fractal and fractional

MDPI

Viewpoint

Fractional-Order Derivatives Defined by Continuous Kernels: Are They Really Too Restrictive?

Jocelyn Sabatier

IMS Laboratory, Bordeaux University, UMR 5218 CNRS, 351 Cours de la Libération, 33405 Talence, France; jocelyn.sabatier@u-bordeaux.fr

Received: 13 June 2020; Accepted: 3 August 2020; Published: 11 August 2020

Abstract: In the field of fractional calculus and applications, a current trend is to propose non-singular kernels for the definition of new fractional integration and differentiation operators. It was recently claimed that fractional-order derivatives defined by continuous (in the sense of non-singular) kernels are too restrictive. This note shows that this conclusion is wrong as it arises from considering the initial conditions incorrectly in (partial or not) fractional differential equations.

Keywords: fractional derivative; continuous kernel; Volterra equation; fractional models' initialization; distributed time delay systems

1. Introduction

Fractional order derivative or integral operators are now widely used in the literature to define various classes of model, often "fractionalisations" of existing classical models [1] (differential equations, partial differential equations, state-space descriptions, transfer functions, etc.). However, several drawbacks associated to these operators and models have been revealed in the last ten years [2].

One of these drawbacks is the singularity of the kernel in the definition of fractional order derivative or integral operators. This singularity has prompted some authors to propose new definitions involving non-singular kernels [3,4]. The publication of these works has sparked much debate in the community [5–7]. For instance, it is claimed in [8] that the kernels used are two restrictive leads to ill-posed initial-value problems, in that they impose a severe restriction on the function $\phi(x)$ that defines the initial condition of the variable $u(t,\ x)$ (t is the time variable and x is the spatial variable) to which a fractional partial differential equation relates to. In [9,10] similar arguments are proposed. However, as explained in the sequel, these claims are not correct.

2. Problem Analysis

Although the mathematical analysis of the work presented in [8] is completely correct, it rests on the erroneous idea that initial conditions can be defined by the knowledge of the variable to which the fractional (partial or not) differential equation relates only at the initial time.

In [8] the following general problem is studied:

$$\hat{D}_t^{\alpha} u(x,t) - \sum_{i,j=1}^{n} p_{ij}(x,t) \frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} q_i(x,t) \frac{\partial u(x,t)}{\partial x_i} + q_i(x,t) u(x,t) = f(x,t) \tag{1}$$

for $(x,t) \in Q\Omega \times]0,T]$.

Boundary conditions are defined by:

$$u(x,t) = \psi(x,t) \text{ for } (x,t) \in \partial\Omega \times]0,T] \tag{2}$$

and initial conditions are defined by:

$$u(x,0) = \phi(x) \text{ for } x \in \Omega \tag{3}$$

In Relation (1), $\hat{D}_t^\alpha$ is the fractional derivative in the Caputo or Riemann-Liouville senses with $\alpha \in]0,1[$. The fractional integral of a function $f(t)$ being defined as:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \text{ with } t > 0,\ \alpha \in]0,1[\tag{4}$$

Caputo or Riemann-Liouville derivatives, taking into account initial conditions, are defined by:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t \frac{\frac{d}{d\tau} f(\tau)}{(t-\tau)^{1+\alpha}} d\tau - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} [f(t)]_{t=0} \tag{5}$$

and

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau \right) - \left[I_t^{1-\alpha} f(t) \right]_{t=0} \tag{6}$$

with $t > 0$, $\alpha \in]0,1[$.

From a mathematical point of view, these definitions are perfectly correct. However, things are different when they are used to define a dynamic model in the form of a (partial or not) differential equation. The initializations proposed in Relations (5) and (6) are then not consistent with the model trajectories.

This problem has been analyzed in the literature on several occasions. The first studies were done by Lorenzo and Hartley [11,12] who proposed to introduce initialization functions to take into account in a convenient way the past of the model in a finite interval, but also to allow the Caputo and Riemann-Liouville definitions to have the same behavior in the presence of initial conditions. The idea that initial functions must be used instead of initial values is reinforced in [13]. Later in [14], the "prehistories" of the functions before the initial instant are introduced to address initial value problems of fractional visco-elastic equations, as these functions are crucial for a unique solution. In [15,16], it was demonstrated on a counter example that initial conditions as defined in Relations (5) and (6) cannot be well taken into account in a dynamical model whether by Caputo or Riemann-Liouville definitions. An impulse response-based representation is thus introduced to define initialization and it is concluded as in [17] that fractional models and fractional derivative initializations are two distinct matters. A similar analysis was carried out for partial differential equations in [18] using an initial time shifting method. Note, a similar time shifting was recently used in [19] to demonstrate that mathematical description of the groundwater flow using time Caputo or Riemann-Liouville fractional partial derivatives is non-objective. The authors of [19] have not taken their analysis of this non-objectivity any further, but that demonstrates the necessity to introduce an initialization function to restore this objectivity.

It is, therefore, possible to claim that the initializations used in the problem studied in [8] and also in [9,10] are not correct and produces the incorrect conclusion on the restriction imposed by non-singular kernels. As discussed in [18,20], to appropriately solve this initialization, fractional integration can be used to replace fractional differentiation (to avoid the choice among plethora of definitions for fractional derivatives [21]) and to take into account all the model past. Instead of relation (1), it is thus better to consider the following model (close to a Volterra equation [22]):

$$u(x,t) = \hat{I}_t^\alpha \begin{bmatrix} f(x,t) + \sum_{i,j=1}^{n} p_{ij}(x,t) \frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} \\ -\sum_{i=1}^{n} q_i(x,t) \frac{\partial u(x,t)}{\partial x_i} - q_i(x,t) u(x,t) \end{bmatrix} + w(x,0) \tag{7}$$

where

$$\hat{I}_t^{\alpha}[g(x,t)] = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{8}$$

with

$$w(x,0) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{0} \frac{1}{(t-\tau)^{1-\alpha}} \left[\begin{array}{c} f(x,\tau) + \sum\limits_{i,j=1}^{n} p_{ij}(x,\tau) \frac{\partial^2 u(x,\tau)}{\partial x_i \partial x_j} \\ - \sum\limits_{i=1}^{n} q_i(x,t) \frac{\partial u(x,t)}{\partial x_i} - q_i(x,\tau) u(x,\tau) \end{array} \right] d\tau \tag{9}$$

Another solution (similar) is to introduce an initialization function in (1) that thus becomes:

$$\begin{array}{c} \hat{D}_t^{\alpha} u(x,t) + \mathrm{w}(x,t) - \sum\limits_{i,j=1}^{n} p_{ij}(x,t) \frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + \sum\limits_{i=1}^{n} q_i(x,t) \frac{\partial u(x,t)}{\partial x_i} + q_i(x,t) u(x,t) \\ = f(x,t) \end{array} \tag{10}$$

such as:

$$\_\hat{D}_t^{\alpha} u(x,t) = \hat{D}_t^{\alpha} u(x,t) + \mathrm{w}(x,t) \tag{11}$$

where $\hat{D}_t^{\alpha} u(x,t)$ denotes the fractional derivative of $u(x,t)$ without taking into account the model past, and where $\_\hat{D}_t^{\alpha} u(x,t)$ denotes the fractional derivative of $u(x,t)$ computed with the model past.

These initializations respect a fundamental property of fractional models: their long (and even infinite) memory. This is not the case with Relation (1) at the initial time which ignores the system past before initial time with Relation (3).

3. Conclusions

The conclusion reached in [8] on the restrictivity property of non-singular kernels in the new definitions of fractional operators is dictated by an incorrect way of considering initial conditions. The same remark can be made about [9,10] (a paper which only focus on two recently studied kernels among the infinity that can be proposed [22,23]). Fractional operators are known for their memory property. With the initialization considered in [8,9] and [10], this property would exist everywhere on the time axis, except at the initial time (which can also be different from $t = 0$). It may be mathematically conceivable and correct, but it does not make sense for a dynamical system. Memory is an intrinsic property that is present at all times (as it is also well explained in the introduction of [6]). It is well known that models involving memory kernels in their definition require an initialization with the knowledge of the past on an interval (not only for one instant) linked to the bounds of the integral on the kernel, as is done for distributed time delay models for instance [24,25]. This remark applies here to [8,9] and [10], but there are hundreds of other articles in the literature that use the same type of initialization and that deserve to be revisited.

Contrary to the criticisms made on the attempts to reformulate fractional operators, the author believes that the solutions to the drawbacks recently highlighted in the literature and related to the use of fractional operators in the definition of dynamical models can be solved by using new kernels in Volterra type operators as done in [3,4,22,23,26–28] and before in [29–37] (but other solutions also exist, as shown in [2,24]).

We cannot ignore that, for a continuous time random walk process with the power-law waiting time, probability density function and jump lengths with finite variance, the corresponding probability density function to find the particle at position x at time t in the Fourier-Laplace domain is the same as the solution of the time fractional diffusion equation with Riemann-Liouville fractional derivative [38,39]. We cannot ignore similar approaches followed with other continuous time random walk processes with different waiting time and jump length probability density functions. However, one has to observe that these results were obtained under spaces of infinite dimension and asymptotic time analysis (time tends towards infinity). This link with a fractional Riemann-Liouville operator (and others) with an infinite space domain can also be found in [15,16] and this is what induces

the distribution of time constants on $]-\infty, 0]$ inherent in fractional Riemann-Liouville and others. This constitutes one of the drawbacks of fractional operators (see [2]) (infinitely slow and infinitely fast time constant, the first inducing an infinite memory) that can be solved by changing kernels used in the Riemann-Liouville operator (and similar operators). Introducing new kernels would also permit one to take into account power law behaviors that take place in a limited frequency or time domain.

Perhaps the kernels used in these papers should be reworked to improve their properties and better reflect real world data [7], but many other kernels are possible as shown in [22] and in [23]. It is thus a path that deserves to be explored, especially since [22] also demonstrates that pseudo-state space descriptions, a fractional model very frequently used in the field of modelling and automatic control, are a particular case of a Volterra equation of the first kind. Note that all the kernels presented in [22,23] met the requirements explained in [6] and exhibit a power law behavior in a given frequency.

Can the resulting operators still claim to belong to the class of derivation and fractional integration and differentiation operators? The important thing is to have models capable of capturing power law type behaviors whose ubiquity is now proven, as fractional models allow, but without the drawbacks associated with them.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Monje, C.A.; Chen, Y.; Vinagre, B.M.; Xue, D.; Batlle, V.F. *Fractional-Order Systems and Controls: Fundamentals and Applications*; Springer: London, UK, 2010.
2. Sabatier, J.; Farges, C.; Tartaglione, V. Some alternative solutions to fractional models for modelling long memory behaviors. *Mathematics* **2020**, *8*, 196. [CrossRef]
3. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
4. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763–769. [CrossRef]
5. Ortigueira, M.D.; Machado, J.A.T. A critical analysis of the Caputo–Fabrizio operator. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *59*, 608–611. [CrossRef]
6. Tarasov, V.E. No nonlocality. No fractional derivative. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *62*, 157–163. [CrossRef]
7. Ortigueira, M.D.; Martynyuk, V.; Fedula, M.; Machado, J.T. The failure of certain fractional calculus operators in two physical models. *Fract. Calc. Appl. Anal.* **2019**, *22*, 255–270. [CrossRef]
8. Stynes, M. Fractional-order derivatives defined by continuous kernels are too restrictive. *Appl. Math. Lett.* **2018**, *85*, 22–26. [CrossRef]
9. Hanyga, A. A comment on a controversial issue: A generalized fractional derivative cannot have a regular kernel. *Fract. Calc. Appl. Anal.* **2020**, *23*, 211–223. [CrossRef]
10. Kai Diethelm, K.; Garrappa, R.; Giusti, A.; Stynes, M. Why fractional derivatives with nonsingular kernels should not be used. *Fract. Calc. Appl. Anal.* **2020**, *23*. [CrossRef]
11. Lorenzo, C.F.; Hartley, T.T. Initialized fractional calculus. *Int. J. Appl. Math.* **2000**, *3*, 249–265.
12. Lorenzo, C.F.; Hartley, T.T. Initialization in fractional order systems. In Proceedings of the European Conference On Control ECC, Porto, Portugal, 4–7 September 2001; pp. 1471–1476.
13. Ortigueira, M.D. On the initial conditions in continuous-time fractional linear systems. *Signal Process.* **2003**, *83*, 2301–2309. [CrossRef]
14. Fukunaga, M.; Shimizu, N. Role of prehistories in the initial value problems of fractional viscoelastic equations. *Nonlinear Dyn.* **2004**, *38*, 207–220. [CrossRef]
15. Sabatier, J.; Merveillaut, M.; Malti, R.; Oustaloup, A. *On a Representation of Fractional Order Systems: Interests for the Initial Condition Problem*, 3rd ed.; IFAC Workshop: Ankara, Turkey, 2008.
16. Sabatier, J.; Merveillaut, M.; Malti, R.; Oustaloup, A. How to impose physically coherent initial conditions to a fractional system? *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 1318–1326. [CrossRef]

17. Ortigueira, M.D.; Coito, F. System initial conditions vs derivative initial conditions. *Comput. Math. Appl.* **2010**, *59*, 1782–1789. [CrossRef]
18. Sabatier, J.; Farges, C. Comments on the description and initialization of fractional partial differential equations using riemann-liouville's and caputo's definitions. *J. Comput. Appl. Math.* **2018**, *339*, 30–39. [CrossRef]
19. Balint, A.M.; Balint, S. Mathematical description of the groundwater flow and that of the impurity spread, which use temporal caputo or riemann-liouville fractional partial derivatives, is non-objective. *Fractal Fract.* **2020**, *4*, 36. [CrossRef]
20. Sabatier, J.; Farges, C.; Trigeassou, J.C. Fractional systems state space description: Some wrong ideas and proposed solutions. *J. Vib. Control* **2014**, *20*, 1076–1084. [CrossRef]
21. Oliveira, E.C.D.; Machado, J.A.T. A review of definitions for fractional derivatives and integral. *Math. Probl. Eng.* **2014**, *2014*. [CrossRef]
22. Sabatier, J. Fractional state space description: A particular case of the volterra equations. *Fractal Fract.* **2020**, *4*, 23. [CrossRef]
23. Sabatier, J. Non-singular kernels for modelling power law type long memory behaviours and beyond. *Cybern. Syst.* **2020**, *51*, 383–401. [CrossRef]
24. Sabatier, J. Power law type long memory behaviors modeled with distributed time delay systems. *Fractal Fract.* **2019**, *4*, 1. [CrossRef]
25. Özbay, H.; Bonnet, C.; Clairambault, J. Stability analysis of systems with distributed delays and application to hematopoietic cell maturation dynamics. In Proceedings of the 47th IEEE Conference on Decision and Control, Cancun, Mexico, 9–11 December 2008; pp. 2050–2055.
26. Zhao, D.; Luo, M. Representations of acting processes and memory effects: General fractional derivative and its application to theory of heat conduction with finite wave speeds. *Appl. Math. Comput.* **2019**, *346*, 531–544.
27. Fernandez, A.; Özarslan, M.A.; Baleanu, D. On fractional calculus with general analytic kernels. *Appl. Math. Comput.* **2019**, *354*, 248–265.
28. Baleanu, D.; Fernandez, A. On fractional operators and their classifications. *Mathematics* **2019**, *7*, 830. [CrossRef]
29. Lutz, E. Fractional langevin equation. *Phys. Rev. E* **2001**, *64*. [CrossRef]
30. Pottier, N. Aging properties of an anomalously diffusing particle. *Phys. Stat. Mech. Appl.* **2003**, *317*, 371–382. [CrossRef]
31. Viñales, A.D.; Despósito, M.A. Anomalous diffusion induced by a Mittag-Leffler correlated noise. *Phys. Rev. E* **2007**, *75*, 042102. [CrossRef]
32. Viñales, A.D.; Wang, K.-G.; Despósito, M.A. Anomalous diffusive behavior of a harmonic oscillator driven by a mittag-leffler noise. *Phys. Rev. E* **2009**, *80*, 011101. [CrossRef]
33. Sandev, T.; Chechkin, A.; Kantz, H.; Metzler, R. Diffusion and fokker-planck-smoluchowski equations with generalized memory kernel. *Fract. Calc. Appl. Anal.* **2015**, *18*, 1006–1038. [CrossRef]
34. Liemert, A.; Sandev, T.; Kantz, H. Generalized langevin equation with tempered memory kernel. *Phys. A Stat. Mech. Appl.* **2017**, *466*, 356–369. [CrossRef]
35. Sandev, T. Generalized langevin equation and the prabhakar derivative. *Mathematics* **2017**, *5*, 66. [CrossRef]
36. Sandev, T.; Deng, W.; Xu, P. Models for characterizing the transition among anomalous diffusions with different diffusion exponents. *J. Phys. A Math. Theor.* **2018**, *51*, 405002. [CrossRef]
37. Sandev, T.; Tomovski, Z.; Dubbeldam, J.L.A.; Chechkin, A. Generalized diffusion-wave equation with memory kernel. *J. Phys. A Math. Theor.* **2018**, *52*, 015101. [CrossRef]
38. Metzler, R.; Joseph Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77. [CrossRef]
39. Barkai, E.; Metzler, R.; Klafter, J. From continuous time random walks to the fractional Fokker-Planck equation. *Phys. Rev. E* **2000**, *61*, 132–138. [CrossRef]