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On q-Uniformly Mocanu Functions

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Abstract: Let f be analytic in open unit disc $E = \{z : |z| < 1\}$ with $f(0) = 0$ and $f^{'}(0) = 1$. The q-derivative of f is defined by: $D_q f(z) = \frac{f(z)-f(qz)}{(1-q)z}$, $q \in (0,1)$, $z \in \mathcal{B} - \{0\}$, where $\mathcal{B}$ is a q-geometric subset of $\mathbb{C}$. Using operator D_q, q-analogue class $k - UM_q(\alpha, \beta)$, k-uniformly Mocanu functions are defined as: For $k = 0$ and $q \to 1^-$, $k-$ reduces to $M(\alpha)$ of Mocanu functions. Subordination is used to investigate many important properties of these functions. Several interesting results are derived as special cases.

Keywords: q-calculus; q-starlike; uniformly convex; subordination; Mocanu functions; q-Ruscheweyh derivative

1. Introduction

Let A denote the class of functions f that are analytic in the open unit disc E and are also normalized by the conditions $f(0) = 0$, $f^{'}(0) = 1$. Let $f, g \in A$. f is said to be subordinate to g (written as $f \prec g$), if there exists a Schwartz function $w(z)$ such that $f(z) = g(w(z))$.

q-calculus is ordinary calculus without a limit, and it has been used recently by many researchers in the field of geometric function theory. q-derivatives and q-integrals play an important and significant role in the study of quantum groups and q-deformed super-algebras, the study of fractal and multi-fractal measures and in chaotic dynamical systems. The name q-calculus also appears in other contexts; see [1,2]. The most sophisticated tool that derives functions in non-integer order is the long-known fractional calculus; see [1–4].

We recall here some basic concepts from q-calculus for which we refer to [5–16] and the references therein.

A subset $\beta \subset \mathbb{C}$ is called q-geometric, if $zq \in \beta$, whenever $z \in \mathcal{B}$, and it contains all the geometric sequences $\{zq^m\}_0^\infty$.

The q-derivative D_q of a function $f \in A$ is defined by:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad (z \in \mathcal{B} - \{0\}) \tag{1}$$

and $D_q f(0) = f^{'}(0)$.

Under this definition, we have the following rules for q-derivative D_q.

(i). $D_q z^m = \frac{1-q^m}{1-q} z^{m-1} = [n,q] z^{m-1}$, where $[m,q] = \frac{1-q^m}{1-q}$.

Let $f(z)$ and $g(z)$ be defined on a q-geometric set $\mathcal{B} \subset \mathbb{C}$ such that q-derivatives of $f(z)$ and $g(z)$ exist for all $z \in \mathcal{B}$. Then, for a, b complex numbers, we have:

(ii). $D_q(af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z)$.

(iii). $D_q(f(z).g(z)) = g(z) D_q f(z) + f(qz) D_q g(z)$.

(iv). $D_q(\frac{f(z)}{g(z)}) = \frac{g(z)D_q f(z) - f(z) D_q g(z)}{g(z) g(qz)}$, $g(z).g(qz) \neq 0$.

(v). $D_q(\log f(z)) = \frac{D_q f(z)}{f(z)}$.

Let $P(z)$ be the class of functions $p(z) = 1 + c_1 z + ...$, analytic in E and satisfying:

$$\left| p(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad (z \in E, q \in (0,1)). \tag{2}$$

It is known [9] that $p \in P(q)$ implies that $p(z) \prec \frac{1+z}{1-qz}$, where $\prec$ denotes subordination, and from this, it easily follows that $Re \quad p(z) > 0, \quad z \in E$.

Now, we have:

Definition 1. *[4,5] Let $f \in A$. Then, it is said to belong to the class $S_q^*(\alpha)$ of q-starlike functions of order α, $0 \leq \alpha \leq 1$, if and only if,*

$$\frac{1}{1-\alpha}\left(\frac{zD_q f(z)}{f(z)} - \alpha \right) \prec \frac{1+z}{1-qz}. \tag{3}$$

We can write (3) as:

$$\left| \frac{zD_q f(z)}{f(z)} - \frac{1-\alpha q}{1-q} \right| \leq \frac{1-\alpha}{1-q}. \tag{4}$$

By taking $a = \frac{1-\alpha}{1-q}$, $b = \frac{1-\alpha}{1-q}$ in 4, it has been shown in [17] that $f \in S_q^*(\alpha)$, if and only if,

$$\frac{zD_q f(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \quad -1 < B < 0 \leq A \leq 1, \tag{5}$$

where $A = 1 - (1+q)\alpha$ and $B = -q$.

As a special case, we note that:

$$\lim_{q \to 1^-} S_q^*(\alpha) = S^*(\alpha) \quad \text{with} \quad A = 1 - 2\alpha,$$

which is the class of starlike functions denoted as $S^*(\alpha)$.

Furthermore, for $\alpha = 0$, we obtain the class S_q^* of q-starlike functions introduced and studied in [10].

Definition 2. *Let $f \in A$ and $k \geq 0, \quad 0 \leq \alpha, \beta \leq 1, q \in (0,1)$. Then,*

$$f \in k - UM_q(\alpha, \beta),$$

if and only if, for $z \in E$,

$$Re \quad \left[(1-\alpha)\frac{zD_q f(z)}{f(z)} + \alpha \frac{D_q(zD_q f(z))}{D_q f(z)} \right] > k \left| (1-\beta)\frac{zD_q f(z)}{f(z)} + \beta \frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right|.$$

Selecting special values of parameters α, β and k and letting $q \to 1^-$, we obtain a number of known classes of analytic functions; see [5,9,18–21]. We list some of these as follows:

(i) Choosing $k = 0$, we get $\lim q \to 1^- M_q(\alpha) = M(\alpha)$, the class of α-convex functions; see [22].

(ii) For $\beta = 0$, $k = 1$, and $q \to 1^-$, we have the class MN; see [23].

(iii) Choosing $\beta = 0, \quad q \to 1^-$, we get the class $k - MN$ introduced in [18,19].

(iv) $k - UM_q(0,0) = k - sT, \quad k - UM_q(1,1) = k - UCV_q$.

Throughout this paper, we shall assume that $q \in (0,1)$, $0 \leq \alpha < 1$ and $z \in E$, unless otherwise mentioned.

2. Preliminary Results

Lemma 1. *[4]. Let $\phi(z)$ be analytic with $\phi(0) = 0$. If $|\phi(z_0)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in E$, then we have:*

$$z_0 D_q \phi(z_0 = m\phi(z_0), \quad m \geq 1, \qquad \textit{real number}.$$

Lemma 2. *[24]. Let $\alpha \geq 0$ and $0 \leq r < 1$. Let $p(z)$ be analytic in E with $p(0) = 1$. If:*

$$\left\{ p(z) + \alpha \frac{zp^{'}(z)}{p(z)} \right\} \prec \frac{1 + (1 - 2r)z}{1 - z},$$

then:

$$p(z) \prec \frac{1 + (1 - 2\S)z}{1 - z},$$

where:

$$\delta = \frac{1}{4}\left[(2r - \alpha) + \sqrt{(2r - \alpha)^2 + 8\alpha}\right].$$

3. Main Results

Theorem 1. *Let $p(z)$ be analytic in E with $p(0) = 1$. Let, for $k > \frac{1+q}{q}$,*

$$Re \left[1 + \frac{zD_q p(z)}{p(z)} \right] > k \left| \frac{zD_q p(z)}{p(z)} \right|, \quad z \in E.$$

Then, $p(z)$ is subordinate to $\frac{1}{1-qz}$, that is, $p(z) \prec \frac{1}{1-qz}$ in E.

Proof. Let $p(z) = \frac{1}{1-q\phi(z)}$. It can easily be seen that $\phi(z)$ is analytic in E and $\phi(0) = 0$. We shall show that $|\phi(z)| < 1$ for all $z \in E$. We suppose on the contrary that there exists a $z_0 \in E$ such that $|\phi(z_0)| = 1$.
Then:

$$Re \left[1 + \frac{z_\circ D_q p(z_\circ)}{p(z_\circ)} \right] - k \left| \frac{z_\circ D_q p(z_\circ)}{p(z_\circ)} \right| = Re \left[1 + \frac{qz_\circ D_q \phi(z_\circ)}{1 - q\phi(z_\circ)} \right] - k \left| \frac{qz_\circ D_q \phi(z_\circ)}{1 - q\phi(z_\circ)} \right|. \tag{6}$$

Now, by Lemma 1, $z_\circ D_q \phi(z_\circ) = m\phi(z_\circ) = me^{i\theta}, \quad m \geq 1$, and we use it in (6) for:

$$Re \left[1 + \frac{mqe^{i\theta}}{1 - qe^{i\theta}} \right] > k \left| \frac{mqe^{i\theta}}{1 - qe^{i\theta}} \right| > k \left| \frac{qe^{i\theta}}{1 - qe^{i\theta}} \right|. \tag{7}$$

From (6), (7), and choosing $\theta = \pi$, we have:

$$Re \left[1 + \frac{z_\circ D_q p(z_\circ)}{p(z_\circ)} \right] - k \left| \frac{z_\circ D_q p(z_\circ)}{p(z_\circ)} \right| \quad = \quad 1 - \frac{mq}{1+q} - \frac{kq}{1+q} < 0 \quad for \quad k > \frac{1+q}{q}.$$

This is a contradiction, and hence, $|\phi(z)| < 1$ for all $z \in E$. This proves that:

$$p(z) \prec \frac{1}{1 - qz} \quad in \quad E.$$

□

We apply Theorem 1 to have the following results.

Corollary 1. *Let $p(z) = f^{'}(z)$, $q \to 1^-$, and $k > 2$. Then, from Theorem 1, it follows that:*

$$Re\ \left[1 + \frac{zf^{''}(z)}{zf'(z)}\right] > k\left|\frac{zf^{''}(z)}{zf'(z)}\right|,$$

which implies $f \in k - UCV$, and so, $Re\ \ f^{'}(z) > \frac{1}{2}$ in E.

Corollary 2. *For $k > \frac{q+1}{q}$, let $f \in k - ST_q$. Then, $\frac{f(z)}{z} \prec \frac{1}{1-qz}$ in E.*
The proof is immediate when we take $p(z) = \frac{f(z)}{z}$ in Theorem 1.

As a special case, when $q \to 1^-$, $k > 2$, $f \in k - ST$ implies $Re\ \frac{f(z)}{z} > \frac{1}{2}$ in E.
Using a similar technique, we can prove the following results.

Theorem 2. *Let $k \geq 0$, $\alpha, \beta \in [0, 1]$, $qk < 1$, and let $p(z)$ be analytic in E with $p(0) = 1$.*

If:

$$Re\ \left[p(z) + \frac{\alpha z D_q p(z)}{p(z)} - \frac{1 - kq}{1 + q}\right] > k\left|p(z) + \frac{\beta z D_q p(z)}{p(z)} - 1\right|, \tag{8}$$

then $p(z) \prec \frac{1}{1-qz}$, $z \in E$.

We can easily deduce some special cases of Theorem 2 as given below.

Corollary 3. *Let $\beta = 0$, $p(z) = \frac{zD_q f(z)}{f(z)}$ in (8). Then:*

$$Re\ \left[(1-\alpha)\frac{zD_q f(z)}{f(z)} + \alpha\frac{D_q(zD_q f(z))}{D_q f(z)} - \frac{1-kq}{1+q}\right] - k\left|\frac{zD_q f(z)}{f(z)} - 1\right| > 0$$

implies:

$$f \in S_q^*(\frac{1}{1+q}),\ \ z \in E.$$

As a special case of this corollary, we observe that $UST \subset S^*(\frac{1}{2})$, when we choose $k = 1$, $\alpha = 0$, and let $q \to 1^-$.

Corollary 4. *Let $q \to 1^-$ and $p(z) = f^{'}(z)$. Then:*

$$\begin{aligned} Re\ \left[f^{'}(z) + \alpha\frac{(zf^{'}(z))^{'}}{f^{'}(z)} - (\alpha + \frac{1-k}{2})\right] &> k\left|f^{'}(z) + \beta\frac{(zf^{'}(z))^{'}}{f^{'}(z)} - (1+\beta)\right| \\ &= k\left|(1+\beta) - f^{'}(z) - \beta\frac{(zf^{'}(z))^{'}}{f^{'}(z)}\right| \\ &\geq Re\ \left[(1+\beta) - f^{'}(z) - \beta\frac{(zf^{'}(z))^{'}}{f^{'}(z)}\right]. \end{aligned}$$

This gives us:

$$Re\ \left[f^{'}(z) + (\frac{\alpha+\beta}{1+k})\frac{(zf^{'}(z))^{'}}{f^{'}(z)}\right] \geq \frac{k(1+\beta) + (\alpha+\gamma)}{1+k} = \eta,\ \ (\gamma = \frac{1-k}{2}).$$

Now, using Lemma 2 together with Theorem 2 when $q \to 1^-$, we obtain the result that:

$$Re\ f'(z) > \delta = \frac{1}{4}[(2\eta - \rho) + \sqrt{(2\eta-\rho)^2 + 8\rho}],\ \rho = \frac{\alpha+\beta}{1+k}.$$

Corollary 5. *In (8), if we take $\beta = 0,\ \alpha = 1,\ k = 1$ and $p(z) = \frac{zD_q f(z)}{f(z)}$, then:*

$$Re\left[\frac{D_q(zD_qf(z))}{D_q(f(z))} - \frac{1-q}{1+q}\right] > \left|\frac{zD_qf(z)}{f(z)} - 1\right|.$$

implies

$$f \in S_q^*(\frac{1}{1+q}) \quad in \quad E.$$

Furthermore, with $\beta = 1,\ \alpha = 1,\ k = 1$ and $p(z) = \frac{zD_qf(z)}{f(z)}$ in (8), it follows that:

$$Re\ \left[\frac{D_q(zD_qf(z))}{D_qf(z)} - \frac{1-q}{1+q}\right] > \left|\frac{D_q(zD_qf(z))}{D_qf(z)} - 1\right|$$

implies $f \in S_q^*(\frac{1}{1+q})$.

Next, we prove the following:

Theorem 3. *Let $p(z)$ be analytic in E with $p(0) = 1$. Let:*

$$Re\left[p(z) + \frac{(zD_qp(z))}{\lambda p(z) + c} - r\right] - k\left|p(z) + \frac{zD_qp(z)}{\lambda p(z)+c} - 1\right| > 0, \tag{9}$$

where $r = \frac{1}{1+q}$, λ, and c are positive real. Then, $p(z) \prec \frac{1}{1-qz}$ in E.

Proof. We shall follow the same procedure to prove this result as was used in Theorem 1. Let $p(z) = \frac{1}{1-q\phi(z)}$. Clearly, $\phi(0) = 0$, and $\phi(z)$ is analytic. We prove that $\phi(z)$ is a Schwartz function, that is $|\phi(z)| < 1,\ \forall z \in E$. Suppose on the contrary that there exists $z_\circ \in E$ such that $|\phi(z_\circ)| = 1 = |e^{i\theta}|,\ \ 0 \le \theta \le 2\pi$.

Now, with some computations, we have:

$$p(z) + \frac{zD_qp(z)}{\lambda p(z) + c} = \frac{1}{1-q\phi(z)} + \frac{(\frac{q}{\lambda})zD_q\phi(z)}{1-q\phi(z)} - \frac{(\frac{q}{\lambda})czD_q\phi(z)}{(\lambda+c) - qc\phi(z)}. \tag{10}$$

We apply Lemma 1 to have $z_\circ D_q\phi(z_\circ) = m\phi(z_\circ),\ \ m \ge 1$, and note that:

$$\begin{aligned} Re\left\{\frac{\frac{q}{\lambda}z_\circ D_q\phi(z_\circ)}{1-q\phi(z_\circ)}\right\} = Re\left\{\frac{\frac{mq}{\lambda}\phi(z_\circ)}{1-q\phi(z_\circ)}\right\} &= Re\left\{\frac{\frac{mq}{\lambda}e^{i\theta}}{1-qe^{i\theta}}\right\} \\ &= \frac{\frac{mq}{\lambda}(cos\theta - q)}{|1-qe^{i\theta}|^2}, \end{aligned} \tag{11}$$

$$Re\left\{\frac{\frac{q}{\lambda}cz_\circ D_q\phi(z_\circ)}{(\lambda+c) - qc\phi(z_\circ)}\right\} = \frac{\frac{q}{\lambda}cm(\lambda+c)cos\theta - \frac{q^2c^2m}{\lambda}}{|(\lambda+c) - qce^{i\theta}|^2}, \tag{12}$$

and:

$$
\begin{aligned}
&\left|\frac{1}{1-qe^{i\theta}} + \{\frac{q}{\lambda}\frac{me^{i\theta}}{(1-qe^{i\theta})}\} - \{\frac{\frac{q}{\lambda}cme^{i\theta}}{(\lambda+c)-qce^{i\theta}} - 1\}\right|_{\theta=\pi} \\
= &\left|\frac{1-q}{1+q} - \frac{\frac{mq}{\lambda}}{1+q} + \frac{\frac{qcm}{\lambda}}{(\lambda+c)+qc}\right| \qquad (13)
\end{aligned}
$$

Using (10), (11), (12), and (13), we get a contradiction to the given hypothesis (9), when we assume $|\phi(z_\circ)| = 1$ for some $z_\circ \in E$. Hence $|\phi(z)| < 1$ for all $z \in E$ and:

$$p(z) \prec \frac{1}{1-qz}, \quad z \in E.$$

This completes the proof. □

In order to develop some applications of Theorem 3, we need the following.
Let the operator $D_q^n : A \to A$ be defined as:

$$
\begin{aligned}
D_q^n f(z) &= F_{n+1,q}(z) * f(z) \\
&= z + \sum_{m=2}^{\infty} \frac{[m+n-1,q]!}{[n,q]![m-1,q]!} a_m z^n, \qquad (14)
\end{aligned}
$$

where:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m,$$

and:

$$F_{n+1,q}(z) = z + \sum \frac{[m+n-1,q]!}{[n,q]![m-1,q]!} z^m.$$

This series is absolutely convergent in E, and $*$ denotes convolution. The operator D_q^n is called the q-Ruscheweyh derivative of order n; see [25].

It can easily be seen that $D_q^\circ f(z) = f(z)$ and $D_q^{'} f(z) = zD_q f(z)$.

The relation (14) can be expressed as:

$$D_q^n f(z) = \frac{zD_q^n(z^{n-1} f(z))}{[n,q]!}, \quad n \in N.$$

Furthermore,

$$lim_{q\to1} D_q^n f(z) = \frac{z}{(1-z)}^{n+1} * f(z) = D^n f(z),$$

which is called the Ruscheweyh derivative of order n; see [25].

Let $f \in A$. Then, f is said to belong to the class $S_q^*(n,\alpha)$, if and only if, $D_q^n f \in S_q^*(\alpha), \ z \in E$.

The following identity can easily be obtained:

$$zD_q(D_q^n f(z)) = \left(1 + \frac{[n,q]}{q^n}\right) D_q^{n+1} f(z) - \frac{[n,q]}{q^n} D_q^n f(z) \qquad (15)$$

We now take $p(z) = \frac{zD_q(D_q^n f(z))}{D_q^n f(z)}$ in relation (9) of Theorem 3 to have:

Theorem 4. *Let $D_q^n f = F_n$ denote the q-Ruscheweyh derivative of order n for $f \in A$. Let:*

$$Re\left[\frac{zD_qF_{n+1}(z)}{F_{n+1}(z)} - \frac{1}{1+q}\right] > k\left|\frac{zD_qF_{n+1}(z)}{F_{n+1}(z)} - 1\right|, \ k \geq 0.$$

Then:

$$\frac{zD_qF_n(z)}{F_n(z)} \prec \frac{1}{1-qz}, \ z \in E.$$

That is, $f \in S_q^*(n, \alpha)$, $\alpha = \frac{1}{1+q}$.

Proof. Let p be analytic in E with $p(0) = 0$, and let:

$$p(z) = \frac{zD_q(D_q^n f(z))}{D_q^n f(z)} = \frac{zD_qF_n(z)}{F_n(z)}.$$

Using identity (15) and some computation, we have:

$$Re\left[p(z) + \frac{zD_qp(z)}{p(z)+n} - \frac{1}{1+q}\right] - k\left|p(z) + \frac{zD_qp(z)}{p(z)} - 1\right| > 0.$$

Now, the required result follows immediately from Theorem 3. □

Corollary 6. *In Theorem 4, we take $k = 0$. Then, it gives us:*

$$S_q^*(n+1, \alpha) \subset S_q^*(n, \alpha) \subset ... \subset S_q^*(\alpha), \ \alpha = \frac{1}{1+q}.$$

When $q \to 1^-$, $\frac{1}{1+q} \to \frac{1}{2}$, and we have:

$$S_q^*(n+1, \frac{1}{2}) \subset S^*(n, \frac{1}{2}) \subset ... \subset S^*(\frac{1}{2}).$$

Corollary 7. *Let $f \in A$, and let:*

$$Re\left[\frac{zD_qf(z)}{f(z)} - \frac{1}{1+q}\right] > k\left|\frac{zD_qf(z)}{f(z)} - 1\right|, \ k \geq 0. \tag{16}$$

Define:

$$L_B(f) = F_c(z) = \frac{[c+1]_q}{z^c}\int_0^z t^{c-1}f(t)d_qt, \ c \in N_\circ. \tag{17}$$

Then:

$$\frac{zD_qF_c(z)}{F_c(z)} \prec \frac{1}{1-qz}, \ z \in E.$$

Proof. The integral operator $L_B : A \to A$ defined in (16) is known as the q-Bernardi integral operator $L_B(f) = F_c$. When $q \to 1^-$, (16) reduces to the well-known Bernardi operator; see [7].

Let,

$$\frac{zD_qF_c(z)}{F_c(z)} = p(z). \tag{18}$$

Then, from (16), (17), (18), and some computations, this leads us to:

$$\begin{aligned} Re \left[\frac{zD_q f(z)}{f(z)} - \frac{1}{1+q}\right] - k\left|\frac{zD_q f(z)}{f(z)} - 1\right| &= Re \left[p(z) + \frac{zD_q p(z)}{p(z) + q^c[c,q]}\right] \\ &- k\left|p(z) + \frac{zD_q p(z)}{p(z) + q^c[c,q]} - 1\right| > 0, z \in E. \end{aligned}$$

We now apply Theorem 3, and it follows that:

$$p(z) = \frac{zD_q F_c(z)}{F_c(z)} \prec \frac{q}{1-qz} \quad \text{in} \quad E.$$

That is, $F_c \in S_q^*(\frac{1}{1+q})$. □

As a special case, when $q \to 1^-$, then $f \in K - UST(\frac{1}{2})$, and then, F_c, defined by 17, belongs to $S^*(\frac{1}{2})$ in E.

4. Conclusions

In this paper, we have used q-calculus, conic domains, and subordination to define and study some new subclasses involving Mocanu functions. Some interesting inclusion and subordination properties of these new classes have been derived. The q-analogue of the Ruscheweyh derivative has been used to obtain a new subordination result for q-Mocanu functions. Some special cases have been discussed as applications of our main results. The technique and ideas of this paper may stimulate further research in this dynamic field.

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