fractal and fractional

Article

k-Fractional Estimates of Hermite–Hadamard Type Inequalities Involving k-Appell's Hypergeometric Functions and Applications

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Abstract: The main objective of this paper is to obtain certain new k-fractional estimates of Hermite–Hadamard type inequalities via s-convex functions of Breckner type essentially involving k-Appell's hypergeometric functions. We also present applications of the obtained results by considering particular examples.

Keywords: convex; s-convex; k-fractional; bounds; Appell's hypergeometric functions

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1. Introduction and Preliminaries

Fractional calculus is a branch of mathematics in which we study the derivatives and integrals of any real or complex order. Over the years it has become a significant tool in various branches of sciences. As a consequence, the classical concepts of fractional calculus have also been extended and generalized in different directions using novel and innovative ideas. In recent years many inequalities experts have utilized the concepts of fractional calculus in obtaining different fractional analogues of classical inequalities. Resultantly fractional inequalities has become a dynamical area of research in mathematical sciences. One of the basic notion of fractional calculus, which was used by Sarikaya and his team [1] in obtaining fractional estimates of well-known Hermite–Hadamard's inequality is Riemann–Liouville fractional integrals and it is defined as:

Definition 1 ([2]). *Let $f \in L_1[a,b]$. Then Riemann–Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(\mu)\mathrm{d}\mu, \quad x > a,$$

and

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\mu-x)^{\alpha-1}f(\mu)\mathrm{d}\mu, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-\mu} x^{\alpha-1} dx,$$

is the gamma function.

Diaz et al. [3] introduced the generalized k-gamma function as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k > 0, x \in \mathbb{C} \setminus k\mathbb{Z}^-, \tag{1}$$

where Γ_k is one parameter deformation of the classical gamma function as $\Gamma_k \to \Gamma$ when $k \to 1$. Γ_k is based on the repeated appearance of the following expression:

$$\phi(\phi+k)(\phi+2k)(\phi+3k)\ldots(\phi+(n-1)k).$$

This above statement is a function of the variable ϕ and is denoted by $(\phi)_{n,k}$. It is known as Pochhammer k-symbol, which reduces to classical Pochhammer symbol $(\phi)_n$ by taking $k = 1$. The integral of Γ_k is given by

$$\Gamma_k(x) = \int_0^\infty \mu^{x-1} e^{-\frac{\mu^k}{k}} \mathrm{d}\mu, \quad \Re(x) > 0. \tag{2}$$

Diaz et al. [3] also introduced k-Beta functions.

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad \Re(x) > 0, \Re(y) > 0. \tag{3}$$

Integral form of k-Beta function is given by

$$B_k(x,y) = \frac{1}{k} \int_0^1 \mu^{\frac{x}{k}-1} (1-\mu)^{\frac{x}{k}-1} \mathrm{d}\mu. \tag{4}$$

From (2) and (4) one can have

$$B_k(x,y) = \frac{1}{k} \beta\left(\frac{x}{k}, \frac{y}{k}\right).$$

Sarikaya et al. [4] introduced and studied a new generalization of Riemann–Liouville fractional integral which are called k-Riemann–Liouville fractional integrals.

Let f be piecewise continuous on $I^* = (0,\infty)$ and integrable on any finite subinterval of $I = [0,\infty]$. Then for $\mu > 0$, we consider the k-Riemann–Liouville fractional integral of f of order α

$${}_kJ_a^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-\mu)^{\frac{\alpha}{k}-1} f(\mu) \mathrm{d}\mu, \quad x > a, k > 0.$$

If $k \to 1$, then k-Riemann–Liouville fractional integrals reduces to classical Riemann–Liouville fractional integral. It is worth mentioning here that the concept of k-Riemann–Liouville fractional integrals is a significant generalization of Riemann–Liouville fractional integrals as for $k \neq 1$ the properties of k-Riemann–Liouville fractional integrals are quite different from the classical Riemann–Liouville fractional integrals. For some recent and interesting details on fractional calculus, special functions and their k-analogues, see [2,4].

The integral representation of k-Appell's series $F_{1,k}$ where $k > 0$ is

$$F_{1,k} = \frac{\Gamma_k(c)}{k\Gamma_k(a')\Gamma_k(c-a')} \int_0^1 \mu^{\frac{a'}{k}-1}(1-\mu)^{\frac{c-a'}{k}-1}(1-kz_1\mu)^{-\frac{b_1}{k}}(1-kz_2\mu)^{-\frac{b_2}{k}}\,\mathrm{d}\mu.$$

For some more details, see [5].

The motivation of this paper is to obtain certain new k-fractional estimates of inequalities of Hermite–Hadamard type via s-convex functions of Breckner type essentially involving k-Appell's hypergeometric functions. We also give applications of the obtained results by considering a particular example.

Before moving further, let us recall some previously known concepts.

Definition 2 (s-convex function of Breckner type)**.** *A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex function of Breckner type, if*

$$f((1-\mu)x + \mu y) \le (1-\mu)^s f(x) + \mu^s f(y), \quad \forall x, y \in [0,\infty), \mu \in [0,1], s \in (0,1].$$

Definition 3 (s-convex function of Godunova–Levin type)**.** *A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex function of Godunova–Levin type, if*

$$f((1-\mu)x + \mu y) \le (1-\mu)^{-s} f(x) + \mu^{-s} f(y), \quad \forall x, y \in [0,\infty), \mu \in (0,1), s \in [0,1].$$

Definition 4 (P-functions)**.** *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be P-function, if*

$$f((1-\mu)x + \mu y) \le f(x) + f(y), \quad \forall x, y \in I, \mu \in [0,1].$$

For some more information on convexity, its generalizations and related inequalities, see [6–9]. We now recall definitions of means of real numbers. For arbitrary real numbers α, β, $\alpha \neq \beta$, we define

1. Arithmetic Mean

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha, \beta \in \mathbb{R}.$$

2. Logarithmic Mean

$$L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|}, \quad \alpha, \beta \in \mathbb{R}\backslash\{0\}.$$

3. Generalized Log-Mean

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{N}, n \ge 1, \alpha, \beta \in \mathbb{R}, \alpha < \beta.$$

For more information, see [10].

2. Auxiliary Result

In this section, we derive a new k-fractional integral identity.

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $f' \in L^1[a,b]$, then*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_k\mathrm{J}^{\alpha}_{a^+}f(b)+{}_k\mathrm{J}^{\alpha}_{b^-}f(a)\right]$$
$$=\frac{b-a}{2^{\frac{\alpha}{k}+2}}\left[\int_0^1\left[(1-\mu)^{\frac{\alpha}{k}}-(1+\mu)^{\frac{\alpha}{k}}\right]f'\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\mathrm{d}\mu\right.$$
$$\left.+\int_0^1\left[(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}\right]f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\mathrm{d}\mu\right]$$

Proof. Let

$$\begin{aligned}
I&=\int_0^1[(1-\mu)^{\frac{\alpha}{k}}-(1+\mu)^{\frac{\alpha}{k}}]f'\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\mathrm{d}\mu\\
&\quad+\int_0^1[(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}]f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\mathrm{d}\mu\\
&=\int_0^1(1-\mu)^{\frac{\alpha}{k}}f'\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\mathrm{d}\mu+\int_0^1(1+\mu)^{\frac{\alpha}{k}}f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\mathrm{d}\mu\\
&\quad-\int_0^1(1-\mu)^{\frac{\alpha}{k}}f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\mathrm{d}\mu-\int_0^1(1+\mu)^{\frac{\alpha}{k}}f'\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\mathrm{d}\mu\\
&=I_1+I_2-I_3-I_4.
\end{aligned}\tag{5}$$

Now

$$\begin{aligned}
I_1&=\int_0^1(1-\mu)^{\frac{\alpha}{k}}f'\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\mathrm{d}\mu\\
&=\int_{-1}^0(1+\mu)^{\frac{\alpha}{k}}f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\mathrm{d}\mu.
\end{aligned}$$

This implies

$$I_1+I_2=\int_{-1}^1(1+\mu)^{\frac{\alpha}{k}}f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\mathrm{d}\mu.$$

Further it implies

$$I_1+I_2=\left(\frac{2}{b-a}\right)^{\frac{\alpha}{k}+1}\int_a^b(x-a)^{\frac{\alpha}{k}}f'(x)\mathrm{d}x.$$

Similarly

$$I_3+I_4=\left(\frac{2}{b-a}\right)^{\frac{\alpha}{k}+1}\int_a^b(b-x)^{\frac{\alpha}{k}}f'(x)\mathrm{d}x.$$

Thus integrating by parts, we have

$$\begin{aligned}
I&=\left(\frac{2}{b-a}\right)^{\frac{\alpha}{k}+1}\left[\int_a^b(x-a)^{\frac{\alpha}{k}}f'(x)\mathrm{d}x-\int_a^b(b-x)^{\frac{\alpha}{k}}f'(x)\mathrm{d}x\right]\\
&=\left(\frac{2}{b-a}\right)^{\frac{\alpha}{k}+1}\left[(b-a)^{\frac{\alpha}{k}}\{f(a)+f(b)\}-\Gamma_k(\alpha+k)\{{}_k\mathrm{J}^{\alpha}_{b^-}f(a)+{}_k\mathrm{J}^{\alpha}_{a^+}f(b)\}\right].
\end{aligned}$$

Multiplying both sides of above equation with $\frac{b-a}{2^{\frac{\alpha}{k}+2}}$ completes the proof. □

3. Main Results

In this section, we derive our main results.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ and $f'\in L^1[a,b]$. If $|f'|$ is s-convex function of Breckner type, then for $k>0$, $s\in(0,1]$, we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_k\mathrm{J}^{\alpha}_{a^+}f(b)+{}_k\mathrm{J}^{\alpha}_{b^-}f(a)\right]\right|\\
&\le\frac{k(b-a)}{2^{\frac{\alpha}{k}+s+2}}\left[\frac{2^{\frac{\alpha}{k}+s+1}-1}{\alpha+ks+k}-2^sB_k(\alpha+k,k)F_{1,k}(\alpha+k,-ks,0,\alpha+2k;\frac{1}{2k},0)\right.\\
&\qquad\left.+2^{\frac{\alpha}{k}}B_k(ks+k,k)F_{1,k}\left(ks+k,-\alpha,0,ks+2k;\frac{1}{2k},0\right)\right][|f'(a)|+|f'(b)|].
\end{aligned}$$

Proof. Using Lemma 1, property of modulus and the given hypothesis of the theorem, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_k\mathrm{J}^{\alpha}_{a^+}f(b)+{}_k\mathrm{J}^{\alpha}_{b^-}f(a)\right]\right|\\
&\le\frac{b-a}{2^{\frac{\alpha}{k}+2}}\left|\left[\int_0^1\left[(1-\mu)^{\frac{\alpha}{k}}-(1+\mu)^{\frac{\alpha}{k}}\right]f'\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\mathrm{d}\mu\right.\right.\\
&\qquad\left.\left.+\int_0^1\left[(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}\right]f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\mathrm{d}\mu\right]\right|\\
&\le\frac{b-a}{2^{\frac{\alpha}{k}+2}}\left[\int_0^1\left|(1-\mu)^{\frac{\alpha}{k}}-(1+\mu)^{\frac{\alpha}{k}}\right|\left|f'\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\right|\mathrm{d}\mu\right.\\
&\qquad\left.+\int_0^1\left|(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}\right|\left|f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\right|\mathrm{d}\mu\right]\\
&\le\frac{b-a}{2^{\frac{\alpha}{k}+2}}\left[\int_0^1\left[(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}\right]\left[\left(\frac{1+\mu}{2}\right)^s|f'(a)|+\left(\frac{1-\mu}{2}\right)^s|f'(b)|\right]\mathrm{d}\mu\right.\\
&\qquad\left.+\int_0^1\left[(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}\right]\left[\left(\frac{1-\mu}{2}\right)^s|f'(a)|+\left(\frac{1+\mu}{2}\right)^s|f'(b)|\right]\mathrm{d}\mu\right]\\
&=\frac{b-a}{2^{\frac{\alpha}{k}+s+2}}\left[\int_0^1\left[(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}\right]\left[(1+\mu)^s|f'(a)|+(1-\mu)^s|f'(b)|\right]\mathrm{d}\mu\right.\\
&\qquad\left.+\int_0^1\left[(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}\right]\left[(1-\mu)^s|f'(a)|+(1+\mu)^s|f'(b)|\right]\mathrm{d}\mu\right]\\
&=\frac{k(b-a)}{2^{\frac{\alpha}{k}+s+2}}\left[\frac{2^{\frac{\alpha}{k}+s+1}-1}{\alpha+ks+k}-2^sB_k(\alpha+k,k)F_{1,k}(\alpha+k,-ks,0,\alpha+2k;\frac{1}{2k},0)\right.\\
&\qquad\left.+2^{\frac{\alpha}{k}}B_k(ks+k,k)F_{1,k}\left(ks+k,-\alpha,0,ks+2k;\frac{1}{2k},0\right)\right][|f'(a)|+|f'(b)|].
\end{aligned}$$

This completes the proof. □

If under the assumptions of Theorem 1, we take $f(x)=x^s$ an s-convex function and $\alpha=1=k$, then

Proposition 1. *Let $a, b \in \mathbb{R}$ with $a < b$ and $0 < s \leq 1$, we have*

$$
\begin{aligned}
&|A(a^s, b^s) - L_s^s(a,b)| \\
&\leq \frac{(b-a)|s|}{2^{s+3}} \left[\frac{2^{s+2}-1}{s+2} - 2^s B_1(2,1) F_{1,1}(2,-s,0,3;\frac{1}{2},0) \right. \\
&\qquad \left. + 2B_1(s+1,1) F_{1,1}\left(s+1,-1,0,s+2;\frac{1}{2},0\right) \right] [|a|^{s-1} + |b|^{s-1}].
\end{aligned}
$$

Corollary 1. *Under the assumptions of Theorem 1, if $|f'|$ is convex function, then for $k > 0$, we have*

$$
\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{k(b-a)}{2^{\frac{\alpha}{k}+3}} \left[\frac{2^{\frac{\alpha}{k}+2}-1}{\alpha+k+k} - 2B_k(\alpha+k,k) F_{1,k}(\alpha+k,-k,0,\alpha+2k;\frac{1}{2k},0) \right. \\
&\qquad \left. + 2^{\frac{\alpha}{k}} B_k(2k,k) F_{1,k}\left(2k,-\alpha,0,k+2k;\frac{1}{2k},0\right) \right] [|f'(a)| + |f'(b)|]
\end{aligned}
$$

Corollary 2. *Under the assumptions of Theorem 1, if $|f'|$ is s-convex function of Godunova–Levin type, then for $k > 0$, we have*

$$
\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{k(b-a)}{2^{\frac{\alpha}{k}-s+2}} \left[\frac{2^{\frac{\alpha}{k}-s+1}-1}{\alpha-ks+k} - 2^{-s} B_k(\alpha+k,k) F_{1,k}(\alpha+k,ks,0,\alpha+2k;\frac{1}{2k},0) \right. \\
&\qquad \left. + 2^{\frac{\alpha}{k}} B_k(k-ks,k) F_{1,k}\left(k-ks,-\alpha,0,2k-ks;\frac{1}{2k},0\right) \right] [|f'(a)| + |f'(b)|].
\end{aligned}
$$

Corollary 3. *Under the assumptions of Theorem 1, if $|f'|$ is P-function, then for $k > 0$, we have*

$$
\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{k(b-a)}{2^{\frac{\alpha}{k}+2}} \left[\frac{2^{\frac{\alpha}{k}+1}-1}{\alpha+k} - B_k(\alpha+k,k) F_{1,k}(\alpha+k,0,0,\alpha+2k;\frac{1}{2k},0) \right. \\
&\qquad \left. + 2^{\frac{\alpha}{k}} B_k(k,k) F_{1,k}\left(k,-\alpha,0,2k;\frac{1}{2k},0\right) \right] [|f'(a)| + |f'(b)|].
\end{aligned}
$$

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$ and $f' \in L^1[a,b]$. If $|f'|^q$ is s-convex function of Breckner type, then for $q > 1$, $k > 0$, $s \in (0,1]$, we have*

$$
\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{b-a}{2^{\frac{\alpha}{k}+\frac{s+1}{q}+1}} \left(\frac{k(q-1)(2^{\frac{q\alpha}{k(q-1)}}-1)}{q\alpha+k(q-1)} \right)^{1-\frac{1}{q}} \\
&\quad \times \left\{ \left(\frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \right\}.
\end{aligned}
$$

Proof. Using Lemma 1, power-mean inequality and the given hypothesis of the theorem, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[{}_k\mathrm{J}^{\alpha}_{a^+}f(b)+{}_k\mathrm{J}^{\alpha}_{b^-}f(a)\right]\right| \\
&\leq \frac{b-a}{2^{\frac{\alpha}{k}+2}}\left|\left[\int_0^1\left[(1-\mu)^{\frac{\alpha}{k}}-(1+\mu)^{\frac{\alpha}{k}}\right]f'\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\mathrm{d}\mu\right.\right. \\
&\qquad\left.\left.+\int_0^1\left[(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}\right]f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\mathrm{d}\mu\right]\right| \\
&\leq \frac{b-a}{2^{\frac{\alpha}{k}+2}}\left[\int_0^1\left|(1-\mu)^{\frac{\alpha}{k}}-(1+\mu)^{\frac{\alpha}{k}}\right|\left|f'\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\right|\mathrm{d}\mu\right. \\
&\qquad\left.+\int_0^1\left|(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}\right|\left|f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\right|\mathrm{d}\mu\right] \\
&\leq \frac{b-a}{2^{\frac{\alpha}{k}+2}}\left[\left(\int_0^1\left|(1-\mu)^{\frac{\alpha}{k}}-(1+\mu)^{\frac{\alpha}{k}}\right|^{\frac{q}{q-1}}\mathrm{d}\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f'\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\right|^q\mathrm{d}\mu\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\int_0^1\left|(1+\mu)^{\frac{\alpha}{k}}-(1-\mu)^{\frac{\alpha}{k}}\right|^{\frac{q}{q-1}}\mathrm{d}\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f'\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\right|^q\mathrm{d}\mu\right)^{\frac{1}{q}}\right] \\
&\leq \frac{b-a}{2^{\frac{\alpha}{k}+\frac{s}{q}+2}}\left[\left(\int_0^1\left[(1+\mu)^{\frac{q\alpha}{k(q-1)}}-(1-\mu)^{\frac{q\alpha}{k(q-1)}}\right]\mathrm{d}\mu\right)^{1-\frac{1}{q}}\right. \\
&\qquad\times\left(\int_0^1\left[(1+\mu)^s|f'(a)|^q+(1-\mu)^s|f'(b)|^q\right]\mathrm{d}\mu\right)^{\frac{1}{q}} \\
&\qquad+\left(\int_0^1\left[(1+\mu)^{\frac{q\alpha}{k(q-1)}}-(1-\mu)^{\frac{q\alpha}{k(q-1)}}\right]\mathrm{d}\mu\right)^{1-\frac{1}{q}} \\
&\qquad\left.\times\left(\int_0^1\left[(1-\mu)^s|f'(a)|^q+(1+\mu)^s|f'(b)|^q\right]\mathrm{d}\mu\right)^{\frac{1}{q}}\right] \\
&= \frac{b-a}{2^{\frac{\alpha}{k}+\frac{s+1}{q}+1}}\left(\frac{k(q-1)(2^{\frac{q\alpha}{k(q-1)}}-1)}{q\alpha+k(q-1)}\right)^{1-\frac{1}{q}} \\
&\qquad\times\left\{\left(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

This completes the proof. □

If under the assumptions of Theorem 2, we take $f(x)=x^s$ an s-convex function and $\alpha=1=k$, then

Proposition 2. *Let $a, b \in \mathbb{R}$ with $a < b$ and $0 < s \leq 1$, we have*

$$\begin{aligned}
&|A(a^s, b^s) - L_s^s(a, b)| \\
&\leq \frac{(b-a)|s|^{\frac{1}{q}}}{2^{\frac{s+1}{q}+2}} \left(\frac{(q-1)(2^{\frac{q}{q-1}} - 1)}{q + (q-1)} \right)^{1-\frac{1}{q}} \\
&\quad \times \left\{ \left(\frac{(2^{s+1}-1)|a|^{q(s-1)} + |b|^{q(s-1)}}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|a|^{q(s-1)} + (2^{s+1}-1)|b|^{q(s-1)}}{s+1} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Corollary 4. *Under the assumptions of Theorem 2, if $|f'|^q$ is convex function, then for $q > 1$, $k > 0$, we have*

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{b-a}{2^{\frac{\alpha}{k} + \frac{2}{q} + 1}} \left(\frac{k(q-1)(2^{\frac{q\alpha}{k(q-1)}} - 1)}{q\alpha + k(q-1)} \right)^{1-\frac{1}{q}} \\
&\quad \times \left\{ \left(\frac{3|f'(a)|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Corollary 5. *Under the assumptions of Theorem 2, if $|f'|^q$ is s-convex function of Godunova–Levin type, then for $q > 1$, $k > 0$, we have*

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{b-a}{2^{\frac{\alpha}{k} + \frac{s+1}{q} + 1}} \left(\frac{k(q-1)(2^{\frac{q\alpha}{k(q-1)}} - 1)}{q\alpha + k(q-1)} \right)^{1-\frac{1}{q}} \\
&\quad \times \left\{ \left(\frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Corollary 6. *Under the assumptions of Theorem 2, if $|f'|^q$ is P-function, then for $q > 1$, $k > 0$, we have*

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}}} \left[{}_k J_{a^+}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a) \right] \right| \\
&\leq \frac{b-a}{2^{\frac{\alpha}{k} + \frac{1}{q}}} \left(\frac{k(q-1)(2^{\frac{q\alpha}{k(q-1)}} - 1)}{q\alpha + k(q-1)} \right)^{1-\frac{1}{q}} \left(|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}}.
\end{aligned}$$

4. Conclusions

In this paper, we have derived some new fractional analogues of Hermite–Hadamard like inequalities using k-fractional calculus approach. We have established these analogues essentially using the classes of s-convex functions of Breckner type, s-convex functions of Godunova–Levin type, P-functions and classical convex functions. We also discussed applications of some main results to means of real numbers. It is expected that the ideas and techniques of the obtained results will inspire interested readers working in the field of inequalities.

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