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Inverse Problem for a Partial Differential Equation with Gerasimov–Caputo-Type Operator and Degeneration

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Abstract: In the three-dimensional open rectangular domain, the problem of the identification of the redefinition function for a partial differential equation with Gerasimov–Caputo-type fractional operator, degeneration, and integral form condition is considered in the case of the $0 < \alpha \leq 1$ order. A positive parameter is present in the mixed derivatives. The solution of this fractional differential equation is studied in the class of regular functions. The Fourier series method is used, and a countable system of ordinary fractional differential equations with degeneration is obtained. The presentation for the redefinition function is obtained using a given additional condition. Using the Cauchy–Schwarz inequality and the Bessel inequality, the absolute and uniform convergence of the obtained Fourier series is proven.

Keywords: inverse problem; Gerasimov–Caputo-type fractional operator; redefinition function; degeneration; integral form condition; one value solvability

MSC: 35A02; 35M10; 35S05

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1. Introduction

When the boundary of a domain of a physical process is impossible to study, a nonlocal condition of integral form can be obtained as additional information sufficient for the unique solvability of the problem. Therefore, in recent years, research has intensified on the nonlocal direct and inverse boundary value problems for differential and integro-differential equations with integral conditions (see, for example, [1–14]). The many problems of gas dynamics, the theory of elasticity, and the theory of plates and shells have been described by high-order partial differential equations.

Fractional calculus plays an important role in mathematical modeling in many scientific and engineering disciplines [15–18]. In [19] where problems of continuum and statistical mechanics are considered. The construction of various models of theoretical physics using fractional calculus is described in ([20], Vol. 4, 5), [21,22]. A detailed review of the application of fractional calculus to solving problems in applied sciences is provided in ([23], Vol. 6–8), [24]. In [25], an inverse problem to determine the right-hand side for a mixed type integro-differential equation with fractional order Gerasimov–Caputo operators is considered. The problem of determining the source function for a degenerate parabolic equation with the Gerasimov–Caputo operator was investigated [26]. In [27], the solvability of the nonlocal boundary problem for a mixed-type differential equation with a fractional-order operator and degeneration is studied. In the applications of fractional derivatives to solving partial differential equations, interesting results have been obtained [28–32].

We recall some materials from the theory of fractional order integro-differentiation. Let $(0;\, T)$ be an interval on the set of non-negative real numbers, $0 < T < \infty$. The Riemann–Liouville $0 < \alpha$-order fractional integral for the function $\eta\,(t)$ has the form

$$I_{0+}^{\alpha}\eta\,(t) = \frac{1}{\Gamma\,(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\eta(s)ds, \quad \alpha > 0, \quad t \in \big(0;\, T\big),$$

where $\Gamma\,(\alpha)$ is the Gamma function.

For the case $n-1 < \alpha \le n, \;\; n \in \mathbb{N}$, the Riemann–Liouville α-order fractional derivative for the function $\eta(t)$ is defined as follows:

$$D_{0+}^{\alpha}\eta\,(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha}\eta\,(t), \quad t \in \big(0;\, T\big).$$

The Gerasimov–Caputo α-order fractional derivative for the function $\eta\,(t)$ is defined by the following formula

$$_*D_{0+}^{\alpha}\eta\,(t) = I_{t_0+}^{n-\alpha}\eta^{(n)}(t) = \frac{1}{\Gamma\,(n-\alpha)}\int\limits_0^t \frac{\eta^{(n)}(s)\,d\,s}{(t-s)^{\alpha-n+1}}, \;\; t \in \big(0;\, T\big).$$

These derivatives are reduced to the nth order derivatives for $\alpha = n \in \mathbb{N}$:

$$D_{0+}^{n}\eta\,(t) = {_*D_{0+}^{n}}\eta\,(t) = \frac{d^n}{dt^n}\eta(t), \quad t \in \big(0;\, T\big).$$

In this paper, for the case of the $0 < \alpha \le 1$ order, we study the regular one value solvability of the inverse boundary value problem for the Gerasimov–Caputo-type fractional partial differential equation with degeneration. This partial differential equation is a fractional-order ordinary differential equation with respect to the first argument and is a higher even-order partial differential equation with respect to spatial arguments. The stability of the solution on the given functions is proved.

So, in the three-dimensional open domain $\Omega = \{(t,\, x,\, y)\,|\, 0 < t < T, \;\, 0 < x,\, y < l\}$, a partial differential equation of the following form is considered

$$D_{t,\,x,\,y}^{\beta,\,\alpha,\,4,\,4}[U\,(t,\, x,\, y)] = a\,(t)\, b\,(x,\, y) \tag{1}$$

with a nonlocal condition on the integral form

$$U\,(T,\, x,\, y) + \Big(I_{0+}^{\rho} U\,(t,\, x,\, y)\Big)|_{t=T} = \varphi(x,\, y), \quad 0 \le x,\, y \le l, \tag{2}$$

where ρ, T, and l are given positive real numbers,

$$D_{t,\,x,\,y}^{\beta,\,\alpha,\,4,\,4}\,[U] = \Bigg[{_*D_{0+}^{\alpha}} + \varepsilon\, D_{0+}^{\alpha}\left(\frac{\partial^{\,4k}}{\partial\, x^{\,4k}} + \frac{\partial^{\,4k}}{\partial\, y^{\,4k}}\right) +$$
$$+\,\omega\, t^{\beta}\left(\frac{\partial^{\,4k}}{\partial\, x^{\,4k}} + \frac{\partial^{\,4k}}{\partial\, y^{\,4k}}\right)\Bigg]\, U\,(t,\, x,\, y),$$

where ω and β are non-negative parameters, ε is a positive parameter, $\varepsilon > \delta > 0$, $\delta = \text{const}$, $0 < \alpha \le 1$, k is a given positive integer, $a\,(t) \in C\,(\Omega_T)$, $\Omega_T \equiv [0;\, T]$, $\Omega_l \equiv [0;\, l]$, $b\,(x,\, y) \in C\left(\Omega_l^2\right)$ is a known function, and $\varphi(x,\, y)$ is a redefinition function, $\Omega_l^2 \equiv \Omega_l \times \Omega_l$. We assume that for the given functions, the following boundary conditions are true

$$\varphi(0,\, y) = \varphi(l,\, y) = \varphi(x,\, 0) = \varphi\,(x,\, l) = 0,$$

$$b\,(0,\,y) = b\,(l,\,y) = b\,(x,\,0) = b\,(x,\,l) = 0.$$

Statement of Problem. We find the pair of functions $\{U\,(t,\,x,\,y);\ \varphi\,(x,\,y)\}$, the first of which satisfies the partial differential Equation (1), nonlocal integral condition (2), and boundary value conditions

$$\begin{aligned} &U\,(t,\,0,\,y) = U\,(t,\,l,\,y) = U\,(t,\,x,\,0) = U\,(t,\,x,\,l) \\ &= \frac{\partial^2}{\partial x^2}U\,(t,\,0,\,y) = \frac{\partial^2}{\partial x^2}U\,(t,\,l,\,y) = \frac{\partial^2}{\partial x^2}U\,(t,\,x,\,0) = \frac{\partial^2}{\partial x^2}U\,(t,\,x,\,l) \\ &= \frac{\partial^2}{\partial y^2}U\,(t,\,0,\,y) = \frac{\partial^2}{\partial y^2}U\,(t,\,l,\,y) = \frac{\partial^2}{\partial y^2}U\,(t,\,x,\,0) = \frac{\partial^2}{\partial y^2}U\,(t,\,x,\,l) = \ldots \\ &= \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,\,0,\,y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,\,l,\,y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,\,x,\,0) = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,\,x,\,l) \\ &= \frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t,0,y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t,l,y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t,x,0) = \frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t,x,l) = 0, \end{aligned} \tag{3}$$

class of functions

$$\left[\begin{array}{l} U\,(t,\,x,\,y) \in C\,(\overline{\Omega}), \\ {}_*D^{\alpha}_{0+}U\,(t,\,x,\,y) \in C^{4k,\,4k}_{x,\,y}(\Omega) \cap C^{4k+0}_{x,\,y}(\Omega) \cap C^{0+4k}_{x,\,y}(\Omega) \end{array}\right. \tag{4}$$

and additional condition

$$U\,(t_1,\,x,\,y) = \psi(x,\,y), \quad 0 < t_1 < T, \;\; 0 \le x,\,y \le l, \tag{5}$$

$\varphi\,(x,\,y) \in C\,[0;\,l]^{\,2}$, where $\psi\,(x,\,y)$ are a given smooth function and

$$\psi(0,\,y) = \psi(l,\,y) = \psi(x,\,0) = \psi(x,\,l) = 0,$$

$C^{4k+0}_{x,\,y}(\Omega)$ is the class of continuous functions $\frac{\partial^{4k}\,U\,(t,\,x,\,y)}{\partial\,x^{4k}}$ on Ω, whereas $C^{0+4k}_{x,\,y}(\Omega)$ is the class of continuous functions $\frac{\partial^{4k}\,U\,(t,\,x,\,y)}{\partial\,y^{4k}}$ on Ω, $\frac{\partial^{4k-2}}{\partial y^{4k-2}}U\,(t,\,x,\,l)$ we understand as $\left.\frac{\partial^{4k-2}}{\partial y^{4k-2}}U\,(t,\,x,\,y)\right|_{y=l}$, $\overline{\Omega} = \{(t,\,x,\,y)\,|\,0 \le t \le T,\; 0 \le x,\,y \le l\}$.

2. Cauchy Problem for a Fractional Ordinary Differential Equation with Degeneration

It is well-known that the two-parametric Mittag–Leffler function is defined as (see, for example, [33])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\,k + \beta)}, \quad \alpha > 0, \quad z,\,\beta \in \mathbb{C}. \tag{6}$$

The generalized Mittag–Lefler (Kilbas–Saigo)-type function was defined for real $\alpha,\,m \in \mathbb{R}$ and complex $l \in \mathbb{C}$ by Kilbas and Saigo in the following form [33]

$$E_{\alpha,\,m,\,l}(z) = \sum_{k=0}^{\infty} c_k z^k, \quad c_0 = 1, \quad c_k = \prod_{j=0}^{k-1} \frac{\Gamma(\alpha\,[jm + l] + 1)}{\Gamma\,(\alpha\,[jm + l + 1] + 1)}, \quad k = 1,\,2,\,\ldots \tag{7}$$

These functions belong to the class of entire functions on the complex plane.

Let us consider the Cauchy problem for an ordinary differential equation of fractional order with degeneration

$$\left\{\begin{array}{l} {}_*D^{\alpha}_{0+}u\,(t) = \lambda\,t^{\beta}u(t) + f(t), \quad t \in (0;\,T), \\ u(0) = u_0, \end{array}\right. \tag{8}$$

where $\beta,\,\lambda,\,u_0 \in \mathbb{R},\ f\,(t)$ is a given continuous function.

Let $\gamma \in [0;\, 1)$. Then, we consider the class of following functions ([34], p. 4, 205):

$$C_\gamma(\Omega_T) = \left\{ g\,(t) : t^\gamma g\,(t) \in C\,(\Omega_T) \right\},$$

$$C_\gamma^\alpha(\Omega_T) = \left\{ g\,(t) \in C\,(\Omega_T) :\, {}_*D_{0+}^\alpha g\,(t) \in C_\gamma(\Omega_T) \right\}.$$

Lemma 1. *Let $\gamma \in [0;\, \alpha]$, $\beta \geq 0$. Then, for all $f(t) \in C_\gamma(\Omega_T)$, there exists a unique solution $u(t) \in C_\gamma^\alpha(\Omega_T)$ of the Cauchy problem (8). This solution has the following form*

$$u\,(t) = u_0 E_{\alpha,\, 1+\frac{\beta}{\alpha},\, \frac{\beta}{\alpha}}\left(\lambda\, t^{\alpha+\beta}\right) + \int\limits_0^t K\,(t,\, \tau)\, f\,(\tau)\, d\tau, \tag{9}$$

where

$$K\,(t,\, \tau) = \sum_{i=1}^{\infty} K_i(t,\, \tau), \tag{10}$$

$$K_0(t,\, \tau) = \frac{1}{\Gamma\,(\alpha)}(t-\tau)^{\alpha-1},\;\; K_i(t,\, \tau) = \frac{\lambda}{\Gamma\,(\alpha)} \int\limits_\tau^t s^\beta (t-s)^{\alpha-1} K_{i-1}(s,\, \tau) ds, \tag{11}$$

where $E_{\alpha,\, 1+\frac{\beta}{\alpha},\, \frac{\beta}{\alpha}}\left(\lambda\, t^{\alpha+\beta}\right)$ is the Kilbas–Saigo function, defined by (7).

Proof. The uniqueness of the solution $u\,(t) \in C_\gamma^\alpha(\Omega_T)$ of the problem (8) was proven in [34], p. 205. In this paper, the existence of solution for the case $f\,(t) = 0$ is also proved. So, we consider the inhomogeneous problem (8) and the solution to this problem as the sum of two functions

$$u\,(t) = v\,(t) + w\,(t), \tag{12}$$

where the functions $v\,(t)$ and $w\,(t)$, respectively, are solutions to the following two problems:

$$\begin{cases} {}_*D_{0+}^\alpha v\,(t) = \lambda\, t^\beta v\,(t), \quad t \in (0;\, T), \\ v\,(0) = u_0, \end{cases} \tag{13}$$

$$\begin{cases} {}_*D_{0+}^\alpha w\,(t) = \lambda\, t^\beta w\,(t) + f(t), \quad t \in (0;\, T), \\ w\,(0) = 0. \end{cases} \tag{14}$$

As implied by [34], p. 233, in their Theorem 4.4, the problem (13) has a unique solution $v\,(t) \in C_\gamma^\alpha(\Omega_T)$ of the form

$$v\,(t) = u_0 E_{\alpha,\, 1+\frac{\beta}{\alpha},\, \frac{\beta}{\alpha}}\left(\lambda\, t^{\alpha+\beta}\right). \tag{15}$$

We consider the problem (14). According to [34] in their Corollary 3.24, p. 202, the problem is equivalent to the one value solvability of theVolterra-type integral equation of the second kind

$$w\,(t) = \lambda\, I_{0+}^\alpha\left(t^\beta w\,(t)\right) + I_{0+}^\alpha f\,(t), \quad t \in (0;\, T). \tag{16}$$

We apply the method of successive approximations to solve the integral Equation (16). In this order, we place

$$\begin{cases} w_0(t) = I_{0+}^\alpha f\,(t), \\ w_m(t) = w_0(t) + \lambda I_{0+}^\alpha\left(t^\beta w_{m-1}(t)\right),\, m = 1, 2, \ldots \end{cases}$$

With the convergence of the iterative process for the integral Equation (16), i.e., the existence of a solution to Equation (16), we can similarly provide the proof of the corresponding part of Theorem 3.25 in [34], p. 202. Hence, it is not difficult to determine that the solution to (16) has the form

$$w(t) = \int_0^t K(t, \tau) f(\tau) d\tau, \tag{17}$$

where the kernel $K(t, \tau)$ is defines by the Formulas (10) and (11). According to (12), from the representations (15) and (17), we have (9). Lemma 1 is proved. □

Lemma 2. *For the case of $\gamma \in [0; \alpha]$, $\beta \geq 0$, there holds the following estimate*

$$\left|(t-\tau)^{1-\alpha} K(t,\tau)\right| \leq E_{\alpha,\alpha}\left(|\lambda| t^{\beta} (t-\tau)^{\alpha}\right). \tag{18}$$

Proof. By virtue of (10) and (11), we obtain that there the following estimates hold:

$$|K(t, \tau)| = \left|\sum_{i=1}^{\infty} K_i(t, \tau)\right| \leq |K_0(t, \tau)| + |K_1(t, \tau)| + |K_2(t, \tau)| + \cdots + |K_i(t, \tau)| + \ldots, \tag{19}$$

$$|K_0(t, \tau)| = \frac{1}{\Gamma(\alpha)} (t-\tau)^{\alpha-1}, \tag{20}$$

$$|K_1(t, \tau)| = \left| \frac{\lambda}{\Gamma(\alpha)} \int_\tau^t s^\beta (t-s)^{\alpha-1} K_0(s, \tau) ds \right| \leq \frac{|\lambda|}{\Gamma(\alpha)} \int_\tau^t s^\beta (t-s)^{\alpha-1} |K_0(s, \tau)| ds$$

$$\leq \frac{|\lambda|}{\Gamma^2(\alpha)} \int_\tau^t s^\beta (t-s)^{\alpha-1} (s-\tau)^{\alpha-1} ds \leq \frac{|\lambda| t^\beta}{\Gamma^2(\alpha)} \int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{\alpha-1} ds. \tag{21}$$

Through the change in variables $s = \tau + (t-\tau)z$, we obtain

$$\int_\tau^t (t-s)^{\rho-1} (s-\tau)^{\sigma-1} ds = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)} (t-\tau)^{\rho+\sigma-1}.$$

Hence, considering $\rho = \sigma = \alpha$, for (21), we derive

$$|K_1(t, \tau)| \leq \frac{|\lambda| t^\beta}{\Gamma^2(\alpha)} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} (t-\tau)^{2\alpha-1} = \frac{|\lambda|}{\Gamma(2\alpha)} t^\beta (t-\tau)^{2\alpha-1}. \tag{22}$$

Now, we estimate the next kernel $K_2(t, \tau)$:

$$|K_2(t, \tau)| = \left| \frac{\lambda}{\Gamma(\alpha)} \int_\tau^t s^\beta (t-s)^{\alpha-1} K_1(s, \tau) ds \right| \leq \frac{|\lambda|}{\Gamma(\alpha)} \int_\tau^t s^\beta (t-s)^{\alpha-1} |K_1(s, \tau)| ds$$

$$\leq \frac{|\lambda|}{\Gamma^2(\alpha)} \frac{|\lambda|}{\Gamma(2\alpha)} \int_\tau^t s^{2\beta} (t-s)^{\alpha-1} (s-\tau)^{2\alpha-1} ds \leq \frac{|\lambda|^2 t^{2\beta}}{\Gamma(\alpha)\Gamma(2\alpha)} \int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{2\alpha-1} ds.$$

Hence, considering $\rho = \alpha$, $\sigma = 2\alpha$, we derive

$$|K_2(t, \tau)| \leq \frac{|\lambda|^2}{\Gamma(3\alpha)} t^{2\beta} (t-\tau)^{3\alpha-1}. \tag{23}$$

Through the induction method, we obtain

$$|K_m(t, \tau)| \leq \frac{|\lambda|^m}{\Gamma((m+1)\alpha)} t^{m\beta} (t-\tau)^{(m+1)\alpha-1}. \tag{24}$$

Considering (6) and the estimates (20), (22)–(24), for (19), we have

$$|K(t,\tau)| \le \frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} + \frac{|\lambda|}{\Gamma(2\alpha)}t^{\beta}(t-\tau)^{2\alpha-1} + \frac{|\lambda|^2}{\Gamma(3\alpha)}t^{2\beta}(t-\tau)^{3\alpha-1}$$

$$+\cdots+\frac{|\lambda|^m}{\Gamma(m\alpha+\alpha)}t^{m\beta}(t-\tau)^{(m+1)\alpha-1}+\ldots$$

$$=(t-\tau)^{\alpha-1}\left[\frac{1}{\Gamma(\alpha)}+\frac{|\lambda|}{\Gamma(2\alpha)}t^{\beta}(t-\tau)^{\alpha}+\frac{|\lambda|^2}{\Gamma(3\alpha)}t^{2\beta}(t-\tau)^{2\alpha}\right.$$

$$\left.+\cdots+\frac{|\lambda|^m}{\Gamma(m\alpha+\alpha)}t^{m\beta}(t-\tau)^{m\alpha}+\ldots\right]=(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(|\lambda|t^{\beta}(t-\tau)^{\alpha}\right).$$

Hence, we obtain the estimate (18). Lemma 2 is proved. □

3. Expansion of the Solution into Fourier Series

Nontrivial solutions of the inverse problem are sought as a Fourier series

$$U(t,x,y)=\sum_{n,m=1}^{\infty}u_{n,m}(t)\,\vartheta_{n,m}(x,y), \tag{25}$$

where

$$u_{n,m}(t)=\int_0^l\int_0^l U(t,x,y)\,\vartheta_{n,m}(x,y)\,dx\,dy, \tag{26}$$

$$\vartheta_{n,m}(x,y)=\frac{2}{l}\sin\frac{\pi n}{l}x\,\sin\frac{\pi m}{l}y, n,\ m=1,2,\ldots$$

We also suppose that the following function is expand to Fourier series

$$b(x,y)=\sum_{n,m=1}^{\infty}b_{n,m}\vartheta_{n,m}(x,y), \tag{27}$$

where

$$b_{n,m}=\int_0^l\int_0^l b(x,y)\,\vartheta_{n,m}(x,y)\,dx\,dy. \tag{28}$$

Substituting Fourier series (25) and (27) into partial differential Equation (1), we obtain the countable system of ordinary fractional differential equations of $0<\alpha<1$-order with degeneration

$${}_*D^{\alpha}_{0+}u_{n,m}(t)+\omega\,\lambda^{2k}_{n,m}\,t^{\beta}u_{n,m}(t)=\frac{a(t)\,b_{n,m}}{1+\varepsilon\,\mu^{4k}_{n,m}}, \tag{29}$$

where

$$\lambda^{2k}_{n,m}=\frac{\mu^{4k}_{n,m}}{1+\varepsilon\,\mu^{4k}_{n,m}},\quad \mu^{k}_{n,m}=\left(\frac{\pi}{l}\right)^{k}\sqrt{n^{2k}+m^{2k}}.$$

According to Lemma 1, the general solution of the countable system of differential Equation (29) has the form

$$u_{n,m}(t)=C_{n,m}E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\omega\,\lambda^{2k}_{n,m}\,t^{\alpha+\beta}\right)+b_{n,m}h_{n,m}(t), \tag{30}$$

where

$$h_{n,m}(t)=\frac{1}{1+\varepsilon\,\mu^{4k}_{n,m}}\int_0^t K(t,\tau)\,a(\tau)\,d\tau,$$

$C_{n,m}$ is arbitrary constant, function $K(t,\tau)$ is defined by the formula (11).

By Fourier coefficients (26), the integral condition (2) is rewritten in the form

$$u_{n,m}(T)+\left(I_{0+}^{\rho}u_{n,m}(t)\right)|_{t=T}$$

$$=\int_0^l\int_0^l\left(U(T,x,y)+\left(I_{0+}^{\rho}U(t,x,y)\right)|_{t=T}\right)\vartheta_{n,m}(x,y)\,dx\,dy$$

$$=\int_0^l\int_0^l\varphi(x,y)\,\vartheta_{n,m}(x,y)\,dx\,dy=\varphi_{n,m}. \tag{31}$$

To find the unknown coefficients $C_{n,m}$ in (30), we use condition (31), and from (30), we have

$$C_{n,m}=\frac{1}{\sigma_{0\,n,m}}\left[\varphi_{n,m}-b_{n,m}\sigma_{1\,n,m}\right], \tag{32}$$

where

$$\sigma_{0\,n,m}=E_{\alpha,1+\beta/\alpha,\beta/\alpha}\left(-\omega\,\lambda_{n,m}^{2k}\,T^{\alpha+\beta}\right)+I_{0+}^{\rho}\left(E_{\alpha,1+\beta/\alpha,\beta/\alpha}\left(-\lambda_{n,m}^{2k}\,\omega\,t^{\alpha+\beta}\right)\right)|_{t=T},$$

$$\sigma_{1\,n,m}=h_{n,m}(T)+I_{0+}^{\rho}h_{n,m}(t)|_{t=T}.$$

Note that $\sigma_{0\,n,m}\neq 0$, which is separate from zero and finite. This easily follows from the complete monotonicity of the function $E_{\alpha,m,l}(-z)$, $z>0$ at $\alpha\leq 1$ and from the finiteness of $\lambda_{n,m}^{2k}$ [35]. Substituting the defined coefficients (32) into presentation (30), we derive that

$$u_{n,m}(t)=\varphi_{n,m}A_{n,m}(t)+b_{n,m}B_{n,m}(t), \tag{33}$$

where

$$A_{n,m}(t)=\frac{1}{\sigma_{0\,n,m}}E_{\alpha,1+\beta/\alpha,\beta/\alpha}\left(-\omega\,\lambda_{n,m}^{2k}\,t^{\alpha+\beta}\right),$$

$$B_{n,m}(t)=h_{n,m}(t)-\frac{\sigma_{1\,n,m}}{\sigma_{0\,n,m}}E_{\alpha,1+\beta/\alpha,\beta/\alpha}\left(-\omega\,\lambda_{n,m}^{2k}\,t^{\alpha+\beta}\right).$$

Substituting the presentation of Fourier coefficients (33) of the main unknown function into Fourier series (25), we obtain

$$U(t,x,y)=\sum_{n,m=1}^{\infty}\vartheta_{n,m}(x,y)\left[\varphi_{n,m}A_{n,m}(t)+b_{n,m}B_{n,m}(t)\right]. \tag{34}$$

We consider the Fourier series (34) as a formal solution of the direct problem (1)–(4).

4. Determination of the Redefinition Function

Using the additional condition (5) and considering presentation (33), we obtain from the Fourier series (34) the following countable system for the Fourier coefficients of the redefinition function

$$\varphi_{n,m}A_{n,m}(t_1)+b_{n,m}B_{n,m}(t_1)=\psi_{n,m}, \tag{35}$$

where

$$\psi_{n,m}=\int_0^l\int_0^l\psi(x,y)\,\vartheta_{n,m}(x,y)\,dx\,dy. \tag{36}$$

From the relation (35), we find the redefinition function as

$$\varphi_{n,m}=\psi_{n,m}\chi_{1\,n,m}+b_{n,m}\chi_{2\,n,m}, \tag{37}$$

where

$$\chi_{1\,n,m} = \frac{1}{A_{n,m}(t_1)}, \quad \chi_{2\,n,m} = -\frac{B_{n,m}(t_1)}{A_{n,m}(t_1)},$$

$$A_{n,m}(t_1) = \frac{1}{\sigma_{0\,n,m}} E_{\alpha,1+\beta/\alpha,\beta/\alpha}\left(-\lambda_{n,m}^{2k}\,\omega\, t_1^{\alpha+\beta}\right) \neq 0, \quad 0 < t_1 < T.$$

Since $\varphi_{n,m}$ are Fourier coefficients defined by (31), we substitute presentation (35) into the Fourier series

$$\varphi(x,y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x,y)[\psi_{n,m}\,\chi_{1\,n,m} + b_{n,m}\,\chi_{2\,n,m}]. \tag{38}$$

We prove the absolute and uniform convergence of the Fourier series (38) of the redefinition function. We need this to use the concepts of the following well-known Banach spaces and the Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ with norm

$$\|\varphi\|_{\ell_2} = \sqrt{\sum_{n,m=1}^{\infty} |\varphi_{n,m}|^2} < \infty.$$

The space $L_2(\Omega_l^2)$ of square-summable functions on the domain $\Omega_l^2 = \Omega_l \times \Omega_l$ has norm

$$\|\vartheta(x,y)\|_{L_2(\Omega_l^2)} = \sqrt{\int_0^l\int_0^l |\vartheta(x,y)|^2 dx\,dy} < \infty.$$

Smoothness conditions. For functions

$$\psi(x,y),\ b(x,y) \in C^{4k}(\Omega_l^2),$$

let there exist piecewise continuous $4k+1$-order derivatives. Then, by integrating the functions (28) and (36) $4k+1$ times over every variable x, y in parts, we obtain the following relations

$$|\psi_{n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|\psi_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \quad |b_{n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \tag{39}$$

$$\left\|\psi_{n,m}^{(8k+2)}\right\|_{\ell_2} \leq \frac{2}{l}\left\|\frac{\partial^{8k+2}\psi(x,y)}{\partial x^{4k+1}\,\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}, \tag{40}$$

$$\left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2} \leq \frac{2}{l}\left\|\frac{\partial^{8k+2} b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}, \tag{41}$$

where

$$\psi_{n,m}^{(8k+2)} = \int_0^l\int_0^l \frac{\partial^{8k+2}\psi(x,y)}{\partial x^{4k+1}\,\partial y^{4k+1}}\,\vartheta_{n,m}(x,y)\,dx\,dy,$$

$$b_{n,m}^{(8k+2)} = \int_0^l\int_0^l \frac{\partial^{8k+2} b(x,y)}{\partial x^{4k+1}\,\partial y^{4k+1}}\,\vartheta_{n,m}(x,y)\,dx\,dy.$$

By obtaining estimates for the solution, we use the above properties of the Kilbas–Saigo function and Lemma 2. Then, it is easy to see that

$$\sigma_2 = \max_{n,m}\{|\chi_{1\,n,m}|;\ |\chi_{2\,n,m}|\} < \infty, \tag{42}$$

where

$$\chi_{1\,n,m} = \frac{1}{A_{n,m}(t_1)}, \quad \chi_{2\,n,m} = -\frac{B_{n,m}(t_1)}{A_{n,m}(t_1)}, \quad 0 < t_1 < T.$$

Theorem 1. *Suppose that the conditions of smoothness are fulfilled. Then, Fourier series (38) is absolute and uniform convergent.*

Proof. We use Formulas (39)–(41) and Estimate (42). Using the Cauchy–Schwartz inequality for series (38), we obtain the estimate

$$\begin{aligned}
|\varphi(x,y)| &\le \sum_{n,m=1}^{\infty} |\vartheta_{n,m}(x,y)| \cdot |\psi_{n,m}\chi_{1\,n,m} + b_{n,m}\chi_{2\,n,m}| \\
&\le \frac{2}{l}\sigma_2 \left[\sum_{n,m=1}^{\infty} |\psi_{n,m}| + \sum_{n,m=1}^{\infty} |b_{n,m}| \right] \\
&\le \frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2} \sigma_2 \left[\sum_{n,m=1}^{\infty} \frac{\left|\psi_{n,m}^{(8k+2)}\right|}{n^{4k+1}m^{4k+1}} + \sum_{n,m=1}^{\infty} \frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1}m^{4k+1}} \right] \\
&\le \frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2} \sigma_2 C_{01} \left[\left\| \psi_{n,m}^{(8k+2)} \right\|_{\ell_2} + \left\| b_{n,m}^{(8k+2)} \right\|_{\ell_2} \right] \\
&\le \gamma_1 \left[\left\| \frac{\partial^{8k+2}\psi(x,y)}{\partial x^{4k+1}\partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} + \left\| \frac{\partial^{8k+2} b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} \right] < \infty,
\end{aligned} \tag{43}$$

where

$$\gamma_1 = \sigma_2 C_{01} \left(\frac{2}{l}\right)^2 \left(\frac{l}{\pi}\right)^{8k+2}, \quad C_{01} = \sqrt{\sum_{n,m=1}^{\infty} \frac{1}{n^{8k+2}\, m^{8k+2}}} < \infty.$$

Estimate (43) implies the absolute and uniform convergence of Fourier series (38). Theorem 1 is proved. □

5. Determination of the Main Unknown Function

So, the redefinition function is determined as a Fourier series (38). Now, the redefinition function is known. Substituting representation (37) into the Fourier series (34), the main unknown function can be presented as

$$U(t,x,y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x,y)[\psi_{n,m}P_{n,m}(t) + b_{n,m}Q_{n,m}(t)], \tag{44}$$

where

$$P_{n,m}(t) = \chi_{1\,n,m}A_{n,m}(t), \quad Q_{n,m}(t) = \chi_{2\,n,m}A_{n,m}(t) + B_{n,m}(t).$$

To establish the uniqueness of the function $U(t,x,y)$, we suppose that there are two functions, U_1 and U_2, satisfying the given conditions (1)–(5). Then, their difference $U = U_1 - U_2$ is a solution of the differential Equation (1), satisfying conditions (2)–(5) with function $\psi(x,y) \equiv 0$. By virtue of relations (36), we have $\psi_{n,m} = 0$. Hence, we obtain from formulas (26), (28), and (44) in the domain Ω that the zero identity follows

$$\int_0^l \int_0^l U(t,x,y)\,\vartheta_{n,m}(x,y)\,dx\,dy \equiv 0.$$

Hence, by virtue of the completeness of the systems of eigenfunctions $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi n}{l}x\right\}$, $\left\{\sqrt{\frac{2}{l}}\sin\frac{\pi m}{l}y\right\}$ in $L_2(\Omega_l^2)$, we deduce that $U(t,x,y)\equiv 0$ for all $x\in\Omega_l^2\equiv[0,l]^2$ and $t\in\Omega_T\equiv[0;T]$.

Since $U(t,x,y)\in C(\overline{\Omega})$, then $U(t,x,y)\equiv 0$ in the domain $\overline{\Omega}$. Therefore, the solution of the problem (1)–(5) is unique in the domain $\overline{\Omega}$.

Theorem 2. *Let the conditions of the Theorem 1 be fulfilled. Then, the series (44) converges. At the same time, their term-by-term differentiation is possible.*

Proof. By virtue of conditions of Theorem 1 and the properties of the Mittag–Leffler and Kilbas–Saigo functions, the functions $P_{n,m}(t)$, $Q_{n,m}(t)$ are uniformly bounded on the segment $[0;T]$. So, for any positive integers n, m, there exists a finite constant σ_3, so the following estimate is true:

$$\max_{n,m}\left\{\max_{0\le t\le T}|P_{n,m}(t)|;\ \max_{0\le t\le T}|Q_{n,m}(t)|\right\}\le\sigma_3. \tag{45}$$

Using estimation Formulas (39)–(41) and (45), analogously to the case of estimate (43), for series (44), we obtain

$$|U(t,x,y)|\le\sum_{n,m=1}^{\infty}|\vartheta_{n,m}(x,y)|\cdot|\psi_{n,m}P_{n,m}(t)+b_{n,m}Q_{n,m}(t)|$$

$$\le\gamma_2\left[\left\|\frac{\partial^{8k+2}\psi(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{46}$$

where $\gamma_2=C_{01}\sigma_3\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{8k+2}$.

The estimate (46) implies the absolute and uniform convergence of the Fourier series (44). We differentiate this function (44) the required number of times

$$\frac{\partial^{4k}}{\partial x^{4k}}U(t,x,y)=\sum_{n,m=1}^{\infty}\left(\frac{\pi n}{l}\right)^{4k}\vartheta_{n,m}(x,y)[\psi_{n,m}P_{n,m}(t)+b_{n,m}Q_{n,m}(t)], \tag{47}$$

$$\frac{\partial^{4k}}{\partial y^{4k}}U(t,x,y)=\sum_{n,m=1}^{\infty}\left(\frac{\pi m}{l}\right)^{4k}\vartheta_{n,m}(x,y)[\psi_{n,m}P_{n,m}(t)+b_{n,m}Q_{n,m}(t)]. \tag{48}$$

The expansions of the following functions into Fourier series are defined similarly

$${}_*D_{0+}^{\alpha}U(t,x,y),\quad \frac{\partial^{4k}}{\partial x^{4k}}\,{}_*D_{0+}^{\alpha}U(t,x,y),\quad \frac{\partial^{4k}}{\partial y^{4k}}\,{}_*D_{0+}^{\alpha}U(t,x,y).$$

We show the convergence of series (47) and (48). As in the case of estimate (43), applying the Cauchy–Schwarz inequality, we obtain

$$\left|\frac{\partial^{4k}}{\partial x^{4k}}U(t,x,y)\right|\le\sum_{n,m=1}^{\infty}\left(\frac{\pi n}{l}\right)^{4k}|u_{n,m}(t)|\cdot|\vartheta_{n,m}(x,y)|$$

$$\le\frac{2}{l}\left(\frac{\pi}{l}\right)^{4k}\sigma_3\left[\sum_{n,m=1}^{\infty}n^{4k}|\psi_{n,m}|+\sum_{n,m=1}^{\infty}n^{4k}|b_{n,m}|\right]$$

$$\le\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3\left[\sum_{n,m=1}^{\infty}\frac{\left|\psi_{n,m}^{(8k+2)}\right|}{n\,m^{4k+1}}+\sum_{n,m=1}^{\infty}\frac{\left|b_{n,m}^{(8k+2)}\right|}{n\,m^{4k+1}}\right]$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{02}\left[\left\|\psi_{n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2}\right]$$

$$\leq \gamma_3\left[\left\|\frac{\partial^{8k+2}\psi(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{49}$$

where $\gamma_3=\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{02},\ C_{02}=\sqrt{\sum_{n,m=1}^{\infty}\frac{1}{n\,m^{8k+2}}}<\infty;$

$$\left|\frac{\partial^{4k}}{\partial y^{4k}}U(t,x,y)\right|\leq\sum_{n,m=1}^{\infty}\left(\frac{\pi m}{l}\right)^{4k}|u_{n,m}(t)|\cdot|\vartheta_{n,m}(x,y)|$$

$$\leq\frac{2}{l}\left(\frac{\pi}{l}\right)^{4k}\sigma_3\left[\sum_{n,m=1}^{\infty}m^{4k}|\psi_{n,m}|+\sum_{n,m=1}^{\infty}m^{4k}|b_{n,m}|\right]$$

$$\leq\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3\left[\sum_{n,m=1}^{\infty}\frac{\left|\psi_{n,m}^{(8k+2)}\right|}{n^{4k+1}m}+\sum_{n,m=1}^{\infty}\frac{\left|b_{n,m}^{(8k+2)}\right|}{n^{4k+1}m}\right]$$

$$\leq \frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{03}\left[\left\|\psi_{n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|b_{n,m}^{(8k+2)}\right\|_{\ell_2}\right]$$

$$\leq \gamma_4\left[\left\|\frac{\partial^{8k+2}\psi(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}b(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{50}$$

where $\gamma_4=\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{4k+2}\sigma_3 C_{03},\ C_{03}=\sqrt{\sum\limits_{n,m=1}^{\infty}\frac{1}{n^{8k+2}m}}<\infty.$

The convergence of the Fourier series for functions

$${}_*D_{0+}^{\alpha}U(t,x,y),\quad \frac{\partial^{4k}}{\partial x^{4k}}\,{}_*D_{0+}^{\alpha}U(t,x,y),\quad \frac{\partial^{4k}}{\partial y^{4k}}\,{}_*D_{0+}^{\alpha}U(t,x,y)$$

is easy to prove because the necessary estimates are obtained similarly for the cases of estimates (43), (49) and (50). Therefore, the function $U(t,x,y)$ belongs to the (4) class of functions. Theorem 2 is proved. □

6. Stability of the Solution $U(t,x,y)$ with Respect to the Given Functions

Theorem 3. *Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function $U(t,x,y)$ as a solution of the problem (1)–(5) is stable with respect to given function $\psi(x,y)$.*

Proof. We show that the solution of the differential Equation (1) $U(t,x,y)$ is stable with respect to a given function $\psi(x,y)$. Let $U_1(t,x,y)$ and $U_2(t,x,y)$ be two different solutions of the inverse boundary value problem (1)–(5), corresponding to two different values of the function $\psi_1(x,y)$ and $\psi_2(x,y)$, respectively.

We state that $|\psi_{1\,n,m}-\psi_{2\,n,m}|<\delta_{n,m}$, where $0<\delta_{n,m}$ is a sufficiently small positive quantity and the series $\sum_{n,m=1}^{\infty}|\delta_{n,m}|$ is convergent. Then, as such, given the conditions of the theorem, from the Fourier series (44), we obtain

$$\|U_1(t,x,y)-U_2(t,x,y)\|_{C(\overline{\Omega})}\leq\frac{2}{l}\sigma_3\sum_{n,m=1}^{\infty}|\psi_{1\,n,m}-\psi_{2\,n,m}|<\frac{2}{l}\sigma_3\sum_{n,m=1}^{\infty}|\delta_{n,m}|<\infty.$$

If we set $\varepsilon=\frac{2}{l}\sigma_3\sum\limits_{n,m=1}^{\infty}|\delta_{n,m}|<\infty$, then, from the last estimate, we derive assertions about the stability of the solution of the differential Equation (1) with respect to a given function $\psi(x,y)$ in (5). Theorem 3 is proved. □

Similarly, we proved that the following two theorems hold.

Theorem 4. *Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function $U(t, x, y)$ as a solution of the problem (1)–(5) is stable with respect to a given function $b(x, y)$ in the right-hand side of Equation (1).*

Theorem 5. *Suppose that all the conditions of Theorem 2 are fulfilled. Then, the function $U(t, x, y)$, as a solution of the problem (1)–(5), is stable with respect to the source function $\varphi(x, y)$.*

7. Conclusions

In three-dimensional domain, an inverse boundary value problem (1)–(5) of the identification of the redefinition function $\varphi(x, y)$ for a partial differential equation with degeneration and integral form condition was studied in this paper. The case of a $0 < \alpha \leq 1$-order Gerasimov–Caputo-type fractional operator was considered. The solution of the partial differential equation was studied in the class of regular functions. Equation (1) depends on three independent arguments: t, x, y. The first argument is the time argument, and with respect to this argument, Equation (1) is a fractional Gerasimov–Caputo-type ordinary differential equation with degeneration. The other two variables x, y are spatial, and Equation (1) with respect to them is a partial differential equation of higher even order. The Fourier series method was used and a countable system of ordinary differential equations was obtained. Using the given additional condition (5), the presentation for the redefinition function was obtained. When conditions of smoothness are fulfilled, then using the Cauchy–Schwarz inequality and the Bessel inequality, the absolute and uniform convergence of the obtained Fourier series was proved. The stability of main unknown function $u(t, x, y)$ of the problem (1)–(5) with respect to given functions $b(x, y)$, $\psi(x, y)$ and redefinition function $\varphi(x, y)$ was studied.

This work is theoretical in nature and develops the theory of differential equations with fractional operators. We studied the unique solvability in the classical sense of a nonlocal inverse problem for a partial differential equation. In the future, we intend to continue our research in the direction of superposition of several fractional-order operators.

Remark 1. *Equation (1) is a generalization of the Barenblatt–Zheltov–Kochina differential equation. The Barenblatt–Zheltov–Kochina equation simulates the filtration of a viscoelastic fluid in fractured porous media [36].*

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