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Homotopy Analysis Method to Solve Two-Dimensional Nonlinear Volterra-Fredholm Fuzzy Integral Equations

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Abstract: The main goal of the paper is to present an approximate method for solving of a two-dimensional nonlinear Volterra-Fredholm fuzzy integral equation (2D-NVFFIE). It is applied the homotopy analysis method (HAM). The studied equation is converted to a nonlinear system of Volterra-Fredholm integral equations in a crisp case. Approximate solutions of this system are obtained by the help with HAM and hence an approximation for the fuzzy solution of the nonlinear Volterra-Fredholm fuzzy integral equation is presented. The convergence of the proposed method is proved and the error estimate between the exact and the approximate solution is obtained. The validity and applicability of the proposed method is illustrated on a numerical example.

Keywords: homotopy analysis method; two-dimensional nonlinear Volterra-Fredholm fuzzy integral equation; convergence; error estimation

MSC: 41A25; 45G10; 65R20

1. Introduction

Fuzzy integral equations are one of the important branches of fuzzy analysis theory and they are applied as an adequate apparatus in mathematical modeling in biology, chemistry, physics, engineering, etc. (see, for example, [1–4]). One of the first applications of fuzzy integration was given by Wu and Ma [5], who investigated the fuzzy Fredholm integral equation of the second kind.

In connection with the application, it is very important to such kind of problems. If methods for solving such problems with uncertainty are developed, then, many real life models in different fields with imprecise variable can be solved easily and accurately. Some fixed point theorems for complete fuzzy metric space are given in [6–8]. In recent years, many mathematicians have studied a solution to fuzzy integral equations by numerical methods [9–14].

Liao employed the basic idea of the homotopy in topology to propose a general analytic method for nonlinear problems, namely HAM (see the monograph [15], and the papers [32], [33], [34]). This method is based on the concept of creating function series. If the series converges, its sum is the solution of this system of equations. Later, HAM has been successfully applied to solve many types of nonlinear problems such as multiple solutions of nonlinear boundary value problems ([16]), Abel fuzzy integral equations ([17]), partial differential equations ([18]), two-dimensional linear Volterra fuzzy integral equations ([19]), and fuzzy linear Volterra integral equations of the second kind ([20]).

The paper presents an application of HAM for solving the following nonlinear Volterra-Fredholm fuzzy integral equations with two variables (2D-NVFFIE)

$$\begin{aligned} u(s,t) =& g(s,t) \oplus (FR) \int_c^t k_1(t,\tau) \odot G_1(u(s,\tau)) d\tau \oplus \\ & \oplus (FR) \int_a^b k_2(s,\xi) \odot G_2(u(\xi,t)) d\xi, \end{aligned} \tag{1}$$

where $g,\ u : A = [a,b] \times [c,d] \to E^1$ are continuous fuzzy-number valued functions, $k_1 : [c,d] \times [c,d] \to \mathbf{R}_+,\ \ k_2 : [a,b] \times [a,b] \to \mathbf{R}_+$ and $G_1,\ G_2 : E^1 \to E^1$ are continuous functions. By E^1 is denoted the set of all fuzzy numbers.

The integral Equation (1) is called the Fredholm equation with respect to the position and Volterra with respect to the time. This equation is used in many problems of mathematical physics, theory of elasticity, contact problems and mixed problems of mechanics of continuous media (see [2,21]). In [22], the Adomian decomposition method is applied for solving this equation.

The structure of this paper is organized as follows: In Section 2, some basic notations used in fuzzy calculus are introduced. In Section 3, we present HAM. In Section 4, we apply HAM for the parametric form of 2D-NVFFIE. In Section 5, we prove the convergence of the proposed method and we give an error estimate. In Section 6, a numerical example is illustrating the application of the presented above procedure for approximately solving of the studied equation.

2. Preliminaries

In this section, we review the fundamental notations of fuzzy set theory to be used throughout this paper.

Definition 1 ([23])**.** *A fuzzy number is a function $u : \mathbf{R} \to [0,1]$ satisfying the following properties:*

(i) *u is upper semi-continuous on $\mathbf{R}$;*
(ii) *$u(x) = 0$ outside of some interval $[c,d]$;*
(iii) *there are the real numbers a and b with $c \le a \le b \le d$, such that u is increasing on $[c,a]$, decreasing on $[b,d]$ and $u(x) = 1$ for each $x \in [a,b]$;*
(iv) *$u(rx + (1-r)y) \ge \min\{u(x), u(y)\}$ for any $x, y \in \mathbf{R},\ r \in [0,1]$.*

By E^1 we denote the set of all fuzzy numbers. Any real number $a \in \mathbf{R}$ can be interpreted as a fuzzy number $\tilde{a} = \chi(a)$ and therefore $\mathbf{R} \subset E^1$.

Denote $\mathbf{R}_+ = (0,\infty)$.

For any $0 < r \le 1$ we denote the r-level set $[u]^r = \{x \in \mathbf{R} : u(x) \ge r\}$ that is a closed interval and $[u]^r = [u^r_-, u^r_+]$ for all $r \in [0,1]$, where u_-, u_+ can be considered as functions $u_-, u_+ : [0,1] \to \mathbf{R}$, such that u_- is increasing and u_+ is decreasing.

For $u, v \in E^1$, $k \in \mathbf{R}$ the addition and the scalar multiplication are defined by $[u \oplus v]^r = [u]^r + [v]^r = [u^r_- + v^r_-, u^r_+ + v^r_+]$ and $[k \odot u]^r = k.[u]^r = \begin{cases} [ku^r_-, ku^r_+], & k \ge 0 \\ [ku^r_+, ku^r_-], & k < 0. \end{cases}$

The neutral element with respect to $\oplus$ in E^1 is denoted by $\tilde{0} = \chi_{\{0\}}$.

According to [24], we can summarize the following algebraic properties:

(i) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ and $u \oplus v = v \oplus u$ for any $u, v, w \in E^1$;
(ii) $u \oplus \tilde{0} = \tilde{0} \oplus u = u$ for any $u \in E^1$;
(iii) with respect to $\tilde{0}$, none $u \in E^1 \setminus \mathbf{R}$, $u \ne \tilde{0}$ has opposite in $(E^1, \oplus)$;
(iv) for any $a, b \in \mathbf{R}$ with $a, b \ge 0$ or $a, b \le 0$ and any $u \in E^1$ we have $(a+b) \odot u = a \odot u \oplus b \odot u$;
(v) for any $a \in \mathbf{R}$ and any $u, v \in E^1$ we have $a \odot (u \oplus v) = a \odot u \oplus a \odot v$;
(vi) for any $a, b \in \mathbf{R}$ and any $u \in E^1$ we have $a \odot (b \odot u) = (ab) \odot u$ and $1 \odot u = u$.

As a distance between fuzzy numbers we use the Hausdorff metric.

Definition 2 ([24]). *For arbitrary fuzzy numbers $u = (u^r_-, u^r_+)$ and $v = (v^r_-, v^r_+)$ the quantity $D(u, v) = \sup_{r\in[0,1]} \max\{|u^r_- - v^r_-|, |u^r_+ - v^r_+|\}$ is the distance between u, v.*

Lemma 1 ([24]). *The following properties of the above distance hold:*
(i) (E^1, D) is a complete metric space;
(ii) $D(u \oplus w, v \oplus w) = D(u, v)$, for all $u, v, w \in E^1$;
(iii) $D(k \odot u, k \odot v) = |k|D(u, v)$, for all $u, v \in E^1$, for all $k \in \mathbf{R}$;
(iv) $D(u \oplus v, w \oplus e) = D(u, w) + D(v, e)$, for all $u, v, w, e \in E^1$;
(v) $D(u \oplus v, \tilde{0}) \leq D(u, \tilde{0}) + D(v, \tilde{0})$, for all $u, v \in E^1$;
(vi) $D(k_1 \odot u, k_2 \odot u) = |k_1 - k_2|D(u, \tilde{0})$, for all $k_1, k_2 \in \mathbf{R}$ with $k_1k_2 \geq 0$ and $u \in E^1$.

For any fuzzy-number-valued function $f : I \subset \mathbf{R} \to E^1$ we define the functions $\underline{f}(., r)$, $\overline{f}(., r) : I \to \mathbf{R}$, by

$$\underline{f}(t, r) = (f(t))^r_- \text{ and } \overline{f}(t, r) = (f(t))^r_+ \text{ for each } t \in I,\ r \in [0, 1].$$

These functions are called the left and right $r-$level functions of f.

We will use the notion of Henstock integral for fuzzy-number-valued functions defined as follows::

Let $f : [a, b] \to E^1$. For $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ a partition of the interval $[a, b]$, we consider the points $\xi_i \in [x_{i-1}, x_i]$, $i = 1, ..., n$, and the function $\delta : [a, b] \to \mathbf{R}_+$.

The partition $P = \{([x_{i-1}, x_i]; \xi_i); i = 1, ..., n\}$ denoted by $P = (\Delta_n, \xi)$ is called δ-fine iff $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.

Definition 3 ([24]). *For $I \in E^1$, the function f is fuzzy-Henstock integrable on $[a, b]$ if for any $\varepsilon > 0$ there is a function $\delta : [a, b] \to \mathbf{R}_+$ such that for any partition δ-fine P, $D(\sum_{i=1}^{n} (x_i - x_{i-1}) \odot f(\xi_i), I) < \varepsilon$. The fuzzy number I is named the fuzzy-Henstock integral of f and will be denoted by $(FH) \int_a^b f(t)dt$.*

When the function $\delta : [a, b] \to \mathbf{R}_+$ is constant, then we obtain the Riemann integrability for fuzzy-number-valued functions ([23]). In this case, $I \in E^1$ is called the fuzzy-Riemann integral of f on $[a, b]$, being denoted by $(FR) \int_a^b f(t)dt$. Consequently, the fuzzy-Riemann integrability is a particular case of the fuzzy-Henstock integrability, and therefore the properties of the integral (FH) will be valid for the integral (FR), too.

Lemma 2 ([25]). *Let $f : [a, b] \to E^1$. Then f is (FH) integrable if and only if $\underline{f}(., r)$ and $\overline{f}(., r)$ are Henstock integrable for any $r \in [0, 1]$. Furthermore, for any $r \in [0, 1]$,*

$$\left[(FH) \int_a^b f(t)dt\right]^r = \left[(H) \int_a^b \underline{f}(t, r)dt,\ (H) \int_a^b \overline{f}(t, r)dt\right].$$

Remark 1. *If $f : [a, b] \to E^1$ is continuous, then $\underline{f}(., r)$ and $\overline{f}(., r)$ are continuous for any $r \in [0, 1]$ and consequently, they are Henstock integrable. According to Lemma 1 we infer that f is (FH) integrable*

For any fuzzy-number-valued function $f : A = [a, b] \times [c, d] \to E^1$ we can define the functions $\underline{f}(., ., r)$, $\overline{f}(., ., r) : A \to \mathbf{R}$, by $\underline{f}(s, t, r) = (f(s, t))^r_-$ and $\overline{f}(s, t, r) = (f(s, t))^r_+$ for all $(s, t) \in A$ and $r \in [0, 1]$. These functions are called the left and right $r-$level functions of f.

Definition 4 ([26]). *A fuzzy-number-valued function $f : A \to E^1$ is called:*

(i) continuous in $(x_0, y_0) \in A$ *if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $(x, y) \in A$ *with* $|x - x_0| < \delta$, $|y - y_0| < \delta$ *it follows that* $D(f(x, y), f(x_0, y_0)) < \varepsilon$.
(ii) continuous on A *if it is continuous in each* $(x, y) \in A$*;*
(iii) bounded if there exists $M \geq 0$ *such that* $D(f(x, y), \tilde{0}) \leq M$ *for all* $(x, y) \in A$.

On the set $C(A, E^1) = \{f : A \to E^1;\ f \text{ is continuous}\}$ the metric is defined by:

$$D^*(f, g) = \max_{(s,t)\in A} D(f(s, t), g(s, t)) \text{ for all } f, g \in C(A, E^1).$$

We see that $(C(A, E^1), D^*)$ is a complete metric space.

3. The Homotopy Analysis Method

We will give a brief overview of the main used method—HAM. Homotopy analysis method transforms the considered equation into the corresponding deformation equation. Using this method, we solve the operator equation

$$N(u(z)) = 0,\ z \in \Omega, \tag{2}$$

where N is the operator, u is the unknown function and Ω is any domain of the variable z.

Define the homotopy operator $\mathcal{H}$ as

$$\mathcal{H}(\Phi, p) \equiv (1 - p)L(\Phi(z; p) - u_0(z)) - phN(\Phi(z; p)),$$

where $p \in [0, 1]$ is an embedded parameter, $h \neq 0$ denotes the convergence control parameter (see, for example [27,28]), u_0 represents the initial approximation of the solution of (2) and L is linear operator with property $L(0) = 0$.

Solving equation $\mathcal{H}(\Phi, p) = 0$, we get the zero-order deformation equation

$$(1 - p)L(\Phi(z; p) - u_0(z)) = phN(\Phi(z; p)), \tag{3}$$

Substitute $p = 0$ in (3) and obtain $L(\Phi(z; 0) - u_0(z)) = 0$. Therefore, $\Phi(z; 0) = u_0(z)$. If $p = 1$, then $N(\Phi(z; p)) = 0$, i.e., $\Phi(z; 1)$ is solution of the Equation (2). In this way, the change of parameter p from zero to one corresponds to the transition from a trivial task to the original task.

Taking the Maclaurin series of function $\Phi(z; p)$ with respect to the parameter p, we obtain

$$\Phi(z; p) = u_0(z) + \sum_{m=1}^{\infty} u_m(z)p^m, \tag{4}$$

where $u_0(z) = \Phi(z; 0)$ and

$$u_m(z) = \frac{1}{m!}\frac{\partial^m \Phi(z; p)}{\partial p^m}|_{p=0},\quad m = 1, 2, 3, ... \tag{5}$$

If the above series converges for $p = 1$, we obtain the required solution

$$u(z) = \sum_{m=0}^{\infty} u_m(z). \tag{6}$$

In order to determine the function u_m we differentiate m-times, with respect to parameter p, the left and right-hand side of Formula (3), then the obtained result is divided by $m!$ and substituted with $p = 0$ which gives the so-called mth-order deformation equation $(m \geq 1)$:

$$L(u_m(z) - \chi_m u_{m-1}(z)) = hR_m(\tilde{u}_{m-1}(z)), \tag{7}$$

where $\tilde{u}_{m-1}(z) = \{u_0(z), u_1(z), ..., u_{m-1}(z)\}$, and

$$\chi_m = \begin{cases} 0, & m = 1 \\ 1, & m \geq 2 \end{cases} \tag{8}$$

and

$$R_m(\tilde{u}_{m-1}(z)) = \frac{1}{(m-1)!}\left(\frac{\partial^{m-1} N[\Phi(z;p)]}{\partial p^{m-1}}\right)|_{p=0}. \tag{9}$$

Apply L^{-1} to both sides of (7) and obtain

$$u_m(z) = \chi_m u_{m-1}(z)) + hL^{-1}(R_m(\tilde{u}_{m-1}(z))). \tag{10}$$

If we are not able to determine the sum of the series in (6), then for the approximate solution of the considered equation we accept the partial sum of this series

$$s_m(z) = \sum_{i=0}^{m} u_i(z). \tag{11}$$

Choosing in an appropriate way the convergence control parameter h, we can influence the convergence region of the created series and the rate of this convergence ([15,29]). One of the methods to select the value of this parameter is the so-called "optimization method" ([15]). In this method, we define the squared residual of the governing equation

$$E_n(h) = \int_{\Omega} (N(s_n(z))^2 dz \tag{12}$$

where $s_n(z)$ and $u_i(z)$ are defined by (11) and (10) respectively and they depend on h.

The optimum value of the convergence control parameter is obtained by determining the minimum of this squared residual. The effective region of the convergence control parameter is additionally defined by

$$R_h = \{h : \lim_{n\to\infty} E_n(h) = 0\}. \tag{13}$$

Choosing a different value of the convergence control parameter than the optimal one, but still belonging to the effective region, we also obtain the convergent series, only the rate of convergence is lower. A version of the method with the above described selection of optimal value the convergence control parameter is called the basic optimal HAM ([15]).

4. Applying HAM to 2D-NVFFIE (1)

In this section we introduce the parametric form of the integral Equation (1) and then we will apply HAM for solving this equation.

Let $u(s,t,r) = (\underline{u}(s,t,r), \overline{u}(s,t,r))$ and $g(s,t,r) = (\underline{g}(s,t,r), \overline{g}(s,t,r))$; $0 \leq r \leq 1$ and $(s,t) \in A$ are parametric form of functions $u(s,t)$ and $g(s,t)$, respectively. So the parametric form of equation (1) is as follows:

$$\underline{u}(s,t,r) = \underline{g}(s,t,r) + \int_c^t \underline{k_1(t,\tau)G_1(u(s,\tau,r))}d\tau + \int_a^b \underline{k_2(s,\xi)G_2(u(\xi,t,r))}d\xi,$$

$$\overline{u}(s,t,r) = \overline{g}(s,t,r) + \int_c^t \overline{k_1(t,\tau)G_1(u(s,\tau,r))}d\tau + \int_a^b \overline{k_2(s,\xi)G_2(u(\xi,t,r))}d\xi.$$

We define

$$\begin{aligned}
H_1(\underline{u}(s,t,r),\overline{u}(s,t,r)) &= min\{G_1(\beta) : \underline{u}(s,t,r) \le \beta \le \overline{u}(s,t,r)\},\\
H_2(\underline{u}(s,t,r),\overline{u}(s,t,r)) &= min\{G_2(\beta) : \underline{u}(s,t,r) \le \beta \le \overline{u}(s,t,r)\},\\
F_1(\underline{u}(s,t,r),\overline{u}(s,t,r)) &= max\{G_1(\beta) : \underline{u}(s,t,r) \le \beta \le \overline{u}(s,t,r)\},\\
F_2(\underline{u}(s,t,r),\overline{u}(s,t,r)) &= max\{G_2(\beta) : \underline{u}(s,t,r) \le \beta \le \overline{u}(s,t,r)\},
\end{aligned}$$

where $(s,t) \in A$. Then,

$$\underline{k_1(t,\tau)G_1(u(s,\tau,r))} = \begin{cases} k_1(t,\tau)H_1(\underline{u}(s,\tau,r),\overline{u}(s,\tau,r)), & \text{if } k_1(t,\tau) \ge 0\\ k_1(t,\tau)F_1(\underline{u}(s,\tau,r),\overline{u}(s,\tau,r)), & \text{if } k_1(t,\tau) < 0, \end{cases}$$

$$\underline{k_2(s,\xi)G_2(u(\xi,t,r))} = \begin{cases} k_2(s,\xi)H_2(\underline{u}(\xi,t,r),\overline{u}(\xi,t,r)), & \text{if } k_2(s,\xi) \ge 0\\ k_2(s,\xi)F_2(\underline{u}(\xi,t,r),\overline{u}(\xi,t,r)), & \text{if } k_2(s,\xi) < 0, \end{cases}$$

$$\overline{k_1(t,\tau)G_1(u(s,\tau,r))} = \begin{cases} k_1(t,\tau)F_1(\underline{u}(s,\tau,r),\overline{u}(s,\tau,r)), & \text{if } k_1(t,\tau) \ge 0\\ k_1(t,\tau)H_1(\underline{u}(s,\tau,r),\overline{u}(s,\tau,r)), & \text{if } k(t,\tau) < 0, \end{cases}$$

$$\overline{k_2(s,\xi)G_2(u(\xi,t,r))} = \begin{cases} k_2(s,\xi)F_2(\underline{u}(\xi,t,r),\overline{u}(\xi,t,r)), & \text{if } k_2(s,\xi) \ge 0\\ k_2(s,\xi)H_2(\underline{u}(\xi,t,r),\overline{u}(\xi,t,r)), & \text{if } k_2(s,\xi) < 0 \end{cases}$$

for $a \le s,\xi \le b$, $c \le \tau \le t \le d$ and $0 \le r \le 1$.

Let for all $a \le s,\xi \le b$, $c \le \tau \le t \le d$ and $0 \le r \le 1$ the functions $G_1(\beta),\ G_2(\beta)$ are increasing for $\beta \in [\underline{u}(s,t,r),\overline{u}(s,t,r)]$ and $k_1(t,\tau) \ge 0,\ k_2(s,\xi) \ge 0$.

Then the parametric form of Equation (1) is

$$\begin{aligned}
\underline{u}(s,t,r) &= \underline{g}(s,t,r) + \int_c^t k_1(t,\tau)G_1(\underline{u}(s,\tau,r))d\tau\\
&+ \int_a^b k_2(s,\xi)G_2(\underline{u}(\xi,t,r))d\xi,\ \ s \in [a,b], t \in [c,d],\ r \in [0,1],
\end{aligned} \tag{14}$$

$$\begin{aligned}
\overline{u}(s,t,r) &= \overline{g}(s,t,r) + \int_c^t k_1(t,\tau)G_1(\overline{u}(s,\tau,r))d\tau\\
&+ \int_a^b k_2(s,\xi)G_2(\overline{u}(\xi,t,r))d\xi,\ \ s \in [a,b], t \in [c,d],\ r \in [0,1].
\end{aligned} \tag{15}$$

We consider the operators

$$L(\underline{u}(s,t,r)) = \underline{u}(s,t,r)$$

and

$$N(\underline{u}(s,t,r)) = \underline{u}(s,t,r) - \underline{g}(s,t,r) - \int_c^t k_1(t,\tau)G_1(\underline{u}(s,\tau,r))d\tau - \int_a^b k_2(s,\xi)G_2(\underline{u}(\xi,t,r))d\xi.$$

Then we get

$$\underline{u}_m(s,t,r) = \chi_m \underline{u}_{m-1}(s,t,r) + hR_m(\underline{\tilde{u}}_{m-1}(s,t,r)), \tag{16}$$

where χ_m is determined by (8).

From (9) for the operator R_m, $m \geq 1$ we obtain

$$R_m(\underline{\tilde{u}}_{m-1}(s,t,r)) = \frac{1}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial p^{m-1}} N(\sum_{i=0}^{\infty} \underline{u}_i(s,t,r)p^i)\right)_{|p=0}$$
$$= \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}\left(\sum_{i=0}^{\infty} \underline{u}_i(s,t,r)p^i - \underline{g}(s,t,r)\right)_{|p=0} -$$
$$- \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}\left(\int_c^t k_1(t,\tau)G_1(\sum_{i=0}^{\infty} \underline{u}_i(s,\tau,r))p^i d\tau + \int_a^b k_2(s,\xi)G_2(\sum_{i=0}^{\infty} \underline{u}_i(\xi,t,r)p^i)d\xi\right)_{|p=0}$$
$$= \underline{u}_{m-1}(s,t,r) - \frac{1-\lambda_m}{(m-1)!}\underline{g}(s,t,r) - \frac{1}{(m-1)!}\int_c^t k_1(t,\tau)\left(\frac{\partial^{m-1}}{\partial p^{m-1}} G_1(\sum_{i=0}^{\infty} \underline{u}_i(s,\tau,r)p^i)\right)_{|p=0} d\tau$$
$$- \frac{1}{(m-1)!}\int_a^b k_2(s,\xi)\left(\frac{\partial^{m-1}}{\partial p^{m-1}} G_2(\sum_{i=0}^{\infty} \underline{u}_i(\xi,t,r))p^i\right)_{|p=0} d\xi,$$

By the definitions of the appropriate operators we obtain

$$\begin{aligned} \underline{u}_1(s,t,r) &= h(\underline{u}_0(s,t,r) - \underline{g}(s,t,r) \\ &- \int_c^t k_1(t,\tau)G_1(\underline{u}_0(s,\tau,r))d\tau - \int_a^b k_2(s,\xi)G_2(\underline{u}_0(\xi,t,r))d\xi), \end{aligned} \tag{17}$$

where $\underline{u}_0 \in C(A \times [0,1], \mathbf{R})$ and for $m \geq 2$

$$\begin{aligned} \underline{u}_m(s,t,r) &= (1+h)\underline{u}_{m-1}(s,t,r) \\ &- \frac{h}{(m-1)!}\int_c^t k_1(t,\tau)\left(\frac{\partial^{m-1}}{\partial p^{m-1}} G_1(\sum_{i=0}^{\infty} \underline{u}_i(s,\tau,r)p^i)\right)_{|p=0} d\tau \\ &- \frac{h}{(m-1)!}\int_a^b k_2(s,\xi)\left(\frac{\partial^{m-1}}{\partial p^{m-1}} G_2(\sum_{i=0}^{\infty} \underline{u}_i(\xi,t,r)p^i)\right)_{|p=0} d\xi. \end{aligned} \tag{18}$$

5. Convergence of the HAM

In this section we prove convergence of HAM and find an error estimate.

We introduce the following conditions:

(i) $g \in C(A, E^1)$, $k_1 \in C([c,d] \times [c,d], \mathbf{R}_+)$, $k_2 \in C([a,b] \times [a,b], \mathbf{R}_+)$ and $M_1 = \max_{t,\tau \in [c,d]} k_1(t,\tau) > 0$ and $M_2 = \max_{t,\tau \in [a,b]} k_2(s,\xi)| > 0$.

(ii) there exist constants $L_k > 0$, $k = 1,2$ such that

$$D(G_k(u), G_k(v)) \leq L_k D(u,v), \text{ for } u,v \in E^1,\ k = 1,2;$$

(iii) The inequality

$$\alpha = M_1 L_1 (d-c) + M_2 L_2 (b-a) < 1 \tag{19}$$

holds.

We will use the following result:

Theorem 1 ([22])**.** *Let the conditions (i)–(iii) be fulfilled. Then the integral Equation (1) has an unique solution.*

Remark 2. *According to Theorem 1 if the conditions (i)–(iii) are fulfilled, then the Equations (14) and (15) have unique solutions.*

Theorem 2. *Let*

1. *the conditions (i)–(iii) are fulfilled.*
2. *the functions $\underline{u}_m(s,t,r)$, $m \geq 1$, are defined by relations (17) and (18).*
3. *$L_k < 1$ for $k = 1,2$*
4. *the series*
$$\sum_{i=0}^{\infty} \underline{u}_i(s,t,r) \tag{20}$$
converges.

Then the sum of the series (20) is the unique solution of Equation (14).

Proof. Introduce the notation

$$\underline{H}_{k,m}(s,t,r) = \frac{1}{m!}\left(\frac{\partial^m}{\partial p^m} G_k(\sum_{i=0}^{\infty} \underline{u}_i(s,t,r)p^i)\right)_{|p=0}, \quad k = 1,2.$$

From [30], if G_k is the contraction mapping and the series (20) converges to $\underline{u}(s,t,r)$, then the series $\sum_{m=0}^{\infty} \underline{H}_{k,m}(s,t,r)$, $k = 1,2$, is respectively convergent to $G_k(\underline{u}(s,t,r))$.

From condition 4, it follows that for any $(s,t,r) \in A \times [0,1]$ we have $\lim_{m\to\infty} \underline{u}_m(s,t,r) = 0$.

By applying the definition of the operator L we can write

$$\begin{aligned}
&\sum_{m=1}^{n} L(\underline{u}_m(s,t,r) - \chi_m \underline{u}_{m-1}(s,t,r)) \\
&= \sum_{m=1}^{n} (\underline{u}_m(s,t,r) - \chi_m \underline{u}_{m-1}(s,t,r)) = \underline{u}_n(s,t,r).
\end{aligned}$$

Hence $\sum_{m=1}^{\infty} L(\underline{u}_m(s,t,r) - \chi_m \underline{u}_{m-1}(s,t,r)) = \lim_{n\to\infty} \underline{u}_n(s,t,r) = 0$.

Therefore, $h \sum_{m=1}^{\infty} R_m(\underline{\tilde{u}}_{m-1}(s,t,r)) = \sum_{m=1}^{\infty} L(\underline{u}_m(s,t,r) - \chi_m \underline{u}_{m-1}(s,t,r))$ and $\sum_{m=1}^{\infty} R_m(\underline{\tilde{u}}_{m-1}(s,t,r)) = 0$.

Then we obtain

$$\begin{aligned}
0 &= \sum_{m=1}^{\infty} R_m(\underline{\tilde{u}}_{m-1}(s,t,r)) = \sum_{m=1}^{\infty} (\underline{u}_{m-1}(s,t,r) \\
&- \frac{1-\lambda_m}{(m-1)!}\underline{g}(s,t,r) - \int_c^t k_1(t,\tau)\left(\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}} G_1(\sum_{i=0}^{\infty} \underline{u}_i(s,\tau,r))p^i\right)_{|p=0} d\tau \\
&- \int_a^b k_2(s,\xi)\left(\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}} G_2(\sum_{i=0}^{\infty} \underline{u}_i(\xi,t,r))p^i\right)_{|p=0} d\xi \\
&= \sum_{m=1}^{\infty} \underline{u}_{m-1}(s,t,r) - \underline{g}(s,t,r) - \int_c^t k_1(t,\tau) \sum_{m=1}^{\infty} \underline{H}_{1,m-1}(s,\tau,r)d\tau \\
&- \int_a^b k_2(t,\tau) \sum_{m=1}^{\infty} \underline{H}_{2,m-1}(\xi,t,r)d\xi.
\end{aligned}$$

Hence $0 = \underline{u}(s,t,r) - \underline{g}(s,t,r) - \int\limits_c^t k_1(t,\tau)G_1(\underline{u}(s,\tau,r))d\tau - \int\limits_a^b k_2(s,\xi)G_2(\underline{u}(\xi,t,r))d\xi.$

□

Theorem 3. *Let the condition (i)–(iii) be fulfilled.*

Then the value of the convergence control parameter h can be selected such that the series (20) converges in A.

Proof. Let $E = (C(A,\mathbf{R}), \|.\|)$ be the Banach space of all continuous functions on A and $\underline{s}_n(s,t,r) = \sum\limits_{i=0}^{n} \underline{u}_i(s,t,r)$ for $(s,t) \in A$, $r \in [0,,1]$.

We will prove that $\{\underline{s}_n\}$ is a Cauchy sequence in E.

We get

$$\|\underline{s}_1 - \underline{s}_0\| = \|\underline{u}_1\| \le |h|(\|\underline{u}_0\| + \|\underline{g}\| + M_1(d-c)\phi_1 + M_2(b-a)\phi_2), \tag{21}$$

where $\phi_1 = \max\limits_{(s,t)\in A} |G_1(\underline{u}_0(s,t,r)|$ and $\phi_2 = \max\limits_{(s,t)\in A} |G_2(\underline{u}_0(s,t,r)|$.

Let $m \ge 1$ and $n > m$. Then we obtain

$$\begin{aligned}
\|\underline{s}_n - \underline{s}_m\| &= \max_{(s,t)\in A} |\underline{s}_n(s,t,r) - \underline{s}_m(s,t,r)| = \max_{(s,t)\in A} |\sum_{i=0}^{n} \underline{u}_i(s,t,r) - \sum_{i=0}^{m} \underline{u}_i(s,t,r)| \\
&= \max_{(s,t)\in A} |\sum_{i=m+1}^{n} \underline{u}_i(s,t,r)| = \max_{(s,t)\in A} |\sum_{i=m}^{n-1} \underline{u}_{i+1}(s,t,r)| \\
&= \max_{(s,t)\in A} |(1+h)\sum_{i=m}^{n-1} \underline{u}_i(s,t,r) - h(\sum_{i=m}^{n-1}(\int_c^t k_1(t,\tau)\underline{H}_{1,i}d\tau + \int_a^b k_2(s,\xi)\underline{H}_{2,i}d\xi))| \\
&= \max_{(s,t)\in A} |(1+h)\sum_{i=m}^{n-1} \underline{u}_i(s,t,r) - h\{\int_c^t k_1(t,\tau)\sum_{i=m}^{n-1}\underline{H}_{1,i}d\tau + \int_a^b k_2(s,\xi)\sum_{i=m}^{n-1}\underline{H}_{2,i}d\xi)\}|
\end{aligned}$$

From [31] we have

$$\sum_{i=m}^{n-1} \underline{H}_{1,i} = G_1(\underline{s}_{n-1}) - G_1(\underline{s}_{m-1}), \quad \text{and} \quad \sum_{i=m}^{n-1} \underline{H}_{2,i} = G_2(\underline{s}_{n-1}) - G_2(\underline{s}_{m-1}).$$

Consequently, from conditions (ii) and (iii) we obtain

$$\begin{aligned}
\|\underline{s}_n - \underline{s}_m\| &= \max_{(s,t)\in A} |(1+h)(\underline{s}_{n-1} - \underline{s}_{m-1}) - h\{\int_c^t k_1(t,\tau)(G_1(\underline{s}_{n-1}) - G_1(\underline{s}_{m-1}))d\tau \\
&+ \int_a^b k_2(s,\xi)(G_2(\underline{s}_{n-1}) - G_2(\underline{s}_{m-1}))d\xi\}| \le |1+h|\|\underline{s}_{n-1} - \underline{s}_{m-1}\| \\
&+ |h|\{M_1L_1(d-c)\|\underline{s}_{n-1} - \underline{s}_{m-1}\| + m_2L_2(b-a)\|\underline{s}_{n-1} - \underline{s}_{m-1}\|\} \\
&= (|1+h| + |h|\alpha)\|\underline{s}_{n-1} - \underline{s}_{m-1}\| = \beta_h\|\underline{s}_{n-1} - \underline{s}_{m-1}\|
\end{aligned}$$

where $\beta_h = |1+h| + |h|\alpha$.

Let $n = m+1$ then

$$\begin{aligned}
&\|\underline{s}_{m+1} - \underline{s}_m\| \le \beta_h\|\underline{s}_m - \underline{s}_{m-1}\| \le {\beta_h}^2\|\underline{s}_{m-1} - \underline{s}_{m-2}\| \\
&\le \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\le {\beta_h}^m\|\underline{s}_1 - \underline{s}_0\| = {\beta_h}^m\|\underline{u}_1\|.
\end{aligned} \tag{22}$$

Using the triangle inequality and (22) we have

$$\begin{aligned}
&\|\underline{s}_n - \underline{s}_m\| \le \|\underline{s}_{m+1} - \underline{s}_m\| + \|\underline{s}_{m+2} - \underline{s}_{m+1}\| + \cdots + \|\underline{s}_n - \underline{s}_{n-1}\| \\
&\le ({\beta_h}^m + {\beta_h}^{m+1} + \ldots + {\beta_h}^{n-1})\|\underline{u}_1\| \le {\beta_h}^m(1 + \beta_h + {\beta_h}^2 + \ldots + {\beta_h}^{n-m-1})\|\underline{u}_1\| \\
&\le {\beta_h}^m \frac{1 - {\beta_h}^{n-m}}{1 - \beta_h}\|\underline{u}_1\|.
\end{aligned}$$

We choose the value of the parameter $h \in (-2, 0)$ such that

$$0 < \alpha < \frac{1 - |1 + h|}{|h|}. \tag{23}$$

Then $\beta_h = |1 + h| + |h|\alpha \in (0, 1)$ and $1 - {\beta_h}^{n-m} < 1$. Therefore, $\|\underline{s}_n - \underline{s}_m\| < \frac{{\beta_h}^m}{1-\beta_h}\|\underline{u}_1\|$

Since $\|\underline{u}_1\| < \infty$ then $\|\underline{s}_n - \underline{s}_m\| \to 0$ as $m \to \infty$,and the sequence $\{\underline{s}_n\}$ is a Cauchy sequence in E. Therefore, the series $\sum\limits_{i=1}^{\infty} \underline{u}_i(s, t, r)$ converges.

Similarly, we have $\{\overline{s}_n\}$ is a Cauchy sequence . □

From the proof of Theorem 3 it follows an upper bound of the error:

Theorem 4. *Let the condition (i)–(iii) be fulfilled.*

Then the error of the approximate solution can be estimated as follows

$$\|\underline{u} - \underline{s}_n\| \le \frac{{\beta_h}^n}{1 - \beta_h}|h|(\|\underline{u}_0\| + \|\underline{g}\| + M_1(d - c)\phi_1 + M_2(b - a)\phi_2),$$

where $\beta_h = |1 + h| + |h|\alpha$ *and* α *is defined by (19).*

6. Numerical Example

In this section, we will illustrate the obtained theoretical results on a numerical example.

Example 1. *Let* $A = [0, 1] \times [0, 1]$. *Consider the Equation (1) in the partial case of*

$$k_1(t, \tau) = \frac{1}{5}(t + \tau), \quad k_2(s, \xi) = \frac{3}{5}s\xi, \quad G_1(u) = \frac{1}{15}u^2, \quad G_2(u) = \frac{1}{5}u$$

and

$$g(s, t, r) = ((\frac{24}{25}st - \frac{7}{900}s^2t^4(1 + r))(1 + r), (\frac{24}{25}st - \frac{7}{900}s^2t^4(3 - r))(3 - r)).$$

Then the exact solution of (1) in this partial case is given by

$$u_{exact}(s, t, r) = (st(1 + r), st(3 - r)).$$

In this case the conditions (i)–(iii) are satisfied with $M_1 = \frac{2}{5}$, $M_2 = \frac{3}{5}$, $L_1 = \frac{2}{5}$, $L_2 = \frac{1}{5}$ *and* $\alpha = \frac{7}{25}$. *We choose the value of parameter* h : $\frac{7}{25} < \frac{1-|1+h|}{|h|}$ *or* $h \in (-1.5625, 0)$. *Numerically determined, the optimal value of the convergence control parameter* h *is equal to* -1.0098.

By using CAS "Wolfram Mathematica" and the proposed above method, we obtain

$$\underline{u}_0(s,t,r) = 0,$$

$$\underline{u}_1(s,t,r) = h(\underline{u}_0(s,t,r) - \underline{g}(s,t,r) - \int_0^t \frac{1}{75}(t+\tau)\underline{u}_0^2(s,\tau,r)d\tau - \int_0^1 \frac{3}{25}s\xi\underline{u}_0(\xi,t,r)d\xi),$$

$$\underline{u}_1(s,t,r) = -h\{\frac{24}{25}st - \frac{7}{900}s^2t^4(1+r)\}(1+r),$$

$$\underline{u}_2(s,t,r) = (1+h)\underline{u}_1(s,t,r) - h(\int_0^t \frac{1}{75}(t+\tau)2\underline{u}_0(s,\tau,r)\underline{u}_1(s,\tau,r)d\tau + \int_0^1 \frac{3}{25}s\xi\underline{u}_1(\xi,t,r)d\xi),$$

$$\begin{aligned}
&\underline{u}_2(s,t,r) \\
&= \{-h(\frac{24}{25}st - \frac{7}{900}s^2t^4(1+r)) - h^2(\frac{24^2}{25^2}st + (\frac{7}{3.10^4}st^4 - \frac{7}{900}s^2t^4)(1+r))\}(1+r),
\end{aligned}$$

$$\begin{aligned}
&\underline{u}_3(s,t,r) = (1+h)\underline{u}_2(s,t,r) \\
&\quad - h(\int_0^t \frac{1}{75}(t+\tau)(\underline{u}_1^2(s,\tau,r) + 2\underline{u}_0(s,\tau,r)\underline{u}_2(s,\tau,r))d\tau + \int_0^1 \frac{3}{25}s\xi\underline{u}_2(\xi,t,r)d\xi),
\end{aligned}$$

$$\begin{aligned}
&\underline{u}_3(s,t,r) = \{-h(\frac{24}{25}st - \frac{7}{900}s^2t^4(1+r)) \\
&\quad - h^2(\frac{1151}{25^2}st + \frac{14}{3.10^4}st^4(1+r) - \frac{7}{450}s^2t^4(1+r)) \\
&\quad - h^3(\frac{13824}{25^3}st + \frac{21}{10^4}st^4(1+r) - \frac{343}{36.25^3}s^2t^4(1+r) \\
&\quad - \frac{26}{27.25^3}s^3t^7(1+r)^2 + \frac{931}{675.900^2}s^4t^{10}(1+r)^3)\}(1+r),
\end{aligned}$$

$$\begin{aligned}
&\underline{s}_3(s,t,r) = \{-3h(\frac{24}{25}st - \frac{7}{900}s^2t^4(1+r)) \\
&\quad - h^2(\frac{1727}{25^2}st + \frac{7}{10^4}st^4(1+r) - \frac{7}{300}s^2t^4(1+r)) \\
&\quad - h^3(\frac{13824}{25^3}st + \frac{21}{10^4}st^4(1+r) - \frac{343}{36.25^3}s^2t^4(1+r) \\
&\quad - \frac{26}{27.25^3}s^3t^7(1+r)^2 + \frac{931}{675.900^2}s^4t^{10}(1+r)^3)\}(1+r).
\end{aligned}$$

Therefore, in this particular case we obtain for the error

$$\underline{\Delta}_3(s,t,0.5) = |\underline{u}_{exact}(s,t,0.5) - \underline{s}_3(s,t,0.5)|.$$

The results are shown in the Table 1.

Table 1. Values of errors $\underline{\Delta}_3(s,t,0.5)$.

	$h = -1$			$h = -1.0098$		
s	t = 0.2	0.4	0.6	t = 0.2	0.4	0.6
0.1	2.0296×10^{-6}	5.5943×10^{-6}	1.4641×10^{-6}	9.6097×10^{-7}	3.3509×10^{-6}	1.0844×10^{-5}
0.3	6.2286×10^{-6}	1.9018×10^{-5}	5.5214×10^{-5}	2.9769×10^{-6}	1.1556×10^{-5}	4.0119×10^{-5}
0.5	1.0627×10^{-5}	3.5632×10^{-5}	1.1181×10^{-4}	5.1318×10^{-6}	2.1982×10^{-5}	8.0508×10^{-5}
0.7	1.5241×10^{-5}	5.5688×10^{-5}	1.8548×10^{-4}	7.4422×10^{-6}	3.4889×10^{-5}	1.3308×10^{-4}
	$h = -0.9$			$h = -1.4$		
s	t = 0.2	0.4	0.6	t = 0.2	0.4	0.6
0.1	7.5640×10^{-5}	1.5374×10^{-4}	2.4064×10^{-4}	1.2216×10^{-3}	2.4483×10^{-3}	3.6934×10^{-3}
0.3	2.2747×10^{-4}	4.7011×10^{-4}	7.6690×10^{-4}	3.6672×10^{-3}	7.3840×10^{-3}	1.1278×10^{-2}
0.5	3.8006×10^{-4}	7.9849×10^{-4}	1.3538×10^{-3}	$1{,}2371\times10^{-2}$	$1{,}9123\times10^{-2}$	$2{,}6976\times10^{-2}$
0.7	5.3341×10^{-4}	1.1391×10^{-3}	2.0022×10^{-3}	8.5681×10^{-3}	1.7409×10^{-2}	2.7227×10^{-2}

Remark 3. *In the standard case $r = 1$ we have $E^1 = \mathbf{R}$ and the Equation (1) is reduced to the exact equation. In this case*

$$g(s,t) = \underline{g}(s,t,1) = \overline{g}(s,t,1) = \frac{48}{25}st - \frac{14}{450}s^2t^4,$$

$$u_{exact}(s,t) = \underline{u}_{exact}(s,t,1) = \overline{u}_{exact}(s,t,1) = 2st,$$

and differently than the "fuzy" case we obtain the following equalities:

$$u_n(s,t) = \underline{u}_n(s,t,1) = \overline{u}_n(s,t,1),$$

$$s_n(s,t) = \underline{s}_n(s,t,1) = \overline{s}_n(s,t,1),$$

$$\Delta_n(s,t) = \underline{\Delta}_n(s,t,1) = \overline{\Delta}_n(s,t,1) = |u_{exact}(s,t) - s_n(s,t)|, \quad n = 0,1,2,....$$

These equalities lead to more simple calculations in the case of $\mathbf{R}$ *comparatively with the "fuzy" case E^1.*

7. Conclusions

In this paper, HAM is applied for solving the two-dimensional nonlinear Volterra-Fredholm fuzzy integral equations, where the solution is found in the form of a series. It is shown that if this series is convergent, its sum is the solution of the considered equation. Sufficient conditions for the convergence of this series are given. Additionally, the error of the approximate solution, taken as the partial sum of generated series, is estimated. The presented example shows that the investigated method is effective in solving the equations of considered kind.

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