fractal and fractional

Article

Exact Solution of Two-Dimensional Fractional Partial Differential Equations

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Abstract: In this study, we examine adapting and using the Sumudu decomposition method (SDM) as a way to find approximate solutions to two-dimensional fractional partial differential equations and propose a numerical algorithm for solving fractional Riccati equation. This method is a combination of the Sumudu transform method and decomposition method. The fractional derivative is described in the Caputo sense. The results obtained show that the approach is easy to implement and accurate when applied to various fractional differential equations.

Keywords: fractional differential equations; fractional generalized biologic population; Sumudu transform; Adomian decomposition method; Caputo fractional derivative

1. Introduction

Fractional calculus has been utilized as an excellent instrument to discover the hidden aspects of various material and physical processes that deal with derivatives and integrals of arbitrary orders [1–4]. The theory of fractional differential equations translates the reality of nature excellently in a useful and systematic manner [5]. Fractional differential equations are viewed as option models to nonlinear differential equations. Varieties of them play important roles and tools, not only in mathematics, but also in physics, dynamical systems, control systems and engineering, to create the mathematical modeling of many physical phenomena. Furthermore, they are employed in social science such as food supplement, climate and economics [6]. The mathematical physics governing by nonlinear partial deferential dynamical equations have applications in physical science. The analytical solutions for these dynamical equations play an important role in many phenomena in optics; fluid mechanics; plasma physics and hydrodynamics [7–10]. In recent years, many authors have investigated partial differential equations of fractional order by various techniques such as homotopy analysis technique [11,12], variational iteration method [13–15], homotopy perturbation method [16], homotopy perturbation transform method [17], Laplace variational iteration method [18–20], reduce differential transform method [21], Laplace decomposition method [22] and other methods [23–27].

There are numerous integral transforms such as the Laplace, Sumudu, Fourier, Mellin and Elzaki to solve PDEs. Of these, the Laplace transformation and Sumudu transformation are the most widely used. The Sumudu transformation method is one of the most important transform methods introduced in the early 1990 [28]. It is a powerful tool for solving many kinds of PDEs in various fields of science and engineering. In addition, various methods are combined with the Sumudu transformation method such as the homotopy perturbation transform method [29] which is a combination of the homotopy perturbation method and the Sumudu transformation method. Another example is the homotopy analysis Sumudu transform method [30], which is a combination of the Sumudu transform method and the homotopy analysis method.

Fractional operators are non-local operators; thus, they are used successfully for describing the phenomena with memory effect. We stress on the fact than by replacing the classical derivative with respect with time by a given fractional operator we change the nature of the partial differential equation from local to a nonlocal one. In this way we can describe better processes with faster of lower velocities, depending on the value of alpha, which in the classical class we cannot do. The domain of the utilized fractional operator and the type, namely local or nonlocal, are other key factors in modeling with high accuracy some real-world phenomena which cannot be described properly by using the classical calculus models. Successful examples of changing the differential operator into the fractional ones can be fined in modeling accurately the fluid mechanics models as well as the mathematical biology models, including the top-level epidemiological models. This article considers the efficiency of fractional Sumudu decomposition method (FSDM) to solve two-dimensional differential equations. The FSDM is a graceful coupling of two powerful techniques, namely ADM and Sumudu transform algorithms and gives more refined convergent series solution.

2. Preliminaries

Some fractional calculus definitions and notation needed [2,16,29] in the course of this work are discussed in this section.

Definition 1. *A real function $\varphi(\mu),\ \mu > 0$, is said to be in the space $C_\vartheta,\ \vartheta \in R$ if there exists a real number $q,\ (q > \vartheta)$, such that $\varphi(\mu) = \mu^q \varphi_1(\mu)$, where $\varphi_1(\mu) \in C[0,\infty)$, and it is said to be in the space C^m_ϑ if $\varphi^{(m)} \in C_\vartheta,\ m \in N$.*

Definition 2. *The Riemann Liouville fractional integral operator of order $\varepsilon \geq 0$, of a function $\varphi(\mu) \in C_\vartheta,\ \vartheta \geq -1$ is defined as*

$$I^\varepsilon \varphi(\mu) = \begin{cases} \frac{1}{\Gamma(\varepsilon)} \int_0^\mu (\mu-\tau)^{\varepsilon-1} \varphi(\tau) d\tau, & \varepsilon > 0,\ \mu > 0, \\ I^0 \varphi(\mu) = \varphi(\mu), & \varepsilon = 0, \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Properties of the operator $I\alpha$, which we will use here, are as follows:

For $\varphi \in C_\vartheta,\ \vartheta \geq -1,\ \varepsilon,\ \epsilon \geq 0$,

1. $I^\varepsilon I^\epsilon \varphi(\mu) = I^{\varepsilon+\epsilon}\varphi(\mu)$.
2. $I^\varepsilon I^\epsilon \varphi(\mu) = I^\epsilon I^\varepsilon \varphi(\mu)$
3. $I^\varepsilon \mu^m = \frac{\Gamma(m+1)}{\Gamma(\varepsilon+m+1)} \mu^{\varepsilon+m}$.

Definition 3. *The fractional derivative of $\varphi(\mu)$ in the Caputo sense is defined as*

$$D^\varepsilon \varphi(\mu) = I^{m-\varepsilon} D^m \varphi(\mu) = \frac{1}{\Gamma(m-\varepsilon)} \int_0^\mu (\mu-\tau)^{m-\varepsilon-1} \varphi^{(m)}(\tau) d\tau, \tag{2.2}$$

for $m-1 < \varepsilon \leq m,\ m \in N,\ \mu > 0,\ \varphi \in C^m_{-1}$.

The following are the basic properties of the operator D^ε:

1. $D^\varepsilon D^\epsilon \varphi(\mu) = D^{\varepsilon+\epsilon}\varphi(\mu)$.
2. $D^\varepsilon \mu^m = \frac{\Gamma(1+m)}{\Gamma(1+m-\varepsilon)} \mu^{m-\varepsilon}$.
3. $D^\varepsilon I^\varepsilon \varphi(\mu) = \varphi(\mu)$.
4. $I^\varepsilon D^\varepsilon \varphi(\mu) = \varphi(\mu) - \sum\limits_{k=0}^{m-1} \varphi^{(k)}(0) \frac{\mu^k}{k!}$.

Definition 4. *The Mittag–Leffler function E_δ with $\varepsilon > 0$ is defined as*

$$E_\varepsilon\ (z) = \sum_{m=0}^{\infty} \frac{z^\varepsilon}{\Gamma(m\varepsilon + 1)} \tag{2.3}$$

Definition 5. *The Sumudu transform is defined over the set of function*

$$A = \left\{\varphi(\tau)/\ \exists M,\ \omega_1, \omega_2 > 0,\ \left|\varphi(\tau)\right| < Me^{|\tau|/\omega_j},\ if\ \tau \in (-1)^j \times [0, \infty)\right\}$$

by the following formula [30,31]:

$$S[\varphi(\tau)] = \int_0^\infty e^{-\tau}\varphi(\omega\tau)d\tau,\ \omega \in (-\omega_1, \omega_2). \tag{2.4}$$

Definition 6. *The Sumudu transform of the Caputo fractional derivative is defined as* [30,31]:

$$S[D_\tau^{m\varepsilon}\varphi(\mu, \gamma, \tau)] = \omega^{-m\varepsilon}S[\varphi(\mu, \gamma, \tau] - \sum_{k=0}^{m-1} \omega^{(-m\varepsilon+k)}\varphi^{(k)}(\mu, \gamma, 0),\ m-1 < m\varepsilon < m. \tag{2.5}$$

3. Fractional Sumudu Decomposition Method (FSDM)

Let us consider a general fractional nonlinear partial differential equation of the form:

$$D_\tau^\varepsilon\varphi(\mu, \gamma, \tau) + L[\varphi(\mu, \gamma, \tau)] + N[\varphi(\mu, \gamma, \tau)] = g(\mu, \gamma, \tau), \tag{3.1}$$

with $n - 1 < \varepsilon \le n$ and subject to the initial condition

$$\frac{\partial^s}{\partial\tau^s}\varphi(\mu, \gamma, 0) = \varphi^{(s)}(\mu, \gamma, 0) = \varphi_s(\mu, \gamma),\ s = 0, 1, \ldots,\ n-1, \tag{3.2}$$

where $\varphi(\mu, \gamma, \tau)$ is an unknown function, $D_\tau^\varepsilon\varphi(\mu, \gamma, \tau)$ is the Caputo fractional derivative of the function $\varphi(\mu, \gamma, \tau)$, L is the linear differential operator, N represents the general nonlinear differential operator and $g(\mu, \gamma, \tau)$ is the source term.

Taking the ST on both sides of (3.1), we have

$$S[D_\tau^\varepsilon\varphi(\mu, \gamma, \tau)] + S[L[\varphi(\mu, \gamma, \tau)]] + S[N[\varphi(\mu, \gamma, \tau)]] = S[g(\mu, \gamma, \tau)]. \tag{3.3}$$

Using the property of the ST, we obtain

$$S[\varphi(\mu, \gamma, \tau)] = \sum_{k=0}^{n-1} \omega^\varepsilon\varphi_k(\mu, \gamma) + \omega^\varepsilon S[g(\mu, \gamma, \tau)] - \omega^\varepsilon S[L[\varphi(\mu, \gamma, \tau)] + N[\varphi(\mu, \gamma, \tau)]]. \tag{3.4}$$

Operating with the ST on both sides of (3.4) gives

$$\begin{aligned} \varphi(\mu, \gamma, \tau) = \ & S^{-1}\left(\sum_{k=0}^{n-1} \omega^\varepsilon\varphi_k(\mu, \gamma)\right) + S^{-1}(\omega^\varepsilon S[g(\mu, \gamma, \tau)]) \\ & - S^{-1}(\ \omega^\varepsilon S[L[\varphi(\mu, \gamma, \tau)] + N[\varphi(\mu, \gamma, \tau)]]). \end{aligned} \tag{3.5}$$

Now, we represent solution as an infinite series given below

$$\varphi(\mu,\gamma,\tau) = \sum_{m=0}^{\infty} \varphi_m(\mu,\gamma,\tau), \tag{3.6}$$

and the nonlinear term can be decomposed as

$$N[\varphi(\mu,\gamma,\tau)] = \sum_{m=0}^{\infty} A_m(\varphi_0,\varphi_1,\ldots,\varphi_m), \tag{3.7}$$

where

$$A_m(\varphi_0,\varphi_1,\ldots,\varphi_m) = \frac{1}{m!}\frac{\partial^m}{\partial\lambda^m}\left[N\left(\sum_{i=0}^{\infty}\lambda^i\varphi_i\right)\right]_{\lambda=0}.$$

Substituting (3.6) and (3.7) in (3.5), we get

$$\begin{aligned}\sum_{m=0}^{\infty}\varphi_m(\mu,\gamma,\tau) = \quad & S^{-1}\left(\sum_{k=0}^{n-1}\omega^{\varepsilon}\varphi_k(\mu,\gamma)\right) + S^{-1}(\omega^{\varepsilon}S[g(\mu,\gamma,\tau)]) \\ & -S^{-1}\left(\omega^{\varepsilon}S\left[L\left[\sum_{m=0}^{\infty}\varphi_m(\mu,\gamma,\tau)\right] + \sum_{m=0}^{\infty}A_m\right]\right)\end{aligned} \tag{3.8}$$

On comparing both sides of the Equation (3.8), we get

$$\begin{aligned}\varphi_0(\mu,\gamma,\tau) &= S^{-1}\left(\sum_{k=0}^{n-1}\omega^{\varepsilon}\varphi_k(\mu,\gamma)\right) + S^{-1}(\omega^{\varepsilon}S[g(\mu,\gamma,\tau)]), \\ \varphi_1(\mu,\gamma,\tau) &= -S^{-1}(\ \omega^{\varepsilon}S[L[\varphi_0(\mu,\gamma,\tau)] + A_0]), \\ \varphi_2(\mu,\gamma,\tau) &= -S^{-1}(\ \omega^{\varepsilon}S[L[\varphi_1(\mu,\gamma,\tau)] + A_1]), \\ &\vdots \\ \varphi_m(\mu,\gamma,\tau) &= -S^{-1}(\ \omega^{\varepsilon}S[L[\varphi_{m-1}(\mu,\gamma,\tau)] + A_{m-1}]),\ m \geq 1.\end{aligned} \tag{3.9}$$

Finally, we approximate the analytical solution $\varphi(\mu,\gamma,\tau)$ by truncated series:

$$\varphi(\mu,\gamma,\tau) = \sum_{m=0}^{\infty}\varphi_n(\mu,\gamma,\tau). \tag{3.10}$$

4. Applications

In this section, we will implement the fractional Sumudu decomposition method for solving two dimensional fractional partial differential equations.

Example 1. *First, we consider the two-dimensional fractional partial differential equations of the form:*

$$D_{\tau}^{\varepsilon}\varphi(\mu,\gamma,\tau) = 2\left(\frac{\partial^2\varphi(\mu,\gamma,\tau)}{\partial\mu^2} + \frac{\partial^2\varphi(\mu,\gamma,\tau)}{\partial\gamma^2}\right), \tag{4.1}$$

with $1 < \varepsilon \leq 2$, *subject to initial condition*

$$\varphi(\mu,\gamma,0) = \sin(\mu)\sin(\gamma). \tag{4.2}$$

From (3.9) and (4.1), the successive approximations are

$$\begin{aligned}\varphi_0(\mu,\gamma,\tau) &= \varphi(\mu,\gamma,0), \\ \varphi_m(\mu,\gamma,\tau) &= S^{-1}\left(\omega^{\delta}S\left[2\left(\frac{\partial^2\varphi_{m-1}(\mu,\gamma,\tau)}{\partial\mu^2} + \frac{\partial^2\varphi_{m-1}(\mu,\gamma,\tau)}{\partial\gamma^2}\right)\right]\right).\end{aligned} \tag{4.3}$$

Then, we have

$$\begin{aligned}
\varphi_0(\mu,\gamma,\tau) &= \sin(\mu)\sin(\gamma),\\
\varphi_1(\mu,\gamma,\tau) &= \boldsymbol{S}^{-1}\Big(\,\omega^{\delta}\boldsymbol{S}\Big[2\Big(\tfrac{\partial^2\varphi_0(\mu,\gamma,\tau)}{\partial\mu^2}+\tfrac{\partial^2\varphi_0(\mu,\gamma,\tau)}{\partial\gamma^2}\Big)\Big]\Big)\\
&= \boldsymbol{S}^{-1}\Big(\,\omega^{\delta}\boldsymbol{S}[-4\sin(\mu)\sin(\gamma)]\Big)\\
&= -4\sin(\mu)\sin(\gamma)\boldsymbol{S}^{-1}(\,\omega^{\varepsilon})\\
&= \tfrac{-4\tau^{\varepsilon}}{\Gamma(\varepsilon+1)}\sin(\mu)\sin(\gamma).\\
\varphi_2(\mu,\gamma,\tau) &= \boldsymbol{S}^{-1}\Big(\,\omega^{\delta}\boldsymbol{S}\Big[2\Big(\tfrac{\partial^2\varphi_1(\mu,\gamma,\tau)}{\partial\mu^2}+\tfrac{\partial^2\varphi_1(\mu,\gamma,\tau)}{\partial\gamma^2}\Big)\Big]\Big)\\
&= \boldsymbol{S}^{-1}\Big(\,\omega^{\delta}\boldsymbol{S}\Big[\tfrac{16\tau^{\varepsilon}}{\Gamma(\varepsilon+1)}\sin(\mu)\sin(\gamma)\Big]\Big)\\
&= 16\sin(\mu)\sin(\gamma)\boldsymbol{S}^{-1}\Big(\,\omega^{2\varepsilon}\Big)\\
&= \tfrac{16\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\sin(\mu)\sin(\gamma).\\
\varphi_3(\mu,\gamma,\tau) &= \boldsymbol{S}^{-1}\Big(\,\omega^{\delta}\boldsymbol{S}\Big[2\Big(\tfrac{\partial^2\varphi_2(\mu,\gamma,\tau)}{\partial\mu^2}+\tfrac{\partial^2\varphi_2(\mu,\gamma,\tau)}{\partial\gamma^2}\Big)\Big]\Big)\\
&= \boldsymbol{S}^{-1}\Big(\,\omega^{\delta}\boldsymbol{S}\Big[-\tfrac{64\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\sin(\mu)\sin(\gamma)\Big]\Big)\\
&= -64\sin(\mu)\sin(\gamma)\boldsymbol{S}^{-1}\Big(\,\omega^{3\varepsilon}\Big)\\
&= \tfrac{-64\tau^{3\varepsilon}}{\Gamma(3\varepsilon+1)}\sin(\mu)\sin(\gamma).\\
&\ \ \vdots\\
\varphi_m(\mu,\gamma,\tau) &= \tfrac{(-4)^m\tau^{m\varepsilon}}{\Gamma(m\varepsilon+1)}\sin(\mu)\sin(\gamma).
\end{aligned}$$

Hence, the solution of (4.1) is given by:

$$\begin{aligned}
\varphi(\mu,\gamma,\tau) &= \sum_{m=0}^{\infty}\varphi_n(\mu,\gamma,\tau).\\
=\sum_{m=0}^{\infty}\tfrac{(-4)^m\tau^{m\varepsilon}}{\Gamma(m\varepsilon+1)}\sin(\mu)\sin(\gamma) &= \sin(\mu)\sin(\gamma)\Big(1-\tfrac{4\tau^{\varepsilon}}{\Gamma(\varepsilon+1)}+\tfrac{4^2\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}-\tfrac{4^3\tau^{3\varepsilon}}{\Gamma(3\varepsilon+1)}+\cdots\Big)\\
&= \sin(\mu)\sin(\gamma)E_{\varepsilon}(-4\tau^{\varepsilon}).
\end{aligned} \tag{4.4}$$

If we put $\varepsilon \to 2$ *in Equation (4.4), we get the exact solution:*

$$\begin{aligned}
\varphi(\mu,\gamma,\tau) &= \sum_{m=0}^{\infty}\tfrac{(-1)^m(2\tau)^{2m}}{\Gamma(2m+1)}\sin(\mu)\sin(\gamma)\\
&= \sin(\mu)\sin(\gamma)\cos(2\tau).
\end{aligned}$$

Example 2. *we consider the fractional generalized biologic population model of the form:*

$$D_{\tau}^{\varepsilon}\varphi(\mu,\gamma,\tau)=\left(\frac{\partial^2\varphi^2(\mu,\gamma,\tau)}{\partial\mu^2}+\frac{\partial^2\varphi^2(\mu,\gamma,\tau)}{\partial\gamma^2}\right)+\varphi(\mu,\gamma,\tau)-r\varphi^2(\mu,\gamma,\tau), \tag{4.5}$$

with $0<\varepsilon\le 1$, *subject to initial condition*

$$\varphi(\mu,\gamma,0)=e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}. \tag{4.6}$$

From (3.9) and (4.6), the successive approximations are

$$\begin{aligned}
\varphi_0(\mu,\gamma,\tau) &= \varphi(\mu,\gamma,0),\\
\varphi_m(\mu,\gamma,\tau) &= \boldsymbol{S}^{-1}\Big(\,\omega^{\delta}\boldsymbol{S}\Big[\Big(\tfrac{\partial^2 A_{m-1}}{\partial\mu^2}+\tfrac{\partial^2 A_{m-1}}{\partial\gamma^2}\Big)+\varphi_{m-1}(\mu,\gamma,\tau)-rA_{m-1}\Big]\Big),
\end{aligned} \tag{4.7}$$

where

$$A_0 = \varphi_0^2$$
$$A_1 = 2\varphi_0\varphi_1$$
$$A_2 = 2\varphi_0\varphi_2 + \varphi_1^2$$
$$A_3 = 2\varphi_0\varphi_3 + 2\varphi_1\varphi_2$$
$$\vdots$$

Then, we have

$$\begin{aligned}
\varphi_0(\mu,\gamma,\tau) &= e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)},\\
\varphi_1(\mu,\gamma,\tau) &= \boldsymbol{S}^{-1}\left(\omega^\delta \boldsymbol{S}\left[\left(\frac{\partial^2 A_0}{\partial\mu^2}+\frac{\partial^2 A_0}{\partial\gamma^2}\right)+\varphi_0(\mu,\gamma,\tau)-rA_0\right]\right)\\
&= \boldsymbol{S}^{-1}\left(\omega^\delta \boldsymbol{S}\left[e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}\right]\right)\\
&= e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}\boldsymbol{S}^{-1}(\omega^\varepsilon)\\
&= \frac{\tau^\varepsilon}{\Gamma(\varepsilon+1)}e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}.\\
\varphi_2(\mu,\gamma,\tau) &= \boldsymbol{S}^{-1}\left(\omega^\delta \boldsymbol{S}\left[\left(\frac{\partial^2 A_1}{\partial\mu^2}+\frac{\partial^2 A_1}{\partial\gamma^2}\right)+\varphi_1(\mu,\gamma,\tau)-rA_1\right]\right)\\
&= \boldsymbol{S}^{-1}\left(\omega^\delta \boldsymbol{S}\left[\frac{\tau^\varepsilon}{\Gamma(\varepsilon+1)}e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}\right]\right)\\
&= e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}\boldsymbol{S}^{-1}\left(\omega^{2\varepsilon}\right)\\
&= \frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}.\\
\varphi_3(\mu,\gamma,\tau) &= \boldsymbol{S}^{-1}\left(\omega^\delta \boldsymbol{S}\left[\left(\frac{\partial^2 A_2}{\partial\mu^2}+\frac{\partial^2 A_2}{\partial\gamma^2}\right)+\varphi_2(\mu,\gamma,\tau)-rA_2\right]\right)\\
&= \boldsymbol{S}^{-1}\left(\omega^\delta \boldsymbol{S}\left[\frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}\right]\right)\\
&= e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}\boldsymbol{S}^{-1}\left(\omega^{3\varepsilon}\right)\\
&= \frac{\tau^{3\varepsilon}}{\Gamma(3\varepsilon+1)}e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}.\\
&\vdots\\
\varphi_m(\mu,\gamma,\tau) &= \frac{\tau^{m\varepsilon}}{\Gamma(m\varepsilon+1)}e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}.
\end{aligned}$$

Hence, the fractional series form of (4.5) is given by

$$\begin{aligned}
\varphi(\mu,\gamma,\tau) &= \sum_{m=0}^{\infty}\frac{\tau^{m\varepsilon}}{\Gamma(m\varepsilon+1)}e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}\\
&= e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}\left(1+\frac{\tau^\varepsilon}{\Gamma(\varepsilon+1)}+\frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}+\frac{\tau^{3\varepsilon}}{\Gamma(3\varepsilon+1)}+\cdots\right)\\
&= e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}E_\varepsilon(\tau^\varepsilon).
\end{aligned} \tag{4.8}$$

If we put $\varepsilon \to 1$ *in Equation (4.8), we get the exact solution*

$$\begin{aligned}
\varphi(\mu,\gamma,\tau) &= \sum_{m=0}^{\infty}\frac{\tau^{m}}{\Gamma(m+1)}e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)}\\
&= e^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\gamma)+t}.
\end{aligned}$$

Example 3. *Consider the nonlinear time-fractional differential equation of the form:*

$$D_\tau^\varepsilon\varphi(\mu,\gamma,\tau) = \left(\frac{\partial^2\varphi^2(\mu,\gamma,\tau)}{\partial\mu^2}+\frac{\partial^2\varphi^2(\mu,\gamma,\tau)}{\partial\gamma^2}\right)+\varphi(\mu,\gamma,\tau), \tag{4.9}$$

with $0 < \varepsilon \leq 1$, *subject to initial condition*

$$\varphi(\mu,\gamma,0) = \sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}. \tag{4.10}$$

From (3.9) and (4.10), the successive approximations are:

$$\begin{aligned} \varphi_0(\mu,\gamma,\tau) &= \varphi(\mu,\gamma,0), \\ \varphi_m(\mu,\gamma,\tau) &= S^{-1}\Big(\omega^\delta S\Big[\Big(\frac{\partial^2 A_{m-1}}{\partial\mu^2} + \frac{\partial^2 A_{m-1}}{\partial\gamma^2}\Big) + \varphi_{m-1}(\mu,\gamma,\tau)\Big]\Big), \end{aligned} \tag{4.11}$$

Then, we have

$$\begin{aligned} \varphi_0(\mu,\gamma,\tau) &= \sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}, \\ \varphi_1(\mu,\gamma,\tau) &= S^{-1}\Big(\omega^\delta S\Big[\Big(\frac{\partial^2 A_0}{\partial\mu^2} + \frac{\partial^2 A_0}{\partial\gamma^2}\Big) + \varphi_0(\mu,\gamma,\tau)\Big]\Big) \\ &= S^{-1}\Big(\omega^\delta S\Big[\sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}\Big]\Big) \\ &= \sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}S^{-1}(\omega^\varepsilon) \\ &= \frac{\tau^\varepsilon}{\Gamma(\varepsilon+1)}\sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}. \\ \varphi_2(\mu,\gamma,\tau) &= S^{-1}\Big(\omega^\delta S\Big[\Big(\frac{\partial^2 A_1}{\partial\mu^2} + \frac{\partial^2 A_1}{\partial\gamma^2}\Big) + \varphi_1(\mu,\gamma,\tau)\Big]\Big) \\ &= S^{-1}\Big(\omega^\delta S\Big[\frac{\tau^\varepsilon}{\Gamma(\varepsilon+1)}\sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}\Big]\Big) \\ &= \sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}S^{-1}\Big(\omega^{2\varepsilon}\Big) \\ &= \frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}. \\ \varphi_3(\mu,\gamma,\tau) &= S^{-1}\Big(\omega^\delta S\Big[\Big(\frac{\partial^2 A_2}{\partial\mu^2} + \frac{\partial^2 A_2}{\partial\gamma^2}\Big) + \varphi_2(\mu,\gamma,\tau)\Big]\Big) \\ &= S^{-1}\Big(\omega^\delta S\Big[\frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}\Big]\Big) \\ &= \sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}S^{-1}\Big(\omega^{3\varepsilon}\Big) \\ &= \frac{\tau^{3\varepsilon}}{\Gamma(3\varepsilon+1)}\sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}. \\ &\vdots \\ \varphi_m(\mu,\gamma,\tau) &= \frac{\tau^{m\varepsilon}}{\Gamma(m\varepsilon+1)}\sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}. \end{aligned}$$

Hence, the fractional series form of (4.5) is given by

$$\begin{aligned} \varphi(\mu,\gamma,\tau) &= \sum_{m=0}^{\infty}\frac{\tau^{m\varepsilon}}{\Gamma(m\varepsilon+1)}\sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)} \\ &= \sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}\Big(1 + \frac{\tau^\varepsilon}{\Gamma(\varepsilon+1)} + \frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)} + \frac{\tau^{3\varepsilon}}{\Gamma(3\varepsilon+1)} + \cdots\Big) \\ &= \sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}E_\varepsilon(\tau^\varepsilon). \end{aligned} \tag{4.12}$$

If we put $\varepsilon \to 1$ *in Equation (4.12), we get the exact solution*

$$\begin{aligned} \varphi(\mu,\gamma,\tau) &= \sum_{m=0}^{\infty}\frac{\tau^m}{\Gamma(m+1)}\sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)} \\ &= \sqrt{\sin(\mu)\sin\mathrm{h}(\gamma)}\, e^t. \end{aligned}$$

5. Conclusions

The coupling of the Adomian decomposition method (ADM) and the Sumudu transform method in the sense of Caputo fractional derivatives proved very effective for solving two-dimensional fractional partial differential equations. The proposed algorithm provides a solution in a series form that converges rapidly to an exact solution if it exists. From the obtained results, it is clear that the FSDM yields very accurate solutions using only a few iterates. As a result, the conclusion that comes through this work is that FSDM can be applied to other fractional partial differential equations of higher order, due to the efficiency and flexibility in the application as can be seen in the proposed examples.

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