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LMI Criteria for Admissibility and Robust Stabilization of Singular Fractional-Order Systems Possessing Poly-Topic Uncertainties

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Abstract: The issue of robust admissibility and control for singular fractional-order systems (FOSs) with polytopic uncertainties is investigated in this paper. Firstly, a new method based on linear matrix inequalities (LMIs) is presented to solve the admissibility problems of uncertain linear systems. Then, a solid criterion of robust admissibility and a corresponding state feedback controller are derived, which overcome the conservatism of the existing results. Finally, for the sake of demonstrating the validity of proposed results, some relevant examples are provided.

Keywords: fractional-order systems; stability criteria; admissibility; robust stabilization; linear matrix inequalities (LMIs)

1. Introduction

In recent years, the application of fractional calculus in the control field has gradually become a research hotspot. The importance of fractional-order systems (FOSs) increases day after day, and FOSs have obtained much attention [1–4]. Physical systems can be well described by using fractional differential equations, and fractional modeling can also be used in robotics, biomedicine, seismic analysis, and other practical fields. Lately, controllers [5–10] have been used to improve the robustness of closed-loop systems and enhance many systems' performance. In contrast to the classical integer-order controller, the fractional derivative has the memory property, so it can provide a powerful tool for modeling.

Stability analysis is the basis of FOSs, which is related to their order [11]. The existing LMI stability criteria are divided into two cases: the (0, 1) case and the [1, 2] case [12], respectively. In practical application, there are some uncertainties in many models, so these uncertainties are taken into account when modeling and performing analysis [13,14]. Uncertainties are considered in many systems; see the details in [15–20]. For example, for fractional-order linear time-invariant (FO-LTI) systems, a robust stability test method for FO-LTI systems with uncertain interval coefficients in the form of a state space was proposed in [17], which uses the method of determining the range of interval eigenvalues by matrix perturbation theory to overcome the difficulty of finding the argument of each eigenvalue. As an improvement of [17], a necessary and sufficient condition of testing the robust stability by using Lyapunov inequality to obtain a maximum eigenvalue of a Hermitian matrix was presented in [18]. A more exact robust stability condition for FOSs was proposed in [19]. Nonetheless, FO-LTI interval systems studied in [17–19] cannot accurately describe the perturbed FO-LTI systems. Therefore, the problem of the robust asymptotic stability of FOSs with structural perturbations was studied in [20], which is more general than the study of FOSs.

Recently, many basic concepts and results on the stability of normal integer-order systems have been successfully extended to singular integer-order systems and FOSs [21–25]. Sufficient conditions

for the robust asymptotic stabilization of uncertain singular FOSs with $0 < \alpha < 2$ were proposed in [21]. However, the method is conservative, which involves the assumption that the system can be normalized. Therefore, this method is essentially a normal system solution method, rather than a singular system solution method. The difference and relation between the fractional-order system and integer-order system were pointed out in [22]. The stability properties of fractional-order differential system were presented in [11], which are the basis of some FOSs' stability analysis. The problem of robust stability and stabilization for fractional interval systems was studied in [13,19]. However, the uncertainties involved do not satisfy the norm bounded condition. Therefore, the stability and stabilization LMI criteria for fractional-order systems were developed based on D-stability [23], and new methods were introduced to deal with FOS control problems. The linear parameter-varying (LPV) discrete-time systems were studied in [24,25]. Under the assumption that all system state space matrices are uncertain parameters, the problem of robust state estimation was studied, and the necessary and sufficient conditions for the admissibility of discrete LPV systems were proposed in [24]. The design of a discrete time LPV system with H_∞ gain-scheduled (GS) state feedback control was proposed for the first time to deal with the admissibility of LPV systems in [24].

Although it has made great contributions in both of the above aspects, up to now, for norm bounded uncertainties, the admissibility and robust stabilization of singular FOSs are still open problems. In the existing research, there are few stability analysis results for polytopic uncertainties. Necessary and sufficient conditions for robust stability with norm-bounded uncertainties were presented in [26]. It is noted that the sufficient condition for robust stability with polytopic-type uncertainties [26] is less conservative. Therefore, another brand-new model with polytopic uncertainties of FOSs was studied in [27]. The discussion of the order of systems considered including $(0, 1)$ and $[1, 2)$ was more comprehensive. Then, the sufficient condition for the robust stability of fractional-order uncertain linear systems was proposed. However, we find that Theorem 1 in [27] is questionable, and its conditions are conservative. Thus, in this paper, new criteria for the stability and robustness of polytopic uncertainties of FOSs are presented, and state feedback controllers are designed. In particular, the new method similar to [28] is used to deal with the admissibility issue of singular FOSs with the polytopic uncertainties, which improves the condition of Theorem 1 in [27] and overcomes the conservativeness. The proofs are also relatively simple. Then, some examples are given to elucidate the validity of the conclusions.

The organizational structure of this paper is as follows: In Section 2, a preliminary definition is introduced, and the model is established, then a related lemma is proposed. Some main results are provided in Section 3. Finally, four examples are given in Section 4 to testify to the significance of the approach. In Section 5, some conclusions are proposed to end the paper .

2. Preliminaries

Definition 1 ([4]). *The Caputo fractional derivative of the function $x(t)$ is proposed as follows:*

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau)d\tau,$$

where the Euler Gamma function is represented as $\Gamma(\cdot)$ and n is an integer that satisfies $n-1 < \alpha \leq n$.

Lemma 1 ([28]). *There are two conditions proposed as follows, which are equivalent:*

1. *Unforced System (1) with order $0 < \alpha < 1$ is admissible.*
2. *Some matrices $X, Y \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{(n-m) \times n}$ exist such that Inequalities (2) and (3) hold or Inequalities (2) and (4) hold,*

$$ED^\alpha x(t) = Ax(t) + Bu(t), \tag{1}$$

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \tag{2}$$

$$\{A(aXE^T - bYE^T + PQ)\} + \{A(aXE^T - bYE^T + PQ)\}^T < 0, \tag{3}$$

$$\{A^T(aXE - bYE + \bar{P}Q)\} + \{A^T(aXE - bYE + \bar{P}Q)\}^T < 0, \tag{4}$$

where a and b represent $\sin(\frac{\alpha\pi}{2})$ *and* $\cos(\frac{\alpha\pi}{2})$, *respectively, and P and $\bar{P}$ are given matrices, which have full column rank and satisfy* $EP = 0$, $E^T\bar{P} = 0$.

Problem Statement

A mathematical model in ordinary physical systems is considered as follows [29]:

$$(A_n + \Delta_n)\, ED^{n\alpha}\tilde{x} + \cdots + (A_1 + \Delta_1)\, ED^\alpha\tilde{x} + (A_0 + \Delta_0)\, \tilde{x} = F, \tag{5}$$

where α is the fractional commensurate order satisfying $0 < \alpha < 1$. $A_i \in \mathbb{R}^{m \times m}$ are known nominal matrices, and $\Delta_i \in \mathbb{R}^{m \times m}$ are unknown. F is a vector denoting a driving source, which is known. $D^\alpha(\tilde{x}), \cdots, D^{n\alpha}(\tilde{x})$ represent the differential of vectors $\tilde{x}$. $A_n + \Delta_n$ is invertible [30].

It is assumed that $z_1(t) = \tilde{x}(t)$, $z_2(t) = ED^\alpha\tilde{x}(t)$, $\cdots, z_n(t) = ED^{(n-1)\alpha}\tilde{x}(t)$. Then, we have the following formulas,

$$\begin{cases} ED^\alpha z_1(t) = z_2(t) \\ \quad\vdots \\ ED^\alpha z_{n-1}(t) = z_n(t) \\ ED^\alpha z_n(t) = (A_n + \Delta_n)^{-1}\left[F - (A_0 + \Delta_0)z_1(t)\cdots - (A_{n-1} + \Delta_{n-1})z_n(t)\right]. \end{cases} \tag{6}$$

The following formula can be derived as [29],

$$-(A_n + \Delta_n)^{-1}(A_i + \Delta_i) = (I_n + A_n^{-1}\Delta_n)^{-1}(-A_n^{-1}A_i - A_n^{-1}\Delta_i). \tag{7}$$

According to Formulas (6) and (7), System (1) can be transformed into a new formula as follows,

$$ED^\alpha x(t) = (I + \Delta_I)[(A + \Delta_A)x(t) + Bu(t)], \tag{8}$$

where $x(t) \in \mathbb{R}^{mn \times mn}$ is the pseudo-state vector and $A \in \mathbb{R}^{mn \times mn}$ and $B \in \mathbb{R}^{mn \times m}$ are the system parameter matrices. $E \in \mathbb{R}^{mn \times mn}$ is a singular matrix with rank(E) = $r < mn$, and D^α is the fractional differential operator of order α. In addition, Δ_I and Δ_A represent the uncertainties with the polytopic structure,

$$\Delta_I = \sum_{i=1}^{P} r_i S_i, \quad \Delta_A = \sum_{i=1}^{q} s_i F_i, \tag{9}$$

where both r_i and s_i represent uncertain parameters, and their range is $|r_i| \le r$ and $|s_i| \le s$, where both r and s are positive numbers and both S_i and F_i are given constant matrices that have appropriate dimensions. Therefore, according to Formula (9), the following inequalities can be obtained clearly,

$$\Delta_I\Delta_I^T \le H, \quad \Delta_A\Delta_A^T \le G, \qquad H = \sum_{i=1}^{P} pr^2 S_i S_i^T, \quad G = \sum_{i=1}^{q} qs^2 F_i F_i^T. \tag{10}$$

System (8) is globally stabilizable where a controller $u(t) = Kx(t)$ is designed,

$$ED^\alpha x(t) = (A + BK + \Delta_A + \Delta_I A + \Delta_I BK + \Delta_I \Delta_A)x(t). \tag{11}$$

If $u(t) = 0$, System (8) is reduced to the following form, whose uncertain parameters are second-order nonlinear,

$$ED^\alpha x(t) = (A + \Delta_A + \Delta_I A + \Delta_I \Delta_A)x(t), \tag{12}$$

where $\Delta_I \Delta_A$ denotes that the second-order uncertain parameter is unknown.

Remark 1. *It is easy to find that the FO-LTI interval uncertain systems studied in [17–20] are described by the system:*

$$D^\alpha x(t) = (A + \Delta_A)x(t) + Bu(t),$$

and these results may not be applicable to FOS (8). For FO-LTI systems with second-order nonlinear uncertain parameters, there are few research results on the robust stability, and the parameters' uncertainties considered in [20] are provided in linear form rather than nonlinear form as System (8).

3. Main Results

In this section, some new criteria for the robust admissibility problems of uncertain FOSs are proposed.

Theorem 1. *Unforced System (8) is admissible if some matrices $X, Y \in \mathbb{R}^{mn \times mn}$, $Q_1 \in \mathbb{R}^{(mn-r) \times mn}$, and some scalars $\varepsilon_1 > 0$ and $\eta_1 > 0$ satisfy Formula (2) and:*

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \star & \Omega_{22} \end{bmatrix} < 0, \tag{13}$$

where:

$$\Omega_{11} = \mathrm{sym}\{A(aXE^T - bYE^T + PQ_1)\} + \varepsilon_1 G + H,$$

$$\Omega_{12} = [\ (aXE^T - bYE^T + PQ_1)^T \quad (aXE^T - bYE^T + PQ_1)^T A^T \quad (aXE^T - bYE^T + PQ_1)^T],$$

$$\Omega_{22} = -\mathrm{diag}(\ \varepsilon_1 I \quad (I - \eta_1 G)I \quad \eta_1 I\ ).$$

Proof. Unforced System (8) is obtained in the following form,

$$ED^\alpha x(t) = \tilde{A}x(t), \tag{14}$$

where $\tilde{A} = A + \Delta_I A + \Delta_A + \Delta_I \Delta_A$.

In light of Lemma 1, we can obtain that Equation (14) is admissible if some matrices $X, Y \in \mathbb{R}^{mn \times mn}$, $Q_1 \in \mathbb{R}^{(mn-r) \times mn}$ satisfy Formula (2) and:

$$\mathrm{sym}\{\tilde{A}(aXE^T - bYE^T + PQ_1)\} = \mathrm{sym}\{(A + \Delta_I A + \Delta_A + \Delta_I \Delta_A)(aXE^T - bYE^T + PQ_1)\} < 0. \tag{15}$$

In order to facilitate the analysis, Formula (15) mentioned above is expanded as follows,

$$\mathrm{sym}\{A(aXE^T - bYE^T + PQ_1) + \Delta_A(aXE^T - bYE^T + PQ_1) \\ + \Delta_I(A + \Delta_A)(aXE^T - bYE^T + PQ_1)\} < 0. \tag{16}$$

For arbitrary $\varepsilon > 0$, we have the following inequality,

$$(\varepsilon^{\frac{1}{2}}X - \varepsilon^{-\frac{1}{2}}Y)^T(\varepsilon^{\frac{1}{2}}X - \varepsilon^{-\frac{1}{2}}Y) = \varepsilon X^T X + \varepsilon^{-1}Y^T Y - Y^T X - X^T Y \geq 0, \tag{17}$$

then we can conclude that,

$$X^T Y + Y^T X \leq \varepsilon X^T X + \varepsilon^{-1} Y^T Y. \tag{18}$$

According to Inequality (18) and Lemma 2 in [31], there exist two scalars ε_i, $i = 1, 2$ such that:

$$\begin{aligned} \mathrm{sym}(\Delta_A (aXE^T - bYE^T + PQ_1)) \\ \leq \varepsilon_1 \Delta_A \Delta_A^T + \varepsilon_1^{-1} (aXE^T - bYE^T + PQ_1)^T (aXE^T - bYE^T + PQ_1) \\ \leq \varepsilon_1 G + \varepsilon_1^{-1} (aXE^T - bYE^T + PQ_1)^T (aXE^T - bYE^T + PQ_1). \end{aligned} \tag{19}$$

Similar to Formula (19), we have:

$$\begin{aligned} \mathrm{sym}\{\Delta_I (A + \Delta_A)(aXE^T - bYE^T + PQ_1)\} \\ \leq \varepsilon_2 H + \varepsilon_2 (aXE^T - bYE^T + PQ_1)^T (A + \Delta_A)^T (A + \Delta_A)(aXE^T - bYE^T + PQ_1). \end{aligned} \tag{20}$$

For arbitrary $\eta_1 > 0$ satisfying $I - \eta_1 G > 0$, from Inequality (10), it is obtained that,

$$I - \eta_1 \Delta_A \Delta_A^T > I - \varepsilon_1 G > 0,$$

which means that:

$$(I - \eta_1 \Delta_A \Delta_A^T)^{-1} < (I - \eta_1 G)^{-1}.$$

From Lemma 2 in [31], it follows that:

$$(A + \Delta_A)^T (A + \Delta_A) \leq A^T (I - \eta_1 \Delta_A \Delta_A^T)^{-1} A + \eta_1^{-1} I < A^T (I - \eta_1 G)^{-1} A + \eta_1^{-1} I. \tag{21}$$

According to Inequalities (20) and (21), for simplicity, setting $\varepsilon_2 = 1$, we have the following inequality,

$$\begin{aligned} \mathrm{sym}\{\Delta_I (A + \Delta_A)(aXE^T - bYE^T + PQ_1)\} \\ < H + (aXE^T - bYE^T + PQ_1)^T \{A^T (I - \eta_1 G)^{-1} A + \eta_1^{-1} I\}(aXE^T - bYE^T + PQ_1). \end{aligned} \tag{22}$$

Substituting Inequalities (19) and (22) into Formula (16) yields:

$$\begin{aligned} \mathrm{sym}\{\widetilde{A}(aXE^T - bYE^T + PQ_1)\} < \mathrm{sym}\{A(aXE^T - bYE^T + PQ_1)\} \\ + \varepsilon_1 G + \varepsilon_1^{-1} (aXE^T - bYE^T + PQ_1)^T (aXE^T - bYE^T + PQ_1) \\ + H + (aXE^T - bYE^T + PQ_1)^T \{A^T (I - \eta_1 G)^{-1} A + \eta_1^{-1} I\}(aXE^T - bYE^T + PQ_1) < 0. \end{aligned} \tag{23}$$

At last, the relationship is equivalent between Formulas (23) and (13). $\square$

Theorem 2. *If matrices X, $Y \in \mathbb{R}^{mn \times mn}$, $Q_2 \in \mathbb{R}^{(mn-r) \times mn}$, and some scalars γ_1, γ_3, and $\eta_2 > 0$ satisfy Inequality (9) and:*

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \star & \Gamma_{22} \end{bmatrix} < 0, \tag{24}$$

where:

$$\Gamma_{11} = \mathrm{sym}\{A(aXE^T - bYE^T + PQ_2) + BZ\} + \gamma_1 G + H + \gamma_3 H,$$

$$\Gamma_{12} = [\ (aXE^T - bYE^T + PQ_2)^T \quad (aXE^T - bYE^T + PQ_2)^T A^T \quad (aXE^T - bYE^T + PQ_2)^T \quad Z^T B^T],$$

$$\Gamma_{22} = -\mathrm{diag}(\ \gamma_1 I \ (I - \eta_2 G)I \ \eta_2 I \ \gamma_3 I \ ).$$

Unforced System (8) is admissible. In addition, a controller is obtained by $K = Z(aXE^T - bYE^T + PQ_2)^{-1}$.

Proof. Consider $\overline{A} = A + BK$, then $\widehat{A} = \overline{A} + \Delta_I \overline{A} + \Delta_A + \Delta_I \Delta_A$, and on the basis of Lemma 1, we can obtain that System (8) is admissible if two matrices $X, Y \in \mathbb{R}^{mn \times mn}$ satisfy Inequality (2) and:

$$
\begin{aligned}
&\text{sym}\{\widehat{A}(aXE^T - bYE^T + PQ_2)\} \\
&= \text{sym}\{(A + BK)(aXE^T - bYE^T + PQ_2) + \Delta_A(aXE^T - bYE^T + PQ_2) \\
&+ \Delta_I(A + \Delta_A)(aXE^T - bYE^T + PQ_2) + \Delta_I BK(aXE^T - bYE^T + PQ_2)\} < 0.
\end{aligned}
\tag{25}
$$

Similar to the proof of Theorem 1, it follows from Inequality (18) and Lemma 2 in [31] that there exist some scalars γ_i, $i = 1, 2, 3$, and $\eta_2 > 0$ such that:

$$
\begin{aligned}
&\text{sym}\{\Delta_A(aXE^T - bYE^T + PQ_2)\} \\
&\leq \gamma_1 G + \gamma_1^{-1}(aXE^T - bYE^T + PQ_2)^T(aXE^T - bYE^T + PQ_2)).
\end{aligned}
\tag{26}
$$

For simplicity, set $\gamma_2 = 1$. The following inequality is the same as Inequality (26),

$$
\begin{aligned}
&\text{sym}\{\Delta_I(A + \Delta_A)(aXE^T - bYE^T + PQ_2)\} \\
&< H + (aXE^T - bYE^T + PQ_2)^T\{A^T(I - \eta_2 G)^{-1}A + \eta_2^{-1}I\}(aXE^T - bYE^T + PQ_2),
\end{aligned}
\tag{27}
$$

Letting $Z = K(aXE^T - bYE^T + PQ_2)$, it yields

$$
\text{sym}(\Delta_I BK(aXE^T - bYE^T + PQ_2)) = \Delta_I BZ + Z^T B^T \Delta_I^T \leq \gamma_3 H + \gamma_3^{-1} Z^T B^T BZ.
\tag{28}
$$

Then, substituting Inequalities (26), (27), and (28) into Formula (25) yields:

$$
\begin{aligned}
&\text{sym}\{\widehat{A}(aXE^T - bYE^T + PQ_2)\} \\
&< \text{sym}\{(A + BK)(aXE^T - bYE^T + PQ_2)\} + \gamma_1 G \\
&+ \gamma_1^{-1}(aXE^T - bYE^T + PQ_2)^T(aXE^T - bYE^T + PQ_2) \\
&+ H + (aXE^T - bYE^T + PQ_2)^T\{A(I - \eta_2 G)^{-1}A + \eta_2 I\}(aXE^T - bYE^T + PQ_2) \\
&+ \gamma_3 H + \gamma_3^{-1} Z^T B^T BZ < 0.
\end{aligned}
\tag{29}
$$

At the end of this proof, using the Schur complement in [32], the relationship between Inequalities (29) and (26) is equivalent. □

Remark 2. *The comparisons in [12,14,27] are provided in Table 1, and we find that their theorems involve many real decision variables $(P_{11}, P_{12}, P_{21}, P_{22})$, under the strict assumptions of $P_{11} = P_{21} > 0$ and $P_{12} = P_{22} = 0$ in [14], which limit the extension to the controller design. The results are conservative.*

Table 1. Comparisons with existing results.

Reference	Variable Kind	Variables	Easy to Solve?	Less Conservative?
[2]	R	1	yes	no
[12]	C	4	no	no
[14]	R	4	no	no
[27]	R	2	no	no
ours	R	2	yes	yes

4. Numerical Examples

In this section, some examples are presented to illustrate the effectiveness of our theoretical results.

Example 1. *In this case, consider an electrical circuit drawn in Figure 1 with given resistances R_i, $i = 1, 2, 3$, inductances L_i, $i = 1, 2, 3$, and source voltages e_1 and e_2 [33].*

Then, we obtain the equations as follows by using Kirchhoff's laws:

$$e_1 = R_1 i_1 + L_1 D^\alpha i_1 + R_3 i_3 + L_3 D^\alpha i_3,$$

$$e_2 = R_2 i_2 + L_2 D^\alpha i_2 + R_3 i_3 + L_3 D^\alpha i_3,$$

$$i_3 = i_1 + i_2,$$

then the following system is obtained,

$$ED^\alpha x(t) = Ax(t) + Bu(t), \tag{30}$$

where:

$$\alpha = 0.2, \quad x(t) = \begin{bmatrix} i_1^T & i_2^T & i_3^T \end{bmatrix}^T, \quad u(t) = \begin{bmatrix} e_1^T & e_2^T \end{bmatrix}^T,$$

$$E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

For simplicity, let $\Delta_A = \Delta_I = 0$, and System (29) is exactly FOS (1). Set:

$$L_i = 1, \ i = 1, 2, 3, \quad R_1 = 1, \ R_2 = 3, \ R_3 = 2.$$

Using the MATLAB LMI toolbox, we obtain that (24) is feasible, which illustrates that System (30) is admissible, and the state feedback gain is obtained as follows,

$$K = \begin{bmatrix} -3.0041 & -2.0744 & 0.9194 \\ 0.6682 & -0.6888 & -1.3623 \end{bmatrix}.$$
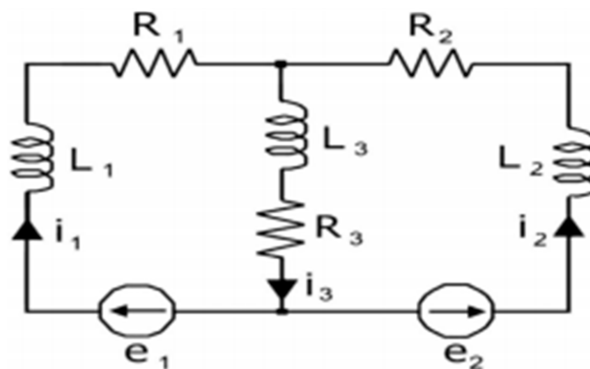

Figure 1. Electronic network.

Example 2. *In this case, consider unforced System (8) with $\alpha = 0.5$ and:*

$$A = \begin{bmatrix} -2 & -1 & -1 \\ -2 & -3 & -1 \\ -1 & -1 & -4 \end{bmatrix}, \quad H = \begin{bmatrix} 0.12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.12 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.16 & 0.16 \\ 0 & 0.16 & 0.16 \end{bmatrix}, \quad P = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

$|\arg(\mathrm{spec}(A))| > \frac{\alpha\pi}{2}$. *The feasible solutions of Inequalities (2) and (13) in Theorem 1 are obtained as follows:*

$$\varepsilon_1 = 0.9214, \quad \eta_1 = 0.9447,$$

$$X = \begin{bmatrix} 0.4872 & -0.3191 & 0.1680 \\ -0.3191 & 0.3740 & -0.2812 \\ 0.1680 & -0.2812 & 0.5251 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -0.0443 & -0.0443 \\ 0.0443 & 0 & -0.0443 \\ 0.0443 & 0.0443 & 0 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} -0.0767 & 0.1154 & -0.0835 \end{bmatrix}.$$

In light of Theorem 1, we conclude that unforced System (8) is admissible.

Example 3. *Consider System (8) with parameters as $\alpha = 0.5$ and:*

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -2 & -4 & 0 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.15 \\ 0.35 \\ 0.55 \end{bmatrix}.$$

Using the MATLAB LMI toolbox, we obtain the feasible solutions of Inequalities (2) and (24) in Theorem 2 as follows:

$$\gamma_1 = 0.9261 \quad \gamma_3 = 0.9280 \quad \eta_2 = 0.9343,$$

$$X = \begin{bmatrix} 0.4389 & -0.3044 & 0.2188 \\ -0.3044 & 0.3453 & -0.3124 \\ 0.2188 & -0.3124 & 0.4309 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -0.0341 & -0.0341 \\ 0.0341 & 0 & -0.0341 \\ 0.0341 & 0.0341 & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} -0.0257 & -0.1704 & -0.5066 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.0579 & -0.0713 & 0.0736 \end{bmatrix},$$

and the state feedback gain is obtained as follows:

$$K = Z(aXE^T - bYE^T + PQ_2)^{-1} = \begin{bmatrix} -4.1971 & -5.1599 & -7.8445 \end{bmatrix}.$$

Thus, System (8) is asymptotically admissible. The transients of the state variables are shown in Figure 2, which confirms the admissibility of System (8).

Example 4. *Consider a polytopic-type uncertain FOS (6) in [27] with the following parameters:*

$$\alpha = 0.5, \quad A = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad H = \begin{bmatrix} 0.018 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.018 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.036 & 0 \\ 0 & 0 & 0.036 \end{bmatrix}.$$

The three eigenvalues of matrix A are -5 and $6 \pm 3j$, respectively, which do not meet the condition of $|\arg(\mathrm{spec}(A))| > \frac{\alpha\pi}{2}$. Using the MATLAB LMI toolbox, we can obtain the best value of t from Inequalities (2) and (13) in Theorem 1. In this present paper, $t = 0.0031 > 0$, which means that these LMI constraints were found to be infeasible. From Lemma 1, it follows that System (6) in [27] is unstable even if $\Delta_I = \Delta_A = 0$. However, the LMI of (17) in [27] for this example has feasible solutions:

$$P_{11} = P_{21} = \begin{bmatrix} 0.0036 & 0.0002 & -0.0003 \\ 0.0002 & 0.0010 & -0.0040 \\ -0.0003 & -0.0040 & 0.1466 \end{bmatrix}, \quad P_{12} = -P_{22} = \begin{bmatrix} 0 & 00.0393 & -0.0014 \\ 0.0393 & 0 & -0.0054 \\ 0.0014 & 0.0054 & 0 \end{bmatrix},$$

$$\varepsilon_1 = 5.3572, \quad \varepsilon_{11} = \varepsilon_{21} = 1.0359, \quad \varepsilon_{12} = \varepsilon_{22} = 1.662.$$

According to Theorem 1 in [27], we can conclude that System (6) in [27] is stable, which is thewrong conclusion.
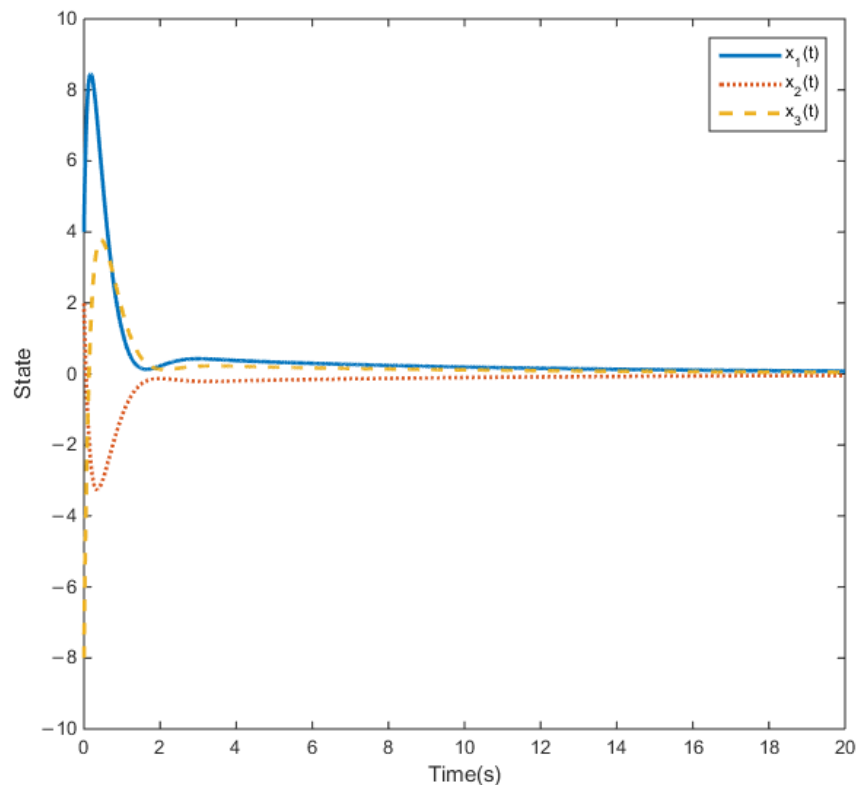

Figure 2. State response curves of System (9) in Example 3.

5. Conclusions

In this paper, the robust admissibility of FOSs with polytopic uncertainties is studied. Sufficient conditions for the robust admissibility of FOSs are proposed based on a new LMI method. Then, the corresponding criteria for the robust stabilization of FOSs with polytopic uncertainties are presented by designing feedback controllers. Finally, some examples are proposed to verify the results. The admissibility of singular FOSs with uncertain description matrix E will be studied in the future work.

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