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Existence of Concentrating Solutions for Critical Hartree Equations in Perforated Domains

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Abstract

In this paper, we study a critical Hartree equation with mixed boundary conditions in a perforated domain. By means of the Lyapunov–Schmidt reduction method, we prove that when the radius ε of the hole is sufficiently small, the problem admits, for each sufficiently small $\varepsilon > 0$, a positive solution u_ε concentrating and blowing up near the center of the hole. Moreover, we characterize its concentration point, scaling parameter and main asymptotic profile.

Keywords: critical Hartree equation; mixed boundary conditions; Lyapunov–Schmidt reduction

MSC: 35A01; 35B44; 35J20

1. Introduction

Let $D_\varepsilon \subset \mathbb{R}^N$ ($N \geq 5$) be a bounded smooth domain, and let $B_\varepsilon(0) \subset D_\varepsilon$ denote the ball centered at the origin with radius $\varepsilon > 0$. We assume that ε is sufficiently small and that

$$\operatorname{dist}(B_\varepsilon(0), \partial D_\varepsilon) \geq c_0 > 0,$$

for some constant c_0 independent of ε. We define the perforated domain by

$$\Omega_\varepsilon := D_\varepsilon \setminus \overline{B_\varepsilon(0)}.$$

where as $\varepsilon \to 0$, D_ε approaches D in a suitable sense which will be specified later.

Hartree-type equations, of which the Choquard equation [1] is a classical example, arise from the Hartree–Fock approximation in quantum many-body theory. In this paper, we investigate the following mixed boundary value problem for a general critical Hartree equation

$$\begin{cases} -\Delta u = \left(\displaystyle\int_{\Omega_\varepsilon} \frac{u^{2^*_\mu}(x)}{|y-x|^\mu} \mathrm{d}x \right) u^{2^*_\mu - 1}, \ u > 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial D_\varepsilon, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_\varepsilon(0), \end{cases} \tag{1}$$

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where $0 < \mu \leq 4$, $2^*_\mu := \dfrac{2N-\mu}{N-2}$ stands for the critical exponent associated with the Hardy–Littlewood–Sobolev (HLS) inequality. The Riesz potential convolution term on the right-hand side encodes a long-range interaction, making the equation intrinsically nonlocal. Here ν represents the outward unit normal vector of $\partial\Omega_\varepsilon$, and the outer boundary ∂D_ε satisfies the Dirichlet condition while the inner boundary $\partial B_\varepsilon(0)$ satisfies the Neumann condition.

We first recall the sharp HLS inequality [2,3], which provides a fundamental basis for our subsequent analysis.

Proposition 1. *Suppose $\theta > 1$, $r > 1$ and $0 < \mu < N$ satisfy*

$$\frac{1}{\theta} + \frac{1}{r} + \frac{\mu}{N} = 2.$$

For any $f \in L^\theta(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$, there exists an optimal constant $C(N,\mu,\theta,r)$ independent of f and g such that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y \leq C(N,\mu,\theta,r)\|f\|_{L^\theta}\|g\|_{L^r}. \tag{2}$$

If $\theta = r = \dfrac{2N}{2N-\mu}$, then

$$C_0(N,\mu) := C(N,\mu,\theta,r) = \pi^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{N-\mu}{2}\right)}{\Gamma\left(N-\frac{\mu}{2}\right)} \left\{\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right\}^{\frac{N-\mu}{N}},$$

where Γ is the Gamma function.

When employing the Lyapunov–Schmidt reduction method, we adopt solutions of the following whole-space equation as approximate solutions

$$-\Delta u = \left(\int_{\mathbb{R}^N} \frac{u^{2^*_\mu}(x)}{|y-x|^\mu}\mathrm{d}x\right) u^{2^*_\mu-1}, \quad u > 0,\ y \in \mathbb{R}^N. \tag{3}$$

Guo et al. [4] verified that all positive solutions to Equation (3) are radially symmetric and monotonically decreasing about some point and further derived the explicit expression of all positive solutions as

$$V_{z,\lambda}(y) := \frac{\Theta\lambda^{\frac{N-2}{2}}}{(1+\lambda^2|y-z|^2)^{\frac{N-2}{2}}}, \qquad y \in \mathbb{R}^N, \tag{4}$$

where $\lambda > 0$, $z \in \mathbb{R}^N$, and

$$\Theta = S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} [C_0(N,\mu)]^{\frac{2-N}{2(N-\mu+2)}} [N(N-2)]^{\frac{N-2}{4}}. \tag{5}$$

Here

$$S := \inf_{u\in\tilde{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}y}{\left(\int_{\mathbb{R}^N}|u|^{2^*}\,\mathrm{d}y\right)^{\frac{2}{2^*}}} = N(N-2)\,\pi\left(\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right)^{\frac{2}{N}}$$

is the best constant in the Sobolev inequality, and $C_0(N,\mu)$ denotes the sharp constant in the HLS inequality given explicitly in Proposition 1. Consequently, the constant Θ in (5) is fully explicit in terms of N and μ. In addition, Du et al. and Li et al. [5,6] proved the nondegeneracy of the positive solutions of (3).

To construct an approximate solution for (1), we use the Dirichlet projection of a standard Hartree bubble onto the bounded domain D_ε. More precisely, let

$$P_\varepsilon : D^{1,2}(\mathbb{R}^N) \to H_0^1(D_\varepsilon)$$

denote the Dirichlet projection operator. For $z \in D_\varepsilon$ and $\lambda > 0$, the function $P_\varepsilon V_{z,\lambda} \in H_0^1(D_\varepsilon)$ is the unique weak solution of

$$\begin{cases} -\Delta P_\varepsilon V_{z,\lambda} &= -\Delta V_{z,\lambda} \quad \text{in } D_\varepsilon, \\ P_\varepsilon V_{z,\lambda} &= 0 \qquad\quad \text{on } \partial D_\varepsilon. \end{cases} \tag{6}$$

In what follows, we restrict $P_\varepsilon V_{z,\lambda}$ to Ω_ε and regard it as an element of $\widetilde{H}_0^1(\Omega_\varepsilon)$, where

$$\widetilde{H}_0^1(\Omega_\varepsilon) := \left\{ u \in H^1(\Omega_\varepsilon) : u = 0 \text{ on } \partial D_\varepsilon \right\}, \tag{7}$$

equipped with the inner product

$$\langle u, v \rangle = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, \mathrm{d}y. \tag{8}$$

By means of the Lyapunov–Schmidt reduction method, we construct a concentrating solution to (1). Our main result is stated below.

Theorem 1. *Assume $N \geq 5$ and $0 < \mu \leq 4$. There exists a constant $\varepsilon_0 > 0$, depending only on D, such that for all $\varepsilon \in (0, \varepsilon_0]$, problem (1) admits a positive solution u of the form*

$$u_\varepsilon = P_\varepsilon V_{z_\varepsilon,\lambda_\varepsilon} + \omega_\varepsilon,$$

where $V_{z_\varepsilon,\lambda_\varepsilon}$ is given by (4), and $P_\varepsilon V_{z_\varepsilon,\lambda_\varepsilon}$ is defined by (6). Moreover, as $\varepsilon \to 0$,

$$\lambda_\varepsilon \varepsilon^{\frac{N}{2(N-1)}} \to c > 0, \qquad \lambda_\varepsilon |z_\varepsilon| \to 0, \qquad \|\omega_\varepsilon\|_{\widetilde{H}_0^1(\Omega_\varepsilon)} \to 0,$$

where $c > 0$ is independent of ε.

Remark 1. *The assumption $N \geq 5$ is of a technical nature and is used at several points of the reduction. For instance, it guarantees that the boundary correction in (34) is of lower order with respect to the leading terms, that the auxiliary exponents such as $2_\mu^* - 2 = \frac{4-\mu}{N-2}$ appearing in Lemma 6 satisfy the required integrability conditions, and that the trace estimate (32) is applicable. Whether the same conclusion holds in dimensions $N = 3, 4$ is not addressed in this paper and would require a separate analysis; we do not claim that the condition $N \geq 5$ is optimal.*

The relationship between the concentration behavior of solutions to critical elliptic equations and the geometry of the underlying domain is a classical theme in critical nonlinear elliptic problems [7,8]. For the Dirichlet Brezis–Nirenberg problem, Rey [8] showed that the concentration locations of positive solutions are related to critical points of the regular part of the Green function; in particular, concentration occurs in the interior of the domain. In perforated geometries, Bandle et al. [9] characterized radial solutions in

annular domains, while further works [10–12] showed that the topology of the domain can play an important role in the existence and multiplicity of positive solutions. Moreover, Li, Yan and Yang [13] constructed, for the local critical equation in domains with small holes, bubble-type concentrating solutions and derived their blow-up rates. In particular, in the mixed-boundary case considered there—Dirichlet condition on the outer boundary and Neumann condition on the boundary of the hole—the solution blows up at the center of the hole. These classical results show that both the global geometry and the local structure created by a small hole may substantially affect concentration phenomena. Related blow-up and concentration questions have also been considered in more general geometric settings, such as manifolds with boundary [14] and Riemannian orbifolds [15].

For critical Hartree equations, the nonlocal Riesz interaction introduces additional difficulties. Indeed, setting $\alpha = N - \mu$, the kernel $|x-y|^{-\mu} = |x-y|^{-(N-\alpha)}$ is, up to a normalization constant, the kernel of the Riesz potential I_α. In the whole-space setting, I_α is a fractional integral operator with Fourier symbol $|\xi|^{-\alpha}$, and is therefore formally identified with $(-\Delta)^{-\alpha/2}$. Thus, although the leading operator in (1) is the classical Laplacian, its Hartree term incorporates a negative-order fractional-potential-type nonlocality. The critical exponent $2^*_\mu = \frac{2N-\mu}{N-2}$, arising from the Hardy–Littlewood–Sobolev inequality [2,3], differs from the Sobolev critical exponent in local problems. The sharp HLS constant, the classification of positive whole-space solutions, and the nondegeneracy of standard bubbles [2,4–6] provide the main analytical ingredients for finite-dimensional reduction. Du et al. [5] and Li et al. [6] proved bubble nondegeneracy. For a critical perturbed Hartree equation with Dirichlet boundary condition, Yang et al. [16] constructed concentrating solutions in bounded domains, and Squassina et al. [17] established their local uniqueness near suitable critical points of the Robin function. In these Dirichlet settings, the boundary plays only a passive role and the concentration point is selected by the Robin function; the interaction of the nonlocal term with a boundary component located inside the domain, remains to the best of our knowledge, unexplored in this framework.

Compared with the local mixed-boundary problem [13] and with the Dirichlet Hartree problems [16,17], the present setting brings new difficulties: the approximate solution $P_D V_{z,\lambda}$ satisfies the Dirichlet condition on ∂D but not the Neumann condition on $\partial B_\varepsilon(0)$, so the reduction has to control the flux on the inner boundary, which produces the leading sign-definite boundary term, and the Hartree convolution over the perforated domain Ω_ε couples the control of the Dirichlet projection, the decay of the bubble and the interaction of the Riesz kernel with the hole. As a consequence, the concentration point is pinned to the center of the hole, in contrast with the Dirichlet Hartree case [16,17], where the blow-up point is selected by the Robin function. Neumann-type conditions in nonlocal frameworks have recently been developed for fractional operators and their superpositions [18,19], offering complementary motivation for the mixed-boundary setting studied here. We address (1) by the Lyapunov–Schmidt reduction method and construct a positive solution concentrating and blowing up near the center of the hole.

The remainder of the paper is organized as follows. Section 2 establishes pointwise and boundary estimates for Hartree bubbles and their Dirichlet projections and introduces the Lyapunov–Schmidt reduction framework. Section 3 performs the finite-dimensional reduction by expanding the energy functional around the approximate solution and deriving refined estimates for the error term and the linearized operator. This reduces the original infinite-dimensional variational problem to a finite-dimensional system. Section 4 uses a topological-degree argument for the reduced system to complete the proof of Theorem 1.

2. Preliminaries

In this section, we first establish several basic estimates for Hartree bubbles and their Dirichlet projections, and then briefly outline the framework of the Lyapunov–Schmidt reduction method.

2.1. Basic Estimates

This subsection provides pointwise and boundary estimates for the Hartree bubble $V_{z,\lambda}$ and its Dirichlet projection $P_\varepsilon V_{z,\lambda}$, and introduces the Dirichlet Green function and its decomposition, which are essential for the subsequent reduction analysis.

Lemma 1. *For any $N \geq 5$, suppose $\overline{\nu}$ denotes the unit outward normal vector of $\partial B_\varepsilon(0)$. Then there exists a positive constant C depending only on N such that*

$$\left|\frac{\partial V_{z,\lambda}}{\partial \overline{\nu}}\right| \leq C\lambda^{\frac{N}{2}}, \quad \left|\frac{\partial V_{z,\lambda}}{\partial \lambda}\right| \leq C\lambda^{\frac{N-4}{2}} \quad \text{and} \quad \left|\frac{\partial V_{z,\lambda}}{\partial z_i}\right| \leq C\lambda^{\frac{N}{2}}.$$

Proof. From the explicit expression of $V_{z,\lambda}$, where Θ is the constant defined in (5), we compute its gradient and normal derivative as follows.

$$\begin{aligned}\nabla V_{z,\lambda}(y) &= \Theta\lambda^{\frac{N-2}{2}}\nabla_y\left(1+\lambda^2|y-z|^2\right)^{-\frac{N-2}{2}} \\ &= -\Theta(N-2)\lambda^{\frac{N+2}{2}}\frac{y-z}{(1+\lambda^2|y-z|^2)^{\frac{N}{2}}},\end{aligned}$$

and

$$\frac{\partial V_{z,\lambda}}{\partial \overline{\nu}} = \nabla V_{z,\lambda}\cdot\overline{\nu} = -\Theta(N-2)\lambda^{\frac{N+2}{2}}\frac{\langle y-z,\overline{\nu}\rangle}{(1+\lambda^2|y-z|^2)^{\frac{N}{2}}}.$$

Taking absolute values and applying the Cauchy–Schwarz inequality $|\langle y-z,\overline{\nu}\rangle| \leq |y-z|$, we obtain

$$\begin{aligned}\left|\frac{\partial V_{z,\lambda}}{\partial \overline{\nu}}\right| &\leq \Theta(N-2)\lambda^{\frac{N+2}{2}}\frac{|y-z|}{(1+\lambda^2|y-z|^2)^{\frac{N}{2}}} \\ &\leq \Theta(N-2)\lambda^{\frac{N+2}{2}}\frac{C'}{\lambda} \\ &\leq C\lambda^{\frac{N}{2}}.\end{aligned}$$

On the other hand, we have

$$\frac{\partial V_{z,\lambda}}{\partial \lambda} = \Theta\frac{N-2}{2}\lambda^{\frac{N-4}{2}}\frac{1-\lambda^2|y-z|^2}{(1+\lambda^2|y-z|^2)^{\frac{N}{2}}}.$$

Using the elementary inequality $|1-a| \leq 1+a$, which holds for all real numbers $a \geq 0$, we deduce the absolute value bound

$$\left|\frac{\partial V_{z,\lambda}}{\partial \lambda}\right| \leq \Theta\frac{N-2}{2}\lambda^{\frac{N-4}{2}}\frac{1}{(1+\lambda^2|y-z|^2)^{\frac{N-2}{2}}} \leq C\lambda^{\frac{N-4}{2}}.$$

Moreover, we prove by a similar argument that $\left|\frac{\partial V_{z,\lambda}}{\partial z_i}\right| \leq C\lambda^{\frac{N}{2}}$. □

Since $\|\omega_\varepsilon\|_{\widetilde{H}_0^1(\Omega_\varepsilon)} \to 0$, the Sobolev embedding yields $\|\omega_\varepsilon\|_{L^{2^*}(\Omega_\varepsilon)} \to 0$. For any fixed $\delta > 0$, the decay of the bubble $V_{z_\varepsilon,\lambda_\varepsilon}$ gives $\|P_\varepsilon V_{z_\varepsilon,\lambda_\varepsilon}\|_{L^\infty(D_\varepsilon\setminus B_\delta(z_\varepsilon))} \to 0$, and hence $\|u_\varepsilon\|_{L^{2^*}(D_\varepsilon\setminus B_\delta(z_\varepsilon))} \to 0$. Since u_ε solves (1), we may write $-\Delta u_\varepsilon = a_\varepsilon u_\varepsilon$ in $D_\varepsilon \setminus B_\delta(z_\varepsilon)$, where

$$a_\varepsilon(x) := \Big(\int_{\Omega_\varepsilon} \frac{u_\varepsilon^{2_\mu^*}(y)}{|x-y|^\mu}\,\mathrm{d}y\Big) u_\varepsilon(x)^{2_\mu^*-2}.$$

By the HLS inequality and the boundedness of $\|u_\varepsilon\|_{L^{2^*}(\Omega_\varepsilon)}$, the convolution term is uniformly bounded in $L^{2N/\mu}$, while $\|u_\varepsilon^{2_\mu^*-2}\|_{L^{2^*/(2_\mu^*-2)}(D_\varepsilon\setminus B_\delta(z_\varepsilon))} \to 0$, hence, by Hölder's inequality, $\|a_\varepsilon\|_{L^{N/2}(D_\varepsilon\setminus B_\delta(z_\varepsilon))} \to 0$. A standard Brezis–Kato-type bootstrap for $-\Delta u = au$ with $\|a\|_{L^{N/2}}$ small then yields

$$\|u_\varepsilon\|_{L^\infty(D_\varepsilon\setminus B_{2\delta}(z_\varepsilon))} \to 0.$$

As a direct consequence, the nonlocal Hartree convolution term $\left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y\right) u^{2_\mu^*-1}$ is negligible on $D_\varepsilon \setminus B_{2\delta}(z_\varepsilon)$. Therefore, we have $-\Delta u_\varepsilon = o(1)$ as $\varepsilon \to 0$ on $D_\varepsilon \setminus B_{2\delta}(z_\varepsilon)$, in the sense that the Hartree convolution term tends to zero uniformly. This motivates us to analyze the Dirichlet Green function associated with the Laplace operator $-\Delta$.

Let $G_{D_\varepsilon}(y,z)$ denote the Dirichlet Green function of the operator $-\Delta$ over the bounded smooth domain $D_\varepsilon \subset \mathbb{R}^N$. It solves the following boundary value problem

$$\begin{cases} -\Delta_y G_{D_\varepsilon}(y,z) &= \delta_z(y) \quad \text{in } D_\varepsilon, \\ \quad G_{D_\varepsilon}(y,z) &= 0 \qquad \text{on } \partial D_\varepsilon. \end{cases} \tag{9}$$

where δ_z is the Dirac delta at z. From the conclusions given in [20], the Green function admits the standard decomposition

$$G_{D_\varepsilon}(y,z) = \gamma_N |y-z|^{2-N} - H_{D_\varepsilon}(y,z),$$

where the constant $\gamma_N = \frac{1}{(N-2)\omega_{N-1}}$ depends solely on the spatial dimension N, and ω_{N-1} represents the surface area of the $(N-1)$-dimensional unit sphere. The function $H_{D_\varepsilon}(y,z)$ is called the regular part of the Green function $G_{D_\varepsilon}(y,z)$, and it satisfies the homogeneous Laplace boundary value problem

$$\begin{cases} -\Delta_y H_{D_\varepsilon}(y,z) &= 0 \qquad\qquad\quad \text{in } D_\varepsilon, \\ \quad H_{D_\varepsilon}(y,z) &= \gamma_N |y-z|^{2-N} \quad \text{on } \partial D_\varepsilon. \end{cases} \tag{10}$$

In particular, the value $H_{D_\varepsilon}(z,z)$ is referred to as the Robin function of the domain D_ε. For simplicity, we write

$$\tau_{D_\varepsilon}(z) := H_{D_\varepsilon}(z,z), \tag{11}$$

for this Robin function.

Set

$$\psi_{z,\lambda} := V_{z,\lambda} - P_\varepsilon V_{z,\lambda}. \tag{12}$$

Then $\psi_{z,\lambda}$ satisfies

$$\begin{cases} -\Delta\psi_{z,\lambda} &= 0, & \text{in } D_\varepsilon, \\ \psi_{z,\lambda} &= V_{z,\lambda}, & \text{on } \partial D_\varepsilon. \end{cases}$$

Thus, $\psi_{z,\lambda} \geq 0$ is a harmonic function, and hence $\psi_{z,\lambda} \in L^\infty(D_\varepsilon)$. Moreover, we have the following expansion for $\psi_{z,\lambda}$.

Lemma 2. *Let $\psi_{z,\lambda}$ be defined by (12). Assume that z stays in a compact subset of D_ε. Then, uniformly for $y \in D_\varepsilon$, it holds that*

$$\psi_{z,\lambda}(y) = \frac{\Theta H_{D_\varepsilon}(y,z)}{\lambda^{\frac{N-2}{2}}} + o\left(\frac{1}{\lambda^{\frac{N-2}{2}}}\right), \quad \text{as } \lambda \to \infty. \tag{13}$$

Proof. Here Θ is the constant defined in (5). For $y \in \partial D_\varepsilon$, one has

$$V_{z,\lambda}(y) = \frac{\Theta}{\lambda^{\frac{N-2}{2}}|y-z|^{N-2}} + o\left(\frac{1}{\lambda^{\frac{N-2}{2}}}\right).$$

Thus, the function

$$w(y) = \psi_{z,\lambda}(y) - \frac{\Theta H_{D_\varepsilon}(z,y)}{\lambda^{\frac{N-2}{2}}}$$

satisfies

$$\begin{cases} \Delta w &= 0, & \text{in } D_\varepsilon, \\ w &= o\left(\frac{1}{\lambda^{\frac{N-2}{2}}}\right), & \text{on } \partial D_\varepsilon. \end{cases}$$

By the maximum principle, we have $|w| = o\left(\frac{1}{\lambda^{\frac{N-2}{2}}}\right)$. Hence the desired result follows. □

Since $\psi_{z,\lambda}$ is harmonic in D_ε with $\psi_{z,\lambda} = V_{z,\lambda}$ on ∂D_ε, differentiating this boundary value problem with respect to λ and z_i shows that $\partial_\lambda \psi_{z,\lambda}$ and $\partial_{z_i}\psi_{z,\lambda}$ are harmonic in D_ε with boundary data $\partial_\lambda V_{z,\lambda}$ and $\partial_{z_i} V_{z,\lambda}$, respectively. Applying the maximum principle together with the explicit expression of $V_{z,\lambda}$, we obtain the following estimates for $\psi_{z,\lambda}$.

Lemma 3. *Let $\psi_{z,\lambda}$ be defined by (12). There exists a constant $C > 0$, independent of z and λ, such that*

$$|\psi_{z,\lambda}(y)| \leq \frac{C}{\lambda^{\frac{N-2}{2}}}, \qquad \left|\frac{\partial \psi_{z,\lambda}(y)}{\partial \lambda}\right| \leq \frac{C}{\lambda^{\frac{N}{2}}}, \qquad \left|\frac{\partial \psi_{z,\lambda}(y)}{\partial z_i}\right| \leq \frac{C}{\lambda^{\frac{N-4}{2}}} \quad (i = 1, \ldots, N).$$

Expanding (13) in a Taylor series about $y = z$, we obtain the following lemma.

Lemma 4. *Let $\psi_{z,\lambda}$ be defined by (12). For y in a small neighborhood of z, we have*

$$\begin{aligned} \psi_{z,\lambda}(y) &= \frac{\Theta}{\lambda^{\frac{N-2}{2}}}\tau_{D_\varepsilon}(z) + \frac{\Theta}{\lambda^{\frac{N-2}{2}}}\sum_{i=1}^{N} \partial_{y_i} H_{D_\varepsilon}(y,z)(y_i - z_i) \\ &\quad + O\left(\frac{|y-z|^2}{\lambda^{\frac{N-2}{2}}}\right) + o\left(\frac{1}{\lambda^{\frac{N-2}{2}}}\right), \end{aligned}$$

where Θ is as in (5) and $\tau_{D_\varepsilon}(z)$ is defined in (11).

The following identity is contained in the proof of [21] ([Theorem 1.2]).

Lemma 5. *For any $z \in \mathbb{R}^N$ and $\lambda > 0$, the standard Hartree bubble $V_{z,\lambda}$ satisfies*

$$\int_{\mathbb{R}^N} \frac{V_{z,\lambda}^{2_\mu^*}(x)}{|y-x|^\mu}\,\mathrm{d}x = C_1(N,\mu)\, V_{z,\lambda}^{2^*-2_\mu^*}(y), \tag{14}$$

where $2^* = \frac{2N}{N-2}$ is the classical Sobolev critical exponent, and $C_1(N,\mu) > 0$ is a constant depending on N and μ.

2.2. Lyapunov–Schmidt Reduction Framework

Lyapunov–Schmidt reduction is a fundamental tool for studying nonlinear elliptic equations of various critical types as well as perturbed elliptic equations near the critical exponent. For such equations, the standard variational approach fails due to the absence of the Palais–Smale compactness condition. The core idea of the reduction is to decompose the infinite-dimensional solution space into the finite-dimensional tangent space of the approximate solution manifold and its orthogonal complement, which reduces the original infinite-dimensional variational problem to a finite-dimensional critical point problem. For more details, see [16,22–24].

In the framework of this paper, we take the Dirichlet-projected Hartree bubbles $P_\varepsilon V_{z,\lambda}$ introduced in Section 1 as the approximate solutions to problem (1). These bubbles form an $(N+1)$-dimensional approximate solution manifold spanned by the concentration point $z \in D$ and the scaling parameter $\lambda > 0$. We orthogonally decompose the solution space $\widetilde{H}_0^1(\Omega_\varepsilon)$ into the tangent space of this manifold and its orthogonal complement.

We seek solutions of the form $u_\varepsilon = P_\varepsilon V_{z_\varepsilon,\lambda_\varepsilon} + \omega_\varepsilon$, where ω_ε belongs to the aforementioned orthogonal complement and represents the higher-order correction term of the approximate solution. After expanding the energy functional around the approximate solution, we first solve for the correction term ω_ε as a function of z and λ by using the invertibility of the linearized operator on the orthogonal complement and the implicit function theorem. The original problem is then reduced to finding critical points of the finite-dimensional reduced functional, which only requires solving an $(N+1)$-dimensional system of equations for the concentration point z and the scaling parameter λ.

3. Reduction

The energy functional corresponding to (1) is defined as

$$I_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2\,\mathrm{d}y - \frac{1}{2\cdot 2_\mu^*}\iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{(u_+)^{2_\mu^*}(x)(u_+)^{2_\mu^*}(y)}{|y-x|^\mu}\,\mathrm{d}x\,\mathrm{d}y, \quad u \in \widetilde{H}_0^1(\Omega_\varepsilon),$$

where $\widetilde{H}_0^1(\Omega_\varepsilon)$ is defined by (7).

Let

$$E_{\varepsilon,z,\lambda} = \left\{\omega \in \widetilde{H}_0^1(\Omega_\varepsilon) \,:\, \left\langle \omega, \frac{\partial P_\varepsilon V_{z,\lambda}}{\partial \lambda}\right\rangle = \left\langle \omega, \frac{\partial P_\varepsilon V_{z,\lambda}}{\partial z_i}\right\rangle = 0,\ i = 1,2,\ldots,N\right\} \tag{15}$$

be the subspace determined by the above orthogonality conditions. Set

$$T_\varepsilon = \left\{(z,\lambda) \,:\, \lambda \in \left[c_1\varepsilon^{-\frac{N}{2(N-1)}}, c_2\varepsilon^{-\frac{N}{2(N-1)}}\right],\ |z| \le c_3\varepsilon^{\frac{(1+\sigma)N}{4(N-1)}}\right\}, \tag{16}$$

where $c_3 > 0$ is a sufficiently large constant, $c_2 > c_1 > 0$, and σ is some small constant. We further define

$$M_\varepsilon = \{(z,\lambda,\omega) \,:\, (z,\lambda) \in T_\varepsilon,\ \omega \in E_{\varepsilon,z,\lambda}\}.$$

Then we introduce

$$J_\varepsilon(z,\lambda,\omega) = I_\varepsilon(P_\varepsilon V_{z,\lambda} + \omega), \quad (z,\lambda,\omega) \in M_\varepsilon. \tag{17}$$

Before proceeding, we record the regularity needed for the reduction below. First, the energy functional I_ε is of class C^2 on $\widetilde{H}_0^1(\Omega_\varepsilon)$. Indeed, since $\mu > 0$, we have $2_\mu^* = \frac{2N-\mu}{N-2} < \frac{2N}{N-2} = 2^*$, and $2_\mu^* \geq 2$ for $0 < \mu \leq 4$, $N \geq 5$. By the Sobolev embedding $\widetilde{H}_0^1(\Omega_\varepsilon) \hookrightarrow L^{2^*}(\Omega_\varepsilon)$, the superposition operator $u \mapsto (u_+)^{2_\mu^*}$ is C^2 from $\widetilde{H}_0^1(\Omega_\varepsilon)$ to $L^{\frac{2N}{2N-\mu}}(\Omega_\varepsilon)$. Composing with the continuous bilinear form

$$B(f,g) := \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{f(x)\, g(y)}{|y-x|^\mu}\, dx\, dy, \qquad |B(f,g)| \leq C \|f\|_{L^{\frac{2N}{2N-\mu}}} \|g\|_{L^{\frac{2N}{2N-\mu}}},$$

whose continuity is guaranteed by Proposition 1, we obtain that the double integral in I_ε is a C^2 function of u. Hence $I_\varepsilon \in C^2(\widetilde{H}_0^1(\Omega_\varepsilon))$. Second, by the explicit formula (4), $(z,\lambda) \mapsto V_{z,\lambda}$ is smooth, and the Dirichlet projection $P_\varepsilon \colon D^{1,2}(\mathbb{R}^N) \to H_0^1(D_\varepsilon)$ is linear and bounded, therefore $(z,\lambda) \mapsto P_\varepsilon V_{z,\lambda}$ is smooth, and $J_\varepsilon(z,\lambda,\omega)$ jointly in (z,λ,ω) on a neighborhood of M_ε.

For each $(z,\lambda) \in T_\varepsilon$, we expand $J_\varepsilon(z,\lambda,\omega)$ around $\omega = 0$ as follows

$$J_\varepsilon(z,\lambda,\omega) = J_\varepsilon(z,\lambda,0) + \langle l_\varepsilon, \omega\rangle + \frac{1}{2}\langle Q_\varepsilon \omega, \omega\rangle + R_\varepsilon(\omega), \tag{18}$$

where the linear term in the expansion (18) is given by the functional $l_\varepsilon \in E_{\varepsilon,z,\lambda}$, which satisfies

$$\langle l_\varepsilon, \omega\rangle = \int_{\Omega_\varepsilon} \nabla(P_\varepsilon V_{z,\lambda}) \cdot \nabla\omega \, \mathrm{d}y - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\, \omega(y)}{|y-x|^\mu} \, \mathrm{d}x\, \mathrm{d}y; \tag{19}$$

the quadratic part of the expansion (18) is governed by the bounded linear operator Q_ε, which satisfies

$$\begin{aligned} \langle Q_\varepsilon \omega, \omega\rangle \;=\; & \int_{\Omega_\varepsilon} |\nabla\omega|^2 \, \mathrm{d}y \\ & - 2_\mu^* \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(x)\, \omega(x)\, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\, \omega(y)}{|y-x|^\mu} \, \mathrm{d}x\, \mathrm{d}y \\ & - \left(2_\mu^* - 1\right) \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}(y)\, \omega^2(y)}{|y-x|^\mu} \, \mathrm{d}x\, \mathrm{d}y; \end{aligned} \tag{20}$$

and the higher-order remainder terms in (18) are collected in $R_\varepsilon(\omega)$, which satisfy

$$R_\varepsilon^j(\omega) = O\left(\|\omega\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}^{\min\{3-j,\, 2\cdot 2_\mu^* - j\}} \right), \quad j = 0,1,2. \tag{21}$$

Based on the above energy expansion, we first estimate the norm of the linear term l_ε, and then prove the uniform invertibility of the linearized operator Q_ε on the orthogonal complement $E_{\varepsilon,z,\lambda}$. Consequently, the implicit function theorem yields a correction term $\omega_{\varepsilon,z,\lambda}$ for each parameter (z,λ), reducing the original problem to a finite-dimensional system.

Lemma 6. *Let $E_{\varepsilon,z,\lambda}$ be as defined in (15), and let l_ε be as defined in (19). Suppose $N \geq 5$ and $\mu \in (0,4]$. Then*

$$\|l_\varepsilon\|_{\widetilde{H}_0^1(\Omega_\varepsilon)} = \varepsilon^{\frac{N}{2}} \lambda^{\frac{N}{2}} O\left(\lambda\varepsilon + \lambda|z| + \frac{1}{\lambda^{N-2}} \right) + O\left(\frac{1}{\lambda^{\frac{N-2}{2}+\sigma}} \right), \tag{22}$$

where $\sigma > 0$ is some constant.

Proof. For any $\omega \in E_{\varepsilon,z,\lambda}$, we obtain

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla (P_\varepsilon V_{z,\lambda}) \cdot \nabla \omega \, \mathrm{d}y - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\,\omega(y)}{|y-x|^\mu}\,\mathrm{d}x\mathrm{d}y \\
= \; & -\int_{\partial B_\varepsilon(0)} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \overline{\nu}} \omega \, \mathrm{d}S \\
& + \int_{\Omega_\varepsilon} \omega(y)\, V_{z,\lambda}^{2_\mu^*-1}(y) \left(\int_{\mathbb{R}^N} \frac{V_{z,\lambda}^{2_\mu^*}(x)}{|y-x|^\mu}\,\mathrm{d}x \right) \mathrm{d}y \\
& - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\,\omega(y)}{|y-x|^\mu}\,\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{23}$$

Here and in the sequel, $\mathrm{d}S$ denotes the surface measure on the corresponding boundary; in the present context, it is the measure on $\partial B_\varepsilon(0)$.

On the one hand, we have the following estimate

$$\begin{aligned}
& \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x) V_{z,\lambda}^{2_\mu^*-1}(y)\omega(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y \\
& \quad - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\omega(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y \\
\le \; & C \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*-1}(x)\psi_{z,\lambda}(x) V_{z,\lambda}^{2_\mu^*-1}(y)\omega(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y \\
& \quad + C \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x) V_{z,\lambda}^{2_\mu^*-2}(y)\psi_{z,\lambda}(y)\,\omega(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{24}$$

The definition of $P_\varepsilon V_{z,\lambda}$ gives $0 \le P_\varepsilon V_{z,\lambda} \le V_{z,\lambda}$, and splitting the domain yields

$$\begin{aligned}
& \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\,\psi_{z,\lambda}(y)\,\omega(y)}{|y-x|^\mu}\,\mathrm{d}y\mathrm{d}x \\
\le \; & \iint_{\mathbb{R}^N \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\,\psi_{z,\lambda}(y)\,\omega(y)}{|y-x|^\mu}\,\mathrm{d}y\mathrm{d}x \\
& - \iint_{(\mathbb{R}^N \setminus D_\varepsilon) \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\,\psi_{z,\lambda}(y)\,\omega(y)}{|y-x|^\mu}\,\mathrm{d}y\mathrm{d}x \\
& - \iint_{B_\varepsilon(0) \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\,\psi_{z,\lambda}(y)\,\omega(y)}{|y-x|^\mu}\,\mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{25}$$

By Lemma 5, we obtain

$$\begin{aligned}
& \iint_{\mathbb{R}^N \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\,\psi_{z,\lambda}(y)\,\omega(y)}{|y-x|^\mu}\,\mathrm{d}y\mathrm{d}x \\
= \; & C_1(N,\mu) \int_{\Omega_\varepsilon} V_{z,\lambda}^{2_\mu^*-2}(y)\,\psi_{z,\lambda}(y)\,\omega(y)\,\mathrm{d}y \\
= \; & \frac{C_1(N,\mu)}{\lambda^{\frac{N-2}{2}}} \left(\int_{\Omega_\varepsilon} \left(V_{z,\lambda}^{2_\mu^*-2} \right)^{\frac{2N}{N+2}} \mathrm{d}y \right)^{\frac{N+2}{2N}} \|\omega\|_{\widetilde{H}_0^1(\Omega_{\varepsilon_n})} \\
= \; & O\left(\frac{1}{\lambda^{\frac{N-2}{2}+\sigma}} \right) \|\omega\|_{\widetilde{H}_0^1(\Omega_{\varepsilon_n})}.
\end{aligned} \tag{26}$$

Then from Proposition 1, it follows that

$$\begin{aligned}
&\iint_{(\mathbb{R}^N\setminus D_\varepsilon)\times\Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \omega(y)}{|y-x|^\mu}\,\mathrm{d}y\mathrm{d}x \\
\le\ & C\left(\int_{\mathbb{R}^N\setminus D_\varepsilon} V_{z,\lambda}^{2_\mu^*}\,\mathrm{d}x\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon}\left(V_{z,\lambda}^{2_\mu^*-2}\,\psi_{z,\lambda}\,\omega\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
\le\ & \frac{C}{\lambda^{\frac{N-2}{2}}}\left(\int_{\mathbb{R}^N\setminus D_\varepsilon} V_{z,\lambda}^{2_\mu^*}\,\mathrm{d}x\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon}\left(V_{z,\lambda}^{2_\mu^*-2}\right)^{\frac{2N}{N-\mu+2}}\mathrm{d}y\right)^{\frac{N-\mu+2}{2N}} \|\omega\|_{\widetilde{H}_0^1(\Omega_{\varepsilon_n})} \\
=\ & O\left(\frac{1}{\lambda^{\frac{3N-\mu-2}{2}}}\right)\|\omega\|_{\widetilde{H}_0^1(\Omega_{\varepsilon_n})},
\end{aligned} \tag{27}$$

where the constant $C = C(N,\mu,d)$ depends on N, μ and d. Similarly, combined with $|B_\varepsilon(0)| = O(\varepsilon^N)$, we have

$$\begin{aligned}
&\iint_{B_\varepsilon(0)\times\Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \omega(y)}{|y-x|^\mu}\,\mathrm{d}y\mathrm{d}x \\
\le\ & \frac{C}{\lambda^{\frac{N-2}{2}}}\left(\int_{B_\varepsilon(0)} V_{z,\lambda}^{2_\mu^*}\,\mathrm{d}x\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon}\left(V_{z,\lambda}^{2_\mu^*-2}\right)^{\frac{2N}{N-\mu+2}}\mathrm{d}y\right)^{\frac{N-\mu+2}{2N}} \|\omega\|_{\widetilde{H}_0^1(\Omega_{\varepsilon_n})} \\
\le\ & \frac{C\,(\lambda\varepsilon)^{\frac{2N-\mu}{2}}}{\lambda^{\frac{N-\mu+2}{2}}}\|\omega\|_{\widetilde{H}_0^1(\Omega_{\varepsilon_n})}.
\end{aligned} \tag{28}$$

Meanwhile, applying Proposition 1, the Sobolev inequality and Hölder's inequality, we deduce

$$\begin{aligned}
&\iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*-1}(x)\, \psi_{z,\lambda}(x)\, V_{z,\lambda}^{2_\mu^*-1}(y)\, \omega(y)}{|y-x|^\mu}\,\mathrm{d}y\mathrm{d}x \\
\le\ & \frac{C}{\lambda^{\frac{N-2}{2}}}\left(\int_{\Omega_\varepsilon}\left(V_{z,\lambda}^{2_\mu^*-1}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}x\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon}\left(V_{z,\lambda}^{2_\mu^*-1}\,\omega\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
\le\ & \frac{C}{\lambda^{\frac{N-2}{2}}}\left(\int_{\Omega_\varepsilon}\left(V_{z,\lambda}^{2_\mu^*-1}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}x\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{\frac{2N(2_\mu^*-1)}{N-\mu+2}}\,\mathrm{d}y\right)^{\frac{N-\mu+2}{2N}} \|\omega\|_{\widetilde{H}_0^1(\Omega_{\varepsilon_n})} \\
\le\ & \frac{C}{\lambda^{N-3}}\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{2^*}\,\mathrm{d}y\right)^{\frac{N-\mu+2}{2N}} \|\omega\|_{\widetilde{H}_0^1(\Omega_{\varepsilon_n})} \\
=\ & O\left(\frac{1}{\lambda^{N-3}}\right)\|\omega\|_{\widetilde{H}_0^1(\Omega_{\varepsilon_n})}.
\end{aligned} \tag{29}$$

By an argument similar to that in (27)–(29), we yield

$$\begin{aligned}
&\iint_{(\mathbb{R}^N\setminus\Omega_\varepsilon)\times\Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-1}(y)\, \omega(y)}{|y-x|^\mu}\,\mathrm{d}y\mathrm{d}x \\
=\ & O\left(\frac{1}{\lambda^N} + (\lambda\varepsilon)^{\frac{2N-\mu}{2}}\right)\|\omega\|_{\widetilde{H}_0^1(\Omega_{\varepsilon_n})}.
\end{aligned} \tag{30}$$

Combining (24)–(30), we obtain

$$\begin{aligned} &\iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-1}(y)\,\omega(y)}{|y-x|^\mu}\,\mathrm{d}y\,\mathrm{d}x \\ &\quad - \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\,\omega(y)}{|y-x|^\mu}\,\mathrm{d}y\,\mathrm{d}x \\ =\;& O\Big(\frac{1}{\lambda^{\frac{N-2}{2}+\sigma}}\Big)\|\omega\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}. \end{aligned} \tag{31}$$

On the other hand, let $\widehat{\omega}_\varepsilon(y) = \omega(\varepsilon(y - z_\varepsilon))$. Then, by the change of variables and the Sobolev trace inequality, one has

$$\begin{aligned} \int_{\partial B_\varepsilon(z)} \omega^2\,\mathrm{d}S &= \varepsilon^{N-1}\int_{\partial B_1(0)} \widehat{\omega}_\varepsilon^2\,\mathrm{d}S \\ &\le C\varepsilon^{N-1}\int_{\mathbb{R}^N\setminus B_1(0)} |\nabla\widehat{\omega}_\varepsilon|^2\,\mathrm{d}y \\ &= \varepsilon^{N-1}\cdot\frac{1}{\varepsilon^{N-2}}\int_{\mathbb{R}^N\setminus B_\varepsilon(z)} |\nabla\omega|^2\,\mathrm{d}y = O(\varepsilon)\|\omega\|^2_{\widetilde{H}_0^1(\Omega_\varepsilon)}. \end{aligned} \tag{32}$$

For any $y \in \partial B_\varepsilon(0)$, we obtain

$$\begin{aligned} \left|\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\overline{\nu}}\right| &\le \left|\frac{\partial V_{z,\lambda}}{\partial\overline{\nu}}\right| + \left|\frac{\partial\psi_{z,\lambda}}{\partial\overline{\nu}}\right| \\ &\le \left|\frac{\partial V_{z,\lambda}}{\partial\overline{\nu}}\right| + C\lambda\psi_{z,\lambda} \le C\frac{\lambda^{\frac{N}{2}}(\lambda|y|+\lambda|z|)}{(1+\lambda^2|y-z|^2)^{\frac{N}{2}}} + \frac{C}{\lambda^{\frac{N-4}{2}}} \\ &\le C\lambda^{\frac{N}{2}}(\lambda\varepsilon+\lambda|z|) + \frac{C}{\lambda^{\frac{N-4}{2}}}. \end{aligned} \tag{33}$$

Using (32) and (33), together with the Cauchy–Schwarz inequality and Lemma 1, we obtain

$$\begin{aligned} &\int_{\partial B_\varepsilon(0)} \frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\overline{\nu}}\omega\,\mathrm{d}S \\ \le\;& \left(\int_{\partial B_\varepsilon(0)}\left|\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\overline{\nu}}\right|^2\mathrm{d}S\right)^{\frac12}\left(\int_{\partial B_\varepsilon(0)}|\omega|^2\,\mathrm{d}S\right)^{\frac12} \\ \le\;& C\varepsilon^{\frac12}\left(\int_{\partial B_\varepsilon(0)}\left|\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\overline{\nu}}\right|^2\mathrm{d}S\right)^{\frac12}\|\omega\|_{\widetilde{H}_0^1(\Omega_\varepsilon)} \\ \le\;& C\varepsilon^{\frac12}\varepsilon^{\frac{N-1}{2}}\left(\lambda^{\frac{N}{2}}(\lambda\varepsilon+\lambda|z|)+\frac{1}{\lambda^{\frac{N-4}{2}}}\right)\|\omega\|_{\widetilde{H}_0^1(\Omega_\varepsilon)} \\ \le\;& C\varepsilon^{\frac{N}{2}}\lambda^{\frac{N}{2}}\left(\lambda\varepsilon+\lambda|z|+\frac{1}{\lambda^{N-2}}\right)\|\omega\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}. \end{aligned} \tag{34}$$

Combining (23), (31) and (34), the desired result follows.

□

We next establish the uniform invertibility of the linearized operator Q_ε on the orthogonal complement $E_{\varepsilon,z,\lambda}$.

Lemma 7. *Let T_ε be as defined in (16), and let Q_ε be the operator defined in (20). For any $(z,\lambda) \in T_\varepsilon$ and any $\omega \in E_{\varepsilon,z,\lambda}$, there exists a constant $M > 0$ independent of ε such that*

$$\|Q_\varepsilon(\omega)\|_{\widetilde{H}_0^1(\Omega_\varepsilon)} \geqslant M\|\omega\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}. \tag{35}$$

Proof. We proceed by contradiction. Assume there exist sequences $\varepsilon_n \to 0$, $(z_n,\lambda_n) \in T_{\varepsilon_n}$ and $\omega_n \in E_{\varepsilon_n,z_n,\lambda_n}$ satisfying

$$\|Q_{\varepsilon_n}(\omega_n)\|_{\widetilde{H}_0^1(\Omega_\varepsilon)} = o(1)\|\omega_n\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}. \tag{36}$$

Without loss of generality, we set $\|\omega_n\|_{\widetilde{H}_0^1(\Omega_\varepsilon)} = 1$.
Define the scaled function

$$\widetilde{\omega}_n(y) = \lambda_n^{-\frac{N-2}{2}}\omega_n\big(\lambda_n^{-1}y + z_n\big).$$

Then

$$\int_{\mathbb{R}^N} |\nabla\widetilde{\omega}_n(y)|^2\,\mathrm{d}y \leqslant C.$$

Passing to a subsequence, and in view of (6), (7) and (15), there exists $\omega \in D^{1,2}(\mathbb{R}^N)$ such that

$$\widetilde{\omega}_n \rightharpoonup \omega \quad \text{in } D^{1,2}(\mathbb{R}^N), \quad n \to \infty.$$

Here and below, the symbol $\rightharpoonup$ always denotes weak convergence.
We shall prove $\omega = 0$. Let

$$\widetilde{V}_n(y) = \lambda_n^{-\frac{N-2}{2}}(P_\varepsilon V_{z_n,\lambda_n})\big(\lambda_n^{-1}y + z_n\big). \tag{37}$$

Then

$$\widetilde{V}_n \rightharpoonup V \quad \text{in } D^{1,2}(\mathbb{R}^N), \quad n \to \infty.$$

Moreover,

$$\begin{aligned}
&\int_{\lambda_n(\Omega_{\varepsilon_n}-z_n)} |\nabla\widetilde{V}_n|^2\,\mathrm{d}y \\
=\ &\int_{\Omega_{\varepsilon_n}} |\nabla(P_\varepsilon V_{z_n,\lambda_n})|^2\,\mathrm{d}y \\
=\ &-\int_{\Omega_{\varepsilon_n}} \Delta V_{z_n,\lambda_n} P_\varepsilon V_{z_n,\lambda_n}\,\mathrm{d}y + \int_{\partial B_{\varepsilon_n}(0)} \frac{\partial(P_\varepsilon V_{z_n,\lambda_n})}{\partial\nu} P_\varepsilon V_{z_n,\lambda_n}\,\mathrm{d}S \\
=\ &\iint_{\mathbb{R}^N\times\Omega_{\varepsilon_n}} \frac{V_{z_n,\lambda_n}^{2_\mu^*}(x)}{|y-x|^\mu} V_{z_n,\lambda_n}^{2_\mu^*-1} P_\varepsilon V_{z_n,\lambda_n}\,\mathrm{d}x\mathrm{d}y + \int_{\partial B_{\varepsilon_n}(0)} \frac{\partial(P_\varepsilon V_{z_n,\lambda_n})}{\partial\nu} P_\varepsilon V_{z_n,\lambda_n}\,\mathrm{d}S,
\end{aligned} \tag{38}$$

where $\lambda_n(\Omega_{\varepsilon_n} - z_n)$ denotes the scaled set $\{\lambda_n(y - z_n) : y \in \Omega_{\varepsilon_n}\}$. By Lemmas 1 and 3, it follows that

$$\begin{aligned}
&\iint_{\mathbb{R}^N\times\Omega_{\varepsilon_n}} \frac{V_{z_n,\lambda_n}^{2_\mu^*}(x)}{|y-x|^\mu} V_{z_n,\lambda_n}^{2_\mu^*-1} P_\varepsilon V_{z_n,\lambda_n}\,\mathrm{d}x\mathrm{d}y \\
=\ &\iint_{\mathbb{R}^N\times\Omega_{\varepsilon_n}} \frac{V_{z_n,\lambda_n}^{2_\mu^*}(x)}{|y-x|^\mu} V_{z_n,\lambda_n}^{2_\mu^*}\,\mathrm{d}x\mathrm{d}y - \iint_{\mathbb{R}^N\times\Omega_{\varepsilon_n}} \frac{V_{z_n,\lambda_n}^{2_\mu^*}(x)}{|y-x|^\mu} V_{z_n,\lambda_n}^{2_\mu^*-1}\psi_{z_n,\lambda_n}\,\mathrm{d}x\mathrm{d}y \\
=\ &\int_{\mathbb{R}^N} |\nabla V|^2\,\mathrm{d}y + o(1).
\end{aligned} \tag{39}$$

Similarly, one has

$$\begin{aligned}
&\int_{\partial B_{\varepsilon_n}(0)} \frac{\partial (P_\varepsilon V_{z_n,\lambda_n})}{\partial \nu} P_\varepsilon V_{z_n,\lambda_n}\, \mathrm{d}S \\
= &\int_{\partial B_{\varepsilon_n}(0)} \frac{\partial V_{z_n,\lambda_n}}{\partial \nu} V_{z_n,\lambda_n}\, \mathrm{d}S - \int_{\partial B_{\varepsilon_n}(0)} \frac{\partial V_{z_n,\lambda_n}}{\partial \nu} \psi_{z_n,\lambda_n}\, \mathrm{d}S \\
&- \int_{\partial B_{\varepsilon_n}(0)} \frac{\partial \psi_{z_n,\lambda_n}}{\partial \nu} V_{z_n,\lambda_n}\, \mathrm{d}S + \int_{\partial B_{\varepsilon_n}(0)} \frac{\partial \psi_{z_n,\lambda_n}}{\partial \nu} \psi_{z_n,\lambda_n}\, \mathrm{d}S.
\end{aligned}$$

By Lemma 1 and the explicit expression of $V_{z,\lambda}$,

$$\left|\frac{\partial V_{z,\lambda}}{\partial \nu}\right| \le C\lambda^{\frac{N}{2}}, \qquad V_{z,\lambda} \le C\lambda^{\frac{N-2}{2}} \quad \text{on } \partial B_{\varepsilon_n}(0),$$

and, by Lemma 3 together with the gradient estimate for the harmonic function $\psi_{z,\lambda}$ on $\partial B_{\varepsilon_n}(0)$,

$$|\psi_{z,\lambda}| \le C\lambda^{-\frac{N-2}{2}}, \qquad \left|\frac{\partial \psi_{z,\lambda}}{\partial \nu}\right| \le C\lambda^{-\frac{N-2}{2}}.$$

Since $|\partial B_{\varepsilon_n}(0)| = O(\varepsilon_n^{N-1})$ and λ is comparable to $\varepsilon^{-\frac{N}{2(N-1)}}$, we obtain

$$\begin{aligned}
&\left|\int_{\partial B_{\varepsilon_n}(0)} \frac{\partial (P_\varepsilon V_{z_n,\lambda_n})}{\partial \nu} P_\varepsilon V_{z_n,\lambda_n}\, \mathrm{d}S\right| \\
\le\ & C\varepsilon_n^{N-1}\lambda_n^{N-1} + C\varepsilon_n^{N-1}\lambda_n + C\varepsilon_n^{N-1} + C\varepsilon_n^{N-1}\lambda_n^{-(N-2)} \qquad (40)\\
=\ & o(1).
\end{aligned}$$

As a result,

$$\int_{\lambda_n(\Omega_{\varepsilon_n} - z_n)} |\nabla \widetilde{V}_n|^2\, \mathrm{d}y = \int_{\mathbb{R}^N} |\nabla V|^2\, \mathrm{d}y + o(1).$$

Thus $\widetilde{V}_n$ converges strongly to V in $D^{1,2}(\mathbb{R}^N)$.
Since $\widetilde{\omega}_n \rightharpoonup \omega$, $\widetilde{V}_n \to V$ and $\omega_n \in E_{\varepsilon_n,z_n,\lambda_n}$, the limit function ω belongs to

$$E := \left\{ v \in D^{1,2}(\mathbb{R}^N) \,:\, \left\langle v, \frac{\partial V}{\partial \lambda} \right\rangle = \left\langle v, \frac{\partial V}{\partial z_i} \right\rangle = 0,\ i = 1, 2, \ldots, N \right\}.$$

Under the contradiction assumption, for all $v \in E_{\varepsilon_n,z_n,\lambda_n}$ we obtain

$$\begin{aligned}
&\langle Q_{\varepsilon_n}\omega_n, v\rangle \\
= &\int_{\Omega_{\varepsilon_n}} \nabla\omega_n \cdot \nabla v\, \mathrm{d}y \\
&- 2^*_\mu \iint_{\Omega_{\varepsilon_n}\times\Omega_{\varepsilon_n}} \frac{(P_\varepsilon V_{z_n,\lambda_n})^{2^*_\mu - 1}(x)\, \omega_n(x)\, (P_\varepsilon V_{z_n,\lambda_n})^{2^*_\mu - 1}(y)\, v(y)}{|y-x|^\mu}\, \mathrm{d}x\, \mathrm{d}y \qquad (41)\\
&- (2^*_\mu - 1) \iint_{\Omega_{\varepsilon_n}\times\Omega_{\varepsilon_n}} \frac{(P_\varepsilon V_{z_n,\lambda_n})^{2^*_\mu}(x)\, (P_\varepsilon V_{z_n,\lambda_n})^{2^*_\mu - 2}(y)\, \omega_n(y)\, v(y)}{|y-x|^\mu}\, \mathrm{d}x\, \mathrm{d}y \\
= &\ o(1)\|v\|_{\widetilde{H}^1_0(\Omega_\varepsilon)}.
\end{aligned}$$

Define the scaled space

$$\begin{aligned}\widetilde{E}_{\varepsilon_n,z_n,\lambda_n} := \Big\{ & v \in \widetilde{H}_0^1(\lambda_n(\Omega_{\varepsilon_n} - z_n)) : \\ & \int_{\lambda_n(\Omega_{\varepsilon_n}-z_n)} \nabla v \cdot \nabla \frac{\partial \widetilde{V}_n}{\partial \lambda}\,\mathrm{d}y = \int_{\lambda_n(\Omega_{\varepsilon_n}-z_n)} \nabla v \cdot \nabla \frac{\partial \widetilde{V}_n}{\partial z_i}\,\mathrm{d}y = 0,\ i = 1,\dots,N \Big\},\end{aligned}$$

where $\widetilde{V}_n$ is defined in (37). Then for all $v \in \widetilde{E}_{\varepsilon_n,z_n,\lambda_n}$, the function $\widetilde{\omega}_n$ satisfies

$$\begin{aligned}& \int_{\lambda_n(\Omega_{\varepsilon_n}-z_n)} \nabla \widetilde{\omega}_n \cdot \nabla v\,\mathrm{d}y \\ & \quad - 2_\mu^* \iint_{\lambda_n(\Omega_{\varepsilon_n}-z_n)\times\lambda_n(\Omega_{\varepsilon_n}-z_n)} \frac{\widetilde{V}_n^{2_\mu^*-1}(x)\,\widetilde{\omega}_n(x)\,\widetilde{V}_n^{2_\mu^*-1}(y)\,v(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ & \quad - (2_\mu^*-1) \iint_{\lambda_n(\Omega_{\varepsilon_n}-z_n)\times\lambda_n(\Omega_{\varepsilon_n}-z_n)} \frac{\widetilde{V}_n^{2_\mu^*}(x)\,\widetilde{V}_n^{2_\mu^*-2}(y)\,\widetilde{\omega}_n(y)\,v(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ = \ & o(1)\|v\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}.\end{aligned} \tag{42}$$

For any $v \in E$, there exist constants $B_n, C_{n,i}$ $(i = 1, 2, \dots, N)$ such that

$$v_n = v - B_n \frac{\partial \widetilde{V}_n}{\partial \lambda} - \sum_{i=1}^N C_{n,i} \frac{\partial \widetilde{V}_n}{\partial z_i} \in \widetilde{E}_{\varepsilon_n,z_n,\lambda_n}.$$

Moreover,

$$\left\langle v, \frac{\partial \widetilde{V}_n}{\partial \lambda} \right\rangle = \left\langle v, \frac{\partial V}{\partial \lambda} \right\rangle + o(1), \quad \left\langle v, \frac{\partial \widetilde{V}_n}{\partial z_i} \right\rangle = \left\langle v, \frac{\partial V}{\partial z_i} \right\rangle + o(1),\ i = 1,\dots,N, \tag{43}$$

where the inner products are defined by (8). This yields $B_n = o(1)$, $C_{n,i} = o(1)$ as $n \to \infty$.

Combining (42) and (43), and letting $n \to \infty$, we obtain

$$\begin{aligned}& \int_{\mathbb{R}^N} \nabla \omega \cdot \nabla v\,\mathrm{d}y - 2_\mu^* \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{V^{2_\mu^*-1}(x)\omega(x)V^{2_\mu^*-1}(y)v(y)}{|y-x|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ & \qquad - (2_\mu^*-1) \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{V^{2_\mu^*}(x)V^{2_\mu^*-2}(y)\omega(y)v(y)}{|y-x|^\mu}\,\mathrm{d}x\,\mathrm{d}y = 0.\end{aligned} \tag{44}$$

This identity holds for every $v \in E$. Since $V_{z,\lambda}$ solves (3) for all z, λ, by differentiating the equation with respect to λ and z_i, and by the symmetry of (44), the identity also holds for $v = \dfrac{\partial V}{\partial \lambda}$ and $v = \dfrac{\partial V}{\partial z_i}$. Thus $\omega \in D^{1,2}(\mathbb{R}^N)$ solves the linearized equation

$$-\Delta\omega - 2_\mu^* \left(\int_{\mathbb{R}^N} \frac{V^{2_\mu^*-1}(x)\omega(x)}{|y-x|^\mu}\,\mathrm{d}x \right) V^{2_\mu^*-1} - (2_\mu^*-1) \left(\int_{\mathbb{R}^N} \frac{V^{2_\mu^*}(x)}{|y-x|^\mu}\,\mathrm{d}x \right) V^{2_\mu^*-2}\omega = 0.$$

By the nondegeneracy of positive radial solutions to the whole-space critical Hartree equation (see [5,6]), we deduce

$$\omega = \alpha_0 \frac{\partial V}{\partial \lambda} + \sum_{i=1}^N \alpha_i \frac{\partial V}{\partial z_i},$$

for some constants α_0, α_i $(i = 1, \dots, N)$. The orthogonality condition $\omega \in E$ then implies $\omega = 0$.

We compute

$$\langle Q_{\varepsilon_n}\omega_n,\omega_n\rangle = \int_{\lambda_n(\Omega_{\varepsilon_n}-z_n)} |\nabla\widetilde{\omega}_n|^2\,\mathrm{d}y - 2^*_\mu\,\Psi_{1,n} - (2^*_\mu-1)\,\Psi_{2,n},$$

where

$$\begin{aligned}
\Psi_{1,n} &= \iint_{\lambda_n(\Omega_{\varepsilon_n}-z_n)\times\lambda_n(\Omega_{\varepsilon_n}-z_n)} \frac{\widetilde{V}_n^{2^*_\mu-1}(x)\,\widetilde{\omega}_n(x)\,\widetilde{V}_n^{2^*_\mu-1}(y)\,\widetilde{\omega}_n(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y,\\
\Psi_{2,n} &= \iint_{\lambda_n(\Omega_{\varepsilon_n}-z_n)\times\lambda_n(\Omega_{\varepsilon_n}-z_n)} \frac{\widetilde{V}_n^{2^*_\mu}(x)\,\widetilde{V}_n^{2^*_\mu-2}(y)\,\widetilde{\omega}_n^2(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Fix a constant $\rho>0$ (to be chosen sufficiently large). For the term $\Psi_{1,n}$, we derive the following estimate

$$\begin{aligned}
\Psi_{1,n} &= \iint_{\lambda_n(\Omega_{\varepsilon_n}-z_n)\times\lambda_n(\Omega_{\varepsilon_n}-z_n)} \frac{\widetilde{V}_n^{2^*_\mu-1}(x)\widetilde{\omega}_n(x)\widetilde{V}_n^{2^*_\mu-1}(y)\widetilde{\omega}_n(y)}{|x-y|^\mu}\,\mathrm{d}x\mathrm{d}y\\
&\le C\left\|\widetilde{V}_n^{2^*_\mu-1}\widetilde{\omega}_n\right\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}\\
&\le C\left\|\widetilde{V}_n^{2^*_\mu-1}\widetilde{\omega}_n\right\|_{L^{\frac{2N}{2N-\mu}}(B_\rho)} + C\left\|\widetilde{V}_n^{2^*_\mu-1}\widetilde{\omega}_n\right\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N\setminus B_\rho)}\\
&\le C\|\widetilde{\omega}_n\|_{L^{\frac{2N}{2N-\mu}}(B_\rho)} + \left\|\widetilde{V}_n\right\|^{2^*_\mu-1}_{L^{2^*_\mu}(\mathbb{R}^N\setminus B_\rho)} = o(1).
\end{aligned}$$

Next we treat $\Psi_{2,n}$. For the case $0<\mu<4$,

$$\begin{aligned}
\Psi_{2,n} &\le C\left\|\widetilde{V}_n^{2^*_\mu}\right\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}\left\|\widetilde{V}_n^{2^*_\mu-2}\widetilde{\omega}_n^2\right\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}\\
&\le C\left\|\widetilde{V}_n\right\|^{2^*_\mu}_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}\left\|\widetilde{V}_n^{2^*_\mu-2}\widetilde{\omega}_n^2\right\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}\\
&\le C\left\|\widetilde{V}_n^{2^*_\mu-2}\widetilde{\omega}_n^2\right\|_{L^{\frac{2N}{2N-\mu}}(B_\rho)} + C\left\|\widetilde{V}_n^{2^*_\mu-2}\widetilde{\omega}_n^2\right\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N\setminus B_\rho)}\\
&\le C\|\widetilde{\omega}_n\|^2_{L^{\frac{4N}{2N-\mu}}(B_\rho)} + C\left\|\widetilde{V}_n\right\|^{2^*_\mu-2}_{L^{2^*_\mu}(\mathbb{R}^N\setminus B_\rho)} = o(1).
\end{aligned}$$

When $\mu=4$, we define

$$W_n(y) := \int_{\mathbb{R}^N}\frac{\widetilde{V}_n^2(x)}{|x-y|^4}\,\mathrm{d}x.$$

By Lemma 5, $W_n(y)\to C\widetilde{V}^{2^*-2}(y)$. The estimate for $\Psi_{2,n}$ then reads

$$\begin{aligned}
\Psi_{2,n} &= \int_{B_\rho} W_n(y)\widetilde{\omega}_n^2(y)\,\mathrm{d}y + \int_{\mathbb{R}^N\setminus B_\rho} W_n(y)\widetilde{\omega}_n^2(y)\,\mathrm{d}y\\
&\le C\|\widetilde{\omega}_n\|^2_{L^2(B_\rho)} + C\|\widetilde{\omega}_n\|_{L^{\frac{2N}{2N-4}}(\mathbb{R}^N\setminus B_\rho)} = o(1).
\end{aligned}$$

It follows that

$$\langle Q_{\varepsilon_n}\omega_n,\omega_n\rangle \ge 1-o(1).$$

This contradicts (36). The proof is completed. □

Using Lemmas 6 and 7, we obtain the following proposition.

Proposition 2. *Let T_ε be as defined in (16) and J_ε as in (17). There exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$, there exists a C^1-mapping $\omega_{\varepsilon,z,\lambda} : T_\varepsilon \to \widetilde{H}_0^1(\Omega_\varepsilon)$ satisfying $\omega_{\varepsilon,z,\lambda} \in E_{\varepsilon,z,\lambda}$. Moreover, there exist constants B and C_i $(i = 1, 2, \ldots, N)$ such that*

$$\frac{\partial J_\varepsilon}{\partial \omega} = B\frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda} + \sum_{i=1}^{N} C_i \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial z_i}, \tag{45}$$

where $\dfrac{\partial J_\varepsilon}{\partial \omega}$ denotes the Fréchet derivative of J_ε with respect to ω at fixed (z, λ).

Furthermore, $\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}$ satisfies the following estimate

$$\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)} = \varepsilon^{\frac{N}{2}} \lambda^{\frac{N}{2}} O\left(\lambda\varepsilon + \lambda|z| + \frac{1}{\lambda^{N-2}}\right) + O\left(\frac{1}{\lambda^{\frac{N-2}{2}+\sigma}}\right). \tag{46}$$

Proof. From expansion (18), finding a critical point of $J_\varepsilon(z, \lambda, \omega)$ in $E_{\varepsilon,z,\lambda}$ is equivalent to solving

$$l_\varepsilon + Q_\varepsilon \omega + R'_\varepsilon(\omega) = 0. \tag{47}$$

For each $(z, \lambda) \in T_\varepsilon$, consider the map

$$F_{\varepsilon,z,\lambda}(\omega) := l_\varepsilon + Q_\varepsilon \omega + R'_\varepsilon(\omega), \qquad \omega \in E_{\varepsilon,z,\lambda}.$$

By the regularity established above, $F_{\varepsilon,z,\lambda}$ is C^1 in ω (indeed C^1 jointly in (z, λ, ω)), with

$$F_{\varepsilon,z,\lambda}(0) = l_\varepsilon, \qquad \partial_\omega F_{\varepsilon,z,\lambda}(0) = Q_\varepsilon$$

By Lemma 7, $Q_\varepsilon \colon E_{\varepsilon,z,\lambda} \to E_{\varepsilon,z,\lambda}$ is uniformly invertible with $\|Q_\varepsilon^{-1}\| \le C$ uniformly for $(z, \lambda) \in T_\varepsilon$. Moreover, since the tangent vectors $\partial_\lambda P_\varepsilon V_{z,\lambda}$ and $\partial_{z_i} P_\varepsilon V_{z,\lambda}$ are linearly independent (by the nondegeneracy of positive radial solutions to the whole-space critical Hartree equation [5,6]) and smooth in (z, λ), the orthogonal projections onto $E_{\varepsilon,z,\lambda}$ vary smoothly with (z, λ). The uniform implicit function theorem therefore yields a C^1 map

$$\omega_{\varepsilon,z,\lambda} \colon T_\varepsilon \to \widetilde{H}_0^1(\Omega_\varepsilon)$$

with $\omega_{\varepsilon,z,\lambda} \in E_{\varepsilon,z,\lambda}$ and

$$F_{\varepsilon,z,\lambda}(\omega_{\varepsilon,z,\lambda}) = 0,$$

for all $(z, \lambda) \in T_\varepsilon$. Moreover, we claim that $\dfrac{\partial J_\varepsilon}{\partial \omega}$ can be expression as a linear combination of the above basis vectors. Indeed, (45) is equivalent to

$$\left\langle \frac{\partial J_\varepsilon}{\partial \omega}, \eta \right\rangle = 0, \quad \text{for all } \eta \in E_{\varepsilon,z,\lambda}.$$

By the definition of $E_{\varepsilon,z,\lambda}$, this implies that $\dfrac{\partial J_\varepsilon}{\partial \omega}$ is orthogonal to all functions η satisfying

$$\left\langle \eta, \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda} \right\rangle = 0 \text{ and } \left\langle \eta, \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial z_i} \right\rangle = 0,$$

for $i = 1, \ldots, N$. Therefore, $\dfrac{\partial J_\varepsilon}{\partial \omega}$ lies in the subspace spanned by

$$\left\{ \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}, \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial z_1}, \ldots, \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial z_N} \right\},$$

i.e., there exist unique constants B and C_i $(i = 1, \dots, N)$ such that

$$\frac{\partial J_\varepsilon}{\partial \omega} = B\frac{\partial (P_D V_{z,\lambda})}{\partial \lambda} + \sum_{i=1}^{N} C_i \frac{\partial (P_D V_{z,\lambda})}{\partial z_i}.$$

In particular, $\omega_{\varepsilon,z,\lambda}$ is C^1 in (z, λ), which justifies differentiating the reduced functional in Section 4.

Finally, from (47), Lemma 7 and (21) we obtain

$$\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)} \le C\|l_\varepsilon\|_{\widetilde{H}_0^1(\Omega_\varepsilon)},$$

and Lemma 6 gives (46). □

According to Proposition 2, solving the original problem is equivalent to finding a suitable point $(z, \lambda) \in T_\varepsilon$ such that the coefficients B and C_i $(i = 1, \dots, N)$ vanish. This finite-dimensional reduction concludes this section, and we will analyze these equations in the next section.

4. Proof of Theorem 1

In this section, we compute the partial derivatives of J_ε with respect to the finite-dimensional parameters λ and z_i; together with the variational derivative $\partial J_\varepsilon/\partial\omega$ already analyzed in Proposition 2, these will yield the reduced finite-dimensional system.

To finish the proof, it suffices to select a suitable pair $(z, \lambda) \in T_\varepsilon$, where T_ε is defined in (16), such that the coefficients B, C_i $(i = 1, 2, \dots, N)$ from Proposition 2 vanish. The conditions $B = 0$ and $C_i = 0$ $(i = 1, 2, \dots, N)$ are therefore equivalent to

$$\begin{cases} \dfrac{\partial J_\varepsilon}{\partial \lambda} = 0, \\ \dfrac{\partial J_\varepsilon}{\partial z_i} = 0, \quad i = 1, \dots, N. \end{cases} \tag{48}$$

where

$$\begin{aligned} \frac{\partial J_\varepsilon}{\partial \lambda} = & \int_{\Omega_\varepsilon} \nabla(P_\varepsilon V_{z,\lambda} + \omega) \cdot \nabla\left(\frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}\right) \mathrm{d}y \\ & - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda}(x) + \omega(x))_+^{2_\mu^*} (P_\varepsilon V_{z,\lambda}(y) + \omega(y))_+^{2_\mu^* - 1} \frac{\partial (P_\varepsilon V_{z,\lambda})(y)}{\partial \lambda}}{|y - x|^\mu} \mathrm{d}x\mathrm{d}y, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial J_\varepsilon}{\partial z_i} = & \int_{\Omega_\varepsilon} \nabla(P_\varepsilon V_{z,\lambda} + \omega) \cdot \nabla\left(\frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial z_i}\right) \mathrm{d}y \\ & - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda}(x) + \omega(x))_+^{2_\mu^*} (P_\varepsilon V_{z,\lambda}(y) + \omega(y))_+^{2_\mu^* - 1} \frac{\partial (P_\varepsilon V_{z,\lambda})(y)}{\partial z_i}}{|y - x|^\mu} \mathrm{d}x\mathrm{d}y, \end{aligned}$$

for $i = 1, \dots, N$.

To solve (48), we first derive estimates for the left-hand sides of the two equations in the system. To this end, we provide the following auxiliary results.

The following lemma is an immediate consequence of the definition of $E_{\varepsilon,z,\lambda}$.

Lemma 8. *For any $\omega_{\varepsilon,z,\lambda} \in E_{\varepsilon,z,\lambda}$, where $E_{\varepsilon,z,\lambda}$ is defined in (15), it holds that*

$$\int_\Omega \nabla\omega_{\varepsilon,z,\lambda} \cdot \nabla\left(\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial \lambda}\right) \mathrm{d}y = 0, \quad \int_\Omega \nabla\omega_{\varepsilon,z,\lambda} \cdot \nabla\left(\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial z_i}\right) \mathrm{d}y = 0. \tag{49}$$

Lemma 9. *For any $(z,\lambda) \in T_\varepsilon$, where T_ε is defined in (16), we obtain*

$$\begin{aligned} &\iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \,\mathrm{d}x\mathrm{d}y \\ = \;& \frac{C_2(N,\mu)\tau_D(z)}{\lambda^{N-1}} + O\left(\frac{1}{\lambda^{N-1+\sigma}}\right), \end{aligned}$$

where

$$C_2(N,\mu) := C_1(N,\mu) \int_{\mathbb{R}^N} V_{0,1}^{2_\mu^*-2} \frac{\partial V_{0,\lambda}}{\partial \lambda}\bigg|_{\lambda=1} \mathrm{d}y < 0,$$

since the integral on the right-hand side is negative.

Proof. By direct computation, we decompose the integral as

$$\begin{aligned} &\iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \,\mathrm{d}y\mathrm{d}x \\ = \;& \iint_{\mathbb{R}^N \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \,\mathrm{d}y\mathrm{d}x \\ &- \iint_{(\mathbb{R}^N \setminus D_\varepsilon) \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \,\mathrm{d}y\mathrm{d}x \\ &- \iint_{B_\varepsilon(0) \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \,\mathrm{d}y\mathrm{d}x \end{aligned} \tag{50}$$

Then, by Lemmas 3–5, we deduce

$$\begin{aligned} &\iint_{\mathbb{R}^N \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \,\mathrm{d}y\mathrm{d}x \\ = \;& C_1(N,\mu) \int_{\Omega_\varepsilon} V_{z,\lambda}^{2^*-2}(y)\, \psi_{z,\lambda}(y)\, \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)\,\mathrm{d}y \\ = \;& C_1(N,\mu) \int_{\Omega_\varepsilon} V_{z,\lambda}^{2^*-2}(y)\, \psi_{z,\lambda}(y)\, \frac{\partial V_{z,\lambda}}{\partial \lambda}(y)\,\mathrm{d}y \\ &- C_1(N,\mu) \int_{\Omega_\varepsilon} V_{z,\lambda}^{2^*-2}(y)\, \psi_{z,\lambda}(y)\, \frac{\partial \psi_{z,\lambda}}{\partial \lambda}(y)\,\mathrm{d}y \\ = \;& \frac{C_1(N,\mu)\tau_D(z)}{\lambda^{N-1}} \int_{\mathbb{R}^N} V_{0,1}^{2^*-2} \frac{\partial V_{0,1}}{\partial \lambda}\bigg|_{\lambda=1} \mathrm{d}y + O\left(\frac{1}{\lambda^{N-1+\sigma}}\right). \end{aligned} \tag{51}$$

On the other hand, by a similar argument as in (28), we obtain

$$\begin{aligned} &\iint_{(\mathbb{R}^N \setminus D_\varepsilon) \times \Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \,\mathrm{d}y\mathrm{d}x \\ \le \;& C\left(\int_{\mathbb{R}^N \setminus D_\varepsilon} V_{z,\lambda}^{2^*} \,\mathrm{d}x\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon} \left(V_{z,\lambda}^{2_\mu^*-2}\, \psi_{z,\lambda}\, \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)\right)^{\frac{2N}{2N-\mu}} \mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\ \le \;& \frac{C}{\lambda^{\frac{N-2}{2}}} \left(\int_{\mathbb{R}^N \setminus D_\varepsilon} V_{z,\lambda}^{2^*} \,\mathrm{d}x\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon} \left(V_{z,\lambda}^{2_\mu^*-2}\, \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}\right)^{\frac{2N}{2N-\mu}}\right)^{\frac{2N-\mu}{2N}} \\ \le \;& O\left(\frac{1}{\lambda^{\frac{4N-\mu-3}{2}}}\right) \end{aligned} \tag{52}$$

Meanwhile, following a similar argument to that of (29), this yields

$$\begin{aligned} & \iint_{B_\varepsilon(0)\times\Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\,\mathrm{d}y\mathrm{d}x \\ \le\ & \frac{C}{\lambda^{\frac{N-2}{2}}}\left(\int_{B_\varepsilon(0)} V_{z,\lambda}^{2_\mu^*}\,\mathrm{d}x\right)^{\frac{2N-\mu}{2N}}\left(\int_{\Omega_\varepsilon}\left(V_{z,\lambda}^{2_\mu^*-2}\,\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}\right)^{\frac{2N}{2N-\mu}}\right)^{\frac{2N-\mu}{2N}} \\ \le\ & \frac{C\,(\lambda\varepsilon)^{\frac{2N-\mu}{2}}}{\lambda^{N-2}}. \end{aligned} \tag{53}$$

Combining (50)–(53), we complete the proof. □

Proposition 3. *For any $(z,\lambda)\in T_\varepsilon$, where T_ε is defined in (16), it follows that*

$$\begin{aligned} & \int_{\Omega_\varepsilon}\nabla(P_\varepsilon V_{z,\lambda})\cdot\nabla\left(\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}\right)\mathrm{d}y \\ & -\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y \\ =\ & \Lambda_0\varepsilon^N\lambda^{N-1}-\frac{\Lambda_1\tau_D(z)}{\lambda^{N-1}}+\varepsilon^N\lambda^{N-1}O((\lambda|z|+\lambda\varepsilon)^2)+O\left(\frac{1}{\lambda^{N-1+\sigma}}\right), \end{aligned}$$

where Λ_0 and Λ_1 are positive constants depending on N and μ.

Proof. Note that the outward unit normal ν to $\partial\Omega_\varepsilon$ on $\partial B_\varepsilon(0)$ is given by $\nu=-\dfrac{y}{|y|}$.

By Lemmas 3 and 9, one has

$$\begin{aligned} & \int_{\Omega_\varepsilon}\nabla(P_\varepsilon V_{z,\lambda})\cdot\nabla\left(\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}\right)\mathrm{d}y \\ & -\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y \\ =\ & -\int_{\partial B_\varepsilon(0)}\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\overline{\nu}}+\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{V_{z,\lambda}^{2_\mu^*}(x)\,V_{z,\lambda}^{2_\mu^*-1}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y \\ & -\Bigg(\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{V_{z,\lambda}^{2_\mu^*}(x)\,V_{z,\lambda}^{2_\mu^*-1}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y \\ & -(2_\mu^*-1)\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{V_{z,\lambda}^{2_\mu^*}(x)\,V_{z,\lambda}^{2_\mu^*-2}(y)\,\psi_{z,\lambda}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y+\mathcal{R}\Bigg), \end{aligned} \tag{54}$$

where

$$\begin{aligned} \mathcal{R} =\ & O\left(\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{V_{z,\lambda}^{2_\mu^*}(x)\,V_{z,\lambda}^{2_\mu^*-3}(y)\,\psi_{z,\lambda}^2(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y\right) \\ & +O\left(\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{V_{z,\lambda}^{2_\mu^*-1}(x)\,\psi_{z,\lambda}(x)\,V_{z,\lambda}^{2_\mu^*-1}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y\right) \\ & +O\left(\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{V_{z,\lambda}^{2_\mu^*-1}(x)\,\psi_{z,\lambda}(x)\,V_{z,\lambda}^{2_\mu^*-2}(y)\,\psi_{z,\lambda}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y\right) \\ & +O\left(\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{V_{z,\lambda}^{2_\mu^*-1}(x)\,\psi_{z,\lambda}(x)\,V_{z,\lambda}^{2_\mu^*-3}(y)\,\psi_{z,\lambda}^2(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y\right). \end{aligned}$$

By the same arguments as in (25)–(29) and Lemma 9, this yields

$$\mathcal{R} = O\left(\frac{1}{\lambda^{N-1+\sigma}}\right).$$

Combining the above estimates, we obtain

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla(P_\varepsilon V_{z,\lambda}) \cdot \nabla\left(\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}\right) \mathrm{d}y \\
&\quad - \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\
= &\quad - \int_{\partial B_\varepsilon(0)} \frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda} \frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\overline{\nu}} \mathrm{d}S \\
&\quad + (2_\mu^* - 1) \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\
&\quad + O\left(\frac{1}{\lambda^{N-1+\sigma}}\right).
\end{aligned} \tag{55}$$

Here $\overline{\nu}$ denotes the outward unit normal of $\partial B_\varepsilon(0)$.
On the other hand, direct computation gives

$$\int_{\partial B_\varepsilon(0)} \mathrm{d}S = O(\varepsilon^{N-1}).$$

Combining this estimate with Lemma 1, Lemma 3 and noting that λ is of the same order as $\varepsilon^{-\frac{N}{2(N-1)}}$, we deduce that

$$\begin{aligned}
&\int_{\partial B_\varepsilon(0)} \frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda} \frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\overline{\nu}} \,\mathrm{d}S \\
= &\int_{\partial B_\varepsilon(0)} \frac{\partial V_{z,\lambda}}{\partial\lambda} \frac{\partial V_{z,\lambda}}{\partial\overline{\nu}} \,\mathrm{d}S - \int_{\partial B_\varepsilon(0)} \frac{\partial\psi_{z,\lambda}}{\partial\lambda} \frac{\partial V_{z,\lambda}}{\partial\overline{\nu}} \,\mathrm{d}S \\
&- \int_{\partial B_\varepsilon(0)} \frac{\partial V_{z,\lambda}}{\partial\lambda} \frac{\partial\psi_{z,\lambda}}{\partial\overline{\nu}} \,\mathrm{d}S + \int_{\partial B_\varepsilon(0)} \frac{\partial\psi_{z,\lambda}}{\partial\lambda} \frac{\partial\psi_{z,\lambda}}{\partial\overline{\nu}} \,\mathrm{d}S \\
= &\int_{\partial B_\varepsilon(0)} \frac{\partial V_{z,\lambda}}{\partial\lambda} \frac{\partial V_{z,\lambda}}{\partial\overline{\nu}} \,\mathrm{d}S + o\left(\frac{1}{\lambda^{N-1}}\right).
\end{aligned} \tag{56}$$

We evaluate the leading boundary integral using the explicit expression of $V_{z,\lambda}$ (with Θ as in (5)) and the change of variables $y \mapsto \lambda y$. Scaling to the unit sphere and expanding the integrand give

$$\begin{aligned}
&\int_{\partial B_\varepsilon(0)} \frac{\partial V_{z,\lambda}}{\partial\lambda} \frac{\partial V_{z,\lambda}}{\partial\overline{\nu}} \,\mathrm{d}S \\
= &-\frac{\Theta^2(N-2)^2}{2} \frac{1}{\lambda} \int_{\partial B_{\lambda\varepsilon}(0)} \frac{\langle y-\lambda z, \overline{\nu}\rangle}{(1+|y-\lambda z|^2)^{\frac{N}{2}}} \frac{1-|y-\lambda z|^2}{(1+|y-\lambda z|^2)^{\frac{N}{2}}} \,\mathrm{d}S \\
= &-\frac{\Theta^2(N-2)^2}{2} \frac{(\lambda\varepsilon)^{N-1}}{\lambda} \int_{\partial B_1(0)} \frac{\langle \lambda\varepsilon y-\lambda z, \overline{\nu}\rangle(1-|\lambda\varepsilon y-\lambda z|^2)}{(1+|\lambda\varepsilon y-\lambda z|^2)^N} \,\mathrm{d}S \\
= &-\frac{\Theta^2(N-2)^2}{2} \frac{(\lambda\varepsilon)^{N-1}}{\lambda} \int_{\partial B_1(0)} \frac{\langle \lambda\varepsilon y-\lambda z, \overline{\nu}\rangle(1-|\lambda z|^2)}{(1+|\lambda z|^2)^N} \,\mathrm{d}S \\
&+ \frac{(\lambda\varepsilon)^N}{\lambda} O((\lambda|z|+\lambda\varepsilon)^2).
\end{aligned} \tag{57}$$

Since $\int_{\partial B_1(0)} y_i \, \mathrm{d}S = 0$, the λz-terms vanish upon integration, the leading contribution comes from $\lambda\varepsilon y$. Hence

$$\begin{aligned} &\int_{\partial B_\varepsilon(0)} \frac{\partial V_{z,\lambda}}{\partial \lambda} \frac{\partial V_{z,\lambda}}{\partial \overline{\nu}} \, \mathrm{d}S \\ = \; & -\frac{\Theta^2 (N-2)^2}{2} \frac{(\lambda\varepsilon)^N}{\lambda} \int_{\partial B_1(0)} \frac{(1-|\lambda z|^2)}{(1+|\lambda z|^2)^N} \, \mathrm{d}S + \frac{(\lambda\varepsilon)^N}{\lambda} O((\lambda|z| + \lambda\varepsilon)^2). \end{aligned} \tag{58}$$

Combining (55), Lemma 9, (56) and (58), and recalling $\lambda|z| = o(1)$, this completes the proof of Proposition 3. □

Proposition 4. *Let T_ε be as defined in (16). For any $(z,\lambda) \in T_\varepsilon$, and let $\omega_{\varepsilon,z,\lambda}$ be the function obtained in Proposition 2. For sufficiently small $\varepsilon > 0$, one has*

$$\begin{aligned} &\frac{\partial J_\varepsilon}{\partial \lambda} \\ = \; & \int_{\Omega_\varepsilon} \nabla\big(P_\varepsilon V_{z,\lambda} + \omega_{\varepsilon,z,\lambda}\big) \cdot \nabla\left(\frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}\right) \mathrm{d}y \\ & - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\big(P_\varepsilon V_{z,\lambda}(x) + \omega_{\varepsilon,z,\lambda}(x)\big)^{2_\mu^*} \big(P_\varepsilon V_{z,\lambda}(y) + \omega_{\varepsilon,z,\lambda}(y)\big)^{2_\mu^*-1} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\ = \; & \Lambda_0 \varepsilon^N \lambda^{N-1} - \frac{\Lambda_1 \tau_D(z)}{\lambda^{N-1}} + \varepsilon^N \lambda^{N-1} O((\lambda|z| + \lambda\varepsilon)^2) + O\left(\frac{1}{\lambda^{N-1+\sigma}}\right), \end{aligned}$$

where Λ_0 and Λ_1 are the positive constants introduced in Proposition 3.

Proof. Expanding the integrand yields

$$\begin{aligned} &\int_{\Omega_\varepsilon} \nabla\big(P_\varepsilon V_{z,\lambda} + \omega_{\varepsilon,z,\lambda}\big) \cdot \nabla\left(\frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}\right) \mathrm{d}y \\ & - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\big(P_\varepsilon V_{z,\lambda}(x) + \omega_{\varepsilon,z,\lambda}(x)\big)^{2_\mu^*} \big(P_\varepsilon V_{z,\lambda}(y) + \omega_{\varepsilon,z,\lambda}(y)\big)^{2_\mu^*-1} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\ = \; & \int_{\Omega_\varepsilon} \nabla(P_\varepsilon V_{z,\lambda}) \cdot \nabla\left(\frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}\right) \mathrm{d}y \\ & - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x) \, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y) \, \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y + \mathrm{I}, \end{aligned} \tag{59}$$

where

$$\begin{aligned} \mathrm{I} \; = \; & \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\big(P_\varepsilon V_{z,\lambda}(x)\big)^{2_\mu^*} \big(P_\varepsilon V_{z,\lambda}(y)\big)^{2_\mu^*-1} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\ & - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\big(P_\varepsilon V_{z,\lambda}(x) + \omega_{\varepsilon,z,\lambda}(x)\big)^{2_\mu^*} \big(P_\varepsilon V_{z,\lambda}(y) + \omega_{\varepsilon,z,\lambda}(y)\big)^{2_\mu^*-1} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y. \end{aligned}$$

From Lemma A2 (Appendix A), one obtains

$$\mathrm{I} \; = \; \varepsilon^N \lambda^{N-1} O((\lambda|z| + \lambda\varepsilon)^2) + O\left(\frac{1}{\lambda^{N-1+\sigma}}\right) \tag{60}$$

On the other hand, from Proposition 3, one has

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla(P_\varepsilon V_{z,\lambda}) \cdot \nabla\left(\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}\right) \mathrm{d}y \\
&\quad - \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\
&= \Lambda_0 \varepsilon^N \lambda^{N-1} - \frac{\Lambda_1 \tau_D(z)}{\lambda^{N-1}} + \varepsilon^N \lambda^{N-1} O((\lambda|z| + \lambda\varepsilon)^2) + O\left(\frac{1}{\lambda^{N-1+\sigma}}\right).
\end{aligned} \tag{61}$$

Substituting (60) and (61) into (59) completes the proof. □

Lemma 10. *For any $(z,\lambda) \in T_\varepsilon$, where T_ε is defined in (16), it holds that*

$$\begin{aligned}
&\iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial z_i}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\
&= \frac{C_1(N,\mu)}{\lambda^{N-2}} \frac{\partial\tau_D(z)}{\partial z_i} \int_{\mathbb{R}^N} y_i\, V^{2_\mu^*-2} \frac{\partial V}{\partial z_i}\, \mathrm{d}y + o\left(\frac{1}{\lambda^{N-2}}\right).
\end{aligned}$$

Proof. Since the proof of Lemma 10 is analogous to that of Lemma 9, we only present the different parts of the argument.

$$\begin{aligned}
&\iint_{\mathbb{R}^N\times\Omega_\varepsilon} \frac{V_{z,\lambda}^{2_\mu^*}(x)\, V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial z_i}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\
&= C_1(N,\mu) \int_{\Omega_\varepsilon} V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial z_i}(y)\, \mathrm{d}y \\
&= C_1(N,\mu) \int_{\Omega_\varepsilon} V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \frac{\partial V_{z,\lambda}}{\partial z_i}(y)\, \mathrm{d}y \\
&\quad - C_1(N,\mu) \int_{\Omega_\varepsilon} V_{z,\lambda}^{2_\mu^*-2}(y)\, \psi_{z,\lambda}(y)\, \frac{\partial\psi_{z,\lambda}}{\partial z_i}(y)\, \mathrm{d}y \\
&= \frac{C_1(N,\mu)}{\lambda^{N-2}} \frac{\partial\tau_D(z)}{\partial z_i} \int_{\mathbb{R}^N} y_i\, V_{0,1}^{2_\mu^*-2} \frac{\partial V_{z,1}}{\partial z_i}\bigg|_{z=0} \mathrm{d}y + o\left(\frac{1}{\lambda^{N-2}}\right),
\end{aligned}$$

where we have used the fact that for $j \neq i$,

$$\int_{\mathbb{R}^N} y_j\, V_{0,1}^{2_\mu^*-2} \frac{\partial V_{z,1}}{\partial z_i}\bigg|_{z=0} \mathrm{d}y = 0.$$

□

Proposition 5. *For any $(z,\lambda) \in T_\varepsilon$, where T_ε is defined in (16), it holds that*

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla(P_\varepsilon V_{z,\lambda}) \cdot \nabla\left(\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial z_i}\right) \mathrm{d}y \\
&\quad - \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial z_i}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\
&= \Lambda_2 \varepsilon^N \lambda^{N+2} z_i + \varepsilon^N \lambda^{N+1} O\Big((\varepsilon\lambda + |\lambda z|)^2\Big) + O\left(\frac{1}{\lambda^{N-3+\sigma}}\right),
\end{aligned}$$

where $\Lambda_2 < 0$ is a constant depending on N and μ, and $i = 1, \ldots, N$.

Proof. Similar to Proposition 3, by Lemma 1, one has

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla(P_\varepsilon V_{z,\lambda}) \cdot \nabla\left(\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial z_i}\right) \mathrm{d}y \\
&\quad - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial z_i}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\
= \quad & -\int_{\partial B_\varepsilon(0)} \frac{\partial V_{z,\lambda}}{\partial z_i} \frac{\partial V_{z,\lambda}}{\partial \overline{\nu}}\, \mathrm{d}S + O\left(\frac{1}{\lambda^{N-3+\sigma}}\right),
\end{aligned} \tag{62}$$

where $\overline{\nu}$ is the outward unit normal of $\partial B_\varepsilon(0)$.

Proceeding similarly as in (57) and (58), we compute the remaining boundary integral by substituting the explicit expression of $V_{z,\lambda}$ (with Θ as in (5)), changing variables to the unit sphere, and using the symmetry properties of the sphere.

$$\begin{aligned}
&\int_{\partial B_\varepsilon(0)} \frac{\partial V_{z,\lambda}}{\partial z_i} \frac{\partial V_{z,\lambda}}{\partial \overline{\nu}}\, \mathrm{d}S \\
= \quad & -\Theta^2(N-2)^2\lambda \int_{\partial B_{\lambda\varepsilon}(0)} \frac{\langle y-\lambda z, \overline{\nu}\rangle (y_i - z_i)}{(1+|y-\lambda z|^2)^N}\, \mathrm{d}S \\
= \quad & -\Theta^2(N-2)^2\lambda(\lambda\varepsilon)^{N-1} \int_{\partial B_1(0)} \frac{\langle \lambda\varepsilon y-\lambda z, \overline{\nu}\rangle (\lambda\varepsilon y_i - \lambda z_i)}{(1+|\lambda\varepsilon y-\lambda z|^2)^N}\, \mathrm{d}S \\
= \quad & \Theta^2(N-2)^2\lambda(\lambda\varepsilon)^N \int_{\partial B_1(0)} \frac{\lambda z_i(1+y_i^2)}{(1+|\lambda z|^2)^N}\, \mathrm{d}S + \lambda(\lambda\varepsilon)^N O\big((\lambda|z|+\lambda\varepsilon)^2\big) \\
= \quad & \Theta^2(N-2)^2\lambda(\lambda\varepsilon)^N \Lambda_2 \lambda z_i + \lambda(\lambda\varepsilon)^N O\big((\lambda|z|+\lambda\varepsilon)^2\big),
\end{aligned} \tag{63}$$

since $\int_{\partial B_1(0)} y_i\, \mathrm{d}S = 0$, $i = 1, \ldots, N$, and $\int_{\partial B_1(0)} y_i y_j\, \mathrm{d}S = 0$ for $i \neq j$, where $\overline{\nu}$ denotes the outward unit normal vector of $\partial B_1(0)$.

Combining (62) and (63) and noting that $\lambda|z| = o(1)$, we obtain the desired results. □

By a similar reasoning as for Proposition 4, we establish the next result.

Proposition 6. *Let T_ε be as defined in (16). For any $(z,\lambda) \in T_\varepsilon$, and let $\omega_{\varepsilon,z,\lambda}$ be the function obtained in Proposition 2. For sufficiently small $\varepsilon > 0$, one has*

$$\begin{aligned}
&\frac{\partial J_\varepsilon}{\partial z_i} \\
= \quad & \int_{\Omega_\varepsilon} \nabla\big(P_\varepsilon V_{z,\lambda} + \omega_{\varepsilon,z,\lambda}\big) \cdot \nabla\left(\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial z_i}\right) \mathrm{d}y \\
& - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\big(P_\varepsilon V_{z,\lambda}(x) + \omega_{\varepsilon,z,\lambda}(x)\big)^{2_\mu^*} \big(P_\varepsilon V_{z,\lambda}(y) + \omega_{\varepsilon,z,\lambda}(y)\big)^{2_\mu^*-1} \dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial z_i}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\
= \quad & \Lambda_2 \varepsilon^N \lambda^{N+2} z_i + \varepsilon^N \lambda^{N+1} O\Big((\varepsilon\lambda + |\lambda z|)^2\Big) + O\left(\frac{1}{\lambda^{N-3+\sigma}}\right),
\end{aligned}$$

where $\Lambda_2 < 0$ is the constant introduced in Proposition 5.

Proof. The proof of this Proposition uses a method analogous to Proposition 4, together with Proposition 3 and Lemma A3. □

With these estimates, we proceed to solve the reduced system.

Proof of Theorem 1. Combining Propositions 4 and 6, system (48) reduces to

$$\begin{cases} \Lambda_0\varepsilon^N\lambda^{N-1} - \dfrac{\Lambda_1\tau_D(z)}{\lambda^{N-1}} &= \varepsilon^N\lambda^{N-1}\lambda O((\lambda|z|+\lambda\varepsilon)^2) + O\left(\dfrac{1}{\lambda^{N-1+\sigma}}\right), \\ \Lambda_2\varepsilon^N\lambda^{N+2}z_i + \varepsilon^N\lambda^{N+1}O\big((\varepsilon\lambda+|\lambda z|)^2\big) &= O\left(\dfrac{1}{\lambda^{N-3+\sigma}}\right), \quad i=1,\dots,N. \end{cases} \tag{64}$$

Introduce the change of variables $\lambda = t\varepsilon^{-\frac{N}{2(N-1)}}$ and $\xi = \lambda z$, where c_1, c_2 are as in (16), $t \in [c_1, c_2]$, and $\xi \in B_{R_\varepsilon}(0)$ with $R_\varepsilon := \varrho\varepsilon^{\frac{\sigma N}{4(N-1)}}$ for some constant $\varrho > 0$. Then (64) is rewritten as

$$\begin{cases} F(t,\xi) := \Lambda_0 t^{N-1} - \dfrac{\Lambda_1\tau_D(0)}{t^{N-1}} + o(1) &= 0, \\ G_i(t,\xi) := \Lambda_2\xi_i + O\left(\left(\varepsilon^{\frac{N-2}{2(N-1)}} + |\xi|\right)^2\right) + O\left(\dfrac{1}{\lambda^\sigma}\right) &= 0, \qquad i=1,\dots,N. \end{cases} \tag{65}$$

If we choose c_1 sufficiently small and c_2 sufficiently large, then

$$\Lambda_0 t^{N-1} - \frac{\Lambda_1\tau_D(0)}{t^{N-1}} + s\,o(1) \neq 0, \quad \text{for all } s \in [0,1],\ t = c_j,\ j = 1,2.$$

Meanwhile, on the lateral boundary $\{t \in [c_1,c_2],\ |\xi| = R_\varepsilon\}$ we argue vectorially rather than componentwise. Write $G(t,\xi) := (G_1(t,\xi),\dots,G_N(t,\xi))$ and $\Lambda_2\xi := (\Lambda_2\xi_1,\dots,\Lambda_2\xi_N)$. Since $R_\varepsilon := \varrho\,\varepsilon^{\frac{\sigma N}{4(N-1)}}$ and $\lambda \asymp \varepsilon^{-\frac{N}{2(N-1)}}$, one has

$$\left(\varepsilon^{\frac{N-2}{2(N-1)}} + R_\varepsilon\right)^2 = o(R_\varepsilon), \qquad \lambda^{-\sigma} = o(R_\varepsilon),$$

so that, for $\xi \in \partial B_{R_\varepsilon}(0)$ (i.e., $|\xi| = R_\varepsilon$),

$$\big|G(t,\xi) - \Lambda_2\xi\big| = o(R_\varepsilon) \quad \text{uniformly in } t \in [c_1,c_2].$$

As $\Lambda_2 < 0$, the reverse triangle inequality gives

$$|G(t,\xi)| \geq |\Lambda_2|\,|\xi| - \big|G(t,\xi) - \Lambda_2\xi\big| \geq \frac{|\Lambda_2|}{2}R_\varepsilon > 0 \quad \text{for all } t \in [c_1,c_2],\ \xi \in \partial B_{R_\varepsilon}(0),$$

i.e., the vector field G has no zero on the lateral boundary. Moreover, for the homotopy

$$G^{(s)}(t,\xi) := (1-s)G(t,\xi) + s\,\Lambda_2\xi, \qquad s \in [0,1],$$

the same bound yields $|G^{(s)}(t,\xi)| \geq \frac{|\Lambda_2|}{2}R_\varepsilon > 0$ uniformly in $s \in [0,1]$, $t \in [c_1,c_2]$, $\xi \in \partial B_{R_\varepsilon}(0)$.

Let deg denote the topological degree. We compute the topological degree by

$$\begin{aligned} &\deg\big((F,G_1,\dots,G_N),\ [c_1,c_2]\times B_{R_\varepsilon}(0),\ 0\big) \\ &= \deg\left(\left(\Lambda_0 t^{N-1} - \frac{\Lambda_1\tau_D(0)}{t^{N-1}},\ \Lambda_2\xi_1,\dots,\Lambda_2\xi_N\right),\ [c_1,c_2]\times B_{R_\varepsilon}(0),\ 0\right) \\ &= \deg\left(\Lambda_0 t^{N-1} - \frac{\Lambda_1\tau_D(0)}{t^{N-1}},\ [c_1,c_2],\ 0\right)\cdot\deg\big(\Lambda_2\xi_1,\dots,\Lambda_2\xi_N,\ B_{R_\varepsilon}(0),\ 0\big). \end{aligned}$$

It follows that there exist $t_\varepsilon \in [c_1, c_2]$ and $\xi_\varepsilon \in B_{R_\varepsilon}(0)$ solving (65). Since $R_\varepsilon \to 0$, we have $\xi_\varepsilon \to 0$ as $\varepsilon \to 0$. Consequently, this yields

$$\lambda_\varepsilon \in \left[c_1 \varepsilon^{-\frac{N}{2(N-1)}},\ c_2 \varepsilon^{-\frac{N}{2(N-1)}}\right], \quad z_\varepsilon \in B_{\frac{\varrho}{c_1}\varepsilon^{\frac{\sigma N}{4(N-1)}}}(0)$$

such that system (48) holds. From the first equation in (65),

$$\varphi(t_\varepsilon) := \Lambda_0 t_\varepsilon^{N-1} - \frac{\Lambda_1 \tau_D(0)}{t_\varepsilon^{N-1}} = o(1).$$

The function φ is strictly increasing on $[c_1, c_2]$, since

$$\varphi'(t) = (N-1)\left(\Lambda_0 t^{N-2} + \Lambda_1 \tau_D(0) t^{-N}\right) > 0$$

and $\varphi(c_1) < 0 < \varphi(c_2)$ by the choice of c_1, c_2. Hence φ has a unique zero $t_0 \in (c_1, c_2)$, and $\varphi(t_\varepsilon) = o(1)$ together with the strict monotonicity yields $t_\varepsilon \to t_0$. Therefore

$$\lambda_\varepsilon \varepsilon^{\frac{N}{2(N-1)}} = t_\varepsilon \to t_0 =: c > 0.$$

Finally, u_ε is positive. Testing the equation

$$-\Delta u = \Big(\int_{\Omega_\varepsilon} \frac{(u_+)^{2^*_\mu}(y)}{|y-x|^\mu}\, dy\Big)(u_+)^{2^*_\mu - 1} \quad \text{in } \Omega_\varepsilon,$$

with $u_\varepsilon^- := \max\{-u_\varepsilon, 0\}$ gives $\int_{\Omega_\varepsilon} |\nabla u_\varepsilon^-|^2\, dy = 0$, hence $u_\varepsilon \geq 0$ in Ω_ε. Since $\|PDV_{z_\varepsilon,\lambda_\varepsilon}\|_{\tilde{H}^1_0} \to \|V\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} > 0$ while $\|\omega_\varepsilon\|_{\tilde{H}^1_0} \to 0$, we have $u_\varepsilon \not\equiv 0$. As $-\Delta u_\varepsilon \geq 0$ in Ω_ε, the strong maximum principle yields $u_\varepsilon > 0$ in Ω_ε. Thus $u_{\varepsilon,+} = u_\varepsilon$, and u_ε solves the problem (1). □

5. Discussion

In this paper we studied the critical Hartree Equation (1) with mixed boundary conditions in the perforated domain Ω_ε, and proved by the Lyapunov–Schmidt reduction the existence of a positive solution concentrating and blowing up near the center of the hole as $\varepsilon \to 0$, with

$$\lambda_\varepsilon \sim c\varepsilon^{-\frac{N}{2(N-1)}}, \qquad \lambda_\varepsilon |z_\varepsilon| \to 0.$$

The interest of the result lies in the mechanism selecting the concentration point. In the Dirichlet Hartree problems [16,17] the boundary is passive and the blow-up point is selected by the Robin function. Here, in contrast, the approximate solution $P_D V_{z,\lambda}$ satisfies the Dirichlet condition on ∂D but not the Neumann condition on $\partial B_\varepsilon(0)$; the resulting flux on the inner boundary produces a leading sign-definite boundary term in the reduced energy expansion, which pins the concentration point to the center of the hole and fixes the scale $\lambda_\varepsilon \sim c\varepsilon^{-N/(2(N-1))}$ by balancing this boundary term against the Robin-function contribution. At the same time, hole-center concentration already occurs in the local mixed-boundary problem [13]. What is new with respect to that setting is the nonlocal Hartree interaction: since the convolution is integrated over the perforated domain, the Riesz kernel interacts with the hole itself, and every estimate must control the Dirichlet projection, the decay of the bubble, and this kernel–hole interaction simultaneously.

The results of this paper open up several interesting directions for future research. A natural question concerns the case of multiple holes: when the domain contains several small holes, can multi-peak solutions be constructed, and how do the interactions among

the holes affect the locations, the concentration scales, and the stability of such solutions? Another direction is to consider different configurations of the boundary conditions—for instance, Dirichlet conditions on the inner boundary or Neumann conditions on the outer boundary—which may lead to substantially different concentration phenomena. Furthermore, questions concerning uniqueness, stability, and multiplicity of solutions for critical Hartree equations in perforated domains remain largely unexplored.

A particularly interesting direction is to study critical Hartree-type equations driven by fractional Laplacians. Such an extension would involve new difficulties arising from the nonlocality of both the principal operator and the Hartree interaction, as well as from the formulation of appropriate Dirichlet-, Neumann-, or mixed-type exterior conditions for fractional operators. It would also be worthwhile to investigate related nonlocal problems in more general geometric settings, including manifolds with boundary and orbifolds.

6. Conclusions

We established the existence of positive solutions to the critical Hartree Equation (1) in a perforated domain with mixed boundary conditions, concentrating and blowing up near the center of the hole with $\lambda_\varepsilon \sim c\varepsilon^{-N/(2(N-1))}$ and $\lambda_\varepsilon |z_\varepsilon| \to 0$ as $\varepsilon \to 0$.

The result shows how the mixed boundary conditions and the small perforation jointly determine the concentration behavior: the Neumann condition on the inner boundary supports the bubble near the hole, while the outer Dirichlet boundary enters through the Dirichlet projection and the regular part of the Green function.

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Appendix A

In this appendix, we present several estimates that are used in the preceding sections.

First, combining the explicit expression of the bubble function in (4) and Lemma 1, we establish the following Lemma.

Lemma A1.

$$\frac{\partial V_{z,\lambda}(x)}{\partial \lambda} = O\left(\frac{V_{z,\lambda}}{\lambda}\right), \qquad \frac{\partial V_{z,\lambda}(x)}{\partial z_j} = O\left(\lambda V_{z,\lambda}\right)$$

Lemma A2. *Let T_ε be as defined in (16). For any $(z,\lambda) \in T_\varepsilon$, and let $\omega_{\varepsilon,z,\lambda}$ be the function obtained in Proposition 2. For sufficiently small $\varepsilon > 0$, one has*

$$\begin{aligned} \mathrm{I} &= \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\left(P_\varepsilon V_{z,\lambda}(x)\right)^{2_\mu^*} \left(P_\varepsilon V_{z,\lambda}(y)\right)^{2_\mu^*-1} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\ &\quad - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\left(P_\varepsilon V_{z,\lambda}(x) + \omega_{\varepsilon,z,\lambda}(x)\right)^{2_\mu^*} \left(P_\varepsilon V_{z,\lambda}(y) + \omega_{\varepsilon,z,\lambda}(y)\right)^{2_\mu^*-1} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\ &= \varepsilon^N \lambda^{N-1} O((\lambda|z| + \lambda\varepsilon)^2) + O\left(\frac{1}{\lambda^{N-1+\sigma}}\right) \end{aligned}$$

Proof. We start with the standard binomial-type estimates

$$\begin{aligned} (a+b)^p &= a^p + O(b^p), \quad p \in (1,2], \\ (a+b)^p &= a^p + pa^{p-1}b + \frac{p(p-1)}{2} a^{p-2}b^2 + O(b^p), \quad p > 2. \end{aligned}$$

Thus we analyze the exponents 2_μ^* and $2_\mu^* - 1$. We have

$$2_\mu^* = \frac{2N-\mu}{N-2} \quad \text{where} \quad \begin{cases} 2_\mu^* > 2, & \mu \in (0,4), \\ 2_\mu^* = 2, & \mu = 4, \end{cases}$$

$$2_\mu^* - 1 = \frac{N-\mu+2}{N-2} \quad \text{where} \quad \begin{cases} 2_\mu^* - 1 \le 2, & N \ge 6-\mu, \\ 2_\mu^* - 1 > 2, & N < 6-\mu. \end{cases}$$

Consequently, when $N \ge 6-\mu$ and $\mu \in (0,4)$,

$$\mathrm{I} = O(\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5 + \mathrm{I}_6 + \mathrm{I}_7 + \mathrm{I}_8 + \mathrm{I}_9 + \mathrm{I}_{10} + \mathrm{I}_{11}). \tag{A1}$$

Define the bilinear integral operator

$$\mathrm{Y}[A,B] = \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{A(x)\,B(y)}{|y-x|^\mu} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y) \mathrm{d}x\mathrm{d}y, \tag{A2}$$

then we decompose the perturbation term into eleven integrals,

$$\begin{aligned} \mathrm{I}_1 &= \mathrm{Y}\left[(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x),\, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}(y)\,\omega_{\varepsilon,z,\lambda}(y)\right], \quad \mathrm{I}_2 = \mathrm{Y}\left[(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x),\, \omega_{\varepsilon,z,\lambda}^{2_\mu^*-1}(y)\right], \\ \mathrm{I}_3 &= \mathrm{Y}\left[(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(x)\,\omega_{\varepsilon,z,\lambda}(x),\, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\right], \\ \mathrm{I}_4 &= \mathrm{Y}\left[(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(x)\,\omega_{\varepsilon,z,\lambda}(x),\, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}(y)\,\omega_{\varepsilon,z,\lambda}(y)\right], \\ \mathrm{I}_5 &= \mathrm{Y}\left[(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(x)\,\omega_{\varepsilon,z,\lambda}(x),\, \omega_{\varepsilon,z,\lambda}^{2_\mu^*-1}(y)\right], \\ \mathrm{I}_6 &= \mathrm{Y}\left[(P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}(x)\,\omega_{\varepsilon,z,\lambda}^2(x),\, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\right], \\ \mathrm{I}_7 &= \mathrm{Y}\left[(P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}(x)\,\omega_{\varepsilon,z,\lambda}^2(x),\, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}(y)\,\omega_{\varepsilon,z,\lambda}(y)\right], \\ \mathrm{I}_8 &= \mathrm{Y}\left[(P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}(x)\,\omega_{\varepsilon,z,\lambda}^2(x),\, \omega_{\varepsilon,z,\lambda}^{2_\mu^*-1}(y)\right], \\ \mathrm{I}_9 &= \mathrm{Y}\left[\omega_{\varepsilon,z,\lambda}^{2_\mu^*}(x),\, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\right], \quad \mathrm{I}_{10} = \mathrm{Y}\left[\omega_{\varepsilon,z,\lambda}^{2_\mu^*}(x),\, (P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}(y)\,\omega_{\varepsilon,z,\lambda}(y)\right], \\ \mathrm{I}_{11} &= \mathrm{Y}\left[\omega_{\varepsilon,z,\lambda}^{2_\mu^*}(x),\, \omega_{\varepsilon,z,\lambda}^{2_\mu^*-1}(y)\right]. \end{aligned} \tag{A3}$$

We begin with the estimate of I_1. By virtue of Proposition 1, one deduces

$$\begin{aligned}
\mathrm{I}_1 &= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}(y)\,\omega_{\varepsilon,z,\lambda}(y)\,\dfrac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y \\
&\le C\left(\int_{\Omega_\varepsilon} P_\varepsilon V_{z,\lambda}^{2^*}\,\mathrm{d}x\right)^{\frac{2N-\mu}{2N}}\left(\int_{\Omega_\varepsilon}(P_\varepsilon V_{z,\lambda})^{\frac{2N(2_\mu^*-2)}{2N-\mu}}\left(\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
&\le O\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{\frac{2N(2_\mu^*-2)}{2N-\mu}}\left(\frac{\partial V_{z,\lambda}}{\partial\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
&\quad +O\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{\frac{2N(2_\mu^*-2)}{2N-\mu}}\left(\frac{\partial\psi_{z,\lambda}}{\partial\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
&\quad +O\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{\frac{2N(2_\mu^*-3)}{2N-\mu}}\left(\frac{\partial V_{z,\lambda}}{\partial\lambda}\psi_{z,\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
&\quad +O\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{\frac{2N(2_\mu^*-3)}{2N-\mu}}\left(\frac{\partial\psi_{z,\lambda}}{\partial\lambda}\psi_{z,\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}}.
\end{aligned} \tag{A4}$$

By Hölder's inequality and the Sobolev embedding theorem, one obtains

$$\begin{aligned}
&\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{\frac{2N(2_\mu^*-2)}{2N-\mu}}\left(\frac{\partial V_{z,\lambda}}{\partial\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
&= O\left(\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{\frac{2N(2_\mu^*-2)}{N-\mu+2}}\left|\frac{\partial V_{z,\lambda}}{\partial\lambda}\right|^{\frac{2N}{N-\mu+2}}\mathrm{d}y\right)^{\frac{N-\mu+2}{2N}}\|\omega_{\varepsilon,z,\lambda}\|_{H_0^1}\right) \\
&\le O\left(\frac{1}{\lambda^{\frac{N-\mu+4}{2}}}\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right).
\end{aligned} \tag{A5}$$

Similarly, combining with Lemma 3, the following estimate holds.

$$\begin{aligned}
&\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{\frac{2N(2_\mu^*-2)}{2N-\mu}}\left(\frac{\partial\psi_{z,\lambda}}{\partial\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
&= o\left(\frac{1}{\lambda^{\frac{N-\mu+4}{2}}}\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right), \\
&\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{\frac{2N(2_\mu^*-3)}{2N-\mu}}\left(\frac{\partial V_{z,\lambda}}{\partial\lambda}\psi_{z,\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
&= o\left(\frac{1}{\lambda^{\frac{N-\mu+4}{2}}}\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right), \\
\text{and}\quad &\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{\frac{2N(2_\mu^*-3)}{2N-\mu}}\left(\frac{\partial\psi_{z,\lambda}}{\partial\lambda}\psi_{z,\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
&= o\left(\frac{1}{\lambda^{\frac{N-\mu+4}{2}}}\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right).
\end{aligned} \tag{A6}$$

Thus the estimate of I_1 follows from (A4)–(A6) and Lemma 6.

We then turn to the other terms in I. Analogous to the estimate of I_1, one deduces

$$\begin{aligned}\mathrm{I}_2 &\leq C\left(\int_{\Omega_\varepsilon} V_{z,\lambda}^{2_\mu^*} dy\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon}\left(\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\right)^{\frac{2N}{2N-\mu}} \omega_{\varepsilon,z,\lambda}^{\frac{2N(2_\mu^*-1)}{2N-\mu}} dy\right)^{\frac{2N-\mu}{2N}} \\ &= O\left(\left(\int_{\Omega_\varepsilon}\left|\frac{\partial V_{z,\lambda}}{\partial\lambda}\right|^{\frac{2N}{N-2}} dy\right)^{\frac{N-2}{2N}} \left(\int_{\Omega_\varepsilon} \omega_{\varepsilon,z,\lambda}^{\frac{2N}{N-2}} dy\right)^{\frac{N-\mu+2}{2N}}\right) \\ &= O\left(\frac{1}{\lambda}\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}^{2_\mu^*-1}\right),\end{aligned}$$

and

$$\begin{aligned}\mathrm{I}_3 &= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(x)\,\omega(x)\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(y)\,\frac{\partial(P_\varepsilon V_{z,\lambda})}{\partial\lambda}(y)}{|y-x|^\mu}\mathrm{d}x\mathrm{d}y \\ &\leq C\left(\int_{\Omega_\varepsilon}\left((P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\right)^{\frac{2N}{2N-\mu}} dy\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon}(P_\varepsilon V_{z,\lambda})^{\frac{2N(2_\mu^*-1)}{2N-\mu}}\omega_{\varepsilon,z,\lambda}^{\frac{2N}{2N-\mu}} dy\right)^{\frac{2N-\mu}{2N}} \\ &= O\left(\frac{1}{\lambda^{\frac{N}{2}} d^{\frac{N}{2}+1}}\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right).\end{aligned}$$

By Hölder's inequality, the Sobolev embedding, Proposition 1 and Lemma 6, the desired estimate follows.

$$\begin{aligned}\mathrm{I}_4 &\leq C\left(\int_{\Omega_\varepsilon}(P_\varepsilon V_{z,\lambda})^{\frac{2N(2_\mu^*-1)}{2N-\mu}}\omega_{\varepsilon,z,\lambda}^{\frac{2N}{2N-\mu}} dy\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon}\left((P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}} dy\right)^{\frac{2N-\mu}{2N}} \\ &\leq O\left(\frac{1}{\lambda^{\frac{N}{2}} d^{\frac{N}{2}+1}}\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}^2\right).\end{aligned}$$

and

$$\begin{aligned}\mathrm{I}_5 &\leq C\left(\int_{\Omega_\varepsilon}\left((P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}} dy\right)^{\frac{2N-\mu}{2N}} \left(\int_{\Omega_\varepsilon}\left(\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\omega_{\varepsilon,z,\lambda}^{2_\mu^*-1}\right)^{\frac{2N}{2N-\mu}} dy\right)^{\frac{2N-\mu}{2N}} \\ &= O\left(\left(\int_{\Omega_\varepsilon}\left|\frac{\partial V_{z,\lambda}}{\partial\lambda}\right|^{\frac{2N}{N-2}} dy\right)^{\frac{N-2}{2N}} \left(\int_{\Omega_\varepsilon}\omega_{\varepsilon,z,\lambda}^{\frac{2N}{N-2}} dy\right)^{\frac{N-\mu+2}{2N}}\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right) \\ &= O\left(\frac{1}{\lambda}\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}^{\frac{2N-\mu}{N-2}}\right).\end{aligned}$$

By estimates analogous to (A5), the corresponding bound follows

$$\begin{aligned}\mathrm{I}_6 &\leq C\left(\int_{\Omega_\varepsilon}\left((P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}\omega_{\varepsilon,z,\lambda}^2\right)^{\frac{2N}{2N-\mu}}\mathrm{d}x\right)^{\frac{2N-\mu}{2N}} \left(\int_\Omega\left((P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}\frac{\partial P_\varepsilon V_{z,\lambda}}{\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}x\right)^{\frac{2N-\mu}{2N}} \\ &= O\left(\frac{1}{\lambda^{N+2-\frac{\mu}{2}}}\|\omega_{\varepsilon,z,\lambda}\|_{\widetilde{H}_0^1(\Omega_\varepsilon)}^2\right).\end{aligned}$$

In view of the preceding estimates, direct computation yields

$$\begin{aligned}
\mathrm{I}_7 &\leq C\left(\int_{\Omega_\varepsilon}(P_\varepsilon V_{z,\lambda})^{\frac{2N(2_\mu^*-2)}{2N-\mu}}\omega_{\varepsilon,z,\lambda}^{\frac{4N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}}\left(\int_{\Omega_\varepsilon}\left((P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}}\\
&= O\left(\frac{1}{\lambda}\left(\int_{\Omega_\varepsilon}V_{z,\lambda}^{\frac{2N}{N-2}}\mathrm{d}y\right)^{\frac{4-\mu}{2N}}\left(\int_{\Omega_\varepsilon}|\omega_{\varepsilon,z,\lambda}|^{\frac{2N}{N-2}}\mathrm{d}y\right)^{\frac{N-2}{N}}\right.\\
&\quad\left.\cdot\left(\int_{\Omega_\varepsilon}V_{z,\lambda}^{\frac{2N}{N-2}}\mathrm{d}y\right)^{\frac{N-\mu+2}{2N}}\left(\int_{\Omega_\varepsilon}|\omega_{\varepsilon,z,\lambda}|^{\frac{2N}{N-2}}\mathrm{d}y\right)^{\frac{N-2}{N}}\right)\\
&= O\left(\frac{1}{\lambda}\|\omega_{\varepsilon,z,\lambda}\|^3_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right).
\end{aligned}$$

Combining Hölder's inequality, the Sobolev embedding with Lemma 6, one deduces

$$\begin{aligned}
\mathrm{I}_8 &\leq C\left(\int_{\Omega_\varepsilon}(P_\varepsilon V_{z,\lambda})^{\frac{2N(2_\mu^*-2)}{2N-\mu}}\omega_{\varepsilon,z,\lambda}^{\frac{4N}{2N-\mu}}dy\right)^{\frac{2N-\mu}{2N}}\left(\int_{\Omega_\varepsilon}\left(\omega_{\varepsilon,z,\lambda}^{2_\mu^*-1}\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\right)^{\frac{2N}{2N-\mu}}dy\right)^{\frac{2N-\mu}{2N}}\\
&= O\left(\frac{1}{\lambda}\left(\int_{\Omega_\varepsilon}V_{z,\lambda}^{\frac{2N}{N-2}}dy\right)^{\frac{N-\mu+2}{2N}}\|\omega_{\varepsilon,z,\lambda}\|^{\frac{2N-\mu}{N-2}}_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right)\\
&= O\left(\frac{1}{\lambda}\|\omega_{\varepsilon,z,\lambda}\|^{\frac{2N-\mu}{N-2}}_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right).
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{I}_9 &\leq C\left(\int_{\Omega_\varepsilon}\left((P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\right)^{\frac{2N}{2N-\mu}}dy\right)^{\frac{2N-\mu}{2N}}\left(\int_{\Omega_\varepsilon}\omega_{\varepsilon,z,\lambda}^{2_\mu^*}dy\right)^{\frac{2N-\mu}{2N}}\\
&\leq C\lambda\left(\int_{\Omega_\varepsilon}V_{z,\lambda}^{2_\mu^*}dy\right)^{\frac{2N-\mu}{2N}}\|\omega_{\varepsilon,z,\lambda}\|^{\frac{2N-\mu}{N-2}}_{\widetilde{H}_0^1(\Omega_\varepsilon)}=O\left(\frac{1}{\lambda}\|\omega_{\varepsilon,z,\lambda}\|^{\frac{2N-\mu}{N-2}}_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right).
\end{aligned}$$

Using reasoning identical to the estimates established for I_1, the same type of estimate holds

$$\begin{aligned}
\mathrm{I}_{10} &\leq C\left(\int_{\Omega_\varepsilon}V_{z,\lambda}^{\frac{2N(2_\mu^*-2)}{2N-\mu}}\left(\frac{\partial V_{z,\lambda}}{\partial\lambda}\omega_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}dy\right)^{\frac{2N-\mu}{2N}}\|\omega_{\varepsilon,z,\lambda}\|^{\frac{2N-\mu}{N-2}}_{\widetilde{H}_0^1(\Omega_\varepsilon)}\\
&\leq O\left(\frac{1}{\lambda^{\frac{N}{2}}d^{\frac{N}{2}+1}}\|\omega_{\varepsilon,z,\lambda}\|^{\frac{3N-\mu+2}{N-2}}_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right).
\end{aligned}$$

Similarly, the corresponding bound follows

$$\begin{aligned}
\mathrm{I}_{11} &\leq C\left(\int_{\Omega_\varepsilon}\omega_{\varepsilon,z,\lambda}^{2_\mu^*}dy\right)^{\frac{2N-\mu}{2N}}\left(\int_{\Omega_\varepsilon}\left(\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\right)^{\frac{2N}{2N-\mu}}\omega_{\varepsilon,z,\lambda}^{\frac{2N(2_\mu^*-1)}{2N-\mu}}dy\right)^{\frac{2N-\mu}{2N}}\\
&= O\left(\left(\int_{\Omega_\varepsilon}\left|\frac{\partial V_{z,\lambda}}{\partial\lambda}\right|^{\frac{2N}{N-2}}dy\right)^{\frac{N-2}{2N}}\left(\int_{\Omega_\varepsilon}\omega_{\varepsilon,z,\lambda}^{\frac{2N}{N-2}}dy\right)^{\frac{N-\mu+2}{2N}}\|\omega_{\varepsilon,z,\lambda}\|^{\frac{2N-\mu}{N-2}}_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right)\\
&= O\left(\frac{1}{\lambda}\|\omega_{\varepsilon,z,\lambda}\|^{\frac{3N-2\mu+2}{N-2}}_{\widetilde{H}_0^1(\Omega_\varepsilon)}\right).
\end{aligned}$$

In contrast, when $N < 6-\mu$ and $\mu \in (0,4)$, we have

$$\mathrm{I} = O(\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5+\mathrm{I}_6+\mathrm{I}_7+\mathrm{I}_8+\mathrm{I}_9+\mathrm{I}_{10}+\mathrm{I}_{11}+\mathrm{I}_{12}+\mathrm{I}_{13}+\mathrm{I}_{14}+\mathrm{I}_{15}),$$

where $\mathrm{I}_1,\ldots,\mathrm{I}_{11}$ are defined in (A3), and the extra terms $\mathrm{I}_{12},\ldots,\mathrm{I}_{15}$ are given by

$$\begin{aligned}
\mathrm{I}_{12} &= \mathrm{Y}\big[(P_\varepsilon V_{z,\lambda})^{2_\mu^*}(x),\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-3}(y)\,\omega^2_{\varepsilon,z,\lambda}(y)\big],\\
\mathrm{I}_{13} &= \mathrm{Y}\big[(P_\varepsilon V_{z,\lambda})^{2_\mu^*-1}(x)\,\omega_{\varepsilon,z,\lambda}(x),\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-3}(y)\,\omega^2_{\varepsilon,z,\lambda}(y)\big],\\
\mathrm{I}_{14} &= \mathrm{Y}\big[(P_\varepsilon V_{z,\lambda})^{2_\mu^*-2}(x)\,\omega^2_{\varepsilon,z,\lambda}(x),\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-3}(y)\,\omega^2_{\varepsilon,z,\lambda}(y)\big],\\
\mathrm{I}_{15} &= \mathrm{Y}\big[\omega^{2_\mu^*}_{\varepsilon,z,\lambda}(x),\,(P_\varepsilon V_{z,\lambda})^{2_\mu^*-3}(y)\,\omega^2_{\varepsilon,z,\lambda}(y)\big],
\end{aligned}$$

where the bilinear integral operator Y is introduced in (A2). Similar to the estimate of I_1, we deduce

$$\begin{aligned}
\mathrm{I}_{12} &= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{\big(P_\varepsilon V_{z,\lambda}(x)\big)^{2_\mu^*}\big(P_\varepsilon V_{z,\lambda}(y)\big)^{2_\mu^*-3}\,\omega^2_{\varepsilon,z,\lambda}(y)\,\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}}{|y-x|^\mu}\,\mathrm{d}x\mathrm{d}y\\
&= O\left(\left(\int_{\Omega_\varepsilon}\left|V_{z,\lambda}^{2_\mu^*-3}\frac{\partial V_{z,\lambda}}{\partial\lambda}\right|^{\frac{2N}{4-\mu}}\mathrm{d}y\right)^{\frac{4-\mu}{2N}}\left(\int_{\Omega_\varepsilon}\omega_{\varepsilon,z,\lambda}^{\frac{2N}{N-2}}\mathrm{d}y\right)^{\frac{N-2}{N}}\right)\\
&\le O\left(\frac{1}{\lambda}\|\omega_{\varepsilon,z,\lambda}\|^2_{\widetilde{H}^1_0(\Omega_\varepsilon)}\right).
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}_{13} &= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{\big(P_\varepsilon V_{z,\lambda}(x)\big)^{2_\mu^*-1}\,\omega_{\varepsilon,z,\lambda}(x)\,\big(P_\varepsilon V_{z,\lambda}(y)\big)^{2_\mu^*-3}\,\omega^2_{\varepsilon,z,\lambda}(y)\,\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}}{|y-x|^\mu}\,\mathrm{d}x\mathrm{d}y\\
&\le C\left(\int_{\Omega_\varepsilon}\big(P_\varepsilon V_{z,\lambda}\big)^{\frac{2N(2_\mu^*-1)}{2N-\mu}}\omega_{\varepsilon,z,\lambda}^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}}\left(\int_{\Omega_\varepsilon}\left(\big(P_\varepsilon V_{z,\lambda}\big)^{2_\mu^*-3}\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\omega^2_{\varepsilon,z,\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}}\\
&= O\left(\frac{1}{\lambda}\|\omega_{\varepsilon,z,\lambda}\|^3_{\widetilde{H}^1_0(\Omega_\varepsilon)}\right).
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}_{14} &= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{\big(P_\varepsilon V_{z,\lambda}(y)\big)^{2_\mu^*-2}\,\omega^2_{\varepsilon,z,\lambda}(y)\,\big(P_\varepsilon V_{z,\lambda}(x)\big)^{2_\mu^*-3}\,\omega^2_{\varepsilon,z,\lambda}(x)\,\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}}{|y-x|^\mu}\,\mathrm{d}x\mathrm{d}y\\
&= O\left(\frac{1}{\lambda}\|\omega_{\varepsilon,z,\lambda}\|^4_{\widetilde{H}^1_0(\Omega_\varepsilon)}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{I}_{15} &= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{\omega^{2_\mu^*}_{\varepsilon,z,\lambda}(y)\,\big(P_\varepsilon V_{z,\lambda}(x)\big)^{2_\mu^*-3}\,\omega^2_{\varepsilon,z,\lambda}(x)\,\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}}{|y-x|^\mu}\,\mathrm{d}x\mathrm{d}y\\
&\le C\left(\int_{\Omega_\varepsilon}\omega^{2_\mu^*}_{\varepsilon,z,\lambda}\,\mathrm{d}y\right)^{\frac{2N-\mu}{2N}}\left(\int_{\Omega_\varepsilon}\left(\big(P_\varepsilon V_{z,\lambda}\big)^{2_\mu^*-3}\,\omega^2_{\varepsilon,z,\lambda}\,\frac{\partial P_\varepsilon V_{z,\lambda}}{\partial\lambda}\right)^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}}\\
&= O\left(\frac{1}{\lambda}\|\omega_{\varepsilon,z,\lambda}\|^{\frac{4N-\mu-4}{N-2}}_{\widetilde{H}^1_0(\Omega_\varepsilon)}\right),
\end{aligned}$$

When $N \geq 5$ and $\mu = 4$, it holds that $2^*_\mu = 2$. In this case,

$$\begin{aligned}
\mathrm{I} &= \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{(P_\varepsilon V_{z,\lambda})^2(x)\,\omega_{\varepsilon,z,\lambda}(y)}{|y-x|^4} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{P_\varepsilon V_{z,\lambda}(x)\,\omega_{\varepsilon,z,\lambda}(x)\,P_\varepsilon V_{z,\lambda}(y)}{|y-x|^4} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + O\Bigg(\iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{P_\varepsilon V_{z,\lambda}(x)\,\omega_{\varepsilon,z,\lambda}(x)\,\omega_{\varepsilon,z,\lambda}(y)}{|y-x|^4} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\omega^2_{\varepsilon,z,\lambda}(x)\,P_\varepsilon V_{z,\lambda}(y)}{|y-x|^4} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\omega^2_{\varepsilon,z,\lambda}(x)\,\omega_{\varepsilon,z,\lambda}(y)}{|y-x|^4} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial \lambda}(y)\,\mathrm{d}x\mathrm{d}y \Bigg) \\
&= \mathrm{I}_{16} + \mathrm{I}_{17} + \mathrm{I}_{18} + \mathrm{I}_{19} + \mathrm{I}_{20}.
\end{aligned}$$

The estimates for I_{16}–I_{20} are analogous to those for I_1–I_{11}.Therefore, combining the above estimates with Lemma 6, we complete the proof. □

Lemma A3. *Let T_ε be as defined in (16). For any $(z,\lambda) \in T_\varepsilon$, and let $\omega_{\varepsilon,z,\lambda}$ be the function obtained in Proposition 2. For sufficiently small $\varepsilon > 0$, one has*

$$\begin{aligned}
\widehat{\mathrm{I}}_i &= \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\left(P_\varepsilon V_{z,\lambda}(x)\right)^{2^*_\mu} \left(P_\varepsilon V_{z,\lambda}(y)\right)^{2^*_\mu - 1} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial z_i}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\
&\quad - \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{\left(P_\varepsilon V_{z,\lambda}(x) + \omega_{\varepsilon,z,\lambda}(x)\right)^{2^*_\mu} \left(P_\varepsilon V_{z,\lambda}(y) + \omega_{\varepsilon,z,\lambda}(y)\right)^{2^*_\mu - 1} \frac{\partial (P_\varepsilon V_{z,\lambda})}{\partial z_i}(y)}{|y-x|^\mu} \mathrm{d}x\mathrm{d}y \\
&= \varepsilon^N \lambda^{N+1} O\Big((\varepsilon\lambda + |\lambda z|)^2 \Big) + O\left(\frac{1}{\lambda^{N-3+\sigma}} \right).
\end{aligned}$$

Proof. The proof of this Lemma follows by arguments similar to those for Lemma A2, combined with Lemma A1. □

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