(t)) \leqslant cE_q(\xi f t^q)\gamma,\ \xi f \geqslant 1,\ t \in [0,T].$$

Definition 4. *Equation (1) is called E_q–Ulam–Hyers stable in case there exist a continuous function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\vartheta(1) = 1$ and a function $v \in C^{\mathbb{E}}[0,T]$ that satisfies the equation in (1) and for all solutions $u \in C^{\mathbb{E}}[0,T]$ of Equation (1) that satisfy the inequality in (3), the following inequality is valid:*

$$D(u(t), v(t)) \leqslant \vartheta(\gamma)E_q(\xi f t^q),\ \xi f \geqslant 1,\ t \in [0,T].$$

Definition 5. *Equation (1) is called E_q–Ulam–Hyers–Rassias stable in case with respect to ζ, when there exist $c_\zeta > 1$ and a function $v \in C^{\mathbb{E}}[0,T]$ that satisfies the equation in (1) such that for* *all $\gamma > 1$ and for all solutions $u \in C^{\mathbb{E}}[0,T]$ of the equation in (1) that satisfy the inequality in (5), the following inequality is valid:*

$$D(u(t), v(t)) \leqslant c_\zeta \gamma \zeta(t) E_q(\xi f t^q),\ \xi f \geqslant 1,\ t \in [0,T].$$

Definition 6. *Equation (1) is called E_q–Ulam–Hyers–Rassias stable in case with respect to ζ if there exist $c_\zeta > 1$ and a function $v \in C^{\mathbb{E}}[0,T]$ that satisfies the equation in (1) such that for all $\gamma > 1$ and for all solutions $u \in C^{\mathbb{E}}[0,T]$ of the equation in (1) that satisfy the inequality in (4), the following inequality is valid:*

$$D(u(t), v(t)) \leqslant c_\zeta \zeta(t) E_q(\xi f t^q),\ \xi f \geqslant 1,\ t \in [0,T].$$

Lemma 5. *The function $u \in C^{\mathbb{E}}[0,T]$ with the property that (H_8) ${}_0^cD_t^q u(t) \ominus [\lambda u(t) \oplus (f(t,u(t))$ $\oplus B(t)C(t))]$ exists in $\mathbb{E}^1$ for all $t \in [0,T]$ satisfies the inequality (3) if and only if there exists a function $h \in C^{\mathbb{E}}[0,T]$ such that*

(i) $D\left(h(t), \hat{0}\right) \leqslant \gamma$, *for all $t \in (0,T]$,*

and the function $u \in C^{\mathbb{E}}[0,T]$ itself satisfies

(ii) ${}_0^cD_t^q u(t) = \lambda u(t) \oplus (f(t,u(t)) \oplus B(t)C(t))$, *for all $t \in (0,T]$.*

Proof. The sufficiency begins obviously, and we will only prove the necessity. From condition (H_8), we observe that the function $t \mapsto h(t), t \in [0,T]$, defined by

$$h(t) = {}_0^c D_t^q u(t) \ominus [\lambda u(t) \oplus (f(t,u(t)) \oplus B(t)C(t))]$$

belongs to $C^{\mathbb{E}}[0,T]$ and that h(t) belongs to $\mathbb{E}^1$ for all $t \in (0,T]$. Therefore, it follows that the equation in (ii) is satisfied. Additionally, we have

$$\begin{aligned} & D({}_0^cD_t^q u(t), \lambda u(t) \oplus (f(t,u(t)) \oplus B(t)C(t))) \\ = \ & D\left({}_0^cD_t^q u(t) \ominus [\lambda u(t) \oplus f(t,u(t)) \oplus B(t)C(t)], \hat{0}\right) = D\left(h(t), \hat{0}\right). \end{aligned}$$

From the inequality (3), it then follows that $D\left(h(t), \hat{0}\right) \leqslant \gamma$, and therefore, (i) is satisfied. This completes the proof of Lemma 5. □

Remark 5. *Similar results as in Lemma 5 can be obtained by using the inequalities in (4) and (5).*

Lemma 6. *Let $u(t)$ be a ${}^c[(i)$-GH]-differentiable function that solves the Cauchy equation in (1) and (2) and satisfies the inequality in (3) and is such that ${}_0^cD_t^q u(0) = u_0$. Let the condition in (H_8) be satisfied. Then, for every $t \in [0,T]$, the function $u(t)$ satisfies the inequality*

$$D(u(t), G_1(f,t)) \leqslant \gamma E_{q,q}(|\lambda| t^q) \int_0^t \frac{\zeta(x)}{(t-x)^{1-q}} dx$$

when $\lambda > 1$, and

$$D(u(t), G_2(f,t)) \leqslant \gamma E_{q,q}(|\lambda| t^q) \int_0^t \frac{\zeta(x)}{(t-x)^{1-q}} dx$$

when $\lambda < 1$, and $t \in [0,T]$. Here, the functions $G_1(f,t)$ and $G_2(f,t)$ are defined by

$$\begin{aligned} G_1(f,t) &= E_{q,1}(\lambda t^q) u_0 \oplus t E_{q,2}(\lambda t^q) u_1 \oplus \int_0^t \frac{E_{q,q}(\lambda (t-x)^q)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}} dx \\ G_2(f,t) &= [Ch_{q,1}(\lambda t^q) u_0 \oplus Sh_{q,1}(\lambda t^q) u_0] \oplus [t Ch_{q,2}(\lambda t^q) u_1 \oplus t Sh_{q,2}(\lambda t^q) u_1] \oplus \\ & \int_0^t \frac{C_q^\lambda(t,x)[f(x,u(x)) + B(x)C(x)] \oplus S_q^\lambda(t,x)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}} dx \end{aligned}$$

Proof. Since the function $u \in \mathcal{C}^{\mathbb{E}}[0,T]$ is a solution to the Cauchy problem (1) and (2), we infer

$$\begin{cases} {}_0^cD_t^{\alpha} u(t) = \lambda u(t) \oplus [f(t,u(t)) + B(t)C(t)], \\ u(0) = u_0 \\ u'(0) = u_1 \end{cases}. \tag{6}$$

Now, regarding clarity, the proof can be divided into two cases.

Case 1.

Suppose $\lambda > 1$. Then, we write

$$C_1(B,C,t) = \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)B(x)C(x)}{(t-x)^{1-q}}dx$$

Observing that u is a ^{c}[(i)-GH]-differentiable solution of Equation (6), then Lemma 4 with $f(t,u(t)) + B(t)C(t)$ instead of $f(t)$ shows the equality

$$\begin{aligned} u(t) &= E_{q,1}(\lambda t^q)u_0 \oplus tE_{q,2}(\lambda t^q)u_1 \oplus \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)(f(x,u(x)) + B(x)C(x))}{(t-x)^{1-q}}dx \\ &= E_{q,1}(\lambda t^q)u_0 \oplus tE_{q,2}(\lambda t^q)u_1 \oplus \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)f(x,u(x))}{(t-x)^{1-q}}dx \\ &\quad \oplus \frac{E_{q,q}(\lambda(t-x)^q)B(x)C(x)}{(t-x)^{1-q}}dx \\ &= G_1(f,t) \oplus C_1(B,C,t) \end{aligned}$$

Now, it follows that

$$\begin{aligned} D(u(t),G_1(f,t)) &= D(u(t) \oplus C_1(B,C,t), G_1(f,t) \oplus C_1(B,C,t)) \\ &= D(u(t) \oplus C_1(B,C,t), u(t)) \\ &= D(C_1(B,C,t), \hat{0}) \\ &= D\left(\int_0^t \frac{E_{q,q}(\lambda(t-x)^q)B(x)C(x), \hat{0}}{(t-x)^{1-q}}\right)dx \\ &= \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)D(B(x)C(x),\hat{0})}{(t-x)^{1-q}}dx \\ &\leqslant \gamma E_{q,q}(|\lambda|t^q)\int_0^t \frac{\zeta(x)}{(t-x)^{1-q}}dx \end{aligned}$$

Case 2.

When $\lambda < 1$, we denote

$$C_2(B,C,t) = \int_0^t \frac{C_q^{\lambda}(t,x)B(x)C(x) \oplus S_q^{\lambda}(t,x)B(x)C(x)}{(t-x)^{1-q}}dx$$

It should be observed that $u(t)$ is a ^{c}[(ii)-GH]-differentiable solution of Equation (6) that obeys the inequality in (3). An application of Lemma 5 then yields

$$\begin{aligned} u(t) &= [Ch_{q,1}(\lambda t^q)u_0 \oplus Sh_{q,1}(\lambda t^q)u_0] \oplus [tCh_{q,2}(\lambda t^q)u_1 \oplus tSh_{q,2}(\lambda t^q)u_1] \oplus \\ &\quad \int_0^t \frac{C_q^{\lambda}(t,x)[f(x,u(x)) + B(x)C(x)] \oplus S_q^{\lambda}(t,x)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx \\ &= G_2(f,t) \oplus C_2(B,C,t). \end{aligned}$$

Now, it follows that

$$\begin{aligned} D(u(t), G_2(f,t)) &= D(C_2(B,C,t), \hat{0}) \\ &\leqslant \int_0^t \frac{D(C_q^{\lambda}(t,x)[B(x)C(x)] \oplus S_q^{\lambda}(t,x)[B(x)C(x)], \hat{0})}{(t-x)^{1-q}} dx \\ &\leqslant \int_0^t \frac{E_{q,q}(|\lambda|t-x^q)D(B(x)C(x)), \hat{0})}{(t-x)^{1-q}} dx \\ &\leqslant \gamma E_{q,q}(|\lambda|t^q) \int_0^t \frac{\zeta(x)}{(t-x)^{1-q}} dx \end{aligned}$$

Now, the proof is completed. □

Lemma 7. *Let $u(t)$ be a ^{c}[(ii)-GH]-differentiable function that solves the Cauchy equation in (1) and (2) and satisfies the inequality (3) and is such that ${}_0^cD_t^q u(0) = u_0$. Let the condition in (H_8) be satisfied. Then, for every $t \in [0,T]$ the function $u(t)$ satisfies the integral inequality*

$$D(u(t), G_3(f,t)) \leqslant \gamma E_{q,q}(|\lambda|t^q) \int_0^t \frac{\zeta(x)}{(t-x)^{1-q}} dx$$

when $\lambda < 1$, and

$$D(u(t), G_4(f,t)) \leqslant \gamma E_{q,q}(|\lambda|t^q) \int_0^t \frac{\zeta(x)}{(t-x)^{1-q}} dx$$

when $\lambda > 1$ and $t \in [0,T]$. Here, the functions $G_3(f,t)$ and $G_4(f,t)$ are defined by

$$\begin{aligned} G_3(f,t) &= E_{q,1}(\lambda t^q)u_0 \ominus (-1)tE_{q,2}(\lambda t^q)u_1 \ominus (-1) \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}} dx \\ G_4(f,t) &= [Ch_{q,1}(\lambda t^q)u_0 \ominus (-1)Sh_{q,1}(\lambda t^q)u_0] \ominus (-1)[tCh_{q,2}(\lambda t^q)u_1 \ominus (-1)tSh_{q,2}(\lambda t^q)u_1] \\ &\ominus(-1) \int_0^t \frac{C_q^{\lambda}(t,x)[f(x,u(x)) + B(x)C(x)] \ominus (-1)S_q^{\lambda}(t,x)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}} dx \end{aligned}$$

Proof. Now, regarding clarity, the proof can be divided into two cases.

Case 1.

When $\lambda < 1$, observe that u is a ^{c}[(ii)-GH]-differentiable solution of Equation (5), then the Lemma 5 with $f(x,u(x)) + B(x)C(x)$ instead of $f(t)$ shows the equality

$$u(t) = G_3(f,t) \ominus (-1)C_1(B,C,t).$$

Now, it follows that

$$\begin{aligned} D(u(t), G_3(f,t)) &= D(\hat{0}, -C_1(g,t)) \\ &\leqslant \int_0^t \frac{E_{q,q}(|\lambda|(t-x)^q)D(B(x)C(x), \hat{0})}{(t-x)^{1-q}} dx \\ &\leqslant \gamma E_{q,q}(|\lambda|t^q) \int_0^t \frac{\zeta(t)}{(t-x)^{1-q}} dx \end{aligned}$$

Case 2.

Suppose $\lambda > 1$. Then, we denote

$$C_3(B,C,t) = \int_0^t \frac{C_q^{\lambda}(t,x)B(x)C(x) \ominus (-1)S_q^{\lambda}(t,x)B(x)C(x)}{(t-x)^{1-q}} dx$$

Observing that $u(t)$ is a ^{c}[(ii)-GH]-differentiable solution of Equation (5), then Lemma 5 with $f(t,u(t)) + B(t)C(t)$ instead of $f(t)$ shows the equality

$$\begin{aligned} u(t) &= [Ch_{q,1}(\lambda t^q)u_0 \ominus (-1)Sh_{q,1}(\lambda t^q)u_0] \ominus (-1)t[Ch_{q,2}(\lambda t^q)u_1 \oplus Sh_{q,2}(\lambda t^q)u_1] \ominus \\ &\quad (-1)\int_0^t \frac{C_q^\lambda(t,x)[f(x,u(x)) + B(x)C(x)] \oplus S_q^\lambda(t,x)[f(x,u(x)) + B(x)C(x)]}{(t-x)^{1-q}}dx \\ &= G_4(f,t) \ominus (-1)C_3(B,C,t). \end{aligned}$$

Now, it follows that

$$\begin{aligned} D(u(t), G_4(f,t)) &= D(C_3(B,C,t), \hat{0}) \\ &\leqslant \int_0^t \frac{D(C_q^\lambda(t,x)[B(x)C(x)] \ominus (-1)S_q^\lambda(t,x)[B(x)C(x)], \hat{0})}{(t-x)^{1-q}}dx \\ &\leqslant \int_0^t \frac{E_{q,q}(|\lambda|t-x^q)D(B(x)C(x)), \hat{0})}{(t-x)^{1-q}}dx \\ &\leqslant \gamma E_{q,q}(|\lambda|t^q)\int_0^t \frac{\zeta(x)}{(t-x)^{1-q}}dx \end{aligned}$$

Now, the proof is completed. □

Remark 6. *We can obtain similar results to those in Lemmas 6 and 7 for inequalities (3) and (4).*

Theorem 5. *Suppose $\lambda \geqslant 1$, condition (H_1)–(H_3) are satisfied, and the following condition holds (H_9); there exists a positive, increasing, and continuous function ζ such that*

$$E_{q,q}(|\lambda|t^q)\int_0^t \frac{\zeta(x)}{(t-x)^{1-q}}dx \leqslant c_\zeta \zeta(t),\ t \in [0,T].$$

Assume further that ^{c}[(i)-GH]-differentiable function u satisfied the inequality (5) with the function ζ in (H_9) and that u satisfies condition (H_8). Then, Equation (1) is E_q–Ulam–Hyers–Rassias stable.

Proof. According to Theorem 1, u is a ^{c}[(i)-GH]-differentiable solution to Cauchy problem (1) and (2). Let u be a ^{c}[(i)-GH]-differentiable solution to Equation (1), which satisfies inequality (5) with $u(0) = u_0$. From Lemma 6, we obtain

$$\begin{aligned} D(u(t), G_1(f,t)) &\leqslant \gamma E_{q,q}(|\lambda|t^q)\int_0^t \frac{\zeta(x)}{(t-x)^{1-q}}dx \\ &\leqslant c_\zeta \gamma \zeta(t), \end{aligned}$$

$t \in [0,T]$. According to condition (H_9), it follows that

$$\begin{aligned} D(u(t), v(t)) &\leqslant D(u(t), G_1(f,t)) + D(G_1(f,t), v(t)) \\ &\leqslant c_\zeta \gamma \zeta(t) + \int_0^t \frac{E_{q,q}(\lambda(t-x)^q)D[(f(x,u(x)) + B(x)C(x)), (f(x,v(x)) + B(x)C(x))]}{(t-x)^{1-q}}dx \\ &\leqslant c_\zeta \gamma \zeta(t) + L\int_0^t \frac{E_{q,q}(\lambda(t-s)^q)D[u(x), v(x)]}{(t-x)^{1-q}}dx \\ &\leqslant c_\zeta \gamma \zeta(t) + LE_{q,q}(\lambda(t-x)^q)\int_0^t \frac{)D[u(x), v(x)]}{(t-x)^{1-q}}dx \end{aligned}$$

By the generalized Gronwall inequality [36], we obtain

$$D(u(t), v(t)) \leqslant c_\zeta \gamma \zeta(t) E_q(LE_{q,q}(|\lambda|T^q)\Gamma(q)t^q).$$

Thus, Equation (1) is E_q–Ulam–Hyers–Rassias stable in view of Definition 5. □

Theorem 6. *Let $\lambda \leqslant 1$ and let the condition (H_1)–(H_4), (H_8) and (H_9) hold for a ^{c}[(ii)-GH]-differentiable function u satisfy inequality (5). Then, Equation (1) is E_q–Ulam–Hyers–Rassias stable.*

Proof. According to Theorem 2, u is a ${}^c[(ii)$-GH]-differentiable solution to Cauchy problem (1) and (2). Let u be a ^{c}[(ii)-GH]-differentiable solution to Equation (1), which satisfies inequality (5) with $u(0) = u_0$. From Lemma 7, we obtain

$$D(u(t), G_3(f,t)) \leqslant c_\zeta \gamma \zeta(t),$$

$t \in [0, T]$. According to condition (H_9) it follows that

$$\begin{aligned} D(u(t), v(t)) &\leqslant D(u(t), G_3(f,t)) + D(G_3(f,t), v(t)) \\ &\leqslant c_\zeta \gamma \zeta(t) + \int_0^t \frac{E_{q,q}(|\lambda|(t-x)^q) D[(f(x,u(x)) + B(x)C(x)), (f(x,v(x)) + B(x)C(x))]}{(t-x)^{1-q}} dx \\ &\leqslant c_\zeta \gamma \zeta(t) + L \int_0^t \frac{E_{q,q}(|\lambda|(t-x)^q) D[u(x), v(x)]}{(t-x)^{1-q}} dx \\ &\leqslant c_\zeta \gamma \zeta(t) + L E_{q,q}(|\lambda|(t-x)^q)\int_0^t \frac{)D[u(x), v(x)]}{(t-x)^{1-q}} dx \end{aligned}$$

By the generalized Gronwall inequality, we obtain

$$D(u(t), v(t)) \leqslant c_\zeta \gamma \zeta(t) E_q(L E_{q,q}(|\lambda| T^q) \Gamma(q) t^q).$$

Thus, Equation (1) is E_q–Ulam–Hyers–Rassias stable in view of Definition 5. □

Theorem 7. *Let $\lambda < 1$, and let the condition (H_1)–(H_3), (H_8) and (H_9) hold for a ^{c}[(i)-GH]-differentiable function u satisfies inequality (5). Then Equation (1) is E_q–Ulam–Hyers–Rassias stable.*

Proof. According to Theorem 3, u is a ^{c}[(i)-GH]-differentiable solution to Cauchy problem (1) and (2). Let u be a ^{c}[(i)-GH]-differentiable solution to Equation (1), which satisfies inequality (5) with $u(0) = u_0$. From Lemma 6, we obtain

$$D(u(t), G_2(f,t)) \leqslant c_\zeta \gamma \zeta(t),$$

$t \in [0, T]$. According to condition (H_9), it follows that

$$\begin{aligned} D(u(t), v(t)) &\leqslant D(u(t), G_2(f,t)) + D(G_2(f,t), v(t)) \\ &\leqslant c_\zeta \gamma \zeta(t) + \int_0^t \frac{E_{q,q}(|\lambda|(t-x)^q) D[(f(x,u(x)) + B(x)C(x)), (f(x,v(x)) + B(x)C(x))]}{(t-x)^{1-q}} dx \\ &\leqslant c_\zeta \gamma \zeta(t) + L \int_0^t \frac{E_{q,q}(|\lambda|(t-x)^q) D[u(x), v(x)]}{(t-x)^{1-q}} dx \\ &\leqslant c_\zeta \gamma \zeta(t) + L E_{q,q}(|\lambda|(t-x)^q)\int_0^t \frac{)D[u(x), v(x)]}{(t-x)^{1-q}} dx \end{aligned}$$

By the generalized Gronwall inequality, we obtain

$$D(u(t), v(t)) \leqslant c_\zeta \gamma \zeta(t) E_q(L E_{q,q}(|\lambda| T^q) \Gamma(q) t^q).$$

Thus, Equation (1) is E_q–Ulam–Hyers–Rassias stable in view of Definition 5. □

Theorem 8. *Let $\lambda > 1$, let the condition (H_1)–(H_3) as well as (H_5)–(H_7), (H_8)–(H_9) hold for a ^{c}[(ii)-GH]-differentiable function u, which satisfies inequality (5). Then, Equation (1) is E_q–Ulam–Hyers–Rassias stable.*

Proof. According to Theorem 4, set u is a ^{c}[(ii)-GH]-differentiable solution to Cauchy problem (1) and (2). let u be a ^{c}[(ii)-GH]-differentiable solution to Equation (1), which satisfies the inequality (5) with $u(0) = u_0$. from Lemma 7, we obtain

$$D(u(t), G_4(f,t)) \leqslant c_\zeta \gamma \zeta(t),$$

$t \in [0, T]$. According to condition (H_9), it follows that

$$\begin{aligned} D(u(t), v(t)) &\leqslant D(u(t), G_4(f,t)) + D(G_4(f,t), v(t)) \\ &\leqslant c_\zeta \gamma \zeta(t) + \int_0^t \frac{E_{q,q}(|\lambda|(t-x)^q) D[(f(x,u(x)) + B(x)C(x)), (f(x,v(x)) + B(x)C(x))]}{(t-x)^{1-q}} dx \\ &\leqslant c_\zeta \gamma \zeta(t) + L \int_0^t \frac{E_{q,q}(|\lambda|(t-x)^q) D[u(x), v(x)]}{(t-x)^{1-q}} dx \\ &\leqslant c_\zeta \gamma \zeta(t) + L E_{q,q}(|\lambda|(t-x)^q) \int_0^t \frac{)D[u(x), v(x)]}{(t-x)^{1-q}} dx \end{aligned}$$

By the generalized Gronwall inequality, we obtain

$$D(u(t), v(t)) \leqslant c_\zeta \gamma \zeta(t) E_q(L E_{q,q}(|\lambda| T^q) \Gamma(q) t^q).$$

Thus, Equation (1) is E_q–Ulam–Hyers–Rassias stable in view of Definition 5. Now, the proof is completed. □

Remark 7. *In view of Definition 6 can be verified as according to the assumption in Theorems 5–8, we assume Equation (1) and inequality (4). It can be verified that Equation (1) is generalized E_q–Ulam–Hyers–Rassias stable with respect to Definition 6.*

Remark 8. *Condition (H_9) weakens $\int_0^t \zeta(x)dx \leqslant c_\zeta \zeta(t) E_2(L E_{2,2}(|1| T^2) \Gamma(2) t^2\ \forall\ t \in [0, T]$ when we assume $q = 1$. This means that certain theorems in [25] are special cases of Theorem 5 and 6 in the present paper.*

Remark 9. *According to the assumptions excluding $(H9)$ in Theorems 5–8, we consider the equation in (1) and inequality in (3). It can be proved that in terms of Definitions 3 and 4, Equation (1) is E_q–Ulam–Hyers.*

5. Examples

In this part, we will show four examples to explain our main results.

Example 1. *Consider the following Cauchy problem in terms of a Fuzzy fractional differential equation*

$$ {}_0^c D_t^{1.5} u(t) = u(t) \oplus 1.2u(t)cos(t) \oplus t^2 e^t F \tag{7}$$

on $(0, 2\pi]$, with initial conditions

$$\begin{array}{l} u(0) = \hat{0} \\ u'(0) = \hat{1} \end{array}. \tag{8}$$

Compared to Equation (1), in the above equations, $q = 1.5$, $\lambda = 2$, $T = 2\pi$, $f(t, u(t)) = 1.2u(t)cos(t) \oplus t^2 e^t F$, and $F = (0, 1, 2) \in \mathbb{E}^1$ is a symmetric triangular fuzzy number. Hence, with $L = 1.3$, the condition (H_1) and (H_2) are satisfied. It is not difficult to prove that condition (H_3) is satisfied. Hence, as a consequence of Theorem 1, the Cauchy problem (7) and (8) has a ^{c}[(i)-GH]-differentiable solution. The numerical solutions with respect to the $q = 1.5$ level are provided by utilizing the Adams–Moultan predictor–corrector method.

Furthermore, for $\varepsilon > 1$, assume that the ^{c}[(i)-GH]-differentiable fuzzy-valued function $u : (0, 2\pi] \to \mathbb{E}^1$ satisfies condition (H_8) and

$$D({}_0^c D_t^{1.5} u(t) = u(t) \oplus 1.2u(t)cos(t) \oplus t^2 e^t F) \leqslant \varepsilon t$$

Assuming $\zeta(t) = 1,\ t \in [0, 2\pi]$ *and* $c_\zeta = \frac{4\sqrt{2\pi}}{3} E_{1.5,1.5}(\sqrt{2\pi})$, *this means that condition* (H_9) *is satisfied. Hence, Equation (7) is* E_q*–Ulam–Hyers–Rassias stable with respect to Theorem 5.*

Example 2. *Consider the following Cauchy problem in terms of a Fuzzy fractional differential equation*

$$ {}_0^cD_t^{1.5}u(t) = -u(t) + t^2 + t + 4 \tag{9}$$

with initial condition

$$\begin{cases} u(0) = u_0 \\ u'(0) = u_1 \end{cases} \tag{10}$$

Compared to equations (1), in the above equation, $q = -1$, $f(t, u(t)) = t^2 + t + 4$, *and* $u_0 = (0, 1, 2) \in \mathbb{E}^1$ *is a symmetric triangular fuzzy number.*

Hence, with $L = 1.3$, *the condition* (H_1)*–*(H_2) *are satisfied. It is not difficult to prove that condition* (H_4) *is satisfied. Hence, by employing Theorem 2, the Cauchy problem (9)–(10) has a different* ^{c}*[(ii)-GH]-differentiable solution. The numerical solution provides with respect to q-level by utilizing the Adams–Moultan predictor–corrector method.*

Furthermore, for $\varepsilon > 1$, *assumes that the* ^{c}*[(ii)-GH]-differentiable fuzzy-valued function* $u : (0, 2\pi] \to \mathbb{E}^1$ *satisfies condition* (H_8) *and*

$$D({}_0^cD_t^{1.5}u(t), -u(t) + t^2 + t + 4) \leqslant \varepsilon t,\ t \in (0, 2\pi]$$

Assuming $\zeta(t) = t,\ t \in [0, 2\pi]$ *and* $c_\zeta = \frac{4\sqrt{2\pi}}{3} E_{1.5,1.5}(\sqrt{2\pi})$, *this means that condition* (H_9) *is satisfied. Hence, Equation (9) is* E_q*–Ulam–Hyers–Rassias stable with respect to Theorem 6.*

Example 3. *Consider the following Cauchy problem in terms of a Fuzzy fractional differential equation*

$$ {}_0^cD_t^{1.5}u(t) = u(t) \oplus 1.2u(t)cos(t) \oplus t^2e^tF \tag{11}$$

on $(0, 2\pi]$, *with initial conditions*

$$\begin{cases} u(0) = \hat{0} \\ u'(0) = \hat{1} \end{cases} \tag{12}$$

Compared to Equation (1), in the above equation, $q = 1.5, \lambda = 2$, $T = 2\pi$, $f(t, u(t)) = -u(t) \oplus 1.2u(t)cos(t) \oplus t^2e^tF$ *and* $F = (0, 1, 2) \in \mathbb{E}^1$ *is a symmetric triangular fuzzy number. Hence, with* $L = 1.3$, *it is not difficult to prove that condition* (H_1) *and* (H_3) *are satisfied. Hence, as a consequence of Theorem 3, the Cauchy problem (11) and (12) has a* ^{c}*[(i)-GH]-differentiable solution. The numerical solutions with respect to* $q = 1.5$ *level are provided by utilizing the Adams–Moultan predictor–corrector method.*

Furthermore, for $\varepsilon > 1$, *assume that the* ^{c}*[(i)-GH]-differentiable fuzzy-valued function* $u : (0, \pi] \to \mathbb{E}^1$ *satisfies condition* (H_8) *and*

$$D({}_0^cD_t^{1.5}u(t), -u(t) \oplus 1.2u(t)cos(t) \oplus t^2e^tF \leqslant \varepsilon t$$

Assuming $\zeta(t) = 1,\ t \in [0, 2\pi]$ *and* $c_\zeta = \frac{4\sqrt{2\pi}}{3} E_{1.5,1.5}(\sqrt{2\pi})$, *this means that condition* (H_9) *satisfied. Hence, Equation (11) is* E_q*–Ulam–Hyers–Rassias stable with respect to Theorem 7.*

Example 4. *Consider the following Cauchy problem in terms of a Fuzzy fractional differential equation*

$$ {}_0^cD_t^{1.5}u(t) = u(t) + t^2 + t + 4,\ t \in (0, 2\pi] \tag{13}$$

with initial condition

$$\begin{cases} u(0) = u_0 \\ u'(0) = u_1 \end{cases} \tag{14}$$

Compared to Equation (1), in the above equations, $q = 1.5$, $f(t, u(t)) = t^2 + t + 4$, *and* $u_0 = (0, 1, 2) \in \mathbb{E}^1$ *is a symmetric triangular fuzzy number.*

Hence, with $L = 1.3$*, the condition* (H_1)*–*(H_3) *are satisfied. Notice* $Ch_{1.5}(x) - Sh_{1.5}(x) > 1$ *and* $Ch_{1.5}(x) - Sh_{1.5}(x) < 1$ *for* $u \in (0, 2\pi]$*. It is not difficult to prove that condition* (H_5)*–*(H_7) *are satisfied. Hence, as a consequence of Theorem 4, the Cauchy problem (13) and (14) has a unique* ^{c}*[(ii)-GH]-differentiable solution. The numerical solutions with respect to the* $q = 1.5$ *level are provided by utilizing the Adams–Moultan predictor–corrector method.*

Furthermore, for $\varepsilon > 1$*, assume that the* ^{c}*[(ii)-GH]-differentiable fuzzy-valued function* $u : (0, 2\pi] \to \mathbb{E}^1$ *satisfies condition* (H_8) *and*

$$D({}_0^cD_t^{1.5}u(t), u(t) + t^2 + t + 4) \leqslant \varepsilon t,\ t \in (0, 2\pi]$$

Assuming $\zeta(t) = t,\ t \in [0, 2\pi]$ *and* $c_\zeta = \frac{4\sqrt{2\pi}}{3}E_{1.5,1.5}(\sqrt{2\pi})$*, this means that condition* (H_9) *satisfied. Hence, Equation (13) is* E_q*–Ulam–Hyers–Rassias stable with respect to Theorem 8.*

6. Graphical Presentation

We used the Adams–Bashforth–Moulton technique to acquire the numerical solution for this fractional differential equation for graphical representation of the solution of the problem presented in Equations (7), (9), (11) and (13). For simulation, the modified predictor–corrector scheme is used to examine the effect and contribution of the time-delayed factor. A graphical representation of the solution with different variations of the time delay factor, as well as other parameters, is made to check and demonstrate the stability of the model under consideration. We are able to see the Ulam–Hyers stability of varied accuracies and delays from the numerical data. The system will attain Ulam–Hyers stability more quickly with greater accuracy. This is also true when the number of delays increases. Figures 1–4 show the stability of the system (7), (9), (11) and (13) for various time delays and fractional derivatives.

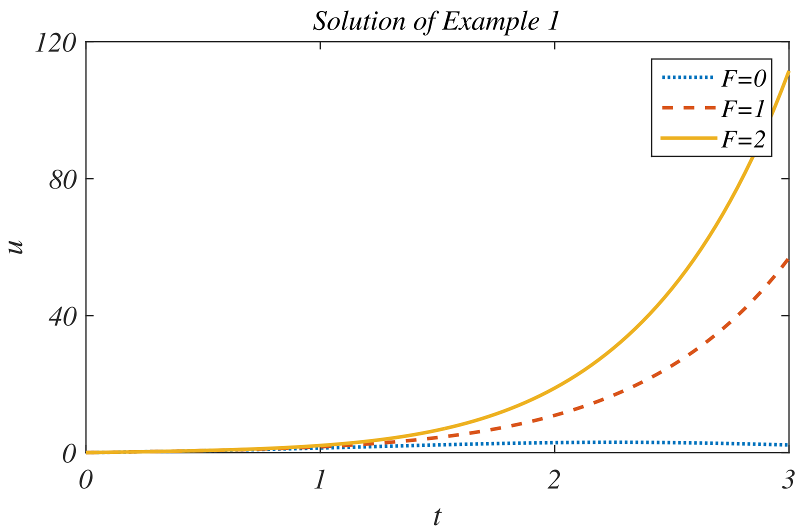

Figure 1. Solution of Problems (7) and (8).

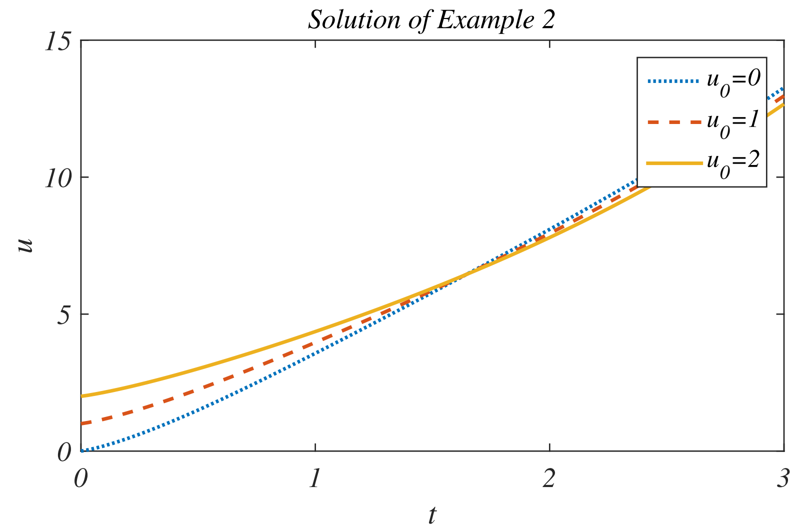

Figure 2. Solution of Problems (9) and (10).

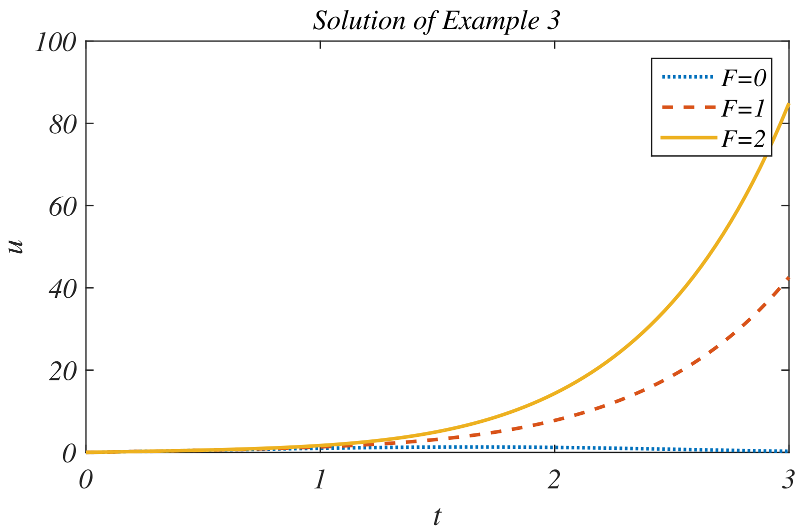

Figure 3. Solution of Problems (11) and (12).

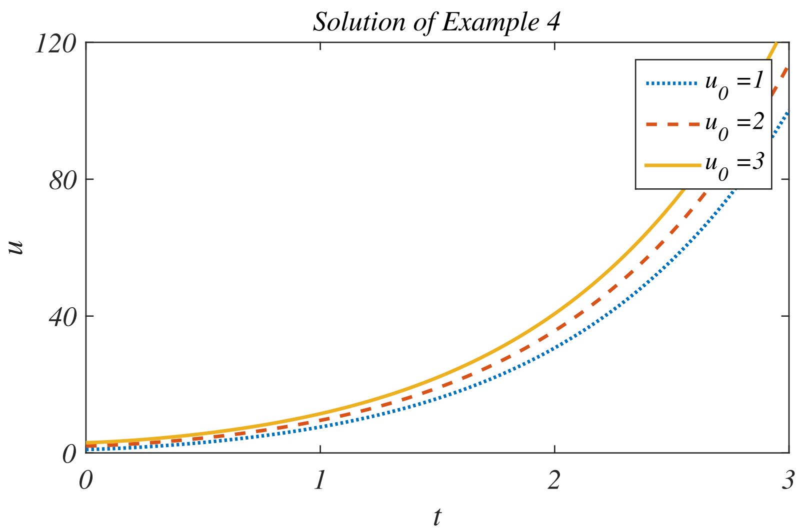

Figure 4. Solution of Problems (13) and (14).

7. Conclusions

This paper aims to define the uniqueness and existence of a group of nonlinear fuzzy fractional differential equation of solutions to the Cauchy problem. Moreover, E_q–Ulam-type stability of Equation (1) is observed by applying the inequality technique. We obtain uniqueness and existence results with the help of nonlocal conditions of the Caputo derivative. Moreover, future work may include broadening the idea indicated in this task and familiarizing observability, and generalize other tasks. Ulam-type stability of fuzzy fractional differential equations, similar to crisp situations for approximate solutions, provides a reliable theoretical basis. This a fruitful area with wide research projects, and it can bring about countless applications and theories. We have decided to devote much attention to this area. Furthermore, it is fruitful to investigate stability problems in a classical sense for the fuzzy fractional differential equation.

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