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Article

Geometric Properties and Applications of a Generalized Bessel–Maitland–Tremblay Function

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Abstract

In this paper, motivated by the modified Tremblay fractional differential operator and the normalized generalized Bessel–Maitland function, we introduce the generalized Bessel–Maitland–Tremblay function via the Hadamard product transformation. This construction established an integrated framework that connects fractional calculus with geometric function theory. By utilizing estimates for the gamma and digamma functions together with monotonicity properties of the associated coefficient sequences, we establish sufficient conditions under which the proposed function belongs to various important subclasses of normalized analytic functions. In particular, criteria are obtained for uniform convexity, starlikeness and convexity of order δ, ν-uniform starlikeness, and ν-uniform convexity, as well as exponential starlikeness, exponential convexity, and lemniscate-type conditions. Several special cases are shown to recover previously known results for the normalized generalized Bessel–Maitland function and the identity mapping. Graphical analysis and numerical examples are presented to show the geometric behavior of the proposed function and to verify the theoretical findings.

Keywords: analytic and univalent functions; generalized Bessel–Maitland function; uniformly convex functions; uniformly starlike functions; Tremblay fractional differential operator; Hadamard product; ν-uniformly convex functions; fractional calculus; ν-uniformly starlike functions

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1. Introduction

Let $\mathbb{D}$ denote the open unit disk

$$\mathbb{D} = \{\omega \in \mathbb{C} : |\omega| < 1\}.$$

Let $\mathcal{A}$ denote the family of normalized analytic functions (AFs) $f : \mathbb{D} \to \mathbb{C}$ of the form

$$f(\omega) = \omega + \sum_{n=2}^{\infty} a_n \omega^n, \qquad \omega \in \mathbb{D}, \tag{1}$$

where $a_n \in \mathbb{C}$ for every $n \geq 2$. The family $\mathcal{A}$ serves as the standard framework for studying normalized univalent functions in geometric function theory. Among its most widely investigated subclasses are the families of starlike and convex functions.

A function $f \in \mathcal{A}$ is said to belong to the class of *starlike functions of order* β, denoted by $\mathcal{S}^*(\beta)$, where $0 \leq \beta < 1$, if

$$\mathrm{Re}\left(\frac{\omega f'(\omega)}{f(\omega)}\right) > \beta, \qquad \omega \in \mathbb{D}. \tag{2}$$

Likewise, $f \in \mathcal{A}$ will be in the class of *convex functions of order* β, denoted by $\mathcal{K}(\beta)$, whenever

$$\mathrm{Re}\left(1 + \frac{\omega f''(\omega)}{f'(\omega)}\right) > \beta, \qquad \omega \in \mathbb{D}. \tag{3}$$

An important relationship between the above subclasses is

$$f \in \mathcal{K}(\beta) \quad \Longleftrightarrow \quad \omega f'(\omega) \in \mathcal{S}^*(\beta). \tag{4}$$

Moreover, a function $f \in \mathcal{A}$ is in the class of *close-to-convex function of order* σ *and type* β if there exists a function $g \in \mathcal{S}^*(\beta)$ such that

$$\mathrm{Re}\left(\frac{\omega f'(\omega)}{g(\omega)}\right) > \sigma, \qquad 0 \le \sigma < 1. \tag{5}$$

Similarly, $f \in \mathcal{A}$ is called *quasi-convex of order* σ if there exists a function $g \in \mathcal{K}(\beta)$ for which

$$\mathrm{Re}\left(\frac{(\omega f'(\omega))'}{g'(\omega)}\right) > \sigma, \qquad \omega \in \mathbb{D}. \tag{6}$$

Further details on these subclasses and their geometric characteristics may be found in [1–3].

A function $f \in \mathcal{A}$ is in the class of uniformly starlike functions, denoted by $\mathcal{UST}$, if

$$\mathrm{Re}\left\{\frac{\omega f'(\omega)}{f(\omega)}\right\} > \left|\frac{\omega f'(\omega)}{f(\omega)} - 1\right|, \quad \omega \in \mathbb{D}. \tag{7}$$

The class $\mathcal{UST}$ was introduced by Goodman [4].

A function $f \in \mathcal{A}$ is in the class of uniformly convex functions (see [5]), denoted by $\mathcal{UCV}$, if

$$\mathrm{Re}\left\{1 + \frac{\omega f''(\omega)}{f'(\omega)}\right\} > \left|\frac{\omega f''(\omega)}{f'(\omega)}\right|, \quad \omega \in \mathbb{D}. \tag{8}$$

Equivalently,

$$\mathcal{UCV} = \left\{f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{\omega f''(\omega)}{f'(\omega)}\right) > \left|\frac{\omega f''(\omega)}{f'(\omega)}\right|, \quad \omega \in \mathbb{D}\right\}. \tag{9}$$

A noteworthy extension of the classical starlike and convex function classes was proposed by Kanas and Wiśniowska [6] through the introduction of the families of ν-uniformly starlike and ν-uniformly convex functions. These classes are associated with conic domains and provide a unified framework that includes many well-known subclasses of class $\mathcal{S}$. Their analytic descriptions are stated as follows.

A function $f \in \mathcal{A}$ is in the class ν–$\mathcal{UCV}$, where $\nu \ge 0$, whenever

$$\mathrm{Re}\left(1 + \frac{\omega f''(\omega)}{f'(\omega)}\right) > \nu\left|\frac{\omega f''(\omega)}{f'(\omega)}\right|, \qquad \omega \in \mathbb{D}. \tag{10}$$

Similarly, $f \in \mathcal{A}$ belongs to the class ν–$\mathcal{ST}$ if

$$\mathrm{Re}\left(\frac{\omega f'(\omega)}{f(\omega)}\right) > \nu\left|\frac{\omega f'(\omega)}{f(\omega)} - 1\right|, \qquad \omega \in \mathbb{D}. \tag{11}$$

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Special functions have a fundamental role in geometric function theory (GFT) owing to their rich analytic structure and their close connections with differential equations, integral transforms, fractional calculus, and mathematical physics. From these, generalized hypergeometric functions constitute one of the most important families. Their importance increased considerably following the resolution of the Bieberbach conjecture, which stimulated extensive research on their geometric behavior. In particular, Balasubramanian et al. [7] analyzed convexity-preserving integral transforms associated with hypergeometric functions, while Miller and Mocanu [8] developed fundamental univalence criteria for the hypergeometric-type special functions by means of differential subordination techniques. Ponnusamy and Rønning [9] studied convolution properties and geometric characteristics of hypergeometric functions, whereas Ponnusamy et al. [10] derived sufficient conditions for the subclasses $\mathcal{S}^*$ and $\mathcal{C}$ by these functions. More recently, considerable attention has shifted from classical hypergeometric functions to Mittag–Leffler-type functions and their several generalizations, due to their fundamental role in fractional calculus, fractional differential equations, and GFT. Recently, Alenazi and Mehrez [11] established new geometric criteria for subclasses of analytic functions by using a generalized Mittag–Leffler-type special function, including various inclusion and univalence results. Taşar et al. [12] introduced convolution operators involving the Mittag–Leffler function to study harmonic starlike function classes and derived various geometric characterizations. More recently, Abubaker et al. [13] employed fractional integral operators related with the four-parameter Mittag–Leffler function to investigate a new criteria for the geometric properties like starlikeness, convexity, and close-to-convexity by means of differential subordination techniques. Furthermore, Matarneh et al. [14] investigated normalized Le Roy-type q-Mittag–Leffler functions and established close-to-convexity criteria, partial sum estimates, and several coefficient inequalities. More recently, research on geometric function theory has expanded to several new families of special functions beyond the classical hypergeometric and Mittag–Leffler functions. Al-Shaikh et al. [15] introduced a normalized Euler–polynomial function and established sufficient conditions for its univalence, and geometric properties by employing coefficient monotonicity, Fejér-type inequalities, and classical geometric criteria. Furthermore, Khan and AbaOud [16] investigated the generalized Marcum Q-function related with Janowski functions and derived several new geometric results involving coefficient estimates, inclusion relations, and harmonic mappings associated with Janowski domains. These recent developments demonstrate that normalized special functions continue to provide a fertile framework for constructing new subclasses of analytic and harmonic functions and for investigating their geometric behavior. Motivated by these advances, the present work introduces a normalized generalized Bessel–Maitland–Tremblay function and investigates its geometric properties through the interaction of generalized Bessel–Maitland functions and fractional differential operators.

Another important family of special functions is the Bessel–Maitland family, which is closely related to generalized hypergeometric functions, Mittag–Leffler functions, fractional calculus, and fractional differential equations. The function

$$J_\kappa^\rho(\omega) = W_{\rho,\kappa+1}(-\omega), \qquad \rho > 0, \quad \kappa > -1, \tag{12}$$

is known as the Bessel–Maitland (BM) function or generalized Bessel function [17]. It admits the series representation

$$J_\kappa^\rho(\omega) = \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!\,\Gamma(\rho n + \kappa + 1)}, \qquad \omega \in \mathbb{C}. \tag{13}$$

A natural generalization of Equation (13) is the generalized Bessel–Maitland function defined by

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$$J^{\rho}_{\kappa,\xi}(\omega) = \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!\,\Gamma(\rho n + \kappa + 1)}\omega^n, \qquad \xi \in \mathbb{C} \setminus \{0\}, \quad \rho > 0, \quad \kappa > -1, \quad \omega \in \mathbb{D}. \tag{14}$$

This function has attracted increasing attention because of its applications in fractional calculus, integral transforms, and mathematical physics. For example, Albayrak et al. [18] introduced an integral transform whose kernel is given by the generalized BM function and studied its analytical properties and applications. Zayed [19] later developed another approach to the generalized Bessel–Maitland function and obtained further properties and applications. These developments motivate the study of geometric characteristics of generalized BM function through fractional differential operators, particularly in connection with normalized analytic functions and subclasses defined by starlikeness, convexity, uniform convexity, exponential starlikeness, and lemniscate-type conditions.

1.1. Fractional Calculus

Fractional calculus has become an indispensable tool in modern mathematical analysis because of its ability to model nonlocal phenomena, hereditary properties, and memory-dependent processes. Beyond its classical applications to fractional differential equations, mathematical physics, and engineering, fractional differential operators have recently attracted considerable attention in geometric function theory, where they provide effective tools for studying subclasses of analytic and univalent functions through coefficient estimates, convolution theory, differential subordinations, superordinations, and admissibility conditions. Comprehensive treatments of the theory of fractional calculus and the classical Riemann–Liouville operators may be found in the monographs of Srivastava and Owa [20], Podlubny [21], Tarasov [22], and Baleanu et al. [23].

Among the fractional differential operators used in GFT, the Tremblay fractional differential operator occupies a prominent position. Originally introduced by Tremblay [24] through the Riemann–Liouville fractional derivative, thereafter, Ibrahim and Jahangiri [25] demonstrated its effectiveness in deriving geometric properties of AFs. Later, Esa et al. [26] proposed a modified complex Tremblay operator and investigated its applications to analytic function classes, while Srivastava et al. [27] employed the Tremblay operator to establish Faber polynomial coefficient estimates for bi-univalent functions, (see for more details [28–30]).

Recent research has further explained the growing importance of fractional operators in geometric function theory. Khan et al. [31] used fractional differential operators to obtain a new family of uniformly q-starlike functions together with coefficient estimates and inclusion relations. Alsoboh and Galıb [32] introduced a q-analogue of the modified Tremblay fractional derivative operator and investigated growth, distortion, convolution, and radius results for the associated analytic function classes. The majorization problems of various subclasses of analytic functions connected with the modified Tremblay fractional operator were investigated by Mohan and Venkataraman [33], while Abdulnaby and Ali [34] developed new applications of generalized Tremblay fractional differential operators for various subclasses of analytic functions. Recent investigations have further studied the applications of the Tremblay fractional derivative operator in GFT. In particular, Tayyah and Atshan [35] proposed new subclasses of bi-Bazilevič and bi-pseudo-starlike functions, while Tayyah, Atshan, and Yalçın [36] established third-order subordination and superordination principles for p-valent analytic functions associated with fractional differential operators. These developments clearly demonstrated that fractional differential operators continue to provide a fruitful framework for defining new family of AFs and investigating their geometric behavior. Motivated by these recent advances, the present paper introduces a normalized generalized Bessel–Maitland–Tremblay function and investigates its geometric properties by combining the modified Tremblay fractional differential operator with the generalized Bessel–Maitland function through the Hadamard product.

Definition 1. *The Riemann–Liouville fractional integral of order δ is defined by*

$$\mathcal{I}^{\delta} f(\omega) = \frac{1}{\Gamma(\delta)} \int_0^{\omega} (\omega - \zeta)^{\delta-1} f(\zeta)\, d\zeta,$$

where f is analytic in a domain simply connected containing the origin, and $\delta > 0$.

Definition 2. *Let $0 \leq \delta < 1$. The Riemann–Liouville fractional derivative of order δ is defined by*

$$\mathcal{D}^{\delta} f(\omega) = \frac{1}{\Gamma(1-\delta)} \frac{d}{d\omega} \int_0^{\omega} \frac{f(\zeta)}{(\omega - \zeta)^{\delta}}\, d\zeta.$$

More generally, if $m \in \mathbb{N}_0$, then

$$\mathcal{D}^{m+\delta} f(\omega) = \frac{d^m}{d\omega^m} \Big(\mathcal{D}^{\delta} f(\omega) \Big).$$

Motivated by the above developments, we recall the definition of the modified Tremblay fractional differential operator.

Definition 3. *Let*

$$0 < \tau \leq 1, \qquad 0 < \mu \leq 1, \qquad 0 \leq \mu - \tau < 1.$$

The normalized modified Tremblay fractional differential operator is defined by

$$\mathfrak{T}_{\mu,\tau} f(\omega) = \frac{\Gamma(\tau+1)}{\Gamma(\mu+1)} \omega^{1-\tau} \mathcal{D}^{\mu-\tau} \Big(\omega^{\mu-1} f(\omega) \Big), \tag{15}$$

where the operator $\mathcal{D}^{\mu-\tau}$ is interpreted as the Riemann–Liouville fractional derivative of order $\mu - \tau$, following the formulation in [25,26].

Applying Equation (15) to the Taylor expansion of $f \in \mathcal{A}$ yields

$$\mathfrak{T}_{\mu,\tau} f(\omega) = \omega + \sum_{n=2}^{\infty} \frac{\Gamma(\tau+1)\Gamma(\mu+n)}{\Gamma(\mu+1)\Gamma(\tau+n)} a_n \omega^n. \tag{16}$$

Using the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

the above expansion may be rewritten as

$$\mathfrak{T}_{\mu,\tau} f(\omega) = \omega + \sum_{n=2}^{\infty} \frac{(\mu+1)_{n-1}}{(\tau+1)_{n-1}} a_n \omega^n. \tag{17}$$

Clearly,

$$\mathfrak{T}_{1,1} f(\omega) = f(\omega),$$

which shows that the normalized modified Tremblay operator reduces to the identity operator.

Furthermore, choosing $\mu = \tau + 1$ gives

$$\mathfrak{T}_{\tau+1,\tau} f(\omega) = \frac{1}{\tau+1} \big(\tau f(\omega) + \omega f'(\omega) \big), \tag{18}$$

which coincides with the modified Tremblay operator introduced by Esa et al. [26].

The operator $\mathfrak{T}_{\mu,\tau}$ satisfies the following recurrence relations (see [37]):

$$\omega\big(\mathfrak{T}_{\mu,\tau}f(\omega)\big)' = (\mu+1)\mathfrak{T}_{\mu+1,\tau}f(\omega) - \mu\,\mathfrak{T}_{\mu,\tau}f(\omega), \tag{19}$$

and

$$\omega\big(\mathfrak{T}_{\mu,\tau+1}f(\omega)\big)' = (\tau+1)\mathfrak{T}_{\mu,\tau}f(\omega) - \tau\,\mathfrak{T}_{\mu,\tau+1}f(\omega). \tag{20}$$

Motivated by recent advances in generalized Bessel–Maitland functions, modified Tremblay fractional differential operators, and their applications in geometric function theory, this work addresses an existing gap in the literature. Although these two concepts have been studied extensively and have produced significant results in coefficient estimates, differential subordinations, and geometric properties of analytic functions, their combined effect has not yet been investigated. To fill this gap, we introduce a new normalized generalized Bessel–Maitland–Tremblay function via the Hadamard product of the modified Tremblay fractional differential operator and the generalized Bessel–Maitland function. Sufficient conditions are then established for this function to belong to the classes of uniformly convex, starlike, convex, ν-uniformly starlike, ν-uniformly convex, exponential starlike, exponential convex, Bernoulli lemniscate starlike, and Bernoulli lemniscate convex functions. The theoretical findings are further illustrated through numerical examples and Figures 1–6, demonstrating the applicability and effectiveness of the proposed framework.

1.2. A Generalized Fractional Bessel–Maitland–Tremblay Function

Motivated by the normalized Tremblay-type fractional differential operator and the normalized generalized Bessel–Maitland function, we introduce the following generalized Bessel–Maitland–Tremblay function.

Definition 4. *Let $f \in \mathcal{A}$. The generalized Bessel–Maitland–Tremblay transform is defined by*

$$\boxed{\left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}f\right)(\omega) = \mathcal{B}^{\rho}_{\kappa,\xi}(\omega) * \big(\mathfrak{T}_{\mu,\tau}f\big)(\omega),} \tag{21}$$

where $*$ denotes the Hadamard product.

Since

$$\mathfrak{T}_{\mu,\tau}f(\omega) = \omega + \sum_{n=2}^{\infty} \tau_u a_n \omega^n,$$

where

$$\tau_n = \frac{\Gamma(\tau+1)\Gamma(\mu+n)}{\Gamma(\mu+1)\Gamma(\tau+n)},$$

and

$$\mathcal{B}^{\rho}_{\kappa,\xi}(\omega) = \omega + \sum_{n=2}^{\infty} \beta_n \omega^n,$$

with

$$\beta_n = \frac{\Gamma(\kappa+1)(-\xi)^{n-1}}{(n-1)!\Gamma(\rho(n-1)+\kappa+1)},$$

it follows from the definition of the Hadamard product that

$$\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi} f\right)(\omega) = \omega + \sum_{n=2}^{\infty} \Theta_n a_n \omega^n. \tag{22}$$

where $\Theta_n = \tau_n \beta_u$. Choosing

$$f(\omega) = \frac{\omega}{1-\omega} = \omega + \sum_{n=2}^{\infty} \omega^n,$$

we obtain

$$\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega) = \omega + \sum_{n=2}^{\infty} \Theta_n \omega^n, \tag{23}$$

where

$$\Theta_n = \frac{\Gamma(\tau+1)\Gamma(\mu+n)}{\Gamma(\mu+1)\Gamma(\tau+n)} \frac{\Gamma(\kappa+1)(-\xi)^{n-1}}{(n-1)!\Gamma(\rho(n-1)+\kappa+1)}, \qquad n \geq 2. \tag{24}$$

Here,

$$0 < \tau \leq 1, \qquad 0 < \mu \leq 1, \qquad 0 \leq \mu - \tau < 1, \qquad \rho > 0, \qquad \kappa > -1, \qquad \xi \in \mathbb{C}.$$

Clearly, $\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi} \in \mathcal{A}$.

Throughout this paper, the function defined by Equation (23) will be referred to as the normalized generalized Bessel–Maitland–Tremblay function.

Remark 1. *1. If $\mu = \tau = 1$, then*

$$\Theta_n = \frac{\Gamma(\kappa+1)(-\xi)^{n-1}}{(n-1)!\Gamma(\rho(n-1)+\kappa+1)},$$

and hence, $\mathfrak{B}_{1,1}^{\rho,\kappa,\xi}(\omega) = \mathcal{B}_{\kappa,\xi}^{\rho}(\omega)$, which reduces to the normalized generalized Bessel–Maitland function.

2. *If, $\xi = 0$, then $\mathfrak{B}_{1,1}^{\rho,\kappa,0}(\omega) = \omega$, which is precisely the identity mapping in the class $\mathcal{A}$.*

2. Set of Lemmas

In this section, we give the well-known lemmas that will be used in establishing the main theorems of the paper.

Lemma 1 ([38], Corollary 2.5, p. 121)**.** *Let $f \in \mathcal{A}$. If*

$$\left|\frac{\omega f''(\omega)}{f'(\omega)}\right| < \frac{1}{2}, \qquad \omega \in \mathbb{D}, \tag{25}$$

then, $f \in \mathcal{UCV}$.

Lemma 2 ([39], Equation (2.2), p. 374)**.** *For $q > 1$, let*

$$\psi(q) = \frac{\Gamma'(q)}{\Gamma(q)}$$

denote the digamma function. Then

$$\log q - \gamma \leq \psi(q) \leq \log q. \tag{26}$$

Lemma 3 ([40], Theorem 2.3, Equation (2.5), p. 648)**.** *Let $f(\omega)$ of the form* (1) *and if*

$$\sum_{n=2}^{\infty}\big(n+\nu(n-1)\big)|a_n|<1, \qquad \nu\geq 0, \tag{27}$$

then $f\in\nu$–$\mathcal{ST}$.

Lemma 4 ([6], Theorem 3.3, Equation (3.15), p. 334)**.** *Let $f(\omega)$ of the form* (1) *and if*

$$\sum_{n=2}^{\infty}n(n-1)|a_n|\leq\frac{1}{\nu+2}, \qquad \nu\geq 0, \tag{28}$$

then $f\in\nu$–$\mathcal{UCV}$.

3. Main Results

Theorem 1. *Let $\kappa\geq 0$, $\rho\geq 1$, $\xi\in\mathbb{C}\setminus\{0\}$, $0<\tau\leq 1$, $\quad 0<\mu\leq 1$, $\quad 0\leq\mu-\tau<1$, where*

$$p=\mu-\tau, \quad K_{\mu,\tau}=\frac{\Gamma(\tau+1)}{\Gamma(\mu+1)}.$$

Furthermore, define

$$A_{\mu,\tau}=\sup_{n\geq 1}\left\{\frac{n(n+1)(n+\tau+1)^p}{n!}\right\},$$

and

$$B_{\mu,\tau}=\sup_{n\geq 1}\left\{\frac{(n+1)(n+\tau+1)^p}{n!}\right\}.$$

If

$$|\xi|<\frac{(\kappa+1)(\kappa+2)}{(\kappa+1)+K_{\mu,\tau}\big(2A_{\mu,\tau}+B_{\mu,\tau}\big)(\kappa+2)},$$

then

$$\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)\in\mathcal{UCV}.$$

Proof. By definition,

$$\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)=\omega+\sum_{n=1}^{\infty}A_n\omega^{n+1},$$

where

$$A_n=\frac{\Gamma(\tau+1)\Gamma(\mu+n+1)}{\Gamma(\mu+1)\Gamma(\tau+n+1)}\frac{\Gamma(\kappa+1)(-\xi)^n}{n!\Gamma(\rho n+\kappa+1)}.$$

Since $p=\mu-\tau\in[0,1)$, the gamma-ratio estimate yields

$$\frac{\Gamma(\mu+n+1)}{\Gamma(\tau+n+1)}\leq(n+\tau+1)^p,$$

and consequently,

$$|A_n|\leq K_{\mu,\tau}\frac{(n+\tau+1)^p|\xi|^n}{n!(\kappa+1)(\kappa+2)\cdots(\kappa+n)}.$$

Hence,

$$\left| \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi} \right)''(\omega) \right| < \frac{K_{\mu,\tau} A_{\mu,\tau} |\xi| (\kappa+2)}{(\kappa+1)(\kappa+2-|\xi|)}. \tag{29}$$

Similarly,

$$\left| \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi} \right)'(\omega) \right| > 1 - \frac{K_{\mu,\tau} B_{\mu,\tau} |\xi| (\kappa+2)}{(\kappa+1)(\kappa+2-|\xi|)}. \tag{30}$$

Combining Equations (29) and (30), we obtain

$$\left| \frac{\omega \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi} \right)''(\omega)}{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi} \right)'(\omega)} \right| < \frac{K_{\mu,\tau} A_{\mu,\tau} |\xi| (\kappa+2)}{(\kappa+1)(\kappa+2-|\xi|) - K_{\mu,\tau} B_{\mu,\tau} |\xi| (\kappa+2)}.$$

The assumed bound on $|\xi|$ implies

$$\left| \frac{\omega \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi} \right)''(\omega)}{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi} \right)'(\omega)} \right| < \frac{1}{2}.$$

Therefore, Lemma 1 yields $\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega) \in \mathcal{UCV}$. □

Example 1. *The following normalized generalized Bessel–Maitland–Tremblay functions satisfy the hypotheses of Theorem 1. Hence each of them belongs to the class $\mathcal{UCV}$.*

(i) $\mathfrak{B}_{1,1}^{1,0,\,0.108}(\omega) \in \mathcal{UCV}$.
(ii) $\mathfrak{B}_{0.95,\,0.20}^{1,0,\,-0.0531}(\omega) \in \mathcal{UCV}$.
(iii) $\mathfrak{B}_{0.90,\,0.10}^{1.1,\,1,\,0.0968}(\omega) \in \mathcal{UCV}$.
(iv) $\mathfrak{B}_{0.80,\,0.05}^{1.1,\,1,\,-0.0981}(\omega) \in \mathcal{UCV}$.

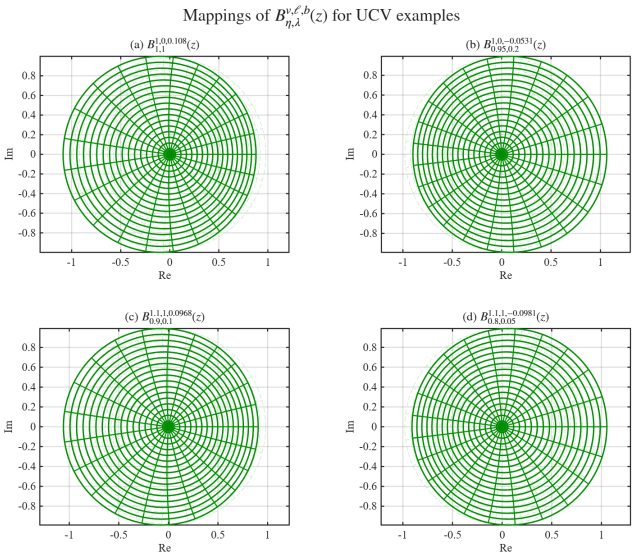

Figure 1. Images of concentric circles and radial lines in the unit disk under the normalized generalized Bessel–Maitland–Tremblay functions listed in Example 1. The selected parameters satisfy the sufficient conditions of Theorem 1. Hence, each mapping belongs to the class UCV.

Corollary 1 ([41])**.** *If $\mu = \tau = 1$, then $\mathfrak{B}_{1,1}^{\rho,\kappa,\xi}(\omega) = J_{\kappa,\xi}^{\rho}(\omega)$. Furthermore, $K_{1,1} = 1, A_{1,1} = 3, B_{1,1} = 2$.*

Consequently, Theorem 1 reduces to

$$|\xi| < \frac{(\kappa+1)(\kappa+2)}{9\kappa+17},$$

which coincides with the sufficient condition for the normalized generalized Bessel–Maitland function to belong to the class $\mathcal{UCV}$ established in [41].

Theorem 2. *Let* $\kappa \geq 0, \quad \rho > 0, \quad \xi \in \mathbb{C} \setminus \{0\}, \quad 0 \leq \delta < 1, 0 < \tau \leq \mu \leq 1.$

Suppose that

$$e^{1+\gamma(\rho+2)} \frac{\mu+2}{\tau+2} < 2(\rho+\kappa+1)^{\rho} \tag{31}$$

and

$$M_{\mu,\tau}|\xi|(2-\delta)\Gamma(\kappa+1) < (1-|\xi|)(1-\delta)\Gamma(\rho+\kappa+1), \tag{32}$$

where $M_{\mu,\tau} = \frac{\mu+1}{\tau+1}$*, then*

$$\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega) \in \mathcal{S}^*(\delta).$$

Proof. It is sufficient to verify that

$$\left| \frac{\omega \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)} - 1 \right| < 1-\delta, \qquad \omega \in \mathbb{D}.$$

Indeed,

$$\left| \frac{\omega \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)} - 1 \right| = \left| \frac{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega) - \dfrac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}}{\dfrac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}} \right|.$$

Therefore,

$$\left| \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega) - \frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega} \right| = \left| \sum_{n=1}^{\infty} n A_n \omega^n \right|,$$

where

$$A_n = \frac{\Gamma(\tau+1)\Gamma(\mu+n+1)}{\Gamma(\mu+1)\Gamma(\tau+n+1)} \frac{\Gamma(\kappa+1)(-\xi)^n}{n!\Gamma(\rho n+\kappa+1)}.$$

Hence,

$$\left| \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega) - \frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega} \right| \leq \Gamma(\kappa+1) \sum_{n=1}^{\infty} d_n |\xi|^n, \tag{33}$$

where

$$d_n = \frac{n}{n!\Gamma(\rho n+\kappa+1)} \frac{\Gamma(\tau+1)\Gamma(\mu+n+1)}{\Gamma(\mu+1)\Gamma(\tau+n+1)}.$$

Consider the function

$$D_1(r) = \frac{r}{\Gamma(r+1)\Gamma(\rho r+\kappa+1)} \frac{\Gamma(\tau+1)\Gamma(\mu+r+1)}{\Gamma(\mu+1)\Gamma(\tau+r+1)}, \qquad r \geq 1.$$

Taking logarithms gives

$$\begin{aligned}
\log D_1(r) = {} & \log r - \log\Gamma(\rho r+\kappa+1) - \log\Gamma(r+1) \\
& + \log\Gamma(\tau+1) + \log\Gamma(\mu+r+1) \\
& - \log\Gamma(\tau+r+1) - \log\Gamma(\mu+1).
\end{aligned}$$

Differentiating with respect to r, we obtain

$$\frac{D_1'(r)}{D_1(r)} = \frac{1}{r} - \psi(r+1) - \rho\psi(\rho r + \kappa + 1) + \psi(\mu + r + 1) - \psi(\tau + r + 1).$$

Therefore,

$$D_1'(r) = D_1(r) D_2(r),$$

where

$$D_2(r) = \frac{1}{r} - \psi(r+1) - \rho\psi(\rho r + \kappa + 1) + \psi(\mu + r + 1) - \psi(\tau + r + 1).$$

By Lemma 2, we have

$$\begin{aligned} D_2(r) \le{}& \frac{1}{r} - \log(r+1) - \rho\log(\rho r + \kappa + 1) \\ &+ \log(\mu + r + 1) - \log(\tau + r + 1) + \gamma(\rho + 2). \end{aligned}$$

Thus,

$$D_2(r) \le D_3(r),$$

where

$$D_3(r) = \frac{1}{r} - \log(r+1) - \rho\log(\rho r + \kappa + 1) + \log\left(\frac{\mu + r + 1}{\tau + r + 1}\right) + \gamma(\rho + 2).$$

Differentiating D_3, we obtain

$$D_3'(r) = -\frac{1}{r^2} - \frac{1}{r+1} - \frac{\rho^2}{\rho r + \kappa + 1} + \frac{\tau - \mu}{(\mu + r + 1)(\tau + r + 1)}.$$

Since $0 < \tau \le \mu \le 1$, we have $\tau - \mu \le 0$. Therefore,

$$D_3'(r) < 0, \qquad r \ge 1.$$

Hence D_3 is decreasing on $[1, \infty)$. Moreover,

$$D_3(1) = 1 - \log 2 - \rho\log(\rho + \kappa + 1) + \log\left(\frac{\mu + 2}{\tau + 2}\right) + \gamma(\rho + 2).$$

Condition (31) is equivalent to $D_3(1) < 0$. Since D_3 is decreasing,

$$D_3(r) \le D_3(1) < 0, \qquad r \ge 1.$$

Thus,

$$D_2(r) < 0, \qquad r \ge 1,$$

and consequently, $D_1'(r) < 0, r \ge 1$. It follows that the sequence $\{d_u\}_{n\ge 1}$ is decreasing. Therefore, $d_u \le d_1$, where

$$\begin{aligned} d_1 &= \frac{1}{\Gamma(\rho + \kappa + 1)} \frac{\Gamma(\tau + 1)\Gamma(\mu + 2)}{\Gamma(\mu + 1)\Gamma(\tau + 2)} \\ &= \frac{M_{\mu,\tau}}{\Gamma(\rho + \kappa + 1)}. \end{aligned}$$

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Using Equation (33), we obtain

$$\left|\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)-\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}\right|<\frac{\Gamma(\kappa+1)M_{\mu,\tau}|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1)}. \tag{34}$$

Next,

$$\left|\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}\right|=\left|1+\sum_{n=1}^{\infty}A_n\omega^n\right|.$$

Hence,

$$\left|\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}\right|\geq 1-\Gamma(\kappa+1)\sum_{n=1}^{\infty}s_n|\xi|^n, \tag{35}$$

where

$$s_n=\frac{1}{n!\Gamma(\rho n+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+n+1)}{\Gamma(\mu+1)\Gamma(\tau+n+1)}.$$

By an argument analogous to the one used above, the sequence $\{s_n\}_{n\geq 1}$ is decreasing. Hence,

$$s_n\leq s_1=\frac{M_{\mu,\tau}}{\Gamma(\rho+\kappa+1)}.$$

Using Equation (35), we obtain

$$\left|\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}\right|\geq 1-\frac{\Gamma(\kappa+1)M_{\mu,\tau}|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1)}. \tag{36}$$

Combining Equations (34) and (36), we have

$$\left|\frac{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)-\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}}{\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}}\right|<\frac{M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1)-M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}<1-\delta.$$

From this, we find condition (32). Consequently,

$$\left|\frac{\omega\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}-1\right|<1-\delta.$$

Therefore,

$$\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)\in\mathcal{S}^*(\delta).$$

This finishes the proof. □

Example 2. *The following functions satisfy the hypotheses of Theorem 2. Hence, each of them belongs to the class $S^*(\delta)$.*

(i) $\mathfrak{B}_{1,1}^{1,0,0.24}(\omega)\in S^*(\delta),\quad 0\leq\delta<0.200000.$
(ii) $\mathfrak{B}_{0.95,0.20}^{1,0,-0.16}(\omega)\in S^*(\delta),\quad 0\leq\delta<0.307864.$
(iii) $\mathfrak{B}_{0.90,0.10}^{1.1,0,0.14}(z)\in S^*(\delta),\quad 0\leq\delta<0.329498.$
(iv) $\mathfrak{B}_{0.80,0.05}^{1.2,0,-0.12}(z)\in S^*(\delta),\quad 0\leq\delta<0.358385.$

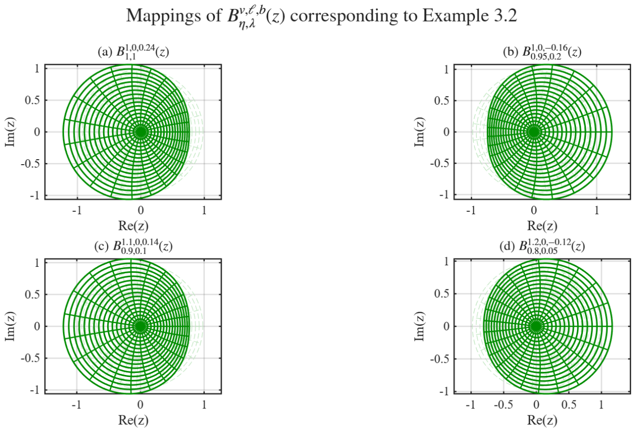

Figure 2. Mappings of the unit disk under the normalized generalized Bessel–Maitland–Tremblay functions corresponding to Example 2. The parameter sets in panels (**a**–**d**) satisfy the hypotheses of Theorem 2. The images preserve a starlike geometry with respect to the origin, while the variations in the parameters μ, τ, ρ, κ, and ξ produce different levels of geometric deformation, thereby illustrating the influence of these parameters on the mapping behavior and confirming the theoretical conclusion that $\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) \in S^*(\delta)$.

Theorem 3. *Suppose that $\kappa \geq 0$, $\rho > 0$, $\xi \in \mathbb{C} \setminus \{0\}$, $0 \leq \delta < 1$, and $0 < \tau \leq \mu \leq 1$. Suppose that*

$$e^{\frac{3}{2}+\gamma(\rho+2)} \frac{\mu+2}{\tau+2} < 2(\rho+\kappa+1)^{\rho} \tag{37}$$

and

$$2M_{\mu,\tau}|\xi|(2-\delta)\Gamma(\kappa+1) < (1-|\xi|)(1-\delta)\Gamma(\rho+\kappa+1), \tag{38}$$

where

$$M_{\mu,\tau} = \frac{\mu+1}{\tau+1}.$$

Then, $\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) \in C(\delta)$.

Proof. To prove that

$$\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) \in C(\delta),$$

it is sufficient to show that

$$\left| \frac{\omega \left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}\right)''(\omega)}{\left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}\right)'(\omega)} \right| < 1-\delta, \qquad \omega \in \mathbb{D}.$$

Now,

$$\left| \omega \left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}\right)''(\omega) \right| = \left| \sum_{n=1}^{\infty} n(n+1) A_n \omega^n \right|.$$

Hence,

$$\left| \omega \left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}\right)''(\omega) \right| \leq \Gamma(\kappa+1) \sum_{n=1}^{\infty} h_n |\xi|^n, \tag{39}$$

where

$$h_n = \frac{n(n+1)}{n!\Gamma(\rho n+\kappa+1)} \frac{\Gamma(\tau+1)\Gamma(\mu+n+1)}{\Gamma(\mu+1)\Gamma(\tau+n+1)}.$$

Consider the function

$$H_1(r) = \frac{r(r+1)}{\Gamma(r+1)\Gamma(\rho r+\kappa+1)} \frac{\Gamma(\tau+1)\Gamma(\mu+r+1)}{\Gamma(\mu+1)\Gamma(\tau+r+1)}, \qquad r \geq 1.$$

Thus

$$H_1'(r) = H_1(r)H_2(r),$$

where

$$H_2(r) = \frac{2r+1}{r(r+1)} - \psi(r+1) - \rho\psi(\rho r+\kappa+1) + \psi(\mu+r+1) - \psi(\tau+r+1).$$

Using Lemma 2, we obtain

$$H_2(r) \leq H_3(r),$$

where

$$H_3(r) = \frac{2r+1}{r(r+1)} - \log(r+1) - \rho\log(\rho r+\kappa+1) + \log\left(\frac{\mu+r+1}{\tau+r+1}\right) + \gamma(\rho+2).$$

Now, differentiating $H_3(r)$ gives

$$H_3'(r) = -\frac{2r^2+2r+1}{r^2(r+1)^2} - \frac{1}{r+1} - \frac{\rho^2}{\rho r+\kappa+1} + \frac{\tau-\mu}{(\mu+r+1)(\tau+r+1)}.$$

Since $0 < \tau \leq \mu \leq 1$, we have

$$H_3'(r) < 0, \qquad r \geq 1.$$

Thus, $H_3(r)$ is decreasing on $[1,\infty)$.

Moreover,

$$H_3(1) = \frac{3}{2} - \log 2 - \rho\log(\rho+\kappa+1) + \log\left(\frac{\mu+2}{\tau+2}\right) + \gamma(\rho+2).$$

Condition (37) is equivalent to $H_3(1) < 0$. Since $H_3(r)$ is decreasing, it follows that

$$H_3(r) \leq H_3(1) < 0, \quad r \geq 1.$$

Thus, $H_2(r) < 0, \quad r \geq 1$.

Consequently, $H_1'(r) < 0, r \geq 1$. Therefore, the sequence $\{h_n\}_{n\geq 1}$ is decreasing. Hence, $h_n \leq h_1$, where

$$h_1 = \frac{2}{\Gamma(\rho+\kappa+1)} \frac{\Gamma(\tau+1)\Gamma(\mu+2)}{\Gamma(\mu+1)\Gamma(\tau+2)} = \frac{2M_{\mu,\tau}}{\Gamma(\rho+\kappa+1)}.$$

Using Equation (39), we obtain

$$\left|\omega\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)''(\omega)\right| < \frac{2M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1)}.$$

Next,

$$\left|\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)\right| = \left|1 + \sum_{n=1}^{\infty}(n+1)A_n\omega^n\right|.$$

Therefore,

$$\left|\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)\right| \geq 1 - \Gamma(\kappa+1)\sum_{n=1}^{\infty} j_n|\xi|^n, \tag{40}$$

where

$$j_n = \frac{n+1}{n!\Gamma(\rho n+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+n+1)}{\Gamma(\mu+1)\Gamma(\tau+n+1)}.$$

In the same way, the sequence $\{j_n\}_{n\geq 1}$ is decreasing. Hence $j_n \leq j_1$, where

$$j_1 = \frac{2}{\Gamma(\rho+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+2)}{\Gamma(\mu+1)\Gamma(\tau+2)} = \frac{2M_{\mu,\tau}}{\Gamma(\rho+\kappa+1)}.$$

Thus,

$$\left|\left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}\right)'(\omega)\right| \geq 1 - \frac{2M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1)}. \tag{41}$$

Combining Equations (40) and (41), we obtain

$$\left|\frac{\omega\left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}\right)''(\omega)}{\left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}\right)'(\omega)}\right| < \frac{2M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1) - 2M_{\mu,\tau}\Gamma(\kappa+1)|\xi|} < 1-\delta.$$

Hence,

$$\frac{2M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1) - 2M_{\mu,\tau}\Gamma(\kappa+1)|\xi|} < 1-\delta.$$

We obtain the condition (38). Therefore,

$$\left|\frac{\omega\left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}\right)''(\omega)}{\left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}\right)'(\omega)}\right| < 1-\delta.$$

Hence, $\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) \in C(\delta)$. This finishes the proof. □

Example 3. *The following functions satisfy the hypotheses of Theorem 3.*

(i) $\mathfrak{B}^{1,0,\,0.48}_{1,1}(\omega) \in C(\delta)$.
(ii) $\mathfrak{B}^{1.05,0,\,-0.42}_{0.90,0.10}(\omega) \in C(\delta)$.
(iii) $\mathfrak{B}^{1.15,0,\,0.36}_{0.75,0.20}(\omega) \in C(\delta)$.
(iv) $\mathfrak{B}^{1.30,0,\,-0.32}_{0.60,0.05}(\omega) \in C(\delta)$.

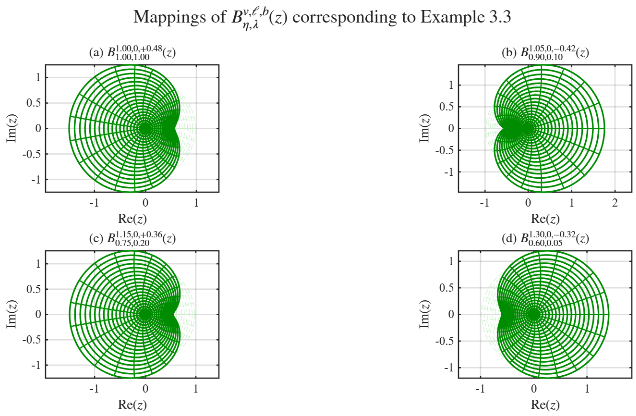

Figure 3. Mapping s of the unit disk under the normalized generalized Bessel–Maitland–Tremblay functions corresponding to Example 3. The parameter sets in panels (**a**–**d**) satisfy the hypotheses of Theorem 3. The images exhibit convex geometric behavior while preserving univalence. The variations in the parameters μ, τ, ρ, κ and ξ produce different geometric deformations of the image domains, illustrating the influence of these parameters on the mapping properties and confirming that $\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega)$.

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Theorem 4. *Let $\kappa \geq 0$, $\rho > 0$, $\xi \in \mathbb{C} \setminus \{0\}$, $\nu \geq 0$, and $0 < \tau \leq \mu \leq 1$. Suppose that*

$$e^{\frac{\nu+1}{\nu+2}+\gamma(\rho+2)}\frac{\mu+2}{\tau+2} < 2(\rho+\kappa+1)^{\rho} \tag{42}$$

and

$$M_{\mu,\tau}|\xi|(\nu+2)\Gamma(\kappa+1) < (1-|\xi|)\Gamma(\rho+\kappa+1), \tag{43}$$

where

$$M_{\mu,\tau} = \frac{\mu+1}{\tau+1}.$$

Then, $\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) \in \nu\text{–}\mathcal{ST}$.

Proof. By definition,

$$\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) = \omega + \sum_{n=2}^{\infty} A_{n-1}\omega^n,$$

where

$$A_{n-1} = \frac{\Gamma(\tau+1)\Gamma(\mu+n)}{\Gamma(\mu+1)\Gamma(\tau+n)}\frac{\Gamma(\kappa+1)(-\xi)^{n-1}}{(n-1)!\Gamma(\rho(n-1)+\kappa+1)}.$$

By Lemma 3, it is sufficient to prove that

$$\sum_{n=2}^{\infty}(n+\nu(n-1))|A_{n-1}| < 1.$$

Thus,

$$\begin{aligned}\sum_{n=2}^{\infty}(n+\nu(n-1))|A_{n-1}| &= \sum_{n=2}^{\infty}((\nu+1)n-\nu)|A_{n-1}| \\ &\leq \Gamma(\kappa+1)\sum_{n=2}^{\infty} q_u|\xi|^{n-1},\end{aligned} \tag{44}$$

where

$$q_n = \frac{(\nu+1)n-\nu}{(n-1)!\Gamma(\rho(n-1)+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+n)}{\Gamma(\mu+1)\Gamma(\tau+n)}.$$

To estimate q_n, put $r = n-1$. Then, $r \geq 1$ and

$$(\nu+1)n-\nu = (\nu+1)(r+1)-\nu = (\nu+1)r+1.$$

Therefore,

$$q_n = \frac{(\nu+1)r+1}{\Gamma(r+1)\Gamma(\rho r+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+r+1)}{\Gamma(\mu+1)\Gamma(\tau+r+1)}.$$

Consider the function

$$Q_1(r) = \frac{(\nu+1)r+1}{\Gamma(r+1)\Gamma(\rho r+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+r+1)}{\Gamma(\mu+1)\Gamma(\tau+r+1)}, \qquad r \geq 1.$$

Differentiating with respect to r, we obtain

$$Q_1'(r) = Q_1(r)Q_2(r),$$

where

$$Q_2(r) = \frac{\nu+1}{(\nu+1)r+1} - \psi(r+1) - \rho\psi(\rho r+\kappa+1) + \psi(\mu+r+1) - \psi(\tau+r+1).$$

Using Lemma 2, we have

$$Q_2(r) \leq \frac{\nu+1}{(\nu+1)r+1} - \log(r+1) - \rho\log(\rho r+\kappa+1)$$
$$+ \log(\mu+r+1) - \log(\tau+r+1) + \gamma(\rho+2).$$

Thus,

$$Q_2(r) \leq Q_3(r),$$

where

$$Q_3(r) = \frac{\nu+1}{(\nu+1)r+1} - \log(r+1) - \rho\log(\rho r+\kappa+1) + \log\left(\frac{\mu+r+1}{\tau+r+1}\right) + \gamma(\rho+2).$$

Again derivative with respect to r, we have

$$Q_3'(r) = -\frac{(\nu+1)^2}{\left((\nu+1)r+1\right)^2} - \frac{1}{r+1} - \frac{\rho^2}{\rho r+\kappa+1} + \frac{\tau-\mu}{(\mu+r+1)(\tau+r+1)}.$$

Since $0 < \tau \leq \mu \leq 1$, we have

$$\tau - \mu \leq 0.$$

Therefore,

$$Q_3'(r) < 0, \qquad r \geq 1.$$

Thus, $Q_3(r)$ is decreasing on $[1, \infty)$. Moreover,

$$Q_3(1) = \frac{\nu+1}{\nu+2} - \log 2 - \rho\log(\rho+\kappa+1) + \log\left(\frac{\mu+2}{\tau+2}\right) + \gamma(\rho+2).$$

Condition (42) is equivalent to $Q_3(1) < 0$. Since $Q_3(r)$ is decreasing on $[1, \infty)$, we obtain

$$Q_3(r) \leq Q_3(1) < 0, \qquad r \geq 1.$$

Hence, $Q_2(r) < 0, r \geq 1$. Consequently, $Q_1'(r) < 0, r \geq 1$. Therefore, the sequence $\{q_n\}_{n\geq 2}$ is decreasing. Thus, $q_n \leq q_2$. For $n = 2$, equivalently $r = 1$, we have

$$q_2 = \frac{\nu+2}{\Gamma(\rho+\kappa+1)} \frac{\Gamma(\tau+1)\Gamma(\mu+2)}{\Gamma(\mu+1)\Gamma(\tau+2)} = \frac{(\nu+2)M_{\mu,\tau}}{\Gamma(\rho+\kappa+1)}.$$

Using this estimate in Equation (44), we obtain

$$\sum_{n=2}^{\infty}(n+\nu(n-1))|A_{n-1}| \leq \Gamma(\kappa+1)\sum_{n=2}^{\infty} q_2|\xi|^{n-1}.$$

Therefore,

$$\sum_{n=2}^{\infty}(n+\nu(n-1))|A_{n-1}| < \Gamma(\kappa+1)q_2\sum_{n=2}^{\infty}|\xi|^{n-1}.$$

Since

$$\sum_{n=2}^{\infty}|\xi|^{n-1} = |\xi| + |\xi|^2 + |\xi|^3 + \cdots = \frac{|\xi|}{1-|\xi|},$$

we find

$$\sum_{n=2}^{\infty}(n+\nu(n-1))|A_{n-1}| < \frac{M_{\mu,\tau}(\nu+2)\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1)}.$$

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By condition (43), we have

$$\frac{M_{\mu,\tau}(\nu+2)\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1)} < 1.$$

Therefore, by Lemma 3,

$$\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) \in \nu\text{–}\mathcal{ST}.$$

This finishes the proof. □

Example 4. *The following normalized generalized Bessel–Maitland–Tremblay functions satisfy the hypotheses of Theorem 4. Hence, they belong to the class ν–$\mathcal{ST}$ in the unit disk $\mathbb{D}$.*

(i) $\mathfrak{B}^{1,0,0.45}_{1,1}(\omega) \in \nu\text{–}\mathcal{ST}$.
(ii) $\mathfrak{B}^{1.10,0,-0.40}_{0.90,0.60}(\omega) \in \nu\text{–}\mathcal{ST}$.
(iii) $\mathfrak{B}^{1.20,0,0.35}_{0.80,0.40}(\omega) \in \nu\text{–}\mathcal{ST}$.
(iv) $\mathfrak{B}^{1.35,0,-0.30}_{0.70,0.30}(\omega) \in \nu\text{–}\mathcal{ST}$.

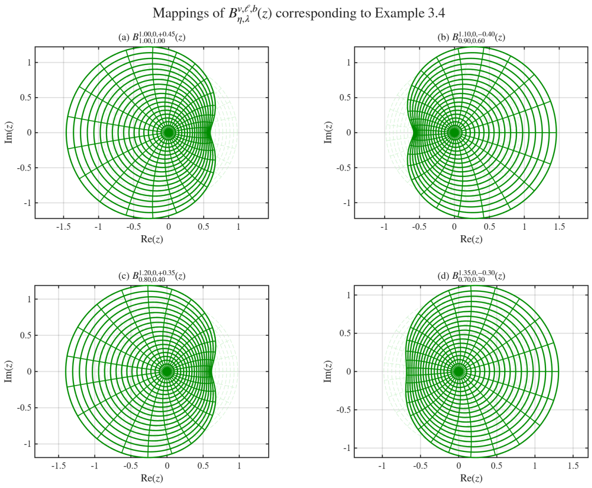

Figure 4. Mappings of the unit disk under the normalized generalized Bessel–Maitland–Tremblay functions for Example 4. Panels (**a**–**d**) correspond to parameter sets satisfying Theorem 4, illustrating the ν-starlike geometry of $\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega)$.

Theorem 5. *Let $\kappa \geq 0$, $\rho > 0$, $\xi \in \mathbb{C} \setminus \{0\}$, $\nu \geq 0$, and $0 < \tau \leq \mu \leq 1$ Suppose that*

$$e^{3+\gamma(\rho+2)}\frac{\mu+2}{\tau+2} < 2(\rho+\kappa+1)^{\rho} \tag{45}$$

and

$$2M_{\mu,\tau}|\xi|(\nu+2)\Gamma(\kappa+1) < (1-|\xi|)\Gamma(\rho+\kappa+1). \tag{46}$$

Then

$$\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) \in \nu\text{–}\mathcal{UCV}.$$

Proof. By Lemma 4, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(n-1)|A_{n-1}| < \frac{1}{\nu+2}.$$

Equivalently, it is enough to show that

$$(\nu+2)\sum_{n=2}^{\infty} n(n-1)|A_{n-1}| < 1.$$

Now,

$$(\nu+2)\sum_{n=2}^{\infty} n(n-1)|A_{n-1}| \leq (\nu+2)\Gamma(\kappa+1)\sum_{n=2}^{\infty} p_n|\xi|^{n-1}, \tag{47}$$

where

$$p_n = \frac{n(n-1)}{(n-1)!\Gamma(\rho(n-1)+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+n)}{\Gamma(\mu+1)\Gamma(\tau+n)}.$$

Put $r = n-1$. Then $r \geq 1$ and

$$n(n-1) = (r+1)r.$$

Hence,

$$p_n = \frac{r(r+1)}{\Gamma(r+1)\Gamma(\rho r+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+r+1)}{\Gamma(\mu+1)\Gamma(\tau+r+1)}.$$

Consider

$$P_1(r) = \frac{r(r+1)}{\Gamma(r+1)\Gamma(\rho r+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+r+1)}{\Gamma(\mu+1)\Gamma(\tau+r+1)}, \qquad r \geq 1.$$

Thus,

$$P_1'(r) = P_1(r)P_2(r),$$

where

$$P_2(r) = \frac{2r+1}{r(r+1)} - \psi(r+1) - \rho\psi(\rho r+\kappa+1) + \psi(\mu+r+1) - \psi(\tau+r+1).$$

Using Lemma 2, we have $P_2(r) \leq P_3(r)$, where

$$P_3(r) = \frac{2r+1}{r(r+1)} - \log(r+1) - \rho\log(\rho r+\kappa+1) + \log\left(\frac{\mu+r+1}{\tau+r+1}\right) + \gamma(\rho+2).$$

Thus, we obtain

$$P_3'(r) = -\frac{2r^2+2r+1}{r^2(r+1)^2} - \frac{1}{r+1} - \frac{\rho^2}{\rho r+\kappa+1} + \frac{\tau-\mu}{(\mu+r+1)(\tau+r+1)}.$$

Since $0 < \tau \leq \mu \leq 1$,

$$P_3'(r) < 0, \qquad r \geq 1.$$

Thus $P_3(r)$ is decreasing on $[1,\infty)$.

Moreover,

$$P_3(1) = \frac{3}{2} - \log 2 - \rho\log(\rho+\kappa+1) + \log\left(\frac{\mu+2}{\tau+2}\right) + \gamma(\rho+2).$$

Therefore,

$$P_3(r) \leq P_3(1) < 0, \qquad r \geq 1.$$

Consequently, $P_2(r) < 0, r \geq 1$, and hence,

$$P_1'(r) < 0, \quad r \geq 1.$$

Thus, the sequence $\{p_n\}_{n\geq 2}$ is decreasing, and so $p_n \leq p_2$. For $n = 2$, that is $r = 1$, we have

$$p_2 = \frac{2}{\Gamma(\rho+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+2)}{\Gamma(\mu+1)\Gamma(\tau+2)} = \frac{2M_{\mu,\tau}}{\Gamma(\rho+\kappa+1)}.$$

Using this in Equation (47), we obtain

$$(\nu+2)\sum_{n=2}^{\infty} n(n-1)|A_{n-1}| < (\nu+2)\Gamma(\kappa+1)p_2\sum_{n=2}^{\infty}|\xi|^{n-1}$$
$$= (\nu+2)\Gamma(\kappa+1)p_2\frac{|\xi|}{1-|\xi|}.$$

Therefore,

$$(\nu+2)\sum_{n=2}^{\infty} n(n-1)|A_{n-1}| < \frac{2M_{\mu,\tau}(\nu+2)\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1)}.$$

By condition (46), we obtain

$$\frac{2M_{\mu,\tau}(\nu+2)\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1)} < 1.$$

Hence,

$$(\nu+2)\sum_{n=2}^{\infty} n(n-1)|A_{n-1}| < 1,$$

which gives

$$\sum_{n=2}^{\infty} n(n-1)|A_{n-1}| < \frac{1}{\nu+2}.$$

Therefore, by Lemma 4, we have

$$\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega) \in \nu\text{–}\mathcal{UCV}.$$

This finishes the proof. □

Example 5. *The following normalized generalized Bessel–Maitland–Tremblay functions satisfy the hypotheses of Theorem 5. Hence, they belong to the class ν–$\mathcal{UCV}$ in the unit disk $\mathbb{D}$.*

(i) *The function* $\mathfrak{B}_{1,1}^{1,0,0.18}(\omega) \in \nu\text{–}\mathcal{UCV}$.
(ii) *The function* $\mathfrak{B}_{0.90,0.60}^{1.10,0,-0.15}(\omega) \in \nu\text{–}\mathcal{UCV}$.
(iii) *The function* $\mathfrak{B}_{0.80,0.40}^{1.20,0,0.13}(\omega) \in \nu\text{–}\mathcal{UCV}$.
(iv) *The function* $\mathfrak{B}_{0.70,0.30}^{1.35,0,-0.10}(\omega) \in \nu\text{–}\mathcal{UCV}$.

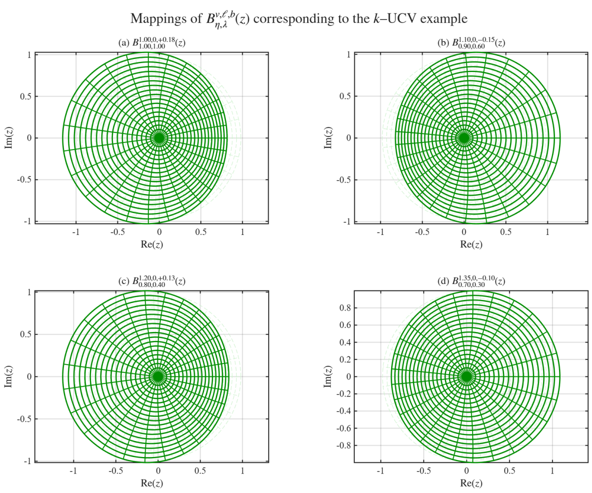

Figure 5. Mappings of the unit disk under the normalized generalized Bessel–Maitland–Tremblay functions for Example 5. Panels (**a**–**d**) satisfy Theorem 5 and illustrate the ν-uniformly convex behavior of $\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega) \in C(\delta)$..

Theorem 6. *Let $\kappa \geq 0$, $\rho > 0$, $\xi \in \mathbb{C} \setminus \{0\}$, $0 < \tau \leq \mu \leq 1$, and $M_{\mu,\tau} = \frac{\mu+1}{\tau+1}$. Suppose that*

(i)

$$e^{1+\gamma(\rho+2)} \frac{\mu+2}{\tau+2} < 2(\rho+\kappa+1)^{\rho},$$

(ii)

$$M_{\mu,\tau}(2e-1)|\xi|\Gamma(\kappa+1) < (e-1)(1-|\xi|)\Gamma(\rho+\kappa+1).$$

Then, $\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega) \in \mathcal{S}_e^*$.

Proof. To prove that $\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega) \in \mathcal{S}_e^*$, it is sufficient to show [42] that

$$\left| \frac{\omega \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)} - 1 \right| < 1 - \frac{1}{e}.$$

Now,

$$\left| \frac{\omega \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)} - 1 \right| = \left| \frac{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega) - \frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}}{\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}} \right|.$$

By the estimates obtained in the proof of Theorem 2, we have

$$\left| \frac{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega) - \frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}}{\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}} \right| < \frac{M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1) - M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}.$$

Since hypothesis (ii) is equivalent to

$$\frac{M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1) - M_{\mu,\tau}\Gamma(\kappa+1)|\xi|} < 1 - \frac{1}{e},$$

we obtain

$$\left| \frac{\omega \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)} - 1 \right| < 1 - \frac{1}{e}.$$

Thus, by the sufficient condition for the class $\mathcal{S}_e^*$ stated in [42], we find

$$\frac{\omega \left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)} \prec e^{\omega}.$$

Consequently,

$$\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega) \in \mathcal{S}_e^*.$$

This finishes the proof. □

Example 6. *The following normalized generalized Bessel–Maitland–Tremblay functions satisfy the hypotheses of Theorem 6. Hence, each of them belongs to the class $\mathcal{S}_e^*$ in the unit disk $\mathbb{D}$.*

(i) *The function $\mathfrak{B}_{1,1}^{5,2,\frac{1}{4}}(\omega) \in \mathcal{S}_e^*$.*

(ii) *The function $\mathfrak{B}_{0.90,0.50}^{4,\frac{5}{4},\frac{1}{3}}(\omega) \in \mathcal{S}_e^*$.*

(iii) *The function* $\mathfrak{B}^{3,\frac{4}{3},\frac{1}{6}}_{0.80,0.40}(\omega) \in \mathcal{S}^*_e$.

(iv) *The function* $\mathfrak{B}^{2,\frac{1}{2},\frac{1}{2}}_{0.70,0.30}(\omega) \in \mathcal{S}^*_e$.

Theorem 7. *Let* $\kappa \geq 0$, $\rho > 0$, $\xi \in \mathbb{C} \setminus \{0\}$, $0 < \tau \leq \mu \leq 1$, *and* $M_{\mu,\tau} = \frac{\mu+1}{\tau+1}$. *Suppose that*

(i)

$$e^{\frac{3}{2}+\gamma(\rho+2)} \frac{\mu+2}{\tau+2} < 2(\rho+\kappa+1)^{\rho},$$

(ii)

$$2M_{\mu,\tau}(2e-1)|\xi|\Gamma(\kappa+1) < (e-1)(1-|\xi|)\Gamma(\rho+\kappa+1).$$

Then,

$$\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) \in \mathcal{C}_e.$$

Proof. Following the same arguments as in the proof of Theorem 3, together with the sufficient condition for the class $\mathcal{C}_e$ given in [42], we obtain

$$\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) \in \mathcal{C}_e.$$

This finishes the proof. □

Example 7. *The following normalized generalized Bessel–Maitland–Tremblay functions satisfy the hypotheses of Theorem 7. Hence, each of them belongs to the class* $\mathcal{C}_e$ *in the unit disk* $\mathbb{D}$.

(i) *The function* $\mathfrak{B}^{5,\frac{4}{3},\frac{1}{3}}_{1,1}(\omega) \in \mathcal{C}_e$.

(ii) *The function* $\mathfrak{B}^{3,2,\frac{1}{4}}_{0.90,0.50}(\omega) \in \mathcal{C}_e$.

(iii) *The function* $\mathfrak{B}^{3,1,\frac{1}{5}}_{0.80,0.40}(\omega) \in \mathcal{C}_e$.

(iv) *The function* $\mathfrak{B}^{5,\frac{1}{3},\frac{1}{6}}_{0.70,0.20}(\omega) \in \mathcal{C}_e$.

Theorem 8. *Let* $\kappa \geq 0$, $\rho > 0$, $\xi \in \mathbb{C} \setminus \{0\}$, $0 < \tau \leq \mu \leq 1$, $M_{\mu,\tau} = \frac{\mu+1}{\tau+1}$, *and*

$$T_{\mu,\tau} = \frac{M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1)}.$$

Suppose that

$$e^{1+\gamma(\rho+2)} \frac{\mu+2}{\tau+2} < 2(\rho+\kappa+1)^{\rho} \tag{48}$$

and

$$T_{\mu,\tau}(T_{\mu,\tau}+2) < \frac{1}{2}. \tag{49}$$

Then, $\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega) \in \mathcal{S}^*_L$.

Proof. It is sufficient to show that

$$\left| \left(\frac{\omega\left(\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}\right)'(\omega)}{\mathfrak{B}^{\rho,\kappa,\xi}_{\mu,\tau}(\omega)} \right)^2 - 1 \right| < 1.$$

From the proof of Theorem 2, we have

$$\left|\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)-\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}\right|<T_{\mu,\tau},$$

and

$$\left|\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}\right|>1-T_{\mu,\tau}.$$

Moreover,

$$\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)+\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}=2+\sum_{n=1}^{\infty}(n+2)A_n\omega^n.$$

Hence,

$$\left|\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)+\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}\right|<2+\Gamma(\kappa+1)\sum_{n=1}^{\infty}l_n|\xi|^n,$$

where

$$l_n=\frac{n+2}{n!\Gamma(\rho n+\kappa+1)}\frac{\Gamma(\tau+1)\Gamma(\mu+n+1)}{\Gamma(\mu+1)\Gamma(\tau+n+1)}.$$

Proceeding exactly as in the proof of Theorem 2, one obtains that $\{l_n\}_{n\geq 1}$ is decreasing. Consequently,

$$l_n\leq l_1=\frac{3M_{\mu,\tau}}{\Gamma(\rho+\kappa+1)},$$

which yields

$$\left|\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)+\frac{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}{\omega}\right|<2+3T_{\mu,\tau}.$$

Therefore,

$$\left|\left(\frac{\omega\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}\right)^2-1\right|<\frac{T_{\mu,\tau}(2+3T_{\mu,\tau})}{(1-T_{\mu,\tau})^2}.$$

Since condition (49) is equivalent to

$$\frac{T_{\mu,\tau}(2+3T_{\mu,\tau})}{(1-T_{\mu,\tau})^2}<1,$$

we conclude that

$$\left|\left(\frac{\omega\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}\right)^2-1\right|<1.$$

Hence,

$$\frac{\omega\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}{\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)}\prec\sqrt{1+\omega},$$

and therefore,

$$\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega)\in\mathcal{S}_L^*.$$

This finishes the proof. □

Example 8. *The following normalized generalized Bessel–Maitland–Tremblay functions satisfy the hypotheses of Theorem 8. Hence, each of them belongs to the class $\mathcal{S}_L^*$ in the unit disk $\mathbb{D}$.*

(i) *The function* $\mathfrak{B}_{1,1}^{2,\frac{3}{5},\frac{1}{3}}(\omega) \in \mathcal{S}_L^*$.

(ii) *The function* $\mathfrak{B}_{0.90,0.50}^{\frac{3}{2},1,\frac{1}{4}}(\omega) \in \mathcal{S}_L^*$.

(iii) *The function* $\mathfrak{B}_{0.80,0.40}^{3,\frac{1}{2},\frac{1}{5}}(\omega) \in \mathcal{S}_L^*$.

(iv) *The function* $\mathfrak{B}_{0.70,0.30}^{\frac{5}{2},2,\frac{1}{6}}(\omega) \in \mathcal{S}_L^*$.

Theorem 9. *Let $\kappa \geq 0$, $\rho > 0$, $\xi \in \mathbb{C} \setminus \{0\}$, $0 < \tau \leq \mu \leq 1$. $M_{\mu,\tau} = \frac{\mu+1}{\tau+1}$, and*

$$R_{\mu,\tau} = \frac{2M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}{(1-|\xi|)\Gamma(\rho+\kappa+1) - 2M_{\mu,\tau}\Gamma(\kappa+1)|\xi|}.$$

Suppose that

(i)

$$e^{\frac{3}{2}+\gamma(\rho+2)}\frac{\mu+2}{\tau+2} < 2(\rho+\kappa+1)^{\rho},$$

(ii)

$$R_{\mu,\tau}\left(R_{\mu,\tau}+2\right) < 1.$$

Then

$$\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega) \in \mathcal{C}_L.$$

Proof. Using the bounds established in the proof of Theorem 3, we have

$$\left|\frac{\omega\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)''(\omega)}{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}\right| < R_{\mu,\tau},$$

and therefore

$$\left|2+\frac{\omega\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)''(\omega)}{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}\right| < 2+R_{\mu,\tau}.$$

Hence,

$$\left|\left(1+\frac{\omega\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)''(\omega)}{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}\right)^2 - 1\right| < R_{\mu,\tau}(R_{\mu,\tau}+2).$$

By hypothesis (ii),

$$\left|\left(1+\frac{\omega\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)''(\omega)}{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)}\right)^2 - 1\right| < 1.$$

Consequently,

$$1+\frac{\omega\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)''(\omega)}{\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega)} \prec \sqrt{1+\omega},$$

which implies that

$$\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}(\omega) \in \mathcal{C}_L.$$

This finishes the proof. □

Example 9. *The following normalized generalized Bessel–Maitland–Tremblay functions satisfy the hypotheses of Theorem 9. Hence each of them belongs to the class $\mathcal{C}_L$ in the unit disk $\mathbb{D}$.*

(i) *The function* $\mathfrak{B}_{1,1}^{5,\frac{4}{3},\frac{1}{3}}(\omega) \in \mathcal{C}_L$.

(ii) *The function* $\mathfrak{B}_{0.90,0.50}^{3,2,\frac{1}{4}}(\omega) \in \mathcal{C}_L$.

(iii) *The function* $\mathfrak{B}_{0.80,0.40}^{3,1,\frac{1}{5}}(\omega) \in \mathcal{C}_L$.

(iv) *The function* $\mathfrak{B}_{0.70,0.20}^{5,\frac{1}{3},\frac{1}{6}}(\omega) \in \mathcal{C}_L$.

Applications

The generalized Bessel–Maitland–Tremblay function investigated in this paper provides a useful analytical framework for the study of fractional-order systems possessing memory and hereditary effects. Since generalized Bessel–Maitland functions and fractional differential operators frequently used in the solutions of fractional differential equations, the established geometric properties may give the qualitative analysis of analytic solutions arising in anomalous diffusion, viscoelastic materials, heat conduction, signal processing, control systems, and biological dynamics. Furthermore, the proposed operator offers a useful tool for investigating nonlocal and fractal models, where fractional calculus plays a main role in describing long-range interactions and memory-dependent phenomena.

To illustrate the parameter-dependent behavior of the proposed generalized Bessel–Maitland–Tremblay function, we consider the exponentially relaxing trajectory

$$\omega(t) = -(1 - e^{-t}), \qquad t \geq 0,$$

that remains entirely inside the unit disk and approaches the boundary point -1 as $t \to \infty$. We define the normalized derivative-response profile by

$$\mathcal{R}(t) = \frac{\left|\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega(t))\right|}{\left|\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(0)\right|} = \left|\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(\omega(t))\right|,$$

since

$$\left(\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}\right)'(0) = 1.$$

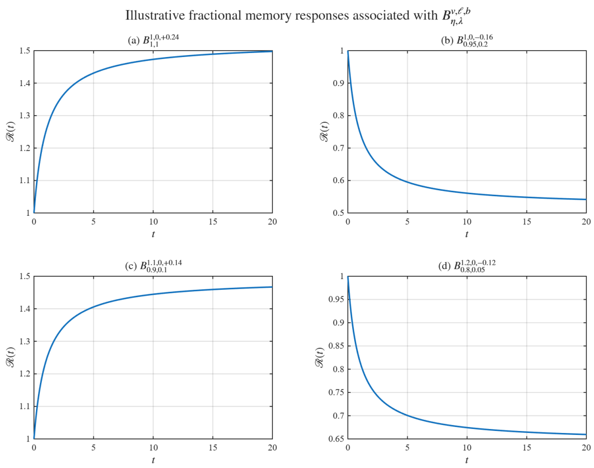

Figure 6. Normalized derivative-response profiles of $\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}$ along the exponentially relaxing path $\omega(t) = -(1 - e^{-t}) \in \mathbb{D}$. Panels (**a**,**c**) show increasing responses for positive ξ, whereas panels (**b**,**d**) exhibit decreasing responses for negative ξ, illustrating the influence of μ, τ, ρ, κ, and ξ on the parameter-dependent behavior of the proposed function.

Figure 6 illustrates the normalized derivative-response profiles of $\mathfrak{B}_{\mu,\tau}^{\rho,\kappa,\xi}$ along the exponentially relaxing trajectory $\omega(t) = -(1 - e^{-t}) \subset \mathbb{D}$. The response evolves smoothly from the origin toward the boundary of the unit disk, resembling the relaxation behavior commonly observed in fractional-order models. As shown in Figure 6a,c, positive values of ξ produce monotone increasing responses approaching stable limiting values, whereas Figure 6b,d exhibit monotone decreasing responses for negative values of ξ. The parameters μ, τ, ρ, and κ regulate the rate of growth or decay without altering the overall stability of the response. These numerical profiles further demonstrate the parameter sensitivity of the proposed generalized Bessel–Maitland–Tremblay function and highlight its potential applicability to the qualitative analysis of fractional-order systems with memory effects.

4. Conclusions

In this paper, we proposed a new normalized generalized Bessel–Maitland–Tremblay function by using the modified Tremblay fractional differential operator with the generalized Bessel–Maitland function through the Hadamard product. This construction established a new connection between fractional differential operators and special functions within the framework of GFT. By using coefficient estimates together with inequalities for the gamma and digamma functions, we established sufficient conditions for the proposed function to belong to the classes of uniformly convex, starlike, convex, ν-uniformly starlike, ν-uniformly convex, exponential starlike, exponential convex, Bernoulli lemniscate starlike, and Bernoulli lemniscate convex functions. The theoretical results were further supported by numerical examples and graphical illustrations, which confirm the geometric behavior and parameter-dependent characteristics of the proposed function.

The proposed framework provides a useful analytical tool for investigating fractional operators associated with special functions and establishes a connection between fractional calculus and GFT. Apart from the sufficient conditions obtained in the present work, several directions appear encouraging for further investigation. Notably, the present study can be extended to q-analogues and quantum-calculus settings, leading to new q-Bessel–Maitland–Tremblay-type operators and their related subclasses of analytic function. Such extensions may give a unifying framework in which fractional and quantum operators are studied together. It would also be interesting to investigate the proposed operator in multivalent, harmonic, and bi-univalent function classes, where the interplay between the operator coefficients and geometric restrictions may lead to new coefficient estimates and sharp inequalities.

Further research may focus on differential subordination and superordination results, including their strong and other generalized forms, related with the proposed fractional operator. Moreover, Fekete–Szegö inequalities, Hankel and Toeplitz determinant estimates, radius problems, distortion and growth results, and integral-transform properties could provide further study about the geometric behavior of the connected analytic functions. From an applied perspective, the fractional nature of the Tremblay operator yields the proposed framework potentially related to fractional differential equations and memory-dependent dynamical processes, where nonlocal effects play a crucial role. Further extensions may therefore explore the operator in fractional dynamical systems, fractal models, and mathematical-physics problems involving hereditary or memory effects. These investigations may also help clarify how the parameters of the generalized Bessel–Maitland–Tremblay operator affect the qualitative properties of the corresponding models and may lead to new analytical and computational applications in fractional and geometric analysis.

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