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Fractals Parrondo's Paradox in Alternated Superior Complex System

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Abstract: This work focuses on a kind of fractals Parrondo's paradoxical phenomenon "disconnected+disconnected=connected" in an alternated superior complex system $z_{n+1} = \beta(z_n^2 + c_i) + (1 - \beta)z_n$, $i = 1, 2$. On the one hand, the connectivity variation in superior Julia sets is explored by analyzing the connectivity loci. On the other hand, we graphically investigate the position relation between superior Mandelbrot set and the Connectivity Loci, which results in the conclusion that two totally disconnected superior Julia sets can originate a new, connected, superior Julia set. Moreover, we present some graphical examples obtained by the use of the escape-time algorithm and the derived criteria.

Keywords: Parrodo' paradox; Mann iteration; Julia set; alternated system; connectivity



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1. Introduction

The natural process has obvious discrete characteristics; therefore, discrete dynamical systems are usually applied for the modeling of actual processes. On the other hand, considering the complexity of nature, more and more attention has been made to the alternate iteration method [1,2], which is more accurate in revealing the complex behaviors in processes than a unique system.

In 1999, Parrondo et al. [3,4] proposed that two games with losing gains can paradoxically become a winning game. This classical "losing + losing = winning" phenomenon was known as Parrondo's paradox, which inspired a new research fever in physics and mathematics areas [5,6] about the combination of two systems with negative expected values. In this theory, the game was divided into process *A* and *B*. As the game goes on, *A* actually changes the distribution of *B* branch, and the overall outcome changes. By analyzing the trajectories of system states, Almeida et al. extended the paradox to the chaos area and exposed the "chaos + chaos = order" phenomenon, which indicated that two chaotic behaviours can reduce to order via alternate iteration.

It should be noted that fractals and chaos are two basic branches in nonlinear science and, to some extent, are closely related to each other. Although the concept of fractal was given in 1975 [7], its basic principle was put forward as early as 1918, when Gaston Julia [8] firstly investigated a simple complex map $z_{n+1} = z_n^2 + c$, $z_n, c \in \mathbb{C}$. Aided by computer technology, Mandelbrot [7] visualized the parameter area where the connected Julia sets' parameter *c* is located. In recent years, there has been much research surrounding the properties of *M-J* sets [9–13], effects of noise disturbance [14–18] and related applications [17–22]. Meanwhile, a few researchers have also focused on the special fractal sets generated from alternated complex maps, superior complex maps, hyper-complex systems, etc. In addition, fractional mathematics is closely related to chaos and fractals. Fractional systems are worth studying from the fractal perspective. In [23,24], researchers investigated the citation profiles of researchers in fractional calculus, and proposed that the application areas of fractional calculus contain the fractal concept. Based on the control theory and method, Wang [25–27] investigates the Julia sets of a fractional Lotka–Volterra model and

realizes its state feedback control. In [28], the numerical simulation of a Boussinesq equation with different fractal dimension and fractional order is carried out. The results show that the correlation model is suitable for groundwater flow in fractured media.

In the various research on fractals, connectivity is one of the most basic and important branches. In physics, Wang indicated that the connectivity of Julia sets can be used to describe particle velocity [20,21]. In biology, Mojica proposed that the cells differentiating real organisms are similar in some features of connected Julia sets [29]. Based on the two-dimensional predator–prey model, Sun et al. applied Julia sets to represent the origin area to ensure the coexistence of two populations. The authors presented that the connectivity of such an origin area is important for the stability of populations [22]. The connectivity investigations mentioned above were mainly concentrated on the Julia sets from a single map. For alternate cases, Danca [1,2] illustrated, in the graphical results, that the connectivity properties have many forms, including connected, disconnected and totally disconnected. Further, Wang [30] compared the connectivity of an alternate iteration Julia set with their original separated Julia sets and gave a preliminary study on the fractals Parrondo's phenomenon of the classical alternate system.

Recently, some of the fractals studies concentrate on Julia sets generated by a more complicated iteration scheme.

For instance, Mandelbrot and Julia sets generated by using the Picard–Mann iteration procedure were introduced in [31]. Based on the Jungck–CR iteration process with s -convexity, authors proved new escape criteria for the generation of Mandelbrot and Julia sets and presented some graphical examples obtained by the use of an escape time algorithm and the derived criteria in [32]. In [33], the authors investigated the biomorphs for certain polynomials by using a more general iteration method and examined their graphical behaviour with respect to the variation in parameters. In [34], the authors adjust algorithms according to the developed conditions and draw some attractive Julia and Mandelbrot sets with iterate sequences from proposed fixed-point iterative methods. Moreover, some results about superior M - J sets was presented by Rani in [13,35]. Further, Rani and Yadav [36] alternated two maps of quadratic family $z_{n+1} = \beta(z_n^2 + c_i) + (1 - \beta)z_n$, $i = 1, 2$ in superior orbit, and indicated that alternate superior Julia sets also show three connectivities: connected, disconnected and totally disconnected. The effects of the superior Mandelbrot set were searched by a new noise criterion in [37]. In [38], researchers found that the superior Julia set showed a higher stability in certain high intensities, and discussed its application in particle dynamics. The effects of dynamic noise in superior Mandelbrot sets were analyzed in [39]. In [40], Mann iteration and superior Julia sets were used for biological morphogenesis algorithm optimization.

Although considerable studies have been made on the superior M - J set, to our best knowledge, little attention was paid to the connectivity investigation. As mentioned above, superior Julia sets have a higher stability than the classical ones, and also show potential application prospects. Thus, it is of interest to seek the Parrondo's paradox in the alternate superior complex system from the perspective of connectivity.

Motivated by the significant investigations mentioned above, the main motivation of this work is to provide a detailed analysis of the connectivity change law of superior Julia sets in an alternated case.

The reminder of this paper is organized as follows. Essential definitions and lemmas are given in Section 2. In Section 3, graphical explorations of alternate superior Julia sets are investigated. Through the use of the escape-time algorithm and image simulation method, "disconnected + disconnected = connected" and "connected + connected = disconnected" phenomena are proved in visual way. Section 4 concludes this work by discussing the potential applications of this fractal's phenomenon and pointing out the prospective research direction.

2. Preliminaries

Definition 1 ([13]). Consider the following Mann iteration, which can be introduced a one-step feedback process

$$x_{n+1} = g(f(x_n), x_n) = \beta f(x_n) + (1 - \beta)x_n.$$

where β lies between 0 and 1, x_n represents input and x_{n+1} is expressed as output. Simplifying the process with invariable β and then consider this process as complex quadratic map

$$P_c : z_{n+1} = \beta(z_n^2 + c) + (1 - \beta)z_n, \quad 0 < \beta \leq 1. \quad (1)$$

When $\beta = 1$, P_c can be seen as a simple complex map: $z_{n+1} = z_n^2 + c$. The filled superior Julia set of system (1) is defined as $K(P_c)$, which satisfies that

$$K(P_c) = \{z_0 | P_c^n(z_0) \not\rightarrow \infty, n \rightarrow \infty\},$$

where P_c^n denote the n -th iteration of z_0 . The superior Julia set of system (1), denoted by SJ is the boundary of $K(P_c)$, i.e., $SJ(P_c) = \partial K(P_c)$.

Definition 2. The Mandelbrot-efficacy set of system (1) is defined as

$$M(P_c) = \{c | \text{The superior Julia sets } SJ(P_c) \text{ is connected}\}.$$

Definition 3 ([13]). Considering two complex quadratic maps alternated in superior orbit

$$P_{c_1, c_2} : z_{n+1} = \begin{cases} \beta(z_n^2 + c_1) + (1 - \beta)z_n, & \text{if } n \text{ is even,} \\ \beta(z_n^2 + c_2) + (1 - \beta)z_n, & \text{if } n \text{ is odd,} \end{cases} \quad (2)$$

The filled alternate superior Julia set of system (2) is denoted as $K(P_{c_1, c_2})$, which satisfies that

$$K(P_{c_1, c_2}) = \{z_0 | P_{c_1, c_2}^n(z_0) \not\rightarrow \infty, n \rightarrow \infty\},$$

where P_{c_1, c_2}^n represents the n -th iteration of the initial point z_0 . The alternate superior Julia set of P_{c_1, c_2} is the boundary of the filled alternate superior Julia set, that is, $SJ(P_{c_1, c_2}) = \partial K(P_{c_1, c_2})$.

Lemma 1 ([36]). For system

$$O_{c_1, c_2} : z_{n+1} = \beta((z_n^2 + c_1)^2 + c_2) + (1 - \beta)(z_n^2 + c_1), \quad z_n, c_1, c_2 \in \mathbb{C}.$$

$SJ(P_{c_1, c_2})$ and $SJ(O_{c_1, c_2})$ are the same for given c_1 and c_2 parameter values.

Lemma 2 ([1,2]). The connectivity properties of superior Julia set for a complex polynomial of degree 2 and $0 < \beta \leq 1$ can be identified based on the following cases:

- (1) Superior Julia set is connected if and only if all the critical orbits are bounded;
- (2) Superior Julia set is totally disconnected, a red Cantor set, if (but not only if) all the critical orbits are unbounded;
- (3) For a polynomial with at least one critical orbit unbounded, the superior Julia set is totally disconnected if and only if all the bounded critical orbits are aperiodic.

3. Graphical Explorations

As can be seen from Figure 1, with the decrease in the value of β , the superior Mandelbrot set $M(P_c)$ expands rapidly. Therefore, we only consider the case of $\beta = 0.9$ in the next simulations.

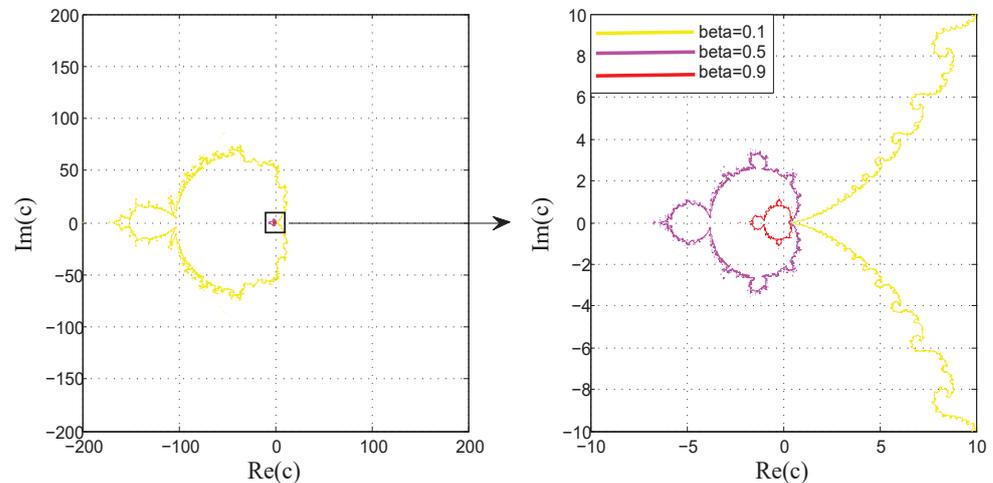


Figure 1. Superior Mandelbrot sets $M(P_c)$ plotted with different β .

Based on the above definitions, system P_{c_1, c_2} originates from the alternation of two single systems P_c , and the superior Julia sets of P_c have only two states, which are determined by the single critical point 0. The Julia sets which are plotted in Figure 2 indicate that the different connectivity relying on weather parameter c belong to the Mandelbrot set.

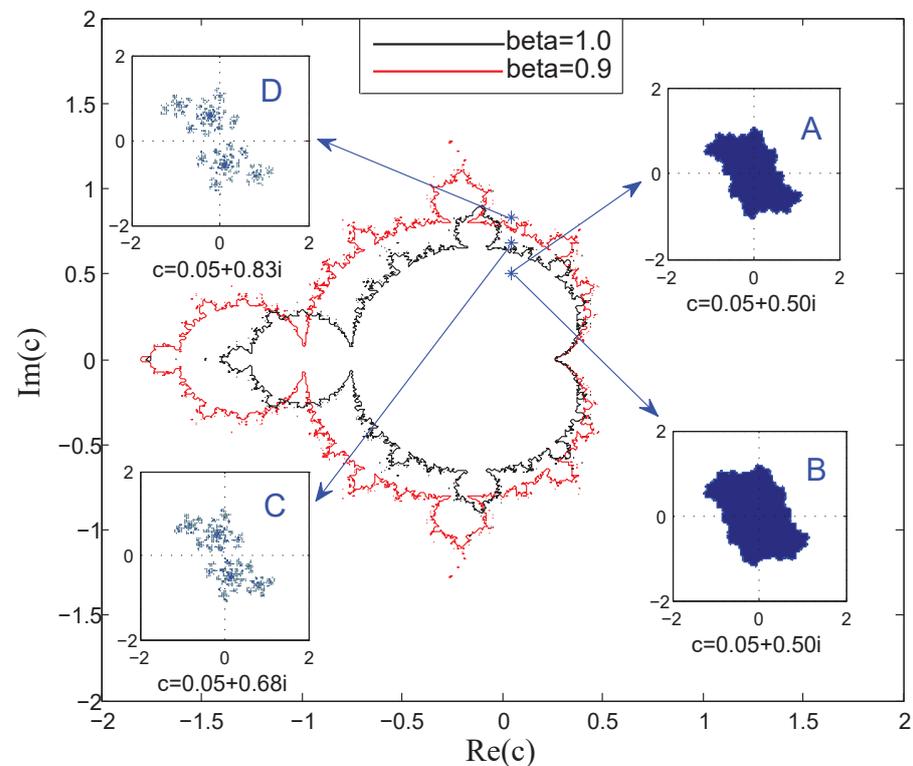


Figure 2. $M(P_c)$, classical Mandelbrot set and (A) Connected Julia set, (B) Connected superior Julia set, (C) Totally disconnected Julia set, and (D) Totally disconnected superior Julia set.

For a single system P_c , $M(P_c)$ can be plotted along two coordinates ($Re\ c$, $Im\ c$). Further, in this study, the whole Connectivity Loci of an alternate superior system P_{c_1, c_2} is defined as $\mathcal{M}(\mathcal{P})$, which is determined by four coordinates ($Re\ c_1$, $Im\ c_1$, $Re\ c_2$, $Im\ c_2$). With the help of the graphical method proposed in [2], this paper visualized its structure via *MATLAB* software. At a certain resolution, fixing the $Im(c_2)$ to 0.3 and screening all $[Re\ c_1, Im\ c_2, Re\ c_2]$ which connect alternated superior Julia sets, we plotted the spatial-Connected Loci in Figure 3. Further, in Figure 4, Fixing the $Re(c_2)$ to 0, and we obtain the planar $\mathcal{M}(\mathcal{P})$ by recognizing the connectivity of the Julia sets corresponding to all $[Im\ c_2, Re\ c_2]$. In a few words, Figure 4 is a slice of a three-dimensional $\mathcal{M}(\mathcal{P})$ and similar slices can be obtained by fixing any two dimensions.

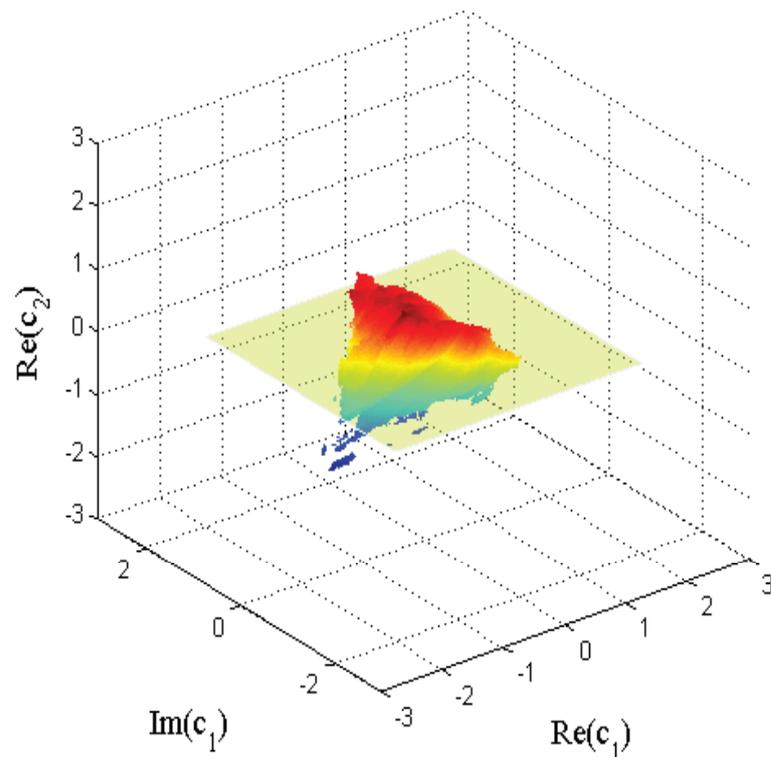


Figure 3. Spatial-Connected Loci ($\mathcal{M}(\mathcal{P})$ without Disconnected Loci) with $Im(c_2) = 0.3$.

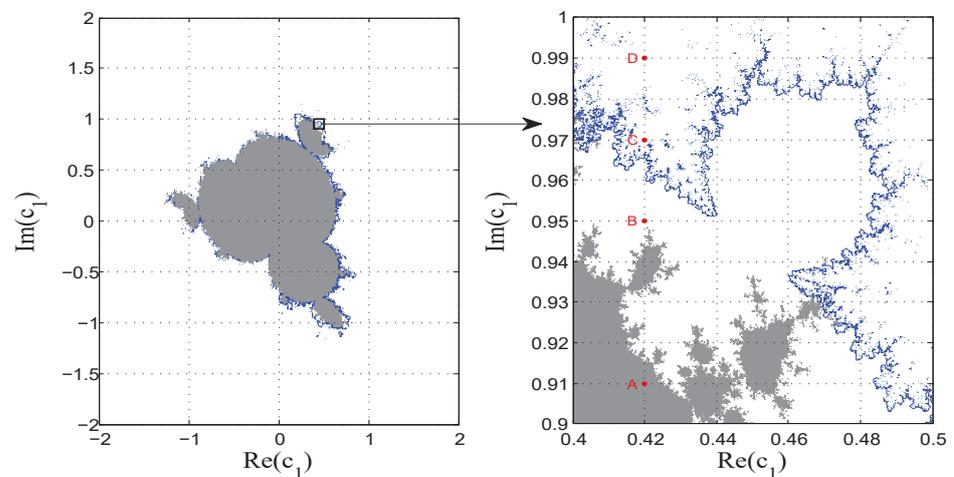


Figure 4. Planar $\mathcal{M}(\mathcal{P})$ with $c_2 = 0 + 0.3i$.

As is shown in Figure 5, the connectivity of four locations in Figure 4 is founded to vary along the Connected Loci, Disconnected Loci and Totally Disconnected Loci. That is, the gray area leads to connected superior Julia sets, the region between the grey boundary and blue line leads to disconnected superior Julia sets, the region outside the blue line gives rise to totally disconnected superior Julia sets.

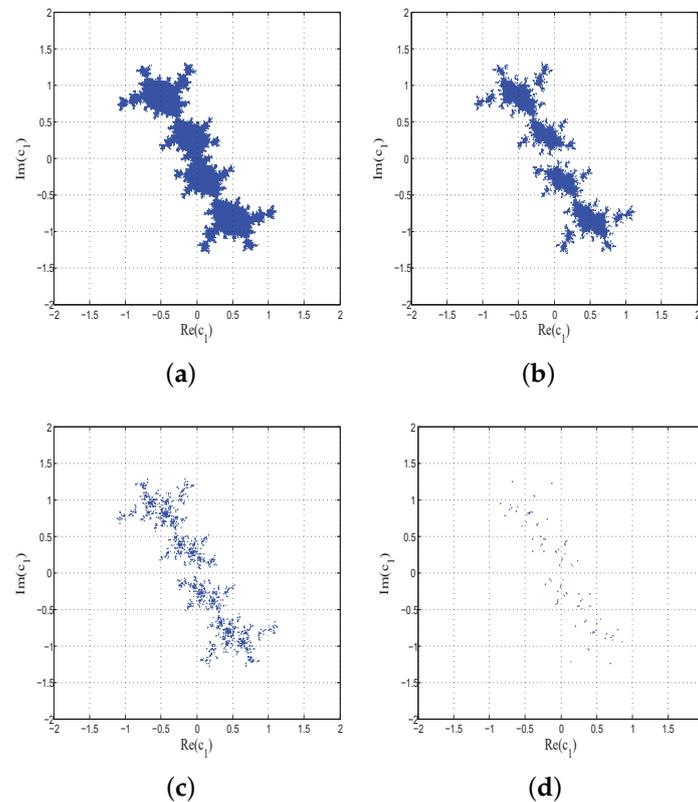


Figure 5. (a) $K(P_{c_1=0.42+0.91i, c_2=0.3i})$; (b) $K(P_{c_1=0.42+0.95i, c_2=0.3i})$; (c) $K(P_{c_1=0.42+0.97i, c_2=0.3i})$; (d) $K(P_{c_1=0.42+0.99i, c_2=0.3i})$.

Now, the next step is to find a pair of parameters c_1, c_2 which make individual superior Julia sets disconnected and alternate superior Julia sets connected. With the help of the relationship between the regions and the connectivity mentioned above, the c_2 which satisfies the phenomenon “disconnected+disconnected=connected” should be outside of the red boundary and inside the grey region.

To verify the analysis mentioned above, our solution is putting the boundary of the $M(P_c)$ cover on planar $\mathcal{M}(\mathcal{P})$. From Figure 6, it can be seen in the location of $c_1 = 0.35 + 0.59i$ that its superior Julia set $SJ(P_{c_1})$ is totally disconnected, the c_2 taken from area outside red boundary can lead to totally disconnected $SJ(P_{c_2})$, the c_2 taken from gray area can lead to connected $SJ(P_{c_1, c_2})$. For example, one c_2 can be set to $0.35 - 0.59i$ (point θ_1 in Figure 7); totally disconnected filled superior Julia set $K(P_{c=0.35+0.59i})$, totally disconnected filled superior Julia set $K(P_{c=0.35-0.59i})$ and connected filled alternate superior Julia set $K(P_{c_1=0.35+0.59i, c_2=0.35-0.59i})$ are shown in Figure 8.

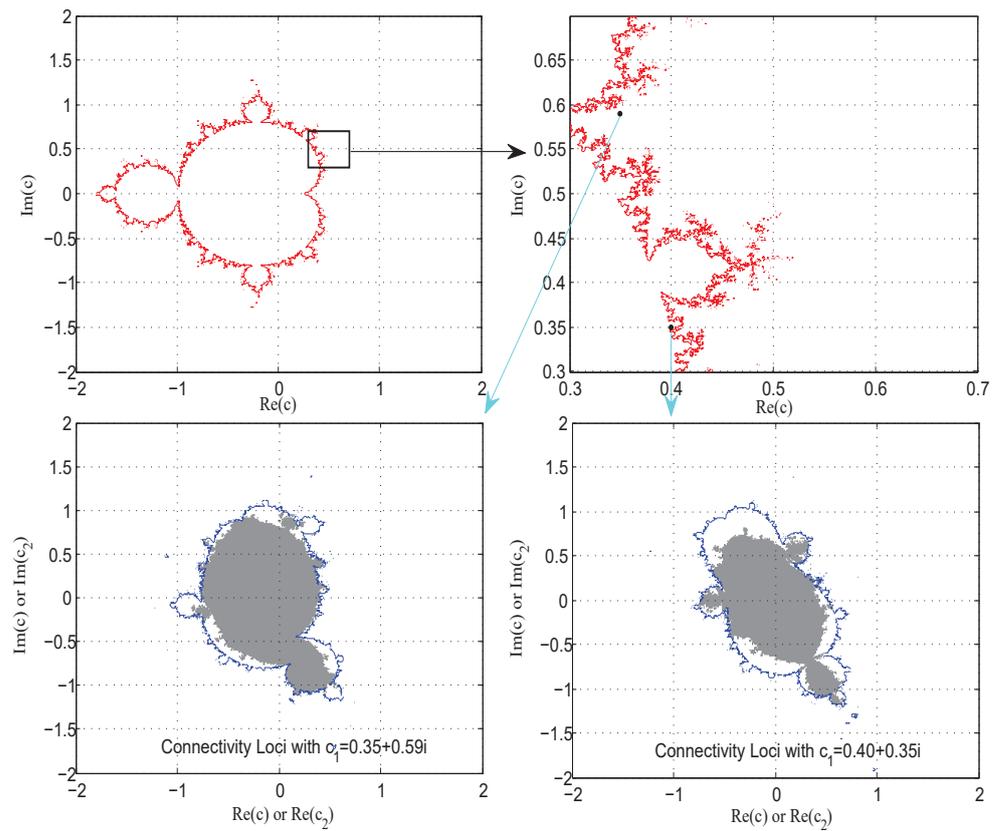


Figure 6. The boundary of $M(P_c)$ and two slices of $M(P)$.

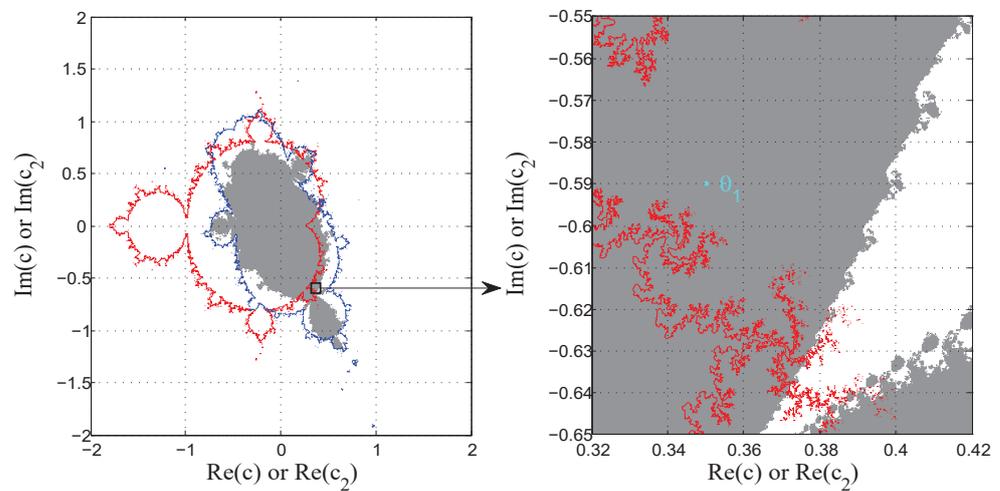


Figure 7. The detail of $M(P)$ with $c_1 = 0.35 + 0.59i$.

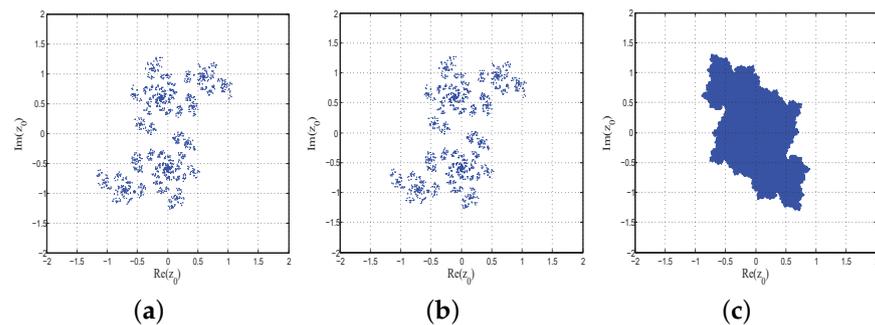


Figure 8. (a) $K(P_{c=0.35+0.59i})$; (b) $K(P_{c=0.35-0.59i})$; (c) $K(P_{c_1=0.35+0.59i, c_2=0.35-0.59i})$.

Based on the above research, in addition to “disconnected+disconnected=connected”, the establishment condition of “connected+connected=disconnected” is that c_2 located between the blue and red boundary. According to the enlarged part in Figure 9, one can choose a proper point $c_2 = 0.39 + 0.32i$ (point θ_2 in Figure 9), connected filled superior Julia set $K(P_{c=0.40+0.35i})$, connected filled superior Julia set $K(P_{c=0.39+0.32i})$ and totally disconnected filled alternate superior Julia set $K(P_{c_1=0.40+0.35i, c_2=0.39+0.32i})$ are shown in Figure 10.

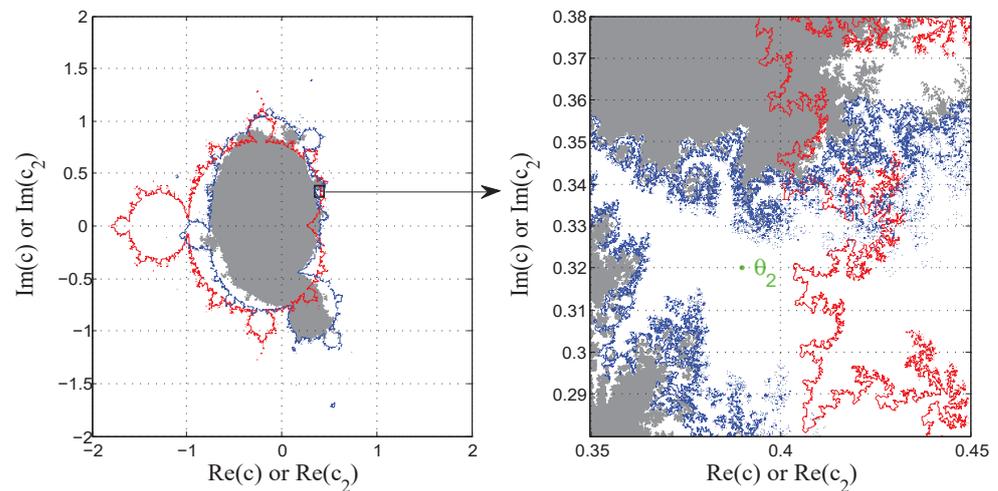


Figure 9. The detail of $\mathcal{M}(\mathcal{P})$ with $c_1 = 0.4 + 0.35i$.

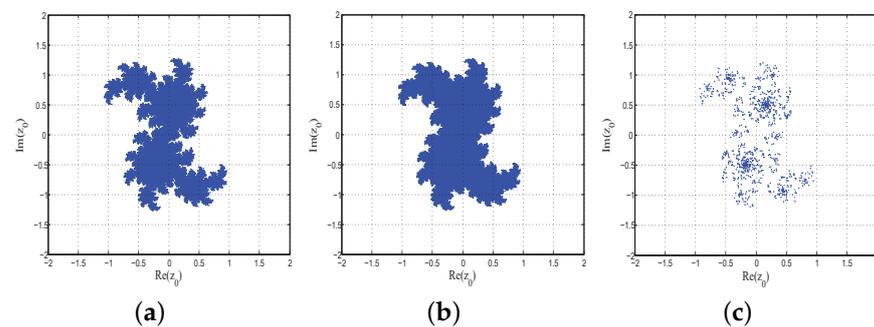


Figure 10. (a) $K(P_{c=0.40+0.35i})$; (b) $K(P_{c=0.39+0.32i})$; (c) $K(P_{c_1=0.40+0.35i, c_2=0.39+0.32i})$.

4. Conclusions

This paper demonstrates that “disconnected+disconnected=connected” and “connected+connected=disconnected” Parrondo’s Paradox phenomena exist in an alternate superior system. As mentioned in the introduction section, superior Julia sets show higher stability in certain situations, and alternate systems have been widely applied to physics, biology, etc. This phenomenon, occurring in alternate superior systems may have potential applications in many fields. We hope that the result of this paper can provide a reference for future research. On the other hand, according to the introduction, there is a close relationship between fractal and fractional; therefore, future research may further expand the Parrondo’s Paradox phenomenon to the fields, combining fractal and fractional.

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Conflicts of Interest: The authors declare no conflict of interest.

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