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Hadamard-Type Fractional Heat Equations and Ultra-Slow Diffusions

Alessandro De Gregorio * and Roberto Garra *

Department of Statistical Sciences, Sapienza, University of Rome. P.le Aldo Moro, 5, 00185 Roma, Italy

* Correspondence: alessandro.degregorio@uniroma1.it (A.D.G.); roberto.garra@uniroma1.it (R.G.)

Abstract: In this paper, we study diffusion equations involving Hadamard-type time-fractional derivatives related to ultra-slow random models. We start our analysis using the abstract fractional Cauchy problem, replacing the classical time derivative with the Hadamard operator. The stochastic meaning of the introduced abstract differential equation is provided, and the application to the particular case of the fractional heat equation is then discussed in detail. The ultra-slow behaviour emerges from the explicit form of the variance of the random process arising from our analysis. Finally, we obtain a particular solution for the nonlinear Hadamard-diffusive equation.

Keywords: anomalous diffusions; Hadamard fractional derivatives; inverse stable subordinators; Lévy processes

1. Introduction

Ultraslow diffusion processes include a wide class of different stochastic processes characterised by a logarithmic growth of the mean squared displacement, namely

$$\langle x^2(t) \rangle \sim \log^\beta t,$$

with $\beta > 0$. A relevant example is the Sinai diffusion for which $\beta = 4$, which is related to a model in which a particle moves in a quenched random force field [1]. Another interesting case is $\beta = 4/3$ in polymer physics (see [2]). There are many different mathematical models that are not equivalent and lead to this behaviour. Different formulations of the macroscopic governing equations have been considered in the literature; for example, those based on heat-type equations involving space or time-dependent diffusivity. We refer to the recent survey in [3] for a complete bibliography of this topic. In a series of relevant papers, the connection between distributed-order fractional differential equations and ultraslow anomalous diffusion has been shown; we refer for example to [4–8]. It was shown that distributed order fractional equations describe a kind of continuous time random walk (CTRW) in which the waiting-time distribution is not a simple power law but a weighted mixture of power law functions.

From the microscopic point of view, an interesting alternative derivation can be based on CTRW models in expanding media (see [9]). Moreover, in this framework, a relevant role is played by the so-called scaled Brownian motion model; see [10] for details. Finally, in a recent paper [11], an ultraslow behaviour emerged in the context of an anomalous diffusion in inhomogeneous media described by a fractional diffusion equation with a space-dependent variable order.

In this paper, we explore a new approach based on the application of the Hadamard-type fractional operator that was introduced in [12] and belongs to the general class of fractional derivatives with regard to a function (see [13]). In particular, motivated by the connection with ultra-slow processes, we first study the abstract Cauchy problem involving the Hadamard-type fractional derivative and supply the stochastic interpretation of



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its solution. For this purpose, we exploit the theory of time-changed processes. Indeed, it is known (see, e.g., [14,15]) that the the solution of the time-fractional heat equation

$$\frac{\partial^\nu u}{\partial t^\nu} = Lu, \quad \nu \in (0, 1),$$

where $\frac{\partial^\nu}{\partial t^\nu}$ represents the Caputo derivative and L is the infinitesimal generator of a C_0 -semigroup, is related to a time-changed Lévy process $X(\mathcal{L}(t)), t \geq 0$. The random time $\mathcal{L}(t)$ is the inverse of a stable subordinator. It is worthwhile to recall that the interplay between anomalous diffusions, fractional partial differential equations and time-changed processes has been extensively investigated in the last two decades. Many papers that appear in the literature are devoted to the study of the fractional dynamics arising from a diffusion process with random time. The reader can also consult, for instance, refs. [16–23] for more information on this topic.

Then, we consider in detail the particular case of heat-type equations based on the Hadamard-type fractional derivative that can be directly related to ultra-slow diffusions. We first show the explicit fundamental solution and then discuss the probabilistic interpretation, as well as some connections with higher-order diffusion equations. We finally provide some simple explicit results for the non-linear diffusive case.

2. Preliminaries about Hadamard-Type Fractional Derivatives

Hadamard time-fractional integrals and derivatives are well-known in the literature about fractional differential equations (we refer for example to the classical monograph [24], Section 2.7). These fractional operators can be obtained essentially by a change of variable $t \rightarrow \ln t$ starting from Riemann–Liouville integrals (a more general definition is discussed in [25]). The connection between ultra-slow diffusion and fractional equations involving Hadamard derivatives has been considered also in a recent review [3]. In [12], motivated by a probabilistic problem, a Hadamard-type fractional-time evolution operator was introduced, namely O_ν^t .

Let $[t_1, t_2]$ be a finite interval such that $-\infty < t_1 < t_2 < \infty$ and let $AC[t_1, t_2]$ be the space of absolutely continuous functions on $[t_1, t_2]$. Let us denote $\delta = \left(\left(\frac{a}{b} + t\right) \frac{d}{dt}\right)$, we define the space

$$AC_\delta^n[t_1, t_2] = \left\{ f: [t_1, t_2] \rightarrow \mathbb{R}: \delta^{n-1}[f(t)] \in AC[t_1, t_2] \right\}. \quad (1)$$

Clearly, $AC_\delta^1[t_1, t_2] \equiv AC[t_1, t_2]$ for $n = 1$.

Definition 1. Let $\nu \in \mathbb{R}^+$ and $n = [\nu] + 1$, where $[\nu]$ is the integer part of ν . The Hadamard-type fractional derivative of order ν applied to the function $f \in AC_\delta^n[t_1, t_2]$, $0 \leq t_1 < t_2 < \infty$, is defined as

$$(O_\nu^t f)(t) = \frac{1}{\Gamma(n - \nu)} \int_{\frac{1-a}{b}}^t \ln^{n-1-\nu} \left(\frac{a + bt}{a + b\tau} \right) \left[\left(\left(\frac{a}{b} + \tau \right) \frac{d}{d\tau} \right)^n f(\tau) \right] \frac{b}{a + b\tau} d\tau,$$

for $n - 1 < \nu < n$, $a \in [0, 1]$ and $b \in (0, 1]$.

This is a generalization of the Hadamard fractional derivative that is recovered for $a = 0$ and is particularly useful for the applications we discuss below. It is possible to prove, by a simple adaptation, that Hadamard-type derivatives exist almost everywhere for a function $f \in AC_\delta^n[t_1, t_2]$ (see [25], Theorem 3.2).

A relevant property of this operator is given by the following result (see [12], p. 1057 for further details):

$$O_\nu^t \ln^\beta(a + bt) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \nu)} \ln^{\beta-\nu}(a + bt) \quad (2)$$

for $\nu \in (0, 1)$ and $\beta > -1 \setminus \{0\}$. Starting from (2), we can show by simple calculations that the composed Mittag–Leffler function

$$E_{\nu,1}(-\ln^\nu(a+bt)), \quad (3)$$

where $E_{\nu,1}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k + 1)}$, $x \in \mathbb{R}$, solves the following integro-differential equation:

$$O_\nu^t f(t) = -f(t). \quad (4)$$

It can be proved that the integer counterpart of this differential operator—for example, for $n = 1$ —becomes

$$(O_\nu^t f)(t) = \left[(a+bt) \frac{\partial}{\partial t} f \right](t).$$

We show in the next sections that this operator can be particularly interesting for ultra-slow diffusions.

Hereafter, we consider for simplicity $a = b = 1$ in Definition 1. We show that this choice is meaningful for our aims, but the more general case can be obtained by direct calculations.

Let us recall that the Caputo fractional derivative of order $\nu > 0$ is defined as

$$\left(\frac{\partial^\nu}{\partial t^\nu} f \right)(x, t) := \begin{cases} \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\nu+1-m}} d\tau, & \text{for } m-1 < \nu < m, \\ \frac{\partial^m}{\partial t^m} f(x, t), & \text{for } \nu = m. \end{cases} \quad (5)$$

Then, we show that by setting $a = b = 1$, the Hadamard-type fractional operator O_ν^t can be obtained by the deterministic time change $t \rightarrow \ln(1+t)$ starting from the Caputo fractional derivatives (5).

We finally observe that these Hadamard-type operators belong to the more general class of fractional derivatives with regard to other functions that were recently studied in detail, for instance, in [13]. These operators are gaining interest both in mathematics and in certain applications.

3. A Fractional Cauchy Problem Involving the Hadamard-Type Derivative

We start our analysis with the abstract fractional Cauchy problem involving the Hadamard-type derivative O_ν^t with $a = b = 1$. First of all, let us recall some basic issues about the theory of the semigroup operators (see, e.g., [26]) and their connection with the stochastic processes.

A family of linear operators $\{T(t), t \geq 0\}$ on a Banach space \mathfrak{X} is called a C_0 -semigroup if: (1) $T(0)f = f$; (2) $T(t)T(s)f = T(t+s)f$; (3) $\|T(t)f - f\| \rightarrow 0$, in the Banach space norm as $t \rightarrow 0$; (4) $\forall t \geq 0, \exists$ a constant $M > 0$ such that $\|T(t)f\| \leq M\|f\|$ for all $f \in \mathfrak{X}$. A C_0 -semigroup such that $\|T(t)f\| \leq \|f\|$ for all $t \geq 0$ and all $f \in \mathfrak{X}$ is called a Feller semigroup.

Every C_0 -semigroup has a generator L defined by the following limit in the norm:

$$Lf = \lim_{t \downarrow 0} \frac{T(t)f - f}{t}, \quad (6)$$

where $f \in \text{Dom}(L)$. Then, we recall that $p := p(x, t) := T(t)f(x)$ is the unique classical solution to the abstract Cauchy problem

$$\frac{\partial p}{\partial t} = Lp, \quad p(x, 0) = f(x), \quad \forall f \in \text{Dom}(L). \quad (7)$$

Let $X := \{X(t), t \geq 0\}$ be a Lévy process defined on a probability space (Ω, \mathcal{F}, P) , with the Lévy symbol $\psi(k)$ and characteristics (a, A, ϕ) ; that is, the characteristic function of X has the following representation (see e.g., [27]):

$$\mathbb{E}[e^{ik \cdot X(t)}] = e^{t\psi(k)}, \quad (8)$$

where

$$\psi(k) := ik \cdot a - \frac{1}{2}k \cdot Ak + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{ik \cdot x} - 1 - \frac{ik \cdot x}{1 + \|x\|^2} \right) \phi(dx), \quad k \in \mathbb{R}^d, \quad (9)$$

where $a \in \mathbb{R}^d$, A is an asymmetric, non-negative, definite $d \times d$ matrix and ϕ is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d \setminus \{0\}} \min\{\|x\|^2, 1\} \phi(dx) < \infty$.

It is well-known (see, e.g., [27]) that X admits a Feller semigroup operator given by

$$T(t)f(x) = \mathbb{E}[f(x - X(t))], \quad f \in L^1(\mathbb{R}^d), \quad (10)$$

with the infinitesimal generator L representing a pseudo-differential operator; that is, let $\hat{f}(k) := \int e^{ik \cdot x} f(x) dx$, and thus we have

$$\widehat{L}f(k) = \psi(k)\hat{f}(k) \quad (11)$$

for $f \in L^1(\mathbb{R}^d)$ such that $\psi(k)\hat{f}(k) \in L^1(\mathbb{R}^d)$. Furthermore, let $W^{2,1}(\mathbb{R}^d)$ be the Sobolev space of the functions $f \in L^1(\mathbb{R}^d)$ having first and second derivatives in $L^1(\mathbb{R}^d)$. We find that (see [14,15])

$$Lf(x) = -a \cdot \nabla f(x) + \frac{1}{2} \nabla \cdot A \nabla f(x) + \int_{\mathbb{R}^d \setminus \{0\}} \left(f(x - y) - f(x) + \frac{\nabla f(x) \cdot y}{1 + \|y\|^2} \right) \phi(dy), \quad (12)$$

for all $f \in \text{Dom}(L) = W^{2,1}(\mathbb{R}^d)$.

Finally, we introduce the process $\mathcal{L}_\nu := \{\mathcal{L}_\nu(t), t \geq 0\}$ representing the inverse of a stable subordinator $\{D(t), t \geq 0\}$ of index $\nu \in (0, 1)$; i.e.,

$$\mathcal{L}_\nu(t) := \inf\{u : D(u) > t\}. \quad (13)$$

Furthermore, \mathcal{L}_ν is defined on the same probability space as X such that X and \mathcal{L}_ν are independent. The density of the inverse of a stable subordinator becomes

$$\ell_\nu(u, t) := \frac{t}{\nu} u^{-1-1/\nu} g_\nu(tu^{-1/\nu}), \quad u > 0, t > 0,$$

where $g_\nu(u)$ is the density of $D(t)$ such that the Laplace transform $\tilde{g}_\nu(s) = \int_0^\infty e^{-su} g_\nu(u) du = e^{-s^\nu}$. For more details on $\{\mathcal{L}_\nu(t), t \geq 0\}$, the reader can consult [15,28].

Now, we present the main theorem of this paper, which allows a stochastic representation of the solution of the Hadamard heat equation. We deal with the infinitesimal generator (11) and (12) associated to a Lévy process X .

Theorem 1. *The unique strong solution $q := q(x, t), x \in \mathbb{R}^d, t > 0$, of the following time-fractional Cauchy problem*

$$O_\nu^t q = Lq, \quad q(x, 0) = f(x), \quad \forall f \in \text{Dom}(L), \nu \in (0, 1), \quad (14)$$

is given by

$$q(x, t) = \mathbb{E}[f(x - X(\mathcal{L}_\nu(\ln(1 + t))))] = \int_0^\infty p(x, u) \ell_\nu(u, \ln(1 + t)) du, \quad (15)$$

where X is the Lévy process generated by L , while $\mathcal{L}_\nu(t), t \geq 0$, is the inverse of the stable subordinator (13) with density $\ell_\nu(u, t)$ and $p(x, t)$ is the solution (10) of the Cauchy problem (7).

Furthermore, if X admits a Lévy symbol which is rotationally invariant—i.e., $\psi(k) = \psi(|k|)$ —we can represent $q(x, t)$ as follows:

$$q(x, t) = \frac{1}{(2\pi)^{d/2}} \frac{1}{||x||^{\frac{d}{2}-1}} \int_0^\infty r^{d/2} E_{\nu,1}(\psi(r) \ln^\nu(1+t)) J_{\frac{d}{2}-1}(r||x||) dr, \quad (16)$$

where the above integral involves the Bessel function $J_\mu(x) = \sum_{k=0}^\infty \frac{(-1)^k (x/2)^{2k+\mu}}{k! \Gamma(\mu+k+1)}$, $x, \mu \in \mathbb{R}$.

Proof. In the literature (see e.g., [14,15,29]), it was proved that, for any C_0 -semigroup, the abstract fractional Cauchy problem involving the Caputo fractional derivative of order $\nu \in (0, 1)$ (here $\partial^\nu / \partial t^\nu$, see (5))

$$\frac{\partial^\nu u}{\partial t^\nu}(x, t) = Lu(x, t), \quad u(x, 0) = f(x), \quad \forall f \in \text{Dom}(L), \quad (17)$$

has solution

$$u(x, t) = \int_0^\infty p(x, u) \ell_\nu(u, t) du, \quad (18)$$

where $p(x, t) = T(t)f(x)$ is the solution (10) of the Cauchy problem (7), while $\ell_\nu(x, t)$ is the density of the inverse of a stable subordinator. Therefore, a stochastic representation of the solution of the fractional Cauchy problem (17) is given by

$$u(x, t) = \mathbb{E}[p(x, \mathcal{L}_\nu(t))]. \quad (19)$$

Furthermore, by exploiting (11), we find that the following eigenfunctions problem

$$\frac{\partial^\nu \hat{u}}{\partial t^\nu}(k, t) = \psi(k) \hat{u}(k, t), \quad \hat{u}(k, 0) = 1,$$

has the unique solution

$$\hat{u}(k, t) = E_{\nu,1}(\psi(k)t^\nu). \quad (20)$$

By means of the result (3.9) in [12], we can conclude that the solution of

$$O_\nu^t \hat{q}(k, t) = \psi(k) \hat{q}(k, t), \quad \hat{q}(k, 0) = 1, \quad (21)$$

is equal to

$$\hat{q}(k, t) = E_{\nu,1}(\psi(k) \ln^\nu(1+t)).$$

Then, from (20), we can conclude that the strong solution of the fractional Cauchy problem (14) is obtained by a time-rescaling $t \rightarrow \ln(1+t)$ of the solution $u(x, t)$ of (17); i.e., $q(x, t) = u(x, \ln(1+t))$.

Now, we prove the result (16) by inverting (20). We use the spherical coordinates transformation, and σ represents the uniform distribution on the unit d -dimensional sphere \mathbb{S}^{d-1} . Therefore,

$$\begin{aligned} q(x, t) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}} e^{-ik \cdot x} \hat{q}(k, t) dx \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}} e^{-ik \cdot x} E_{\nu,1}(\psi(|k|) \ln^\nu(1+t)) dx \\ &= \frac{1}{(2\pi)^{d/2}} \int_0^\infty r^{d-1} E_{\nu,1}(\psi(r) \ln^\nu(1+t)) \left(\int_{\mathbb{S}^{d-1}} e^{-ir \cdot x} \sigma(d\theta) \right) dr \\ &= \frac{1}{(2\pi)^{d/2}} \int_0^\infty r^{d-1} E_{\nu,1}(\psi(r) \ln^\nu(1+t)) \frac{J_{\frac{d}{2}-1}(r||x||)}{(r||x||)^{\frac{d}{2}-1}} dr, \end{aligned}$$

where in the last step we have used the following well-known result

$$\int_{\mathbb{S}^{d-1}} e^{-ir \cdot x} \sigma(d\theta) = (2\pi)^{d/2} \frac{J_{\frac{d}{2}-1}(r||x||)}{(r||x||)^{\frac{d}{2}-1}}.$$

□

Therefore, from Theorem 1, we can conclude that the stochastic model linked to (14) results in the time-changed process $\{X\mathcal{L}_\nu(t), t \geq 0\}$.

Corollary 1. For $\nu = 1/2$, we obtain that

$$X(\mathcal{L}_{1/2}(\ln(1+t))) \stackrel{d}{=} X(|B(\ln(1+t))|), \quad (22)$$

where $\{B(t), t \geq 0\}$ is a Brownian motion independent of X .

Proof. We observe that the characteristic function of the process $X(\mathcal{L}_{1/2}(\ln(1+t))), t \geq 0$, is given by

$$\hat{u}(k, t) = E_{1/2,1}(\psi(k) \ln(1+t)^{1/2})$$

and coincides with the characteristic function of the process $X(|B(\ln(1+t))|), t \geq 0$, which reads

$$\frac{1}{\sqrt{\pi \ln(1+t)}} \int_0^\infty e^{-\frac{w^2}{4 \ln(1+t)}} e^{\psi(k)w} dw.$$

Indeed we have that

$$\begin{aligned} \frac{1}{\sqrt{\pi \ln(1+t)}} \int_0^\infty e^{-\frac{w^2}{4 \ln(1+t)}} e^{\psi(k)w} dw &= \frac{1}{\sqrt{\pi \ln(1+t)}} \sum_{n=0}^\infty \frac{(\psi(k))^n}{n!} \int_0^\infty e^{-\frac{w^2}{4 \ln(1+t)}} w^n dw \\ &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(\psi(k))^n (4 \ln(1+t))^{n/2}}{n!} \int_0^\infty e^{-z} z^{\frac{n}{2} + \frac{1}{2} - 1} dz \\ &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(\psi(k))^n (4 \ln(1+t))^{n/2}}{n!} \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) \\ &= E_{1/2,1}(\psi(k) \ln(1+t)^{1/2}), \end{aligned}$$

where in the last passage we used the following well-known result on the Gamma function

$$\Gamma\left(\frac{n}{2} + \frac{1}{2}\right) = \frac{2^{1-n} \sqrt{\pi} \Gamma(n)}{\Gamma(\frac{n}{2})}.$$

Therefore, the above equality leads to the result claimed in the statement of the theorem. □

Example 1. A relevant case is given by the space-fractional Laplacian

$$Lu = -(-\Delta)^{\beta/2} u := -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ik \cdot x} \|k\|^\beta \hat{u}(k) dk \quad (23)$$

that is the generator of a d -dimensional, β -stable process, namely $S_d^\beta(t)$. Then, we find that the Fourier transform of the fundamental solution of the fractional equation

$$\partial_t^v q = -(-\Delta)^{\beta/2} q, \quad v \in (0, 1), \quad (24)$$

is given by

$$\hat{q}(k, t) = E_{v,1}(-\|k\|^\beta \ln^v(1+t)) \quad (25)$$

and coincides with the characteristic function of the process $S_d^\beta(\mathcal{L}_\nu(\ln(1+t)))$. In the special case $\beta = 2$, we recover the Hadamard-type fractional diffusion equation that is related to ultra-slow diffusions. A full discussion of this model-equation is presented in the next section.

Example 2. If

$$L = H_{m,\alpha} := m - \left(m^{\frac{2}{\alpha}} - \Delta\right)^{\frac{\alpha}{2}}, \quad \alpha \in (0, 2), \quad m > 0, \quad (26)$$

we obtain the infinitesimal generator of the relativistic stable process. In [30], the time-fractional generalization was considered by replacing the first-order time-derivative with a Caputo time-fractional derivative. Then, we find (according to Theorem 3.1) that the Fourier transform of the fundamental solution of the fractional equation

$$O_\nu^t q = H_{m,\alpha} q, \quad \nu \in (0, 1), \quad (27)$$

is given by

$$\hat{q}(k, t) = E_{\nu,1} \left(-[(m^{2/\alpha} + |k|^2)^{\alpha/2} - m] \ln^\nu(1+t) \right) \quad (28)$$

and coincides with the characteristic function of the process $R(\mathcal{L}_\nu(\ln(1+t)))$, where $R(t)$ is a relativistic stable subordinator.

From [31], Example 3.2, we have that

$$\text{Var}(X(\mathcal{L}_\nu(\ln(1+t)))) = \ln^\nu(1+t) \frac{\text{Var}(X(1))}{\Gamma(1+\nu)} + \ln^{2\nu}(1+t) \frac{(\mathbb{E}X(1))^2}{\nu} \left(\frac{1}{\Gamma(2\nu)} - \frac{1}{\nu(\Gamma(\nu))^2} \right)$$

and let $t' := \ln(1+t)$ and $s' := \ln(1+s)$

$$\begin{aligned} \text{Cov}(X(\mathcal{L}_\nu(t')), X(\mathcal{L}_\nu(s'))) &= \min\{t', s'\}^\nu \frac{\text{Var}(X(1))}{\Gamma(1+\nu)} \\ &+ \frac{(\mathbb{E}X(1))^2}{\Gamma(1+\nu)^2} \left(\nu \min\{t', s'\}^{2\nu} B(\nu, \nu+1) + \nu \max\{t', s'\}^{2\nu} B\left(\nu, \nu+1; \frac{\min\{t', s'\}}{\max\{t', s'\}}\right) - (t's')^\nu \right), \end{aligned}$$

where $B(\nu, \nu+1)$ and $B(\nu, \nu+1; x)$ are the beta function and the incomplete beta function, respectively. Let $t \geq s$ and then $t' \geq s'$. Fixed s and s'

$$\text{Cov}(X(\mathcal{L}_\nu(t')), X(\mathcal{L}_\nu(s'))) \longrightarrow (s')^\nu \frac{\text{Var}(X(1))}{\Gamma(1+\nu)} + (s')^{2\nu} \frac{(\mathbb{E}X(1))^2}{\Gamma(1+2\nu)}$$

as $t \rightarrow \infty$. Therefore, the previous variance and the covariance increase as $\ln^\nu(1+t)$, and this confirm the ultra-slow behaviour of the stochastic processes related to fractional equations involving Hadamard-type derivatives.

4. The Hadamard-Type Fractional Heat Equation

Here, we consider analytical and probabilistic results regarding the Hadamard-type fractional heat equation obtained by setting $L = \frac{d^2}{dx^2}$ in (14). We first observe that the obtained heat-type equation is, in fact, the generalization of the diffusion equation

$$\frac{\partial u}{\partial t} = D(t) \frac{\partial^2 u}{\partial x^2}, \quad (29)$$

with $D(t) = 1/(1+t)$. This is the governing equation of a scaled Brownian motion; see [3] Equations (81) and (82), where the time-dependence of the diffusion coefficient is considered. We obtain the following interesting result.

Theorem 2. The fundamental solution of the time-fractional heat equation

$$O_\nu^t u(x, t) = \frac{\partial^2 u}{\partial x^2}, \quad \nu \in (0, 1), \quad x \in \mathbb{R}, \quad t > 0, \quad (30)$$

is given by

$$u_\nu(x, t) = \frac{1}{2} \frac{1}{(\ln(1+t))^{\nu/2}} M_{-\frac{\nu}{2}, 1-\frac{\nu}{2}} \left(\frac{|x|}{(\ln(1+t))^{\nu/2}} \right),$$

where

$$M_{-\frac{\nu}{2}, 1-\frac{\nu}{2}}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\frac{\nu}{2}n + 1 - \frac{\nu}{2})}, \quad z \in \mathbb{C}, \quad 0 < \nu < 1,$$

is the M-Wright function.

Proof. By taking the Fourier transform of (30), we obtain

$$O_\nu^t \hat{u}_\nu(k, t) = -k^2 \hat{u}_\nu(k, t),$$

whose solution is given by

$$\hat{u}_\nu(k, t) = E_\nu \left[-k^2 (\ln(1+t))^\nu \right]. \quad (31)$$

Recalling that (see [32])

$$\int_{\mathbb{R}} e^{ikx} \frac{1}{2} \frac{1}{(\lambda)^{\nu/2}} M_{-\frac{\nu}{2}, 1-\frac{\nu}{2}} \left(\frac{|x|}{(\lambda)^{\nu/2}} \right) dx = E_\nu \left(-k^2 \lambda^\nu \right), \quad (32)$$

it is easy to see that the inverse Fourier transform of (31) leads to the claimed result. \square

As already seen, in the non-fractional case—namely for $\nu = 1$ —the equation with a time-varying diffusion coefficient considered here is directly related to ultra-slow scaled Brownian motion (see [10] for a full discussion). We must then consider the fractional counterpart? In order to proceed, we apply the general Theorem 1 directly and observe that the fractional operator involved in the governing equation can be obtained from the Caputo derivative by means of the deterministic time-change $t' = \ln(1+t)$. Thus, we obtain the following result; that is, the direct generalization of the well-known result for the stochastic interpretation of the time-fractional diffusion equation.

Corollary 2. The solution to the Cauchy problem

$$\begin{cases} O_\nu^t u(x, t) = \frac{\partial^2 u}{\partial x^2}, & \nu \in (0, 1), \quad x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = \delta(x), \end{cases} \quad (33)$$

coincides with the probability law of the process

$$B(\mathcal{L}^\nu(\ln(1+t))),$$

where B is a Brownian motion independent of $\mathcal{L}^\nu(t)$.

Remark 1. Observe that for $\nu = 1$, as expected, we find that

$$u(x, t) = \frac{1}{\sqrt{4\pi} \sqrt{\ln(1+t)}} \exp \left(-\frac{x^2}{4(\ln(1+t))} \right);$$

that is, the fundamental solution of the heat-type equation

$$\frac{\partial u}{\partial t} = D(t) \frac{\partial^2 u}{\partial x^2},$$

where the time-dependent diffusivity coefficient is given by

$$D(t) = \frac{1}{1+t}.$$

By using our approach, we obtain the composition of the classical solution of the time-fractional diffusion equation with a new time variable $t' = \ln(1+t)$. Therefore, in the non-fractional case, this is the governing equation of a scaled diffusion equation with a time-variable diffusion coefficient. On the other hand, the relevance of our approach lies in the ability to obtain a simple solution to a fractional problem with time-varying diffusivity.

Remark 2. We observe that (Theorem 2.1 in [23] with our time-scaling) the fundamental solution to the Cauchy problem (33) becomes

$$u(x, t) = \frac{1}{\sqrt{\pi \ln(1+t)}} \int_0^{+\infty} e^{-\frac{z^2}{4 \ln(1+t)}} \bar{u}(x, z) dz,$$

where $\bar{u}(x, z)$ denotes the fundamental solution for

$$O_{2\nu}^z u = \frac{\partial^2 u}{\partial x^2}, \quad \nu \in (0, 1/2).$$

Therefore, for $\nu = 1/2$, we have that

$$u(x, t) = 2 \int_0^{+\infty} \frac{e^{-\frac{z^2}{4 \ln(1+t)}}}{\sqrt{4\pi \ln(1+t)}} \frac{e^{-\frac{x^2}{4 \ln(1+z)}}}{\sqrt{4\pi \ln(1+z)}} dz,$$

which coincides with the probability density of the time-scaled iterated Brownian motion, namely $B_1(|B_2(\ln(1+t))|)$, $t \geq 0$, where $B_1(t)$ and $B_2(t)$ are independent Brownian motions. Moreover, according to Remark 3.1, we find that the characteristic function of $B_1(|B_2(\ln(1+t))|)$ coincides with that of $B(\mathcal{L}_{1/2}(\ln(1+t)))$.

We observe that, by using similar methods, it is possible to prove that the fundamental solution of the generalized fractional equation of order $\nu = \frac{1}{2^n}$

$$O_{\frac{1}{2^n}}^t u(x, t) = \frac{\partial^2 u}{\partial x^2}, \quad n \geq 1. \quad (34)$$

coincides with the density function of the n -times iterated Brownian motion with a scaled-time change $t \rightarrow \ln(1+t)$ (see [23] for details).

Corollary 3. The fundamental solution of the fractional Equation (30) for $\nu = 2/3$ is given by

$$u(x, t) = \frac{3^{2/3}}{2 \cdot \sqrt[3]{\ln(1+t)}} Ai\left(\frac{|x|}{\sqrt[3]{3 \ln(1+t)}}\right),$$

which corresponds to the solution of the linearized KdV equation with the time-varying coefficient

$$\frac{\partial u}{\partial t} = -\frac{1}{1+t} \frac{\partial^3 u}{\partial x^3},$$

where $Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{z^3}{3} + xz\right) dz$, $x \in \mathbb{R}$, represents the so-called Airy function.

Proof. The last Corollary is a consequence of the well-known fact that

$$M_{-1/3, 1-1/3}(z) = 3^{2/3} Ai(z/3^{1/3}).$$

□

5. The Nonlinear Case: Some Explicit Results

In this section, we provide some results regarding the nonlinear diffusive counterpart of the fractional Equation (30)

$$O_v^t u = \frac{\partial^2(u^m)}{\partial x^2}, \quad m > 0. \quad (35)$$

This is a generalized time-fractional porous medium equation with time-varying diffusivity. One can observe that, in the recent literature, there are many relevant studies about the time-fractional porous medium equation; see, e.g., Dipierro et al. in [33].

In this case, we obtain the following result.

Theorem 3. Let $v \in (0, 1)$, $m \in \mathbb{R}^+ \setminus \{1\}$, then Equation (35) admits a solution of the form

$$u(x, t) = \left[\frac{(m-1)^2}{2m(m+1)} \frac{\Gamma(1 - \frac{v}{m-1})}{\Gamma(1 - \frac{v}{m-1} - v)} \right]^{\frac{1}{m-1}} \left[\frac{x^2}{(\ln(1+t))^v} \right]^{\frac{1}{m-1}}, \quad t > 0. \quad (36)$$

Proof. We can search a particular solution in the separating variable form

$$u(x, t) = f(t)x^{\frac{2}{m-1}}, \quad (37)$$

based on the fact that

$$\frac{\partial^2}{\partial x^2} \left(x^{\frac{2}{m-1}} \right)^m = \left(\frac{2m}{m-1} \right) \left(\frac{2m}{m-1} - 1 \right) x^{\frac{2}{m-1}}. \quad (38)$$

By substitution, it follows that $f(t)$ has to satisfy the following equation:

$$O_v^t f(t) = \left(\frac{2m}{m-1} \right) \left(\frac{2m}{m-1} - 1 \right) f^m(t). \quad (39)$$

We now assume that (39) admits a solution of the form

$$f(t) = C_1(\ln(1+t))^\gamma, \quad (40)$$

where γ and C_1 are two as-yet unknown functions of m and v .

By substituting (40) in (39) and by using the property (2) of the fractional operator appearing in (35), we find that

$$\begin{aligned} O_v^t C_1(\ln(1+t))^\gamma &= C_1 \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-v)} (\ln(1+t))^{\gamma-v} \\ &= \left(\frac{2m}{m-1} \right) \left(\frac{2m}{m-1} - 1 \right) C_1^m (\ln(1+t))^{\gamma m} \end{aligned} \quad (41)$$

and this equality is clearly satisfied only if

$$\begin{aligned} \gamma &= -\frac{v}{m-1}, \\ C_1 &= \left[\frac{1}{\left(\frac{2m}{m-1} \right) \left(\frac{2m}{m-1} - 1 \right)} \frac{\Gamma(1 - \frac{v}{m-1})}{\Gamma(1 - \frac{v}{m-1} - v)} \right]^{\frac{1}{m-1}} \end{aligned}$$

as claimed. \square

Remark 3. Observe that the obtained solution can be considered as a fractional counterpart of the similarity solution

$$u(x, t) \sim \left(\frac{x^2}{\ln(1+t)} \right)^{\frac{1}{m-1}}$$

of the nonlinear diffusive equation

$$\frac{\partial u}{\partial t} = D(t) \frac{\partial^2 (u^m)}{\partial x^2}. \quad (42)$$

with

$$D(t) \sim \frac{1}{t+1}$$

Remark 4. The sign of the solution obtained above clearly depends on m and v . Moreover, this similarity-type solution is peculiar for non-linear diffusive equations, as m clearly has to be non-vanishing.

6. Conclusions

In this paper, we have introduced an ultraslow stochastic diffusion scattering as $\ln(1+t)$. We have solved a fractional-time abstract heat equation involving the Hadamard-time derivative and the infinitesimal generator L of a C_0 -semigroup. The solution of this equation is interpreted as a time-changed Lévy process. Furthermore, the Hadamard derivative leads to a deterministic rescaling time in our model as well. Particularly interesting is the case $L = \frac{d^2}{dx^2}$, which is examined in depth. Finally, we have shown that a nonlinear Hadamard-type fractional heat equation has a particular solution. This mathematical approach can be applied in future studies; for example, for the analysis of Hadamard-type fractional diffusions in a bounded domain. Moreover, we observe that sub-diffusion equations with Caputo fractional derivatives with respect to another function are attracting interest and should be objects of further research (see [34]).

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