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Circle Criterion for Multi-Order Fractional System Control

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Abstract

The paper investigates the asymptotic stability for the control of systems described by multi-order fractional differential equations. By utilizing generalized Lyapunov functions and the Kalman–Yakubovich–Popov lemma, frequency-domain criteria are derived to evaluate asymptotic stability. The formulated criteria are similar to the ‘Popov Circle Criterion,’ but the circle parameters are determined by the system’s fractional order and the control parameters. Additionally, the asymptotic stability condition requires that all polar plots associated with the multi-fractional-order system lie inside the circle defining the criterion. Human–Robot System applications highlight the investigation techniques and the particularities of the presented criteria.

Keywords: multi-variable fractional-order system; control; stability; frequency criteria; circle criterion

1. Introduction

An important element in frequency-domain investigation methods lies in the capability to utilize graphical characteristics via Fourier or Laplace transforms, such as Nyquist or Bode plots. These characteristics highlight the parameters of linear systems and easily provide solutions for achieving superior control performance. A major milestone in the investigation of systems with nonlinear components was achieved through the Popov criteria, illustrated by ‘Popov’s Criterion’ or ‘Popov’s line’. Culminating in specific criteria via the Kalman–Yakubovich–Popov (KYP) lemma, these methods offer more powerful tools for investigating control techniques. There is a vast body of literature in this field, and an exhaustive review of it would be beyond the scope of this paper. We will, however, attempt to highlight several papers that have shaped this research effort. An early foundation can be found in the foundational papers [1–5], where several fundamental concepts of study were formulated. The application of the Nyquist and Popov stability criteria to linearized systems is treated in [6,7]. The extension of the Circle Criterion using Lyapunov functions is presented in [8,9]. In [10–12] is treated the usefulness of the Popov stability criterion, Circle Criterion and Popov’s Line, in the analysis of nonlinear power systems. Uncertain systems and robust control are studied in [13–15]. Techniques for applying Popov’s Circle to complex nonlinearities are discussed in [16,17]. The stability of fuzzy systems with nonlinearities defined by rules or Takagi–Sugeno models is analyzed through Popov criteria in [18–20]. The extension of the YKP lemma for system stabilization and the determination of stability criteria are discussed in [21,22]. Generalizations and new interpretations of the KYP lemma for systems described by fractional-order equations are presented in [23,24]. In [25–28], the stability of multiple fractional-order systems is analyzed



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using Lyapunov functions and the comparison method. Significant results in this field are formulated in [29,30]. Necessary and sufficient conditions for the stability and stabilization of fractional-order interval systems are discussed in [31–33]. A synthetic overview of this research effort is presented in Table 1.

Table 1. Synthetic overview of Circle Criterion techniques.

Paper	System Model	Methods/Investigation Techniques	Nonlinearity	Graphical Representations
[2,3]	IOS	Transfer function/Lyapunov techniques	Nonlinear Sector	Popov Circle/Popov Line Bode Diagram
[8,33]	IOS, FOS	Lyapunov functions	Nonlinear Sector	Polar Plots/Popov Circle
[10]	IOS	Transfer function	Nonlinear Sector	Popov Line
[11]	IOS	Lyapunov techniques	Linearized Approximation	Circle Popov
[12,13]	IOS/FOS	Transfer function	Nonlinear Sector	-
[14]	IOS	Transfer function	Nonlinear Sector	Popov Circle
[15]	IOS	Transfer function	Nonlinear Sector	-
[16,17]	IOS	Transfer function	Approximation by Multiple Sectors	Popov Circle
[18–20]	Fuzzy models	Transfer functions	Linearized Approximation	Popov Circle
[21,22]	IOS	KYP lemma	Nonlinear Sector	-
[23,24]	IOS/FOS	Generalized KYP lemma	–	–
[25,26]	FOS	KYP lemma	–	–
[27,28]	MFOS	Lyapunov functions	–	–
[31,32]	FOS	Lyapunov functions	Nonlinear Sector	–
[34]	FOS	Transfer function	Nonlinear Sector	Bode Diagram

The present paper introduces several frequency-domain criteria for the control of certain classes of linear systems, as well as systems with nonlinear components, described by multiple fractional orders. These criteria are presented through graphical representations similar to the Circle Criterion, although the parameters defining the criterion differ from the classical one. To the best of the authors' knowledge, these criteria represent the first graphical representation for multiple fractional-order systems and provide, we hope, a more efficient tool for evaluating the asymptotic stability of these complex systems. Main contributions of this paper can be summarized as:

- New stability criteria, frequency criteria, represented graphically by polar plots similar to familiar Popov plots are obtained by the application of the Kalman–Yakubovich–Popov lemma.

Theorem 3 is “Circle Criterion” for γ_i -multi-order fractional systems ($0 < \gamma_i \leq 1$). In Theorem 3, the circle C is determined by the γ_i -fractional order and the control gain constraint σ . Additionally, the asymptotic stability condition requires that the polar plots of all transfer functions associated with the multi-fractional-order system to lie entirely within the interior of this defining disk.

- These results are extended for γ_i -multi-order fractional systems with nonlinear components.
- Human–Robot System applications highlight the investigation techniques and the particularities of the presented criteria.

The paper is structured as follows: Section 2 presents mathematical preliminaries and control criteria, Section 3 discusses numerical applications; Section 4 is dedicated to Discussion and Conclusions.

2. Materials and Methods

2.1. Preliminaries

Notations: The bold letters denote the vector or matrix and non-bold letters denote scalar variables.

Definition 1. The system

$$D^{\gamma_i} \mathbf{z}_i(t) = \sum_{j=1}^N \mathbf{A}_{ij} \mathbf{z}_j(t), 0 < \gamma_i \leq 1; i = 1, 2, \dots, N \quad (1)$$

where $\mathbf{z} = [\mathbf{z}_1 \mathbf{z}_2 \dots \mathbf{z}_i \dots \mathbf{z}_N]^T$ is the state vector, $D^{\gamma_i} \mathbf{z}_i(t)$ is the **Caputo** derivative of $\mathbf{z}_i$ for $\gamma_i \in (0, 1]$, $\mathbf{z}_i = [z_{i1} z_{i2} \dots z_{iN_i}]^T$, $\mathbf{z} \in \mathbb{R}^{\sum N_i}$, and $\mathbf{z}_i \in \mathbb{R}^{N_i}$, $\mathbf{A}_{ij} \in \mathbb{R}^{N_i \times N_j}$ is the linear **multivariable fractional-order system (MFOS)**.

Definition 2. The generalized Lyapunov function of MFOS is

$$V(\mathbf{z}) = \sum_{i=1}^N V_i(\mathbf{z}_i) \quad (2)$$

where $V_i(\mathbf{z}_i) = \mathbf{z}_i^T \mathbf{P}_i \mathbf{z}_i$, $\mathbf{P}_i = \mathbf{P}_i^T > 0$. The multivariable fractional-order derivative of $V(\mathbf{z})$ is ($D^\gamma V$ is the notation),

$$D^\gamma V(\mathbf{z}) = \sum_{i=1}^N D^{\gamma_i} (\mathbf{z}_i^T \mathbf{P}_i \mathbf{z}_i) \quad (3)$$

Theorem 1. MFOS is asymptotically stable if the generalized Lyapunov function satisfies the following conditions:

$$\eta_1 \|\mathbf{z}\|^2 \leq V(\mathbf{z}) \leq \eta_2 \|\mathbf{z}\|^2 \quad (4)$$

$$D^\gamma V(\mathbf{z}) \leq -\eta_3 V(\mathbf{z}), \eta_i > 0, i = 1, 2, 3. \quad (5)$$

Proof of Theorem 1. See Appendix A. $\square$

Main Remark. If the polar plot $G(j\omega)$ of the system is inside the circle C defined by the diameter $[-a, b]$ then (Figure 1):

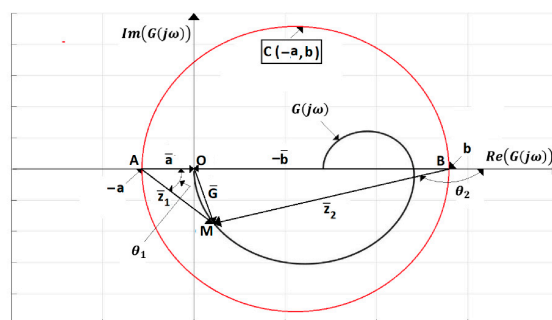


Figure 1. Circle Criterion.

$$a + \operatorname{Re}(G(j\omega)) > 0 \quad (6)$$

$$-b + \operatorname{Re}(G(j\omega)) < 0 \quad (7)$$

Proof of Main Remark. See Appendix B. $\square$

2.2. Main Result

2.2.1. Linear Fractional System

Consider the system

$$D^\gamma \mathbf{z}(t) = \mathbf{A} \mathbf{z}(t) - \mathbf{b} u(t) \quad (8)$$

$$\mathbf{y}(t) = \mathbf{c}^T \mathbf{z}(t) \quad (9)$$

where $\mathbf{z} = [z_1 z_2 \dots z_i \dots z_N]^T$ is the state vector, $\mathbf{z} \in \mathbb{R}^N$, $D^\gamma \mathbf{z}(t)$ is the Caputo derivative of $\mathbf{z}(t)$ for $\gamma \in (0, 1]$, u is the input, $u \in \mathbb{R}^1$, $\mathbf{y} \in \mathbb{R}^1$ is the output, $\mathbf{A} \in \mathbb{R}^{N \times N}$, and $\mathbf{b}, \mathbf{c} \in \mathbb{R}^N$. Assume that the systems (6) and (7) are unstable, if at least one eigenvalue satisfies the condition $|\operatorname{Arg}(\lambda_i)| < \gamma \frac{\pi}{2}$. Assume that the following control law

$$u(t) = k y(t), \quad k \in [k_\gamma, 2k_H]; k < \sigma; \sigma > 0 \quad (10)$$

where the control gain k ensures as the matrix $\mathbf{A}$ either γ —fractional-order stable ($k = k_\gamma$),

$$\mathbf{A}^{(\gamma)} = \mathbf{A} - k_\gamma \mathbf{b} \mathbf{c}^T \quad (11)$$

or Hurwitz stable ($k = k_H$),

$$\mathbf{A}^H = \mathbf{A} - k_H \mathbf{b} \mathbf{c}^T \quad (12)$$

Define the coefficient

$$\mu = \frac{k_H}{k_\gamma} \quad (13)$$

where μ satisfies the condition,

$$\mu \gamma \frac{\pi}{2} > \frac{\pi}{2}; \mu > \frac{1}{\gamma} > 1; \quad (14)$$

The following relations can be readily inferred,

$$k_\gamma = \frac{k_H}{\mu} = k_H - \mu^*; \quad (15)$$

$$\mu^* = k_H \frac{\mu - 1}{\mu} \quad (16)$$

Substituting the control law (10) for $k = k_\gamma$, $k = 2k_H$, respectively, into the systems (8) and (9), and taking (13)–(16) into account, yields,

$$D^\gamma \mathbf{z}(t) = (\mathbf{A} - k_H \mathbf{b} \mathbf{c}^T) \mathbf{z}(t) + \mu^* \mathbf{b} \mathbf{c}^T \mathbf{z}(t) \quad (17)$$

$$D^\gamma \mathbf{z}(t) = (\mathbf{A} - k_H \mathbf{b} \mathbf{c}^T) \mathbf{z}(t) - k_H \mathbf{b} \mathbf{c}^T \mathbf{z}(t) \quad (18)$$

Consider the systems (8) and (9) with the control law (10).

Theorem 2 (γ -Fractional-Order System Circle Criterion). The systems (8) and (9) with the control law (10) are asymptotically stable if the polar plot $G(j\omega)$ is inside the circle C defined by the diameter $\left[-\frac{2}{\sigma}, \frac{\gamma}{\sigma(1-\gamma)}\right]$ where

$$G(j\omega) = \mathbf{c}^T (j\omega \mathbf{I} - \mathbf{A}^H)^{-1} \mathbf{b}. \quad (19)$$

Proof of Theorem 2. Let $V(\mathbf{z}) = \mathbf{z}^T \mathbf{P} \mathbf{z}$ be the Lyapunov function that satisfies the condition (4). The γ -fractional order derivative of $V(\mathbf{z})$ is (Appendix B)

$$D^\gamma V(t) = D^\gamma (\mathbf{z}^T \mathbf{P} \mathbf{z}) \leq (\mathbf{D}^\gamma \mathbf{z})^T \mathbf{P} \mathbf{z} + \mathbf{z}^T \mathbf{P} (\mathbf{D}^\gamma \mathbf{z}) \quad (20)$$

This inequality becomes (see Appendix C),

$$D^\gamma V(\mathbf{z}) \leq -\mathbf{z}^T \mathbf{Q} \mathbf{z} \quad (21)$$

if the following inequalities are verified:

$$-\frac{\gamma}{\sigma(1-\gamma)} + \operatorname{Re}(G(j\omega)) < 0, \quad \omega \in [0, +\infty) \quad (22)$$

$$\frac{2}{\sigma} + \operatorname{Re}(G(j\omega)) > 0, \quad \omega \in [0, +\infty) \quad (23)$$

where

$$\mathbf{Q} = \begin{cases} \mathbf{Q}_\gamma = (\mathbf{q} + \frac{k_\gamma \mathbf{c}}{\sqrt{\sigma}})(\mathbf{q} + \frac{k_\gamma \mathbf{c}}{\sqrt{\sigma}})^T, & \text{for } k = k_\gamma \\ \mathbf{Q}_H = (\mathbf{q} + \frac{k_H \mathbf{c}}{\sqrt{\sigma}})(\mathbf{q} + \frac{k_H \mathbf{c}}{\sqrt{\sigma}})^T, & \text{for } k = k_H \end{cases} \quad (24)$$

The plot of $G(j\omega)$ is inside circle C $(-\frac{2}{\sigma}, \frac{\gamma}{\sigma(1-\gamma)})$ and according to Main Remark, the inequalities (22) and (23) are satisfied. Consequently, inequality (21) becomes

$$D^\gamma V(\mathbf{z}) \leq -\eta \mathbf{z}^T \mathbf{P} \mathbf{z} \quad (25)$$

where

$$\eta = \min \left(\frac{\lambda_{\min}(\mathbf{Q}_\gamma)}{\lambda_{\max}(\mathbf{P})}, \frac{\lambda_{\min}(\mathbf{Q}_H)}{\lambda_{\max}(\mathbf{P})} \right), \quad \eta > 0 \quad (26)$$

$$D^\gamma V(\mathbf{z}) \leq -\eta V(\mathbf{z}) \quad (27)$$

According to Theorem 1, the systems (8) and (9) with the control law (10) are asymptotically stable. $\square$

Corollary 1. Theorem 2 is named the “Circle Criterion” for γ -fractional-order systems $(0 < \gamma \leq 1)$. The parameters defining the “Circle” are determined by the γ -fractional order and the control gain constraint σ .

2.2.2. Multi-Order Fractional Linear Systems

Consider the following MFOS

$$D^{\gamma_i} \mathbf{z}_i(t) = \sum_{j=1}^N \mathbf{A}_{ij} \mathbf{z}_j(t) - \mathbf{b}_i u_i(t), \quad (28)$$

$$y_i(t) = \mathbf{c}_i^T \mathbf{z}_i(t), \quad i = 1, 2, \dots, N \quad (29)$$

where $\mathbf{z} = [z_1 z_2 \dots z_i \dots z_N]^T$ is the state vector, $D^{\gamma_i} z_i(t)$ is the Caputo derivative of $z_i(t)$ for $\gamma_i \in (0, 1]$, $z_i \in \mathbb{R}^{N_i}$, $u_i, y_i \in \mathbb{R}^1$, $A_{ij} \in \mathbb{R}^{N_i \times N_j}$, $\mathbf{b}_i, \mathbf{c}_i \in \mathbb{R}^{N_i}$, and the matrices A_{ij} satisfy the conditions,

$$A_{ij} A_{ij}^T \leq \rho_{ij} \mathbf{I}_{N_i}, \rho_{ij} > 0; \rho_{ij} = \rho_{ji}; i \neq j, \mathbf{I}_{N_i} \text{ is identity matrix, } \mathbf{I}_{N_i} \in \mathbb{R}^{N_i \times N_i} \quad (30)$$

The following control laws are applied

$$u_i(t) = k_i y_i(t); i = 1, 2, \dots, N \quad (31)$$

$$k_i \in [k_{\gamma_i}, 2k_{Hi}] \quad (32)$$

where k_{γ_i}, k_{Hi} , determined as A_i , become the γ -fractional-order stable and Hurwitz stable, respectively. Let μ_i be a positive constant that satisfies inequality

$$\mu_i \gamma_i \frac{\pi}{2} > \frac{\pi}{2}; \mu_i > \frac{1}{\gamma_i} > 1; \quad (33)$$

We define the coefficients

$$k_{Hi} = \mu_i k_{\gamma_i}; k_{Hi} > 1; i = 1, 2, \dots, N \quad (34)$$

The following relations can be inferred,

$$k_{\gamma_i} = \frac{k_{Hi}}{\mu_i} = k_{Hi} - \mu_i^*; \quad (35)$$

$$\mu_i^* = k_{Hi} \frac{\mu_i - 1}{\mu_i} \quad (36)$$

If the control law (31), for $k_i = k_{\gamma_i}$, applied to the systems (28) and (29), and taking (33)–(36) into account, yields,

$$D^{\gamma_i} z_i(t) = A_{ii}^H z_i(t) + \mu_i^* \mathbf{b}_i \mathbf{c}_i^T z_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^N A_{ij} z_j(t) \quad (37)$$

where $A_{ii}^H = A_{ii} - k_{Hi} \mathbf{b}_i \mathbf{c}_i^T$ is Hurwitz. Also, for $k_i = k_{Hi}$, it results in

$$D^{\gamma_i} z_i(t) = A_{ii}^H z_i(t) - k_{Hi} \mathbf{b}_i \mathbf{c}_i^T z_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^N A_{ij} z_j(t) \quad (38)$$

Consider MFOS (28) and (29) with the control laws (31) and (32).

Theorem 3 (Multi-Order Fractional System Circle Criterion). *The systems (28)–(30) with the control laws (31) and (32) are asymptotically stable if the polar plots $G_i(j\omega)$, $i = 1, 2, \dots, N$ are inside the circle C defined by the diameter $\left[-\frac{2}{\sigma^*}, \frac{\gamma^*}{\sigma^*(1-\gamma^*)}\right]$, where $G_i(j\omega) = \mathbf{c}_i^T (j\omega \mathbf{I} - A_i^H)^{-1} \mathbf{b}_i$ and $\sigma^* = \max_i \sigma_i$; $\gamma^* = \min_i \frac{\gamma_i}{(1-\gamma_i)}$ and the following conditions are satisfied:*

$$\eta_{\gamma_i} = \frac{\lambda_{\min}(Q_{\gamma_i})}{\lambda_{\max}(P_{\gamma_i})} - \frac{1}{\lambda_{\min}(P_{\gamma_i})} ((N-1)\lambda_{\max}(P_{\gamma_i}^2) + \rho_i) > 0; \eta_{\gamma} = \min_i \eta_{\gamma_i} \quad (39)$$

$$\eta_{Hi} = \frac{\lambda_{\min}(Q_{Hi})}{\lambda_{\max}(P_{Hi})} - \frac{1}{\lambda_{\min}(P_{Hi})} ((N-1)\lambda_{\max}(P_{Hi}^2) + \rho_i) > 0; \eta_H = \min_i \eta_{Hi} \quad (40)$$

Proof of Theorem 3. Consider the generalized Lyapunov function (2),

$V(t) = \sum_{i=1}^N V_i(z_i(t))$, $V_i(z_i) = z_i^T P_i z_i$, $P_i = P_i^T > 0$, that satisfies the condition (4). According to Definition 2, the multivariable fractional-order derivative of $V(x)$ is (Appendix C),

$$D^\gamma V(t) = \sum_{i=1}^N D^{\gamma_i} (z_i^T P_i z_i) \leq \sum_{i=1}^N \left((D^{\gamma_i} z_i^T) P_i z_i + z_i^T P_i (D^{\gamma_i} z_i) \right) \quad (41)$$

According to Main Remark, if the polar plots $G_i(j\omega)$, $i = 1, 2, \dots, N$, are inside the circle C defined by the diameter $\left[-\frac{2}{\sigma^*}, \frac{\gamma^*}{\sigma^*(1-\gamma^*)} \right]$, the inequalities (A40) and (A50) (Appendix E)

$$\operatorname{Re}(G_i(j\omega)) < \frac{\gamma_i}{\sigma_i(1-\gamma_i)}, \quad i = 1, 2, \dots, N \quad (42)$$

$$\operatorname{Re}(G_i(j\omega)) > -\frac{2}{\sigma_i}, \quad i = 1, 2, \dots, N, \omega \in [0, +\infty) \quad (43)$$

are satisfied. Consequently, according to Appendix E, the inequality (41) becomes

$$D^\gamma V(t) \leq -\eta V(t) \quad (44)$$

$$\eta = \min(\eta_\gamma, \eta_H) \quad (45)$$

According to Theorem 1, the systems (28)–(30) with the control laws (31) and (32) are asymptotically stable. $\square$

Corollary 2. We can consider that Theorem 3 represents the “Circle Criterion” for

Multi – Order Fractional Systems, $(0 < \gamma_i \leq 1)$.

2.2.3. Multi-Order Fractional Systems with Nonlinear Components

Consider the following **MFOS**

$$D^{\gamma_i} z_i(t) = \sum_{j=1}^N (A_{ij} z_j(t) + f_i(z_i) - b_i u_i(t)) \quad (46)$$

$$y_i(t) = c_i^T z_i(t), \quad i = 1, 2, \dots, N \quad (47)$$

where A_{ij} , b_i , c_i , z_j , z , u_i , y_i are defined as in previous sections and nonlinear components $f_i = [f_{i1} f_{i2} \dots f_{iN_i}]^T$ satisfies the inequalities

$$\|f_i(z_i)\|^2 \leq \sum_{j=1}^{N_i} l_{ij} (z_{ij})^2, \quad l_{ij} > 0 \quad (48)$$

The control laws (31) and (32) are used as the matrices A_{ii} defined for $k_i = k_{Hi}$ are Hurwitz.

Consider **MFOS** (46) and (47) with nonlinear component (48) and the control laws (31) and (32).

Theorem 4. The systems (46)–(48) with the control laws (31) and (32) are asymptotically stable if the polar plots $G_i(j\omega)$, $i = 1, 2, \dots, N$, are inside the circle C defined by the diameter $\left[-\frac{2}{\sigma^*}, \frac{\gamma^*}{\sigma^*(1-\gamma^*)} \right]$, and the following conditions are satisfied:

$$\eta_{\gamma i} = \frac{\lambda_{\min}(Q_{\gamma i})}{\lambda_{\max}(P_{\gamma i})} - \frac{1}{\lambda_{\min}(P_{\gamma i})} ((N-1)\lambda_{\max}(P_{\gamma i}^2) + \rho_i + l_i) > 0; \quad \eta_\gamma = \min_i \eta_{\gamma i} \quad (49)$$

$$\eta_{Hi} = \frac{\lambda_{\min}(\mathbf{Q}_{Hi})}{\lambda_{\max}(\mathbf{P}_{Hi})} - \frac{1}{\lambda_{\min}(\mathbf{P}_{Hi})}((N-1)\lambda_{\max}(\mathbf{P}_{Hi}^2) + \rho_i + l_i) > 0; \eta_H = \min_i \eta_{Hi} \quad (50)$$

where $l_i = \max_j l_{ij}$, $G_i(j\omega) = \mathbf{c}_i^T(j\omega\mathbf{I} - \mathbf{A}_{ii}^H)^{-1}\mathbf{b}_i$ and the other variables are defined as for the previous Theorem.

Proof of Theorem 4. Following a similar procedure as for the previous Theorem 3 and Appendix E, it follows that,

$$D^\gamma V(t) \leq -\eta V(t) \quad (51)$$

$$\eta = \min(\eta_\gamma, \eta_H) \quad (52)$$

□

Corollary 3. Consider the system:

$$D^{\gamma_i} \mathbf{z}_i(t) = \sum_{j=1}^{N_i} (\mathbf{A}_{ij} \mathbf{z}_j(t) - \mathbf{b}_i u_i(t)) + \mathbf{d}_i f_i(y_i) \quad (53)$$

$$y_i(t) = \mathbf{c}_i^T \mathbf{z}_i(t), \quad i = 1, 2, \dots, N \quad (54)$$

where the nonlinear component satisfies the sector condition,

$$f_i < \zeta_i y_i; \quad \zeta_i > \sigma_i > 0; \quad \zeta^* = \max_i(\zeta_i) \quad (55)$$

Following a similar procedure as for previous Theorems, we obtain the following:

The systems (53)–(55) with the control laws (31) and (32) are asymptotically stable if the polar plots $G_i(j\omega)$, $i = 1, 2, \dots, N$, are inside the circle C defined by the diameter $\left[-\frac{2}{\zeta^*}, \frac{\gamma^*}{\zeta^*(1-\gamma^*)}\right]$, and the following conditions are satisfied:

$$\eta_{\gamma i} = \frac{\lambda_{\min}(\mathbf{Q}_{\gamma i})}{\lambda_{\max}(\mathbf{P}_{\gamma i})} - \frac{1}{\lambda_{\min}(\mathbf{P}_{\gamma i})}((N-1)\lambda_{\max}(\mathbf{P}_{\gamma i}^2) + \rho_i) > 0; \quad \eta_\gamma = \min_i \eta_{\gamma i} \quad (56)$$

$$\eta_{Hi} = \frac{\lambda_{\min}(\mathbf{Q}_{Hi})}{\lambda_{\max}(\mathbf{P}_{Hi})} - \frac{1}{\lambda_{\min}(\mathbf{P}_{Hi})}((N-1)\lambda_{\max}(\mathbf{P}_{Hi}^2) + \rho_i) > 0; \quad \eta_H = \min_i \eta_{Hi} \quad (57)$$

Corollary 4. The following linear approximation of $f_i(y_i)$ is considered

$$f_i(y_i) = \xi_i^T \mathbf{z}_i, \quad \xi_i = [\xi_{i1}, \xi_{i2}, \dots, \xi_{iN_i}]^T \quad (58)$$

The system (53) can be rewritten as

$$D^{\gamma_i} \mathbf{z}_i(t) = \mathbf{A}_{ii}^* + \sum_{\substack{j=1 \\ j \neq i}}^{N_i} (\mathbf{A}_{ij} \mathbf{z}_j(t) - \mathbf{b}_i u_i(t)) \quad (59)$$

where

$$\mathbf{A}_{ii}^* = \mathbf{A}_{ii} + \mathbf{d}_i \xi_i^T \quad (60)$$

The stability conditions are determined by Theorem 3 for

$$G_i^*(j\omega) = \mathbf{c}_i^T (j\omega\mathbf{I} - \mathbf{A}_{ii}^{*H})^{-1} \mathbf{b}_i \quad (61)$$

According to Theorem 3, polar plots of approximated $G_i^*(j\omega)$, $i = 1, 2, \dots, N$ must be inside the circle determined by the same parameters: $\left[-\frac{2}{\sigma}, \frac{\gamma}{\sigma(1-\gamma)}\right]$. Verifying these conditions through analytical methods is highly challenging; hence, graphical solutions would be recommended.

3. Results

3.1. Example 1

Consider the system

$$D^{0.8}z(t) = Az(t) + bu(t) \quad (62)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -13 & 6 \end{bmatrix}; b = \begin{bmatrix} 0 \\ 3 \end{bmatrix}; c = \begin{bmatrix} 0.1 \\ 1.2 \end{bmatrix}; \quad (63)$$

The matrix A is 0.8 – fractional-order unstable with $\lambda_A = 1.56 \pm j 3.287$. The control laws (9) and (10) ensure that matrix A is either 0.8-stable for $k_{0.8} = 1.1$ ($\lambda_{A^{0.8}} = 1.02 \pm j 3.5$) or Hurwitz stable for $k_H = 2.5$ ($\lambda_{A^H} = -1.5 \pm j 3.39$). According to (36)–(39), this yields $\mu = 2.27$, $\mu^* = 1.40$. Consider the control law defined by $\sigma = 20$. The pairs (A^H, b) , (A^H, c) are completely controllable and observable, respectively, and the polar plot of $G(j\omega)$ is inside the circle $C(-0.1, 0.2)$, as seen in Figure 2. Solving in MATLAB (R 2020 a) the systems (A21) and (A29) yields,

$$\eta = 0.27 \quad (64)$$

$$D^{0.8}V(z) \leq -0.27 V(x) \quad (65)$$

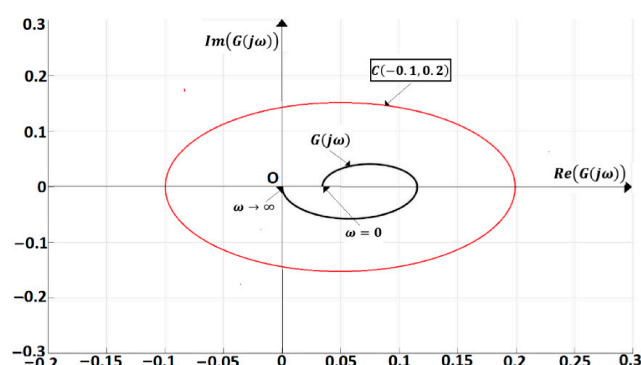


Figure 2. Circle Criterion for Example 1.

The conditions of Theorem 2 are satisfied, and the systems (62) and (63) are asymptotically stable. Figure 3 shows the trajectories of the system for $k_{0.8} = 0.8$. This evolution confirms the stability of the system.

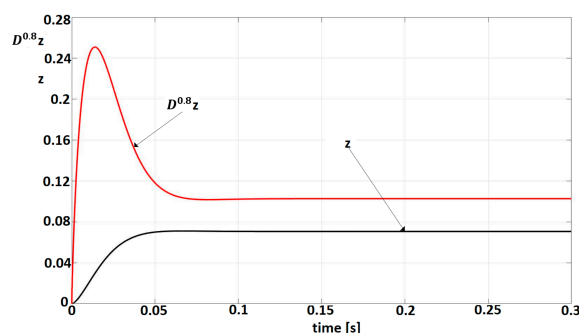


Figure 3. Trajectories of the systems (53) and (54) for $k_{0.8} = 0.8$.

3.2. Example 2

Consider the system,

$$D^{0.5}z_1(t) = A_{11}z_1(t) + A_{12}z_2(t) + b_1u_1(t) \quad (66)$$

$$D^{0.7}z_2(t) = A_{21}z_1(t) + A_{22}z_2(t) + b_2u_2(t) \quad (67)$$

$$y_1(t) = c_1^T z_1(t) \quad (68)$$

$$y_2(t) = c_2^T z_2(t) \quad (69)$$

where

$$A_{11} = \begin{bmatrix} 0 & 1 \\ -34 & 10 \end{bmatrix}; A_{12} = \begin{bmatrix} -0.017 & 0 \\ 0 & 0.011 \end{bmatrix}; b_1 = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}; c_1 = \begin{bmatrix} 2 \\ 2.2 \end{bmatrix}; \quad (70)$$

$$A_{22} = \begin{bmatrix} 0 & 1 \\ -25 & 6 \end{bmatrix}; A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; c_2 = \begin{bmatrix} 1.8 \\ 2.0 \end{bmatrix}; \quad (71)$$

The matrix A_{11} is 0.5 – fractional-order unstable with $\lambda_{A_{11}} = 5 \pm j 3$. The control laws (34) and (35) ensure that matrix A_{11} is 0.5-stable for $k_{0.5} \in [1.4 : 2]$ and for $k_{H,1} \in [3:3.5]$, it becomes Hurwitz stable. According to (36) and (39), if we consider $k_{0.5} = 1.7$ and $k_{H,1} = 3$ yields $\mu_1 = 1.6$, $\mu_1^* = 3$. Also, the matrix A_{22} is 0.7 – fractional-order unstable with $\lambda_{A_{22}} = 3 \pm j 4$. The control laws (34) and (35) ensure that matrix A_{22} is 0.7-stable for $k_{0.7} \in [1.2 : 1.8]$ or Hurwitz stable for $k_{H,2} \in [1.9:4]$. Consider $k_{0.7} = 1.4$ and $k_{H,2} = 3.5$, $\sigma = 8$ and obtain $\mu_2 = 2.67$, $\mu_2^* = 2.5$. The pairs (A_{11}^H, b_1) , (A_{11}^H, c_1) and the pairs (A_{22}^H, b_2) , (A_{22}^H, c_2) are completely controllable and observable, respectively, and the polar plots $G_i(j\omega)$, $i = 1, 2$ are inside the circle $C(-0.25, 0.125)$, as seen in Figure 4. Solving in MATLAB the systems (A37) (A38), (A48) and (A49) yields: $\eta_\gamma = 6.54$; $\eta_H = 0.53$. According to Theorem 3, this results in

$$D^\gamma V(t) \leq -0.53 V(t) \quad (72)$$

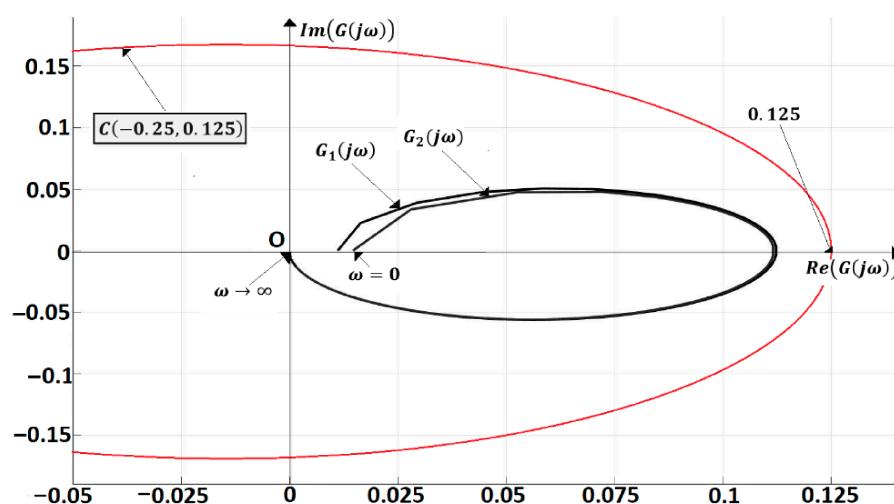


Figure 4. Circle Criterion for Example 2.

All conditions of Theorem 3 (Circle Criterion) are verified, and the systems (66)–(71) are asymptotically stable. Figure 5 shows the trajectories of the system for $k_{0.5} = 1.7$, $k_{0.7} = 1.5$. This evolution confirms the stability of the system.

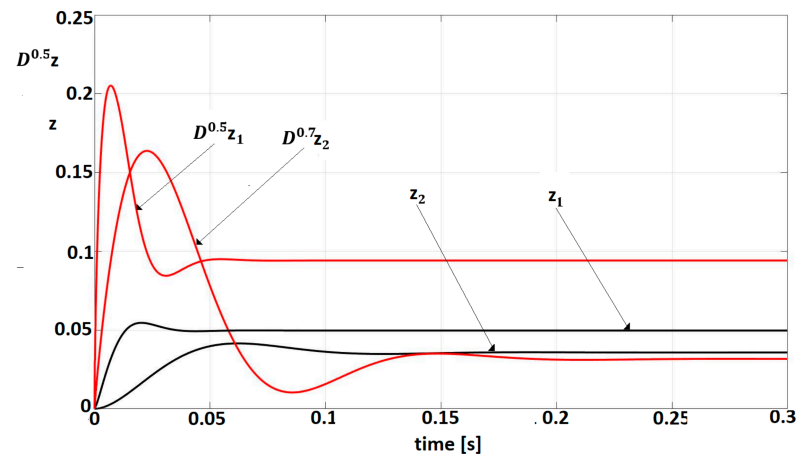


Figure 5. Trajectories of the systems (57)–(62) for $k_{0.5} = 1.7$, $k_{0.7} = 1.5$

3.3. Human–Robot System

The Human–Robot System is analyzed as a multi-fractional-order system where the human operator is described by the fractional-order system and the robot is determined by the classical second-order differential equations. From [33–35], the human operator is described by the following equation

$$D^{1.4}z_h(t) + a_{h1}D^{0.7}z_h(t) + a_{h2}z_h(t) = w_h(t) \quad (73)$$

$$y_h(t) = z_h(t) \quad (74)$$

where z_h defines the human state, y_h is the human operator's output and the parameters a_{h1} , a_{h2} define the human characteristics. The input w_h represents the human operator's capacity to evaluate how the robot performs meaningful work by determining the robot terminal position z_F . Dynamic models (73) and (74) can be rewritten as

$$D^{0.7}z_h(t) = A_h z_h(t) + d_h w_h(t) \quad (75)$$

$$y_h(t) = c_h^T z_h(t) \quad (76)$$

where $z_h(t) = [z_h(t), D^{0.7}z_h(t)]^T$ and $z_h(0) = [0; 0]^T$

$$A_h = \begin{bmatrix} 0 & 1 \\ -a_{h2} & -a_{h1} \end{bmatrix}; c_h = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; d_h = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (77)$$

The operator's output y_h represents the command that generates the robot motion to the operating position q_o

$$q_o = d_T y_h, d_T \in R^{6 \times 1} \quad (78)$$

The operator's visual input w_h is processed by the operator and provides information regarding the position Z_F of the robot, where

$$z_F = J(q)z_q; J \in R^{3 \times 6} \quad (79)$$

Consider the dynamic model of a six-axis articulated robot, as seen in Figure 6 [36],

$$M(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau \quad (80)$$

where $q = [q_1, q_2, \dots, q_6]^T$ represents the state vector, $q \in R^6$, q_i are angles that define joint positions, $q_i \in Q_i$, Q_i , represent the admissibility domain of the angles around a posture,

$M(q)$ is the inertial matrix, $M(q) \in R^{6 \times 6}$, $C(q, \dot{q})$ is the Coriolis term, $C(q, \dot{q}) \in R^{6 \times 1}$, $G(q)$ is the nonlinear term representing gravitational components, $G(q) \in R^{6 \times 1}$, and τ denotes the actuator torques. We assume that around the equilibrium point, for $\dot{q} \sim 0$, the Coriolis components are negligible. Consider that the robot operates inside the envelope $Q = \cup Q_i$. Using the standard linearized dynamic model about a nominal operating point $(q_o, \dot{q}_o, \ddot{q}_o, \tau_o)$, define

$$z_q = q_0 - q \quad (81)$$

$$u = \Delta \tau \quad (82)$$

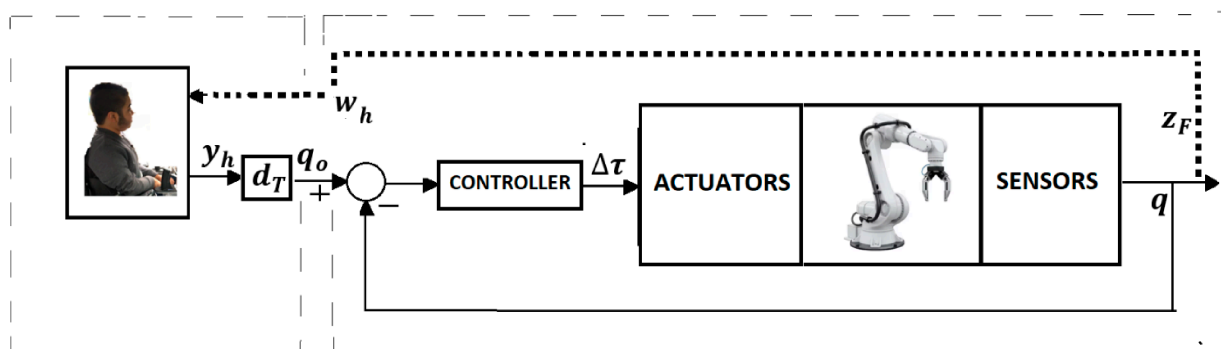


Figure 6. The Human–Robot System.

The decoupled dynamic model becomes,

$$\dot{z}_{qi} = A_i z_{qi} + \Delta h_i(q) + b_i u_i; y_{qi} = c_i^T z_{qi} \quad (83)$$

$$z_{qi} = [z_{qi} \ \dot{z}_{qi}]^T, i = 1, 2, \dots, 6 \quad (84)$$

$$A_i = \begin{bmatrix} 0 & 1 \\ -\frac{c_{qi}}{M_{ii}} & -\frac{c_{vi}}{M_{ii}} \end{bmatrix}; b_i = \begin{bmatrix} 0 \\ \frac{1}{M_{ii}} \end{bmatrix}, i = 1, 2, \dots, 6 \quad (85)$$

$$\Delta h_i^T(q) \Delta h_i(q) < \rho_i \|z_{qi}\|^2, q \in Q; \rho_i > 0, i = 1, 2, \dots, 6 \quad (86)$$

The control law

$$u_i = -k_i(q_{oi} - q_i) = -k_i z_{qi}; k_i < \sigma_i \quad (87)$$

ensures that the decoupled component A_i becomes Hurwitz stable. The 2-DOF planar robot (Appendix F) with

$$J(q)J^T(q) < \xi_1 \|z_q\|^2; q \in Q; \quad (88)$$

is used. The control law (87) with $k_1 = 0.2$, $k_2 = 0.5$ and $\sigma_1 = 0.5$; $\sigma_2 = 1$ is applied. The circle is defined as $C\left(-\frac{2}{\sigma^*}, \frac{\gamma^*}{\sigma^*(1-\gamma^*)}\right) = C(-2, 2.33)$. Consider the generalized Lyapunov function

$$V(x) = z_h^T P_h z_h + \sum_{i=1}^6 z_{qi}^T P_i z_{qi} \quad (89)$$

The same procedure as described in the preceding sections will be followed.

$$A_h^T P_h + P_h A_h = -Q_h \quad (90)$$

$$A_i^{HT} P_i + P_i A_i^H = -Q_i \quad (91)$$

The polar plots $G_h(j\omega)$, $G_i(j\omega)$, $i = 1, 2$ are inside the circle (Figure 7), hence

$$\text{Re}\left(c_h^T(j\omega I - A_h)^{-1} d_h\right) < 2.33 \quad (92)$$

$$\operatorname{Re}\left(\mathbf{c}_i^T(j\omega\mathbf{I} - \mathbf{A}_i^H)^{-1}\mathbf{b}_i\right) > -2, \quad i = 1, 2 \quad (93)$$

$$\eta = \min_i(\eta_h, \eta_i) \quad (94)$$

$$\eta_h = \frac{\lambda_{\min}(\mathbf{Q}_h)}{\lambda_{\max}(\mathbf{P}_h)} - \frac{\lambda_{\max}(\mathbf{P}_h^2)}{\lambda_{\min}(\mathbf{P}_h)} \quad (95)$$

$$\eta_i = \frac{\lambda_{\min}(\mathbf{Q}_i)}{\lambda_{\max}(\mathbf{P}_i)} - \frac{\rho_{ii}}{\lambda_{\min}(\mathbf{P}_i)} - \frac{\xi_1\xi_2}{2\lambda_{\min}(\mathbf{P}_i)}, \quad i = 1, 2 \quad (96)$$

$$D^\gamma V(t) \leq -1.45 V(t) \quad (97)$$

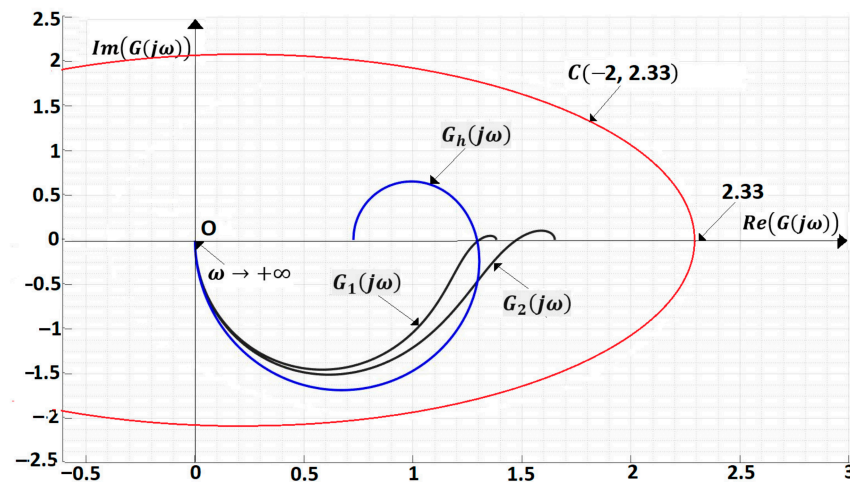


Figure 7. Circle Criterion of Human–Robot System.

Therefore, according to Theorem 4, the systems (75), (76), and (83)–(86) are asymptotically stable. This conclusion is also confirmed by the evolution of the system trajectories in Figure 8.

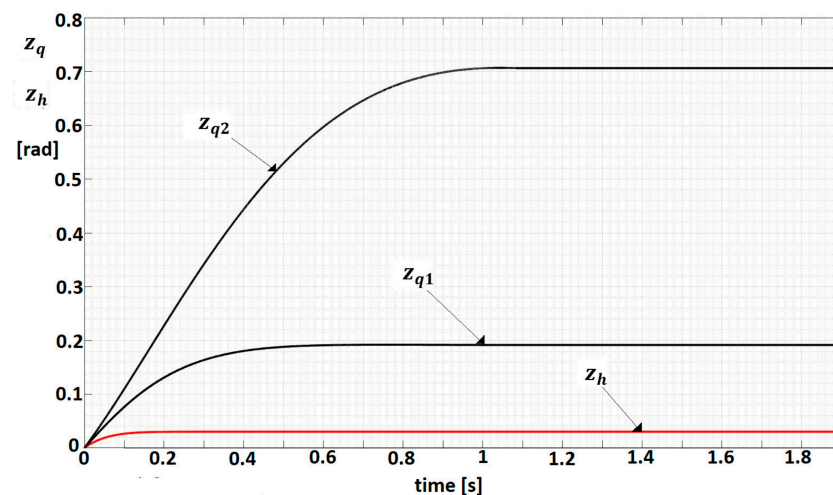


Figure 8. Trajectories of the Human–Robot System.

Remark. For the robotic system where $\gamma = 1$, (74), the condition “the plot of $G(j\omega)$ is inside the circle C ” reduces to “the plot of $G(j\omega)$ lies in the half-plane $\operatorname{Re}(G(j\omega)) > -2$.”

4. Discussion/Conclusions

The significant impact that frequency-domain graphical representations have had on classical automatic control systems theory is well known. The Nyquist polar plot and Bode diagram have represented, and continue to represent, efficient tools for investigating the stability of control systems, determining control parameters, and, to a certain extent, evaluating performance. A novel and efficient step in this direction was achieved through the Popov criteria. Based on a specific interpretation of the KYP lemma, Popov's Criterion and Popov's Line were derived, constituting powerful tools for investigating the asymptotic stability of systems with nonlinear components.

The present paper contributes to these research efforts by evaluating the stability of systems described by fractional-order differential equations of order γ . Using the YKP lemma as a starting point, a frequency-domain stability criterion was obtained (Theorem 2), which graphically requires the polar plot associated with the system to lie inside a circle determined by the γ -fractional order and the control constraint. In this regard, this criterion can be considered an adaptation of the Circle Criterion, although the parameters defining the circle are different, representing the characteristics of the fractional-order system in the case of Theorem 2. This result is extended via generalized Lyapunov functions to γ_i -multi-order fractional systems (γ_i -MOFS). The result is substantiated by Theorem 3, which requires the polar plots associated with each γ_i -order to lie inside a circle determined by parameters similar to those formulated in Theorem 2. Theorem 4 formulates similar stability criteria for γ_i -MOFS with nonlinear components. The two Numerical Examples presented highlight the implementation techniques of the formulated criteria, particularly the challenges in solving the equations derived from the YKP lemma. In the final application, the Human–Robot System, human behavior is analyzed through a fractional-order system of order $\gamma = 0.7$. The robotic system is studied as a 6-DOF model using linearization and decoupling techniques and is numerically implemented on a specific case of a planar 2-DOF robot.

The discussion above aimed to clarify the specific features of the results obtained in the present paper and the approach taken to establish stability evaluation criteria for MOFS that are similar to classical investigation methods. To the best of the authors' knowledge, no similar results were found in the literature to allow for a direct comparison. Although, as presented in Section 1, papers [21–32] address the stability of multiple fractional-order systems using Lyapunov functions or Fractional Generalized KYP Lemma for Fractional-Order Systems—with significant results culminating in frequency-domain or algebraic criteria—they do not result in graphical representations that permit adequate comparisons.

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Appendix A

Proof of Theorem 1.

Consider the generalized Lyapunov function $V(a)$ of MFOS that satisfies (3) and (4).

Taking (1) and (2) into account, inequality (4) can be rewritten as

$$\begin{bmatrix} D^{\gamma_1} V_1(z_1) \\ D^{\gamma_2} V_2(z_2) \\ \vdots \\ D^{\gamma_N} V_N(z_N) \end{bmatrix} \leq -\eta \begin{bmatrix} V_1(z_1) \\ V_2(z_2) \\ \vdots \\ V_N(z_N) \end{bmatrix} \quad (\text{A1})$$

$$\begin{bmatrix} D^{\gamma_1} V_1(z_1) \\ D^{\gamma_2} V_2(z_2) \\ \vdots \\ D^{\gamma_N} V_N(z_N) \end{bmatrix} \leq \mathcal{M} \begin{bmatrix} V_1(z_1) \\ V_2(z_2) \\ \vdots \\ V_N(z_N) \end{bmatrix} \quad (\text{A2})$$

where $\mathcal{M} \in \mathbb{R}^{N \times N}$,

$$\mathcal{M} = -\eta \mathbf{I}_N \quad (\text{A3})$$

is Hurwitz matrix. According to [26–29] and Lemma 3.1. (fractional comparison principle) and Theorem 4.1), the system (MFOS) is asymptotically stable. $\square$

Appendix B

Proof of Main Remark

Let $G(j\omega)$ be the polar plot of the system. Assume that the polar plot is inside the circle C defined by the diameter $[-a, b]$, as seen in Figure 1. The following relations can be easily inferred:

$$z_1 = a + G(j\omega) \quad (\text{A4})$$

$$z_2 = -b + G(j\omega) \quad (\text{A5})$$

where z_1, z_2 are complex numbers

$$z_1 = r_1 e^{j\theta_1}; z_2 = r_2 e^{j\theta_2}; \quad (\text{A6})$$

Proof that the $G(j\omega)$ is inside the circle C, $\theta_1 < \frac{\pi}{2}$, $\theta_2 > \frac{\pi}{2}$, is as follows:

$$\operatorname{Re}(z_1) = r_1 \cos(\theta_1) > 0 \quad (\text{A7})$$

$$\operatorname{Re}(z_2) = r_2 \cos(\theta_2) < 0 \quad (\text{A8})$$

$$a + \operatorname{Re}(G(j\omega)) > 0 \quad (\text{A9})$$

$$-b + \operatorname{Re}(G(j\omega)) < 0 \quad (\text{A10})$$

$\square$

(The inequalities (A4), (A5)) can also be easily inferred by simple visual inspection of Figure A1).

Appendix C

Proof of the inequality (20).

Consider the system

$$D^{\gamma_i} z(t) = A z(t) \quad (\text{A11})$$

Let $V(z) = z^T P z$ be Lyapunov function. The Caputo fractional derivative of $V(z)$ is [26–30]

$$D^{\gamma_i} (z^T P z) \leq 2z^T P (D^{\gamma_i} z) = 2z^T P A z \quad (\text{A12})$$

and, using properties of the transpose, yields:

$$z^T P A z = z^T A^T P z \quad (\text{A13})$$

Consequently, (A11) can be rewritten as

$$D^{\gamma_i} (z^T P z) \leq z^T P A z + z^T A^T P z = z^T (P A + A^T P) z \quad (\text{A14})$$

□

Appendix D

Substituting (9) into (6) for $k = k_\gamma$ yields (for the sake of simplicity, throughout the remainder of this section, $z(t)$ is denoted as z).

$$D^{\gamma} V(z) \leq z^T (A^{HT} P + P A^H) z + 2z^T P \mu^* b c^T z \quad (\text{A15})$$

Considering that A^H is Hurwitz, and adding and subtracting the term uy , after simple additional manipulations, it follows that,

$$D^{\gamma} V(z) \leq -z^T q q^T z + 2z^T P \frac{\mu^*}{k_\gamma} b u + u y - u y \quad (\text{A16})$$

where

$$A^{HT} P + P A^H = -q q^T \quad (\text{A17})$$

Considering that

$$u < \sigma y, \sigma > 0 \quad (\text{A18})$$

$$-u y < -\frac{1}{\sigma} u^2 \quad (\text{A19})$$

the inequality (A16) becomes

$$D^{\gamma} V(z) \leq -z^T q q^T z - 2z^T \left(-P \frac{\mu^*}{k_\gamma} b - \frac{c}{2} \right) u - \frac{1}{\sigma} u^2 \quad (\text{A20})$$

According to the Kalman–Yakubovich–Popov (KYP) lemma [3,23,24], if A^H is Hurwitz stable, the pairs (A^H, b) , (A^H, c) are completely controllable and observable, respectively, a symmetric matrix P and the matrix $q q^T$, solution of (A17) and

$$-P \frac{\mu^*}{k_\gamma} b - \frac{c}{2} = \frac{1}{\sqrt{\sigma}} q \quad (\text{A21})$$

exist if and only if

$$\frac{1}{\sigma} + 2 \left(-\frac{\mu^*}{k_\gamma} \right) \operatorname{Re} \left(\frac{c^T}{2} (j\omega I - A^H)^{-1} b \right) > 0, \omega \in [0, +\infty) \quad (\text{A22})$$

or

$$\operatorname{Re} \left(c^T (j\omega I - A^H)^{-1} b \right) < \frac{k_\gamma}{\sigma \mu^*}, \omega \in [0, +\infty) \quad (\text{A23})$$

Taking (14)–(15) into account, this inequality becomes,

$$\operatorname{Re}\left(\mathbf{c}^T\left(j\omega\mathbf{I}-\mathbf{A}^H\right)^{-1}\mathbf{b}\right)<\frac{\gamma}{\sigma(1-\gamma)}, \quad \omega \in [0, +\infty) \quad (\text{A24})$$

If all these conditions are satisfied, (A15) becomes

$$D^\gamma V(\mathbf{z}) \leq -\mathbf{z}^T \mathbf{q} \mathbf{q}^T \mathbf{z} - 2\mathbf{z}^T \mathbf{q} \frac{u}{\sqrt{\sigma}} - \frac{1}{\sigma} u^2 \quad (\text{A25})$$

$$D^\gamma V(\mathbf{z}) \leq -\mathbf{z}^T \left(\mathbf{q} + \frac{k_\gamma \mathbf{c}}{\sqrt{\sigma}} \right) \left(\mathbf{q} + \frac{k_\gamma \mathbf{c}}{\sqrt{\sigma}} \right)^T \mathbf{z} \quad (\text{A26})$$

Similarly, for $k = 2k_H$, $k < \sigma$, from (20), yields,

$$D^\gamma V(\mathbf{z}) \leq \mathbf{z}^T \left(\mathbf{A}^{HT} \mathbf{P} + \mathbf{P} \mathbf{A}^H \right) \mathbf{z} - \mathbf{z}^T \mathbf{P} \mathbf{b} u \quad (\text{A27})$$

Considering the inequality (10), adding and subtracting the term uy results in,

$$D^\gamma V(\mathbf{z}) \leq \mathbf{z}^T \left(\mathbf{A}^{HT} \mathbf{P} + \mathbf{P} \mathbf{A}^H \right) \mathbf{z} - 2\mathbf{z}^T \left(\mathbf{P} \frac{\mathbf{b}}{2} - \frac{\mathbf{c}}{2} \right) u - \frac{1}{\sigma} u^2 \quad (\text{A28})$$

The following KYP lemma substitution is used,

$$\mathbf{P} \frac{\mathbf{b}}{2} - \frac{\mathbf{c}}{2} = \frac{1}{\sqrt{\sigma}} \mathbf{q} \quad (\text{A29})$$

if the following conditions are satisfied

$$\frac{1}{\sigma} + 2\operatorname{Re}\left(\frac{\mathbf{c}^T}{2}\left(j\omega\mathbf{I}-\mathbf{A}^H\right)^{-1}\mathbf{b}\right) > 0, \quad \omega \in [0, +\infty) \quad (\text{A30})$$

the inequality (A28) becomes,

$$D^\gamma V(\mathbf{z}) \leq -\mathbf{z}^T \mathbf{q} \mathbf{q}^T \mathbf{z} - 2\mathbf{z}^T \mathbf{q} \frac{u}{\sqrt{\sigma}} - \frac{1}{\sigma} u^2 \quad (\text{A31})$$

$$D^\gamma V(\mathbf{z}) \leq -\mathbf{z}^T \left(\mathbf{q} + \frac{k_H}{\sqrt{\sigma}} \mathbf{c} \right) \left(\mathbf{q} + \frac{k_H}{\sqrt{\sigma}} \mathbf{c} \right)^T \mathbf{z} \quad (\text{A32})$$

□

Appendix E

By following the same procedure as described in the preceding section, substituting (28) and (29) into (41) for $k_i = k_{\gamma i}$, yields

$$D^\gamma V(t) \leq \sum_{i=1}^N \left(\mathbf{z}_i^T \left(\mathbf{A}_{ii}^{HT} \mathbf{P}_i + \mathbf{P}_i \mathbf{A}_{ii}^H \right) \mathbf{z}_i + 2\mathbf{z}_i^T \mathbf{P}_i \mu_i^* \mathbf{b}_i \mathbf{c}_i^T \mathbf{z}_i \right) + 2 \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{x}_j^T \mathbf{A}_{ij}^T \mathbf{P}_i \mathbf{z}_i \quad (\text{A33})$$

Using inequalities

$$2\mathbf{z}_i^T \mathbf{A}_{ij}^T \mathbf{P}_i \mathbf{z}_i \leq \mathbf{z}_i^T \mathbf{P}_i \mathbf{P}_i \mathbf{z}_i + \mathbf{z}_j^T \mathbf{A}_{ij}^T \mathbf{A}_{ij} \mathbf{z}_j \quad (\text{A34})$$

$$-u_i y_i < -\frac{1}{\sigma_i} u_i^2 \quad (\text{A35})$$

after simple additional manipulations, it follows that,

$$D^\gamma V(t) \leq \sum_{i=1}^N \left(z_i^T \left(A_{ii}^{HT} P_i + P_i A_{ii}^H \right) z_i - 2z_i^T \left(-P_i \frac{\mu_i^*}{k_{\gamma i}} b_i - \frac{c_i}{2} \right) u_i - \frac{1}{\sigma_i} u_i^2 + x_i^T P_i^2 z_i \right) + \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \rho_{ij} x_j^T z_i \quad (\text{A36})$$

According to the Kalman–Yakubovich–Popov lemma [3], if A_{ii}^H is Hurwitz stable, the pairs (A_{ii}^H, b_i) , (A_{ii}^H, c_i) are completely controllable and observable, respectively, a symmetric matrix $P_{\gamma i}$, the matrix $q_{\gamma i} q_{\gamma i}^T$, solution of

$$A_{ii}^{HT} P_{\gamma i} + P_{\gamma i} A_{ii}^H = -q_{\gamma i} q_{\gamma i}^T = -Q_{\gamma i} \quad (\text{A37})$$

$$-P_{\gamma i} \frac{\mu_i^*}{k_{\gamma i}} b_i - \frac{c_i}{2} = \frac{1}{\sqrt{\sigma_i}} q_{\gamma i}, i = 1, 2, \dots, N \quad (\text{A38})$$

exists if and only if

$$\frac{1}{\sigma_i} + 2 \operatorname{Re} \left(\frac{c_i^T}{2} \left(j\omega \mathbf{I} - A_{ii}^H \right)^{-1} \left(-\frac{\mu_i^*}{k_{\gamma i}} b_i \right) \right) > 0 \quad (\text{A39})$$

or

$$\operatorname{Re} \left(c_i^T \left(j\omega \mathbf{I} - A_{ii}^H \right)^{-1} b_i \right) < \frac{k_{\gamma i}}{\sigma_i \mu_i^*} < \frac{\gamma_i}{\sigma_i (1 - \gamma_i)}, i = 1, 2, \dots, N \quad (\text{A40})$$

If (A40) is satisfied, the inequality (A36) becomes,

$$D^\gamma V(t) \leq \sum_{i=1}^N \left(-\lambda_{\min}(Q_{\gamma i}) z_i^T z_i + \lambda_{\max}(P_{\gamma i}^2) z_i^T z_i \right) + \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \rho_{ij} z_j^T z_i \quad (\text{A41})$$

or, taking (2) into account,

$$D^\gamma V(t) \leq \sum_{i=1}^N \left(-\frac{\lambda_{\min}(Q_{\gamma i})}{\lambda_{\max}(P_{\gamma i})} + (N-1) \frac{\lambda_{\max}(P_{\gamma i}^2)}{\lambda_{\min}(P_{\gamma i})} + \frac{\rho_i}{\lambda_{\min}(P_{\gamma i})} \right) z_i^T P_{\gamma i} z_i \quad (\text{A42})$$

where

$$\rho_i = \sum_{\substack{j=1 \\ j \neq i}}^N \rho_{ij} \quad (\text{A43})$$

If we denote by,

$$\eta_{\gamma i} = \frac{\lambda_{\min}(Q_{\gamma i})}{\lambda_{\max}(P_{\gamma i})} - \frac{1}{\lambda_{\min}(P_{\gamma i})} ((N-1) \lambda_{\max}(P_{\gamma i}^2) + \rho_i) > 0; \eta_\gamma = \min_i \eta_{\gamma i} \quad (\text{A44})$$

The inequality (A42) becomes,

$$D^\gamma V(t) \leq -\eta_\gamma V(t) \quad (\text{A45})$$

Similarly, for $k_i = k_{Hi}$, we obtain,

$$D^\gamma V(t) \leq -\eta_H V(t) \quad (\text{A46})$$

where

$$\eta_{Hi} = \frac{\lambda_{\min}(Q_{Hi})}{\lambda_{\max}(P_{Hi})} - \frac{1}{\lambda_{\min}(P_{Hi})} ((N-1) \lambda_{\max}(P_{Hi}^2) + \rho_i) > 0; \eta_H = \min_i \eta_{Hi} \quad (\text{A47})$$

matrix P_{Hi} , the matrix $q_{Hi}q_{Hi}^T$ are solutions of

$$A_i^{HT}P_{Hi} + P_{Hi}A_i^H = -q_{Hi}q_{Hi}^T = -Q_{Hi} \quad (A48)$$

$$P_{Hi}\frac{b_i}{2} - \frac{c_i}{2} = \frac{1}{\sqrt{\sigma_i}}q_{Hi}, i = 1, 2, \dots, N \quad (A49)$$

and the following conditions are satisfied,

$$\operatorname{Re}\left(c_i^T(j\omega I - A_i^H)^{-1}b_i\right) > -\frac{2}{\sigma_i}, i = 1, 2, \dots, N, \omega \in [0, +\infty) \quad (A50)$$

□

Appendix F

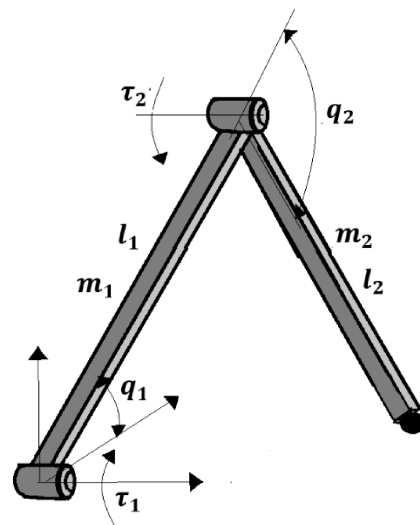


Figure A1. 2-DOF planar robot.

The 2-DOF planar robot (Figure A1) is characterized by the following parameters:

$$Q_1 = [0, 5\frac{\pi}{3}]; Q_2 = [-\frac{\pi}{3}, 5\frac{\pi}{3}]; c_{e1} = 1500 \frac{\text{Nm}}{\text{rad}}; c_{e2} = 1800 \frac{\text{Nm}}{\text{rad}}; c_{v1} = 0.422 \frac{\text{Nm s}}{\text{rad}}; c_{v2} = 0.213 \frac{\text{Nm s}}{\text{rad}};$$

$$l_1 = 0.8 \text{ m}; l_2 = 1 \text{ m}; m_1 = 2.4 \text{ kg}; m_2 = 2.8 \text{ kg}$$

$$M_{11} = l_1 + l_2 + m_1c_1^2 + m_2(l_1^2 + c_2^2) = 4.98$$

$$M_{22} = l_2 + m_2c_2^2 = 2.08$$

$$b_1 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}; b_2 = \begin{bmatrix} 0 \\ 0.48 \end{bmatrix} \quad (A51)$$

$$h_1(q) = \begin{bmatrix} 0 \\ (m_1c_1 + m_2l_1)g \cos q_1 + m_2c_2 \cos(q_1 + q_2) / M_{11} \end{bmatrix} \quad (A52)$$

$$h_2(q) = \begin{bmatrix} 0 \\ (m_2c_2 \cos(q_1 + q_2) / M_{22}) \end{bmatrix} \quad (A53)$$

$$\Delta h_i^T \Delta h_i < \rho_i z_{qi}^2; \rho_1 = 0.12; \rho_2 = 0.35 \quad (A54)$$

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