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Article

On the Volterra-Type Fractional Integro-Differential Equations Pertaining to Special Functions

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Abstract: In this article, we apply an integral transform-based technique to solve the fractional order Volterra-type integro-differential equation (FVIDE) involving the generalized Lorenzo-Hartely function and generalized Lauricella confluent hypergeometric function in terms of several complex variables in the kernel. We also investigate and introduce the Elazki transform of Hilfer-derivative, generalized Lorenzo-Hartely function and generalized Lauricella confluent hypergeometric function. In this article, we have established three results that are present in the form of lemmas, which give us new results on the above mentioned three functions, and by using these results we have derived our main results that are given in the form of theorems. Our main results are very general in nature, which gives us some new and known results as a particular case of results established here.

Keywords: Volterra-type fractional integro-differential equation; Hilfer fractional derivative; Lorenzo-Hartely function; generalized Lauricella confluent hypergeometric function; Elazki transform

1. Introduction

From last three decade the fractional calculus have experienced significant observation to solve the mathematics, science & engineering and mathematical physics problems [1–11]. Fractional calculus plays a vital role to derive the solution of various kinds of differential and integral equations of fractional order arising in fractal geometry, propagation of seismic waves and diffusion problems for these we can cite the following works mentioned in [2,4,11–14]. In this connection Boyadjiev et al. [15] studied the non-homogeneous fractional integro-differential equation of Volterra-type (FIDEV) and obtained the solution in closed form in terms of Kummer functions and incomplete gamma function (IGF). Al-shammery et al. [16] studied the unsaturated behavior of the freeelectron lesser (FEL) and developed an analytical and numerical treatment of fractional generalization of the FEL equation. Further, Al-Shammery et al. [17] studied the arbitrary order generalization of the FEL equation and expressed their solution in terms of Kummer confluent hypergeometric functions (KCHF) as well as analyzed the behavior of FEL and it is governed by first-order IDEV. After this Saxena and Kalla [18] further generalized the first-order IDEV, which was an extension of the work done by Al-Shammery et al. [16,17]. In continuation of solution of FIDEV Kilbas et al. [19] consecutively studied and further generalized the work done by Saxena and Kalla [18] and established the solution in terms of generalized Mittag-Leffler function. Motivated by current work done by several authors on significant generalization of FIDEV with the help of fractional operator, Saxena and Kalla [20] investigated the solution of Cauchy-type generalized FIDEV involving a generalized Lauricella CHF in the kernels. At the same time Srivastava and Saxena [21] further derived the solution of FIDEV by using multivariable CHF in the kernel. Recently, Singh et al. [22] derived the solution of FEL containing Hilfer-Prabhakar derivative operator by using Elzaki transform in terms of Mittag-Leffler type function. Many authors have been work in the solution of fractional differential and integral equations refer to the work mentioned in [16,23–33]. In the literature of fractional differentiations and integrations there are several integral transforms like Laplace, Fourier, Mellin, Sumudu etc. Recently Elzaki introduced a new integral transform whose name is Elzaki transform [34,35], which is a modified form of classical Laplace and Sumudu transform and have some quality features. Elzaki transform has been effectively used to solve the integral equations as well as ordinary and partial differential equations in fractional calculus [36].

Primarily our objective of this paper is to investigate the formulae of Elzaki transform of functions which have been mentioned earlier and these results will be used to solve the generalized fractional integro-differential equations established here.

2. Definitions and Preliminaries

In this portion, we study a few important fundamental definitions associated to fractional calculus, Elzaki transform and special function to understand the further results, lemmas and application.

2.1. Elzaki Transform

Let $h(t)$ belong to a class K, where $K = \left\{h(t)\ :\ 3\,N,\ \mathrm{p}_1,\ \mathrm{p}_2 > 0\ \text{ such that } \left|h(t)\right| < Ne^{\frac{|t|}{p_i}}\ \text{ if } t\,\epsilon\,(-i)^j x[0,\ \infty)\right\}$.

Elzaki transform [34,35] of function $h(t)$ introduced by Tarig M. Elzaki is defined as

$$E[\mathrm{h(t)}] = u\int_0^{\infty} e^{-\frac{t}{u}} h(t)dt = \mathrm{T}(u\ ),\ \mathrm{t} > 0,\ u\ \epsilon\left(-\mathrm{p}_1, \mathrm{p}_2\right). \tag{1}$$

Convolution Property

The Elzaki transform of the convolution of $f(t)$ *and* $g(t)$ is given by

$$E[(f*g)(t)] = \frac{1}{u}F(u)G(u), \tag{2}$$

where $F(u)$ *and* $G(u)$ are the Elzaki transform of $f(t)$ *and* $g(t)$ *resp*., and

$$(f*g)(t) = \int_0^t f(t)g(t-u)du.$$

2.2. Generalized Lorenzo-Hartley Function

Generalized Lorenzo-Hartley [37] is defined as:

$$G_{\nu,\mu,\sigma}(a,\omega) = \sum\nolimits_{J=0}^{\infty} \frac{(\sigma)_j\ (a)^j\ \omega^{(\sigma+j)\nu-\mu-1}}{\Gamma(j+1)\Gamma(\sigma+j)\nu-\mu},\ \Re(\sigma\nu-\mu) > 0. \tag{3}$$

2.3. Hilfer Derivative Operator

The Hilfer derivative [38] of order α is defined as:

$$D_{a^+}^{\beta,\,\gamma}y(x)=\left(I_{a^+}^{\gamma(1-\beta\,)}\frac{d}{dt}I_{a^+}^{(1-\gamma)(1-\beta)}y\right)(x). \tag{4}$$

2.4. Generalized Lauricella Confluent Hypergeometric Function

A special case of generalized Lauricella function in several complex variables, proposed by Srivastava and Daoust [39] (p. 454) in terms of a multiple series express in the following manner:

$$F^{\,0:\overbrace{1;\ldots;1}^{n}}_{\,1:\underbrace{0;\ldots;0}_{n}}\left[\begin{array}{c}-:(\beta_1:1);\ldots;(\beta_n:1);-\\ \tau_1,\ldots,\tau_n\\ (\alpha:\delta_1,\ldots,\delta_n):-:-\end{array}\right]=\sum_{k_{1,\ldots},\,k_n=0}^{\infty}\frac{(\beta_1)_{k_1}\ldots\,(\beta_n)_{k_n}\tau_1{}^{k_1}\ldots\tau_n{}^{k_n}}{\Gamma[\alpha+\delta_1k_1+\ldots+\delta_nk_n](k_1)!\,\ldots\,(k_n)!}, \tag{5}$$

where $\alpha,\,\delta_j,\,\beta_j,\,\tau_j\in\mathbb{C}$ *and* $j=1,\ldots,n$.

As per convergence condition mentioned by Srivastava and Doust [40] (p. 157) for the generalized Lauricella series in several variables, the series given in (5) converges for Re $(\delta_j)>0$ *for* $j=1,\ldots,n$.

3. Elzaki Transform of Generalized Lorenzo-Hartley Function, Hilfer Derivative & Generalized Lauricella Confluent Hypergeometric Function

In this portion, we introduce a formula of Elzaki transform of generalized Lorenzo-Hartely function, Hilfer derivative & generalized Lauricella confluent hypergeometric function.

Lemma 1. *The Elzaki transform of generalized Lorenzo-Hartely function is given by*

$$E\Big[G_{\nu,\mu,\sigma}(a,\omega)\Big]=u^{\sigma\nu-\mu+1}[1-(au^{\nu})]^{-\sigma},\,\Re(\sigma\nu-\mu)>0. \tag{6}$$

Proof. Elzaki transform of generalized Lorenzo-Hartely function defined by (3) is given by

$$E\Big[G_{\nu,\mu,\sigma}(a,\omega)\Big]=E\left[\sum\nolimits_{J=0}^{\infty}\frac{(\sigma)_j\,(a)^j\,\omega^{(\sigma+j)\nu-\mu-1}}{\Gamma(j+1)\Gamma(\sigma+j)\nu-\mu}\right]=\sum\nolimits_{J=0}^{\infty}\frac{(\sigma)_j\,(a)^j}{\Gamma(j+1)\Gamma(\sigma+j)\nu-\mu}E\Big[\omega^{(\sigma+j)\nu-\mu-1}\Big]. \tag{7}$$

Now, applying the formula of the Elazki transform in (7), we arrive at

$$E\Big[G_{\nu,\mu,\sigma}(a,\omega)\Big]=\sum\nolimits_{J=0}^{\infty}\frac{\mathrm{u}^{(\sigma+j)\nu-\mu+1}\,\Gamma(\sigma+\mathrm{j})\,(a)^j}{\Gamma(j+1)\,\Gamma(\sigma)}.$$

After this, we are rearranging the terms to convert the above equation in binomial function form

$$E\Big[G_{\nu,\mu,\sigma}(a,\omega)\Big]=\;\mathrm{u}^{\sigma\nu-\mu+1}\sum\nolimits_{J=0}^{\infty}\frac{\Gamma(\sigma+\mathrm{j})(au^{\nu})^j}{\Gamma(j+1)\,\Gamma(\sigma)}.$$

Finally, we get the desired result

$$E\Big[G_{\nu,\mu,\sigma}(a,\omega)\Big]=u^{\sigma\nu-\mu+1}[1-(au^{\nu})]^{-\sigma}.$$

□

Lemma 2. *The Elzaki transform of Hilfer derivative of fractional order defined in (4) is given by*

$$E\Big[D_{a^+}^{\beta,\gamma} y(x)\Big](u) = u^{-\beta} E[y(x)](u) - u^{-\gamma(\beta-1)+1}\Big(I^{(1-\beta)(1-\gamma)} y\Big)(0+) \tag{8}$$

Proof. The Hilfer-derivative is defined as

$$D_{a^+}^{\beta,\gamma} y(x) = \Big(I_{a^+}^{\gamma(1-\beta)} (D_{a^+}^{\beta+\gamma-\beta\gamma} y)\Big)(x). \tag{9}$$

Applying the integral operator ($I_{a^+}^{\beta}$) on both side (see for instance [14]), we have

$$I_{a^+}^{\beta} (D_{a^+}^{\beta,\gamma} y)(x) = \Big(I_{a^+}^{\gamma(1-\beta)+\beta} (D_{a^+}^{\beta+\gamma-\beta\gamma} y)\Big)(x),$$

by using the definition of Riemann-Liouville integral operator [4]

$$\begin{aligned} &\frac{1}{\Gamma(\beta)} \int_a^t (t-x)^{\beta-1} (D_{a^+}^{\beta,\gamma} y)(x) dx \\ &= y(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{k-(n-\beta)(1-\gamma)}}{\Gamma[k-(n-\beta)(1-\gamma)+1]} \lim_{x\to a} \frac{d^k}{dx^k} \Big(I_{a^+}^{(n-\beta)(1-\gamma)} y\Big)(x). \end{aligned} \tag{10}$$

Applying Elzaki transform and also using convolution property of Elzaki transform on above equation, we get

$$\begin{aligned} &\frac{1}{\mathrm{u}\,\Gamma(\beta)} E\Big[t^{\beta-1}\Big]\, E\Big[D_{a^+}^{\beta,\gamma} y(x)\Big](u) = \\ &E[y(x)](u) - \textstyle\sum_{k=0}^{(n-1)} \Bigg[\Big\{\lim_{x\to a} \frac{d^k}{dx^k} \Big(I_{a^+}^{(n-\beta)(1-\gamma)} y\Big)(x)\Big\} E\Big\{\frac{(x-a)^{k-(n-\beta)(1-\gamma)}}{\Gamma[k-(n-\beta)(1-\gamma)+1]}\Big\}\Bigg]. \end{aligned}$$

Using formula of Elzaki transform, we arrive at

$$\begin{aligned} &u^{\beta}\, E\Big[D_{a^+}^{\beta,\gamma} y(x)\Big](u) = \\ &E[y(x)](u) - \sum_{k=0}^{n-1} \Big[(u-a)^{k-(n-\beta)(1-\gamma)+2} \Big\{\lim_{x\to a} \frac{d^k}{dx^k} \Big(I_{a^+}^{(n-\beta)(1-\gamma)} y\Big)(x)\Big\}\Big]. \end{aligned} \tag{11}$$

Multiplying by $u^{-\beta}$ both side and taking $a = 0$ in Equation (11), we arrive at

$$\begin{aligned} &E\Big[D_{0+}^{\beta,\gamma} y(x)\Big](u) = \\ &u^{-\beta} E[y(x)](u) - \sum_{k=0}^{n-1} \Big[(u)^{k-n+\gamma(n-\beta)+2} \Big\{\lim_{x\to 0} \frac{d^k}{dx^k} \Big(I_{0+}^{(n-\beta)(1-\gamma)} y\Big)(x)\Big\}\Big], \end{aligned} \tag{12}$$

where $(n-1) < \beta \le n$.

For $n = 1$, the above equation becomes

$$E\Big[D_{0+}^{\beta,\gamma} y(x)\Big](u) = u^{-\beta} E[y(x)](u) - (u)^{-\gamma(\beta-1)+1}\Big(I_{0+}^{(1-\beta)(1-\gamma)} y\Big)(0+).$$

This is the Elzaki transform formula of Hilfer-derivative. We use this result to solve fractional integro-differential equation. □

Lemma 3. *The Elzaki transform of generalized Lauricella confluent hypergeometric function in several complex variables defined in (5) is given by*

$$E^{-1}\left\{u^{\alpha+1}\prod_{j=1}^{n}\left(1-\tau_j u^{\delta_j}\right)^{-\beta_j}\right\} = \omega^{\alpha-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1:1);\ldots;(\beta_n:1);- \\ \tau_1\omega^{\delta_1},\ldots,\tau_n\omega^{\delta_n} \\ (\alpha:\delta_1,\ldots,\delta_n):-:- \end{array}\right] \tag{13}$$

where $\alpha, \delta_j, \beta_j, \tau_j \in \mathbb{C}$, $\Re\,(u) > 0$, $\max_{1\le j\le n}\left|\tau_j u^{\delta_j}\right| < 1$, $\min_{1\le j\le n}\Re\,(\delta_j) > 0$, $\Re(\alpha) > 0$.

Proof. The Equation (13) can be easily solve by taking Elzaki transform of the function given in left hand side of (13), we have

$$E\left\{\omega^{\alpha-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1:1);\ldots;(\beta_n:1);- \\ \tau_1\omega^{\delta_1},\ldots,\tau_n\omega^{\delta_n} \\ (\alpha:\delta_1,\ldots,\delta_n):-:- \end{array}\right]\right\}$$
$$= u\int_0^\infty \omega^{\alpha-1}\left(\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{(\beta_1)_{k_1}\cdots(\beta_n)_{k_n}\tau_1 k_1\ldots\tau_n k_n}{\Gamma[\alpha+\delta_1k_1+\ldots+\delta_nk_n](k_1)!\ldots(k_n)!}\right)e^{-\frac{\omega}{u}}d\omega,$$

interchanging the order of summations and integration, which is permissible under the conditions stated with Lemma 3, after rearrangement of the terms it is possible to express the above equation in the form

$$= u\left(\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{(\beta_1)_{k_1}\cdots(\beta_n)_{k_n}}{\Gamma[\alpha+\delta_1k_1+\ldots+\delta_nk_n](k_1)!\ldots(k_n)!}\right)$$
$$\times\int_0^\infty \omega^{\alpha-1}\tau_1k_1\omega^{\delta_1k_1}\ldots\tau_nk_n\omega^{\delta_nk_n}e^{-\frac{\omega}{u}}d\omega,$$

and above equation can be written as

$$=\left(\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{(\beta_1)_{k_1}\ldots(\beta_n)_{k_n}}{\Gamma[\alpha+\delta_1k_1+\ldots+\delta_nk_n](k_1)!\ldots(k_n)!}\right)E\left\{\tau_1k_1\ldots\tau_nk_n\,\omega^{\alpha+\delta_1k_1+\ldots+\delta_nk_n-1}\right\}. \tag{14}$$

Now, using the formula of the Elazki transform in (14), we have

$$E\left\{\omega^{\alpha-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1:1);\ldots;(\beta_n:1);- \\ \tau_1\omega^{\delta_1},\ldots,\tau_n\omega^{\delta_n} \\ (\alpha:\delta_1,\ldots,\delta_n):-:- \end{array}\right]\right\} =$$
$$\sum_{k_1,\ldots,k_n=0}^{\infty}u^{\alpha+1}\frac{(\beta_1)_{k_1}\left(\tau_1u^{\delta_1}\right)^{k_1}}{(k_1)!}\cdots\frac{(\beta_n)_n\left(\tau_nu^{\delta_n}\right)^{k_n}}{(k_n)!}.$$

We express the above result in the form of product of binomial functions as

$$E\left\{\omega^{\alpha-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1:1);\ldots;(\beta_n:1);- \\ \tau_1\omega^{\delta_1},\ldots,\tau_n\omega^{\delta_n} \\ (\alpha:\delta_1,\ldots,\delta_n):-:- \end{array}\right]\right\} = u^{\alpha+1}\left(1-\tau_1u^{\delta_1}\right)^{-\beta_1}\ldots\left(1-\tau_nu^{\delta_n}\right)^{-\beta_n}.$$

It can be written as

$$E\left\{\omega^{\alpha-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1:1);\ldots;(\beta_n:1);- \\ \tau_1\omega^{\delta_1},\ldots,\tau_n\omega^{\delta_n} \\ (\alpha:\delta_1,\ldots,\delta_n):-:- \end{array}\right]\right\} = u^{\alpha+1}\prod_{j=1}^{n}\left(1-\tau_j u^{\delta_j}\right)^{-\beta_j}.$$

Finally, we arrive at the desired result. □

4. Solution of Generalized Fractional Integro-Differential Equations

Theorem 1. *Let us consider the following generalized fractional integro-differential equation of Volterra-type:*

$$D_{0+}^{\beta,\,\gamma}[y(x)] = \rho f(x) + \lambda\int_0^x G_{\nu,\mu,\sigma}(a,\omega)y(x-\omega)d\omega, \tag{15}$$

where $\nu,\ \mu,\ \sigma,\ \lambda,\ \rho \in \mathbb{C},\ 0 \le x \le 1,\ \beta \in (0,1),\ \gamma \in [0,1]$ *and* $\Re(\nu) > 0,\ \Re(\mu) > 0, \Re(\nu-\mu) > 0,$ *with the initial condition* $\left(I_{0+}^{(1-\beta)(1-\gamma)}y\right)(0+) = \mathrm{C}$, *and* $f(x)$ *is assumed to be continuous on every finite closed interval* $[0,\ X](0 < X < \infty),$ *has its solution given by*

$$y(x) = \mathrm{C}\ \phi(\mathrm{x}) + \rho\int_0^x f(\omega)\ \xi(x-\omega)d\omega, \tag{16}$$

where

$$\phi(\mathrm{x}) = \sum_{m=0}^{\infty}\lambda^m G_{\nu,\ (\mu-\beta)m+\gamma(\beta-1)-\beta,\sigma m}\ (a,x), \tag{17}$$

and

$$\xi(x) = \sum_{m=0}^{\infty}\lambda^m G_{\nu,\ (\mu-\beta)m-\beta,\sigma m}\ (a,\ x). \tag{18}$$

Proof. Applying Elzaki transform on Equation (15) and using (6) and (8), we have

$$\begin{aligned} u^{-\beta}\ Y(u) - u^{-\gamma(\beta-1)+1}\left(I^{(1-\beta)(1-\gamma)}y\right)(0+) \\ = \rho\ F(u) + \lambda\tfrac{1}{u}\ Y(u)u^{\sigma\nu-\mu+1}[1-(au^{\nu})]^{-\sigma}. \end{aligned}$$

After rearranging the terms, the above equation can be written as

$$u^{-\beta}\ Y(u)\left[1-\frac{\lambda u^{\sigma\nu-\mu+\beta}}{[1-(au^{\nu})]^{\sigma}}\right] = \rho\ F(u) + C\ u^{-\gamma(\beta-1)+1},$$

it can be written as follow

$$Y(u) = u^{\beta}\rho\ F(u)\left[1-\frac{\lambda u^{\sigma\nu-\mu+\beta}}{[1-(au^{\nu})]^{\sigma}}\right]^{-1} + C\ u^{-\gamma(\beta-1)+\beta+1}\left[1-\frac{\lambda u^{\sigma\nu-\mu+\beta}}{[1-(au^{\nu})]^{\sigma}}\right]^{-1}.$$

By virtue of binomial formula, we get

$$Y(u) = u^{\beta}\rho\ F(u)\sum_{m=0}^{\infty}\frac{\lambda^m u^{(\sigma\nu-\mu+\beta)m}}{[1-(au^{\nu})]^{\sigma m}} + C\ u^{-\gamma(\beta-1)+\beta+1}\sum_{m=0}^{\infty}\frac{\lambda^m u^{(\sigma\nu-\mu+\beta)m}}{[1-(au^{\nu})]^{\sigma m}}. \tag{19}$$

Now, inverting the Elzaki transform, we have

$$y(x) = \rho\, E^{-1}\left[\sum_{m=0}^{\infty} \lambda^m u^{(\sigma\nu-\mu+\beta)m+\beta}[1-(au^{\nu})]^{-\sigma m}F(u)\right] + CE^{-1}\left[\sum_{m=0}^{\infty} \lambda^m u^{(\sigma\nu-\mu+\beta)m-\gamma(\beta-1)+\beta+1}[1-(au^{\nu})]^{-\sigma m}\right].$$

Again using binomial result, we arrive at

$$y(x) = \rho\, E^{-1}\left[\sum_{m=0}^{\infty} \lambda^m \left\{\sum_{j=0}^{\infty} \frac{(\sigma m)_j (a)^j u^{(\sigma m+j)\nu-[(\mu-\beta)m-\beta]+1-1}}{\Gamma(j+1)}\right\} F(u)\right] + CE^{-1}\left[\sum_{m=0}^{\infty} \lambda^m \left\{\sum_{j=0}^{\infty} \frac{(\sigma m)_j (a)^j u^{(\sigma m+j)\nu-[(\mu-\beta)m+\gamma(\beta-1)-\beta]+1}}{\Gamma(j+1)}\right\}\right].$$

This leads us to the following equation

$$y(x) = \rho \sum_{m=0}^{\infty} \lambda^m \left[\sum_{j=0}^{\infty} \frac{(\sigma m)_j (a)^j}{\Gamma(j+1)} E^{-1}\{\frac{1}{u}\, E\left(\frac{x^{(\sigma m+j)\nu-[(\mu-\beta)m-\beta]-1}}{\Gamma(\sigma m+j)\nu-[(\mu-\beta)m-\beta]}\right) E\,\{f(x)\}\}\right] + C \sum_{m=0}^{\infty} \lambda^m \left[\sum_{j=0}^{\infty} \frac{(\sigma m)_j (a)^j}{\Gamma(j+1)} \frac{x^{(\sigma m+j)\nu-[(\mu-\beta)m+\gamma(\beta-1)-\beta]-1}}{\Gamma(\sigma m+j)\nu-[(\mu-\beta)m+\gamma(\beta-1)-\beta]}\right]. \tag{20}$$

Now, we applying the convolution property of the Elzaki transform in (20) and using (3), we find that

$$y(x) = \rho \sum_{m=0}^{\infty} \lambda^m \left[\sum_{j=0}^{\infty} \frac{(\sigma m)_j (a)^j}{\Gamma(j+1)} \left\{\int_0^x f(\omega) \frac{(\,x-\omega)^{(\sigma m+j)\nu-[(\mu-\beta)m-\beta]-1}}{\Gamma(\sigma m+j)\nu-[(\mu-\beta)m-\beta]}\, d\omega\right\}\right] + C \sum_{m=0}^{\infty} \lambda^m G_{\nu,\,(\mu-\beta)m+\gamma(\beta-1)-\beta,\sigma m}\,(a,x),$$

or

$$y(x) = C \sum_{m=0}^{\infty} \lambda^m G_{\nu,\,(\mu-\beta)m+\gamma(\beta-1)-\beta,\sigma m}\,(a,x) + \rho \sum_{m=0}^{\infty} \lambda^m \left[\int_0^x f(\omega) \sum_{j=0}^{\infty} \frac{(\sigma m)_j (a)^j}{\Gamma(j+1)} \frac{(\,x-\omega)^{(\sigma m+j)\nu-[(\mu-\beta)m-\beta]-1}}{\Gamma(\sigma m+j)\nu-[(\mu-\beta)m-\beta]}\, d\omega\right].$$

Finally, we arrive at the solution given by (15)

$$y(x) = C \sum_{m=0}^{\infty} \lambda^m G_{\nu,\,(\mu-\beta)m+\gamma(\beta-1)-\beta,\sigma m}\,(a,x) + \rho\left[\int_0^x f(\omega) \sum_{m=0}^{\infty} \lambda^m G_{\nu,\,(\mu-\beta)m-\beta,\sigma m}\,(a,\ x-\omega) d\omega\right].$$

We can also display the above result in this way

$$y(x) = \mathrm{C}\,\phi(\mathrm{x}) + \rho \int_0^x f(\omega)\ \xi(x-\omega) d\omega. \tag{21}$$

Here $\phi(\mathrm{x})$ and $\xi(x-\omega)$ are given by Equations (17) and (18). □

Theorem 2. *Let us consider the following generalized fractional integro-differential equation of Volterra-type:*

$$D_{0+}^{\beta,\,\gamma}[y(x)] = \rho f(x) + \lambda \int_0^x \omega^{\alpha-1} y(x-\omega) F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}} \left[\begin{array}{c} -:(\beta_1:1);\ldots;(\beta_n:1);- \\ \tau_1\omega^{\delta_1},\ldots,\tau_n\omega^{\delta_n} \\ (\alpha:\delta_1,\ldots,\delta_n):-:- \end{array} \right] d\omega, \tag{22}$$

where $\lambda\ ,\alpha,\ \rho,\ \beta_j,\ \delta_j,\ \tau_j \in \mathbb{C}\ ;\ 0 \le x \le 1;\ \max_{1\le j\le n}\left|\tau_j\omega^{\delta_j}\right| < \infty;\ \Re\ (\alpha) > 0;\ \beta \in\ (0,1),\ \gamma \in [0,1];$ $\min_{1\le j\le n}\Re\left(\delta_j\right) > 0\ (j = 1,\ldots,n)$, *with the initial condition* $\left(I_{0+}^{(1-\beta)(1-\gamma)}y\right)(0+) = C$, *and* $f(x)$ *is assumed to be continuous on every finite closed interval* $[0,\ X](0 < X < \infty)$, *has its solution given by*

$$y(x) = \rho \int_0^x f(\omega)\ \varphi(x-\omega)\ d\omega + C\psi(x), \tag{23}$$

where

$$\varphi(x) = x^{\beta-1} \sum_{r=0}^{\infty} \lambda^r x^{(\alpha+\beta)r} F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}} \left[\begin{array}{c} -:(\beta_1 r:1);\ldots;(\beta_n r:1);- \\ \tau_1 x^{\delta_1},\ldots,\tau_n x^{\delta_n} \\ (\beta+(\alpha+\beta)r\ :\delta_1,\ldots,\delta_n):-:- \end{array} \right], \tag{24}$$

and

$$\psi(x) = x^{\beta-\gamma(\beta-1)-1} \sum_{r=0}^{\infty} \lambda^r x^{(\alpha+\beta)r} F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}} \left[\begin{array}{c} -:(\beta_1 r:1);\ldots;(\beta_n r:1);- \\ \tau_1 x^{\delta_1},\ldots,\tau_n x^{\delta_n} \\ ((\alpha+\beta)r+\beta-\gamma(\beta-1):\delta_1,\ldots,\delta_n):-:- \end{array} \right]. \tag{25}$$

Proof. Applying the Elzaki transform on Equation (22), we have

$$E\left\{ D_{0+}^{\beta,\,\gamma}[y(x)]\right\} = \rho E[f(x)] + \lambda\ E\left\{ \int_0^x \omega^{\alpha-1} F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}} \left[\begin{array}{c} -:(\beta_1:1);\ldots;(\beta_n:1);- \\ \tau_1\omega^{\delta_1},\ldots,\tau_n\omega^{\delta_n} \\ (\alpha:\delta_1,\ldots,\delta_n):-:- \end{array} \right] y(x-\omega)d\omega \right\}. \tag{26}$$

Now, using (8) & (13) in (26), we found that

$$u^{-\beta}Y(u)\left[1-\lambda u^{\alpha+\beta}\left\{\prod_{j=1}^{n}\left(1-\tau_j u^{\delta_j}\right)^{-\beta_j}\right\}\right] = \rho F(u) + u^{-\gamma(\beta-1)+1}.C, \tag{27}$$

where $Y(u)$ *and* $F(u)$ represent, respectively the Elzaki transform of the function $y(x)$ *and* $f(x)$.

Solving Equation (27), we find that

$$\begin{aligned} Y(u) &= \rho u^{\beta} F(u)\left[1-\lambda u^{\alpha+\beta}\left\{\prod_{j=1}^{n}\left(1-\tau_j u^{\delta_j}\right)^{-\beta_j}\right\}\right]^{-1} \\ &+ u^{\beta-\gamma(\beta-1)+1}.C\left[1-\lambda u^{\alpha+\beta}\left\{\prod_{j=1}^{n}\left(1-\tau_j u^{\delta_j}\right)^{-\beta_j}\right\}\right]^{-1}, \end{aligned} \tag{28}$$

where we have tacitly assumed that

$$\left|\lambda u^{\alpha+\beta}\left\{\prod_{j=1}^{n}\left(1-\tau_j u^{\delta_j}\right)^{-\beta_j}\right\}\right| < 1\,, \tag{29}$$

by virtue of binomial formula, we obtain

$$\begin{aligned} Y(u) &= \rho F(u)\sum_{r=0}^{\infty}\lambda^r u^{\beta+(\alpha+\beta)r}\left\{\prod_{j=1}^{n}\left(1-\tau_j u^{\delta_j}\right)^{-\beta_j r}\right\} \\ &+C\sum_{r=0}^{\infty}\lambda^r u^{(\alpha+\beta)r+\beta-\gamma(\beta-1)+1}\left\{\prod_{j=1}^{n}\left(1-\tau_j u^{\delta_j}\right)^{-\beta_j r}\right\}. \end{aligned} \tag{30}$$

Now, inverting the Elzaki transform and using the formula defined in (13) once again, we find from (30) that

$$\begin{aligned} y(x) &= \\ &\rho\, E^{-1}\left[\sum_{r=0}^{\infty}\lambda^r\, E\left\{f(x)*x^{\beta+(\alpha+\beta)r-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1 r:1);\ldots;(\beta_n r:1);- \\ \tau_1 x^{\delta_1},\ldots,\tau_n x^{\delta_n} \\ (\beta+(\alpha+\beta)r\ :\delta_1,\ldots,\delta_n):-:- \end{array}\right]\right\}\right] \\ &+C\sum_{r=0}^{\infty}\lambda^r x^{(\alpha+\beta)r+\beta-\gamma(\beta-1)-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1 r:1);\ldots;(\beta_n r:1);- \\ \tau_1 x^{\delta_1},\ldots,\tau_n x^{\delta_n} \\ (\alpha+\beta)r+\beta-\gamma(\beta-1):(\delta_1,\ldots,\delta_n):-:- \end{array}\right]. \end{aligned} \tag{31}$$

Using the convolution property of Elazki transform in the above equation, we have

$$\begin{aligned} y(x) &= \\ &\rho E^{-1}\left[\sum_{r=0}^{\infty}\lambda^r E\left\{\int_0^x f(\omega)(x-\omega)^{\beta+(\alpha+\beta)r-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1 r:1);\ldots;(\beta_n r:1);- \\ \tau_1(x-\omega)^{\delta_1},\ldots,\tau_n(x-\omega)^{\delta_n} \\ (\beta+(\alpha+\beta)r\ :\delta_1,\ldots,\delta_n):-:- \end{array}\right]d\omega\right\}\right] \\ &+C\sum_{r=0}^{\infty}\lambda^r x^{(\alpha+\beta)r+\beta-\gamma(\beta-1)-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1 r:1);\ldots;(\beta_n r:1);- \\ \tau_1 x^{\delta_1},\ldots,\tau_n x^{\delta_n} \\ (\alpha+\beta)r+\beta-\gamma(\beta-1):(\delta_1,\ldots,\delta_n):-:- \end{array}\right]. \end{aligned} \tag{32}$$

Finally, after little simplification, we find that

$$\begin{aligned} y(x) &= \\ &\rho\left[\left\{\int_0^x\sum_{r=0}^{\infty}\lambda^r f(\omega)(x-\omega)^{\beta+(\alpha+\beta)r-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1 r:1);\ldots;(\beta_n r:1);- \\ \tau_1(x-\omega)^{\delta_1},\ldots,\tau_n(x-\omega)^{\delta_n} \\ (\beta+(\alpha+\beta)r\ :\delta_1,\ldots,\delta_n):-:- \end{array}\right]d\omega\right\}\right] \\ &+C\sum_{r=0}^{\infty}\lambda^r x^{(\alpha+\beta)r+\beta-\gamma(\beta-1)-1}F^{\overbrace{0:1;\ldots;1}^{n}}_{\underbrace{1:0;\ldots;0}_{n}}\left[\begin{array}{c} -:(\beta_1 r:1);\ldots;(\beta_n r:1);- \\ \tau_1 x^{\delta_1},\ldots,\tau_n x^{\delta_n} \\ (\alpha+\beta)r+\beta-\gamma(\beta-1):(\delta_1,\ldots,\delta_n):-:- \end{array}\right]. \end{aligned} \tag{33}$$

We see that the above expression can be demonstrated in the form (23). Therefore this completes the proof of Theorem 2. □

5. Conclusions

In this work, we have applied efficient and interesting transform (Elzaki transform) to obtain the close form solution of generalized fractional integro-differential equation of Volterra-type involving the generalized Lorenzo-Hartely function and generalized Lauricella series function in terms of function itself. We also derived novel results such as Elzaki transform of Hilfer-derivative, generalized Lorenzo-Hartely function as well as generalized Lauricella confluent hypergeometric function. If we assign particular value to the parameters involve in (16) and (23), then our results established here are particular cases of various results derived by numbers of authors. We can use this transform to solve numerous problems, such as problems occurring in mathematics can be solve without utilizing a novel frequency domain, ODE, Non-homogenous equations, fractional integral and differential equations, one of the important aspect of this transform is that it can change the system of equations (differential & Integral) into algebraic equations.

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