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Almost Sure Power-Law Consensus for Fractional Inhomogeneous Hegselmann-Krause Systems with Multiplicative Noise

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Abstract

This paper investigates how opinion dynamics with memory effects and random communication uncertainties achieve consensus in an inhomogeneous Hegselmann–Krause (H-K) model. The Caputo fractional derivative is introduced to describe the influence of historical opinions, while multiplicative noise captures random disturbances in communication. We establish sufficient conditions under which all followers almost surely converge to the leader’s opinion. Moreover, we characterize the algebraic (power-law) convergence behavior determined by the fractional order. The analysis is based on fractional Lyapunov techniques and fractional Grönwall estimates, which enable the treatment of the nonlinear stochastic system with leadership. The results reveal how memory effects, leadership strength and stochastic perturbations jointly influence consensus formation. Numerical simulations are provided to verify the theoretical results and illustrate the effects of the model parameters on the convergence dynamics.

Keywords: inhomogeneous Hegselmann–Krause model; Caputo fractional derivative; multiplicative noise; almost sure consensus; power-law convergence

1. Introduction

The rapid proliferation of social networks has stimulated extensive research on opinion dynamics, a field that investigates how individual attitudes, perceptions or cultural traits evolve through peer interactions. In its broadest sense, the term “opinion” admits various interpretations: it may represent an individual’s stance on a specific issue, a set of cultural characteristics, or even a measure of an agent’s skill in a particular domain, as discussed in recent surveys [1]. Among the numerous mathematical frameworks proposed to model such collective phenomena, the Hegselmann–Krause (H-K) model [2] has attracted particular attention due to its inherently self-organizing nature and the psychologically plausible assumption of bounded confidence, which posits that individuals are immune to influence from peers whose opinions differ excessively from their own. Since its introduction as a discrete-time model, the H-K type model has undergone substantial extensions, including continuous-time formulations and various modifications tailored to specific applications. In particular, Blondel et al. [3] studied continuous-time opinion



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dynamics with opinion-dependent communication and characterized the preservation of the average opinion. Canuto et al. [4] developed an Eulerian approach for the analysis of continuum versions of the Krause model, thereby providing a framework for studying the evolution of opinion distributions. Choi et al. [5] investigated the H-K model with time delay and established consensus results in the presence of delayed information. Haskovec [6] gave a direct proof of unconditional asymptotic consensus for a model with transmission-type delay, while Haskovec [7] provided a simplified consensus proof for the H-K and Cucker–Smale models with normalization and delay. Vasca et al. [8] analyzed practical consensus in bounded-confidence opinion dynamics, which is relevant when convergence is understood up to a prescribed tolerance. Haskovec and Rodriguez Cartabia [9] derived an optimal condition for asymptotic consensus when information propagates at finite speed. Zheng and Zhou [10] studied mean-square bipartite consensus in discrete-time multi-agent systems with cooperation–competition interactions. Collectively, these studies clarify how communication dependence, continuum descriptions, delay, bounded confidence, finite propagation speed and stochastic effects influence consensus, and they motivate the present extension to an inhomogeneous fractional H-K model with multiplicative noise and memory. A key development in this lineage is the transition from homogeneous to inhomogeneous settings, where asymmetric interaction structures, such as the presence of opinion leaders, are incorporated to reflect hierarchical social realities [11–13]. Concurrently, the recognition that real-world opinion formation is inevitably subject to random communication uncertainty has motivated the inclusion of stochastic perturbations into the dynamics. Notably, Du et al. [14] recently provided the first rigorous mathematical treatment of this combined scenario by proposing inhomogeneous H-K models with two distinct types of noise and establishing the corresponding consensus conditions, thereby laying a solid foundation for further theoretical exploration.

Although the inhomogeneous stochastic H-K models established by Du et al. [14] provide a rigorous framework for analyzing consensus under random perturbations, they remain confined to integer-order dynamics, which implicitly assume that opinion evolution depends only on the current state. In reality, however, social interactions are inherently history-dependent: an individual’s present judgment is often shaped by past experiences, long-standing relationships and accumulated trust or distrust. Fractional calculus offers a natural mathematical tool to encode such memory and hereditary effects (see e.g., refs [15–17]). Among various definitions of fractional derivatives, the Caputo formulation is particularly appealing for opinion dynamical systems, since it intrinsically incorporates the historical states of the system into its current evolution through a nonlocal memory kernel and admits initial conditions of the same form as integer-order differential equations, thereby preserving the physical interpretability of the state variables. It is well known that fractional-order systems often exhibit algebraic relaxation due to the long-memory property of fractional operators. However, whether such behavior persists in nonlinear stochastic opinion dynamics with bounded-confidence interactions and leadership is a nontrivial question. The present work aims to address this issue by establishing rigorous almost sure consensus conditions for a fractional stochastic H-K model.

We note that almost sure consensus in stochastic multi-agent systems has been studied extensively from various perspectives. For instance, Shang [18] proposed a median-based resilient consensus strategy for multi-agent systems over time-varying directed random networks, and also developed an opinion formation model featuring both private and expressed opinions over dynamical networks [19]. Zhang et al. [20] investigated mean square and almost sure consensus of heterogeneous multi-agent systems under multiplicative noise. These studies typically focus on graph-theoretic conditions or general multi-agent frameworks to achieve consensus under stochastic or adversarial perturbations.

However, the effect of historical interactions on stochastic opinion evolution remains insufficiently understood. On the other hand, although fractional stochastic differential equations have been extensively studied from the viewpoints of existence, stability and solution properties [21], their application to nonlinear bounded-confidence opinion dynamics with leadership remains insufficiently explored. The stochastic-calculus framework of Mao [22], the anticipated backward stochastic analysis of Hao et al. [23], and the qualitative fractional stochastic analysis of Mohammed Djaouti and Imran Liaqat [24] provide complementary analytical background for this setting.

Motivated by these considerations, we propose a fractional-order inhomogeneous H-K model with multiplicative noise, where the Caputo derivative accounts for the long-term memory of social interactions, while the multiplicative noise captures communication uncertainties that scale with the extremity of opinions. Unlike the aforementioned resilient consensus literature, which primarily addresses robustness against malicious agents, our work focuses on the interplay between fractional memory and communication randomness within the bounded-confidence framework, where uncertainty arises naturally from transmission errors rather than adversarial attacks. By means of fractional Lyapunov inequalities and fractional Grönwall-type estimates, we establish a sufficient condition for almost sure consensus and reveal a distinct power-law decay rate of the consensus error. The established condition theoretically corroborates the intuitive notion that larger groups with stronger leadership are more resilient to severe communication noise. In addition, we characterize the algebraic convergence behavior of the consensus error in this nonlinear stochastic setting. Although algebraic decay is a well-known asymptotic property of fractional dynamics, establishing such behavior for a nonlinear H-K model with bounded-confidence interactions, multiplicative noise, and leadership requires a dedicated stochastic analysis. Numerical simulations are conducted to verify the analytical results and to explore the sensitivity of convergence dynamics to the fractional order and confidence parameters.

The remainder of this paper is organized as follows. Section 2 recalls some basic notations, definitions and preliminary results on fractional calculus and Mittag-Leffler functions. In Section 3, we propose the fractional-order stochastic H-K model and prove the main consensus theorem. Section 4 presents numerical experiments to validate the theoretical analysis. Finally, Section 5 concludes the paper with a summary and discussion of future directions.

2. Preliminaries

Throughout this paper, let $C^1[t_0, T]$ denote the space of continuously differentiable functions on $[t_0, T]$. The symbol $\Gamma(\cdot)$ denotes the Gamma function and ${}^C\mathcal{D}_t^\alpha$ represents the Caputo fractional derivative of order $0 < \alpha < 1$.

Next, we recall some basic definitions and lemmas from fractional calculus, which will be used in the proof of the main results. For more details, we refer the reader to [15–17,25,26].

Definition 1 ([15]). Let $0 < \alpha < 1$ and $x(t) \in C^1[t_0, T]$. The Caputo fractional derivative is defined as

$${}^C\mathcal{D}_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{\dot{x}(s)}{(t-s)^\alpha} ds.$$

Definition 2 ([16]). For $\alpha > 0$, the Mittag-Leffler function is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

and the Gamma function is defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad z > 0.$$

Lemma 1 ([16]). For $0 < \alpha < 1$ and $a > 0$,

$$E_\alpha(-at^\alpha) \sim \frac{1}{a\Gamma(1-\alpha)} t^{-\alpha}, \quad t \rightarrow \infty.$$

Consequently,

$$\lim_{t \rightarrow \infty} E_\alpha(-at^\alpha) = 0.$$

Lemma 2 ([26]). Let $V(x)$ be a continuously differentiable convex function. Then

$${}^C\mathcal{D}_t^\alpha V(x(t)) \leq \left(\frac{\partial V(x)}{\partial x} \right)^T {}^C\mathcal{D}_t^\alpha x(t), \quad 0 < \alpha < 1.$$

Lemma 3 ([17]). Let $y(t)$ be a nonnegative continuous function satisfying

$${}^C\mathcal{D}_t^\alpha y(t) \leq -ay(t), \quad a > 0, \quad 0 < \alpha < 1.$$

Then

$$y(t) \leq y(0) E_\alpha(-at^\alpha).$$

3. Fractional Inhomogeneous H–K Model and Consensus Analysis

With the necessary preliminaries on fractional calculus at hand, we now proceed to formulate the fractional-order Hegselmann–Krause model with multiplicative noise and leadership. We first present the model in detail, together with the assumptions on the communication functions and the adaptive control mechanism. Then, by means of the fractional Lyapunov inequality and the fractional Grönwall inequality, we establish a sufficient condition for the almost sure consensus behavior towards the leader's opinion.

3.1. Model Formulation

Let us consider a community consisting of one leader and N followers. Let $x_0(t) \in \mathbb{R}^d$ denote the leader's opinion and $x_i(t) \in \mathbb{R}^d$ the opinion of the i -th follower. The fractional-order stochastic dynamics are governed by the following equations:

$$\begin{cases} {}^C\mathcal{D}_t^\alpha x_0(t) = u(t) + \sqrt{2\sigma} x_0(t) \frac{dB_t}{dt}, \\ {}^C\mathcal{D}_t^\alpha x_i(t) = \frac{1}{N} \sum_{j=1}^N a_{ij}(t)(x_j(t) - x_i(t)) + \gamma \phi_{i0}(t)(x_0(t) - x_i(t)) + \sqrt{2\sigma} x_i(t) \frac{dB_t}{dt}, \end{cases} \quad (1)$$

for $i = 1, \dots, N$, where ${}^C\mathcal{D}_t^\alpha$ denotes the Caputo fractional derivative of order $\frac{1}{2} < \alpha < 1$. The symbol $\{B_t\}_{t \geq 0}$ stands for a one-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the term $\frac{dB_t}{dt}$ is used only as a formal notation for stochastic forcing (since Brownian motion is nowhere differentiable almost surely, thus this expression does not represent a classical derivative), and $\sigma > 0$ measures the intensity of the multiplicative noise.

Remark 1. The proposed fractional stochastic system is rigorously interpreted through its equivalent fractional stochastic convolution formulation [21]:

$$\begin{aligned} x_0(t) &= x_0(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + \frac{\sqrt{2\sigma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x_0(s) dB_s, \\ x_i(t) &= x_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\frac{1}{N} \sum_{j=1}^N a_{ij}(s)(x_j(s) - x_i(s)) + \gamma \phi_{i0}(s)(x_0(s) - x_i(s)) \right] ds \\ &\quad + \frac{\sqrt{2\sigma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x_i(s) dB_s, \end{aligned} \quad (2)$$

where the stochastic terms are interpreted as fractional stochastic convolutions with respect to the Brownian motion. More precisely, the kernel $(t-s)^{\alpha-1}/\Gamma(\alpha)$ is induced by the fractional integral operator associated with the Caputo derivative. Therefore, these terms are different from standard Itô stochastic integrals without memory effects. All subsequent theoretical analyses are conducted based on this fractional stochastic integral formulation.

Remark 2. In this work, the same Brownian motion is adopted for all agents to represent common stochastic disturbances arising from shared environmental factors or global information fluctuations. This assumption enables us to analyze the influence of fractional memory and stochastic perturbations on consensus behavior in a tractable framework. More general cases involving agent-dependent or correlated stochastic noises will be considered in future studies.

Remark 3. In the subsequent analysis, we restrict our attention to the case $\alpha \in (1/2, 1)$, which is the standard regime for applying Burkholder–Davis–Gundy type estimates to fractional stochastic convolutions.

We now interpret each term in the model to clarify its sociological and mathematical meaning.

The leader’s equation. The first equation in (1) describes the evolution of the leader’s opinion. The term $u(t)$ represents the influence exerted by the followers on the leader, which is designed to be weak so that the leader remains relatively stable. Its explicit form will be given shortly. The stochastic term $\sqrt{2\sigma} x_0(t) dB_t/dt$ captures communication uncertainty in a multiplicative form: the magnitude of the random fluctuation is proportional to the leader’s current opinion value. This reflects the empirically plausible scenario that more extreme or prominent opinions are subject to greater distortion during transmission.

The followers’ equations. For each follower i , the right-hand side consists of three competing forces. The first term, $\frac{1}{N} \sum_j a_{ij}(t)(x_j - x_i)$, is the classic H–K consensus driver: each follower adjusts its opinion toward neighbors whose opinions lie within a confidence bound. The weight function $a_{ij}(t) = a(\|x_j(t) - x_i(t)\|)$, which is determined by the distance $\|x_j(t) - x_i(t)\|$ and is non-increasing in this distance. Specifically, we set

$$a(r) = \begin{cases} 1, & 0 \leq r \leq \delta, \\ \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{1}{r-\delta} + \frac{1}{r-(\delta+\epsilon)}\right), & \delta < r < \delta + \epsilon, \\ 0, & r \geq \delta + \epsilon, \end{cases} \quad (3)$$

where $\delta > 0$ and $\epsilon > 0$ are the confidence threshold and the transition width, respectively. Thus, followers whose opinions are sufficiently close (within distance δ) interact with full strength; those whose opinions are farther apart than $\delta + \epsilon$ exert no direct influence on each other; and in the intermediate region the influence decays smoothly.

The second term, $\gamma \phi_{i0}(t)(x_0 - x_i)$, is the leader's attractive force acting on follower i . The parameter $\gamma > 0$ quantifies the strength of the leader's influence. The weight $\phi_{i0}(t) = \phi(\|x_0(t) - x_i(t)\|)$ measures how effectively the leader can influence follower i ; it is assumed to be strictly positive and bounded away from zero:

$$0 < \phi_m \leq \phi_{i0}(t) \leq \phi_M \leq 1. \quad (4)$$

This assumption is crucial: it guarantees that the leader is always audible to every follower, regardless of how far apart their opinions are. In sociological terms, the leader's voice never completely fades away, although its intensity may decrease with distance.

The third term, $\sqrt{2\sigma} x_i(t) dB_t/dt$, is the same multiplicative noise that also affects the leader. Notably, all agents share the same Brownian motion B_t , meaning that the random perturbation acts as a common environmental fluctuation rather than independent individual noise. This choice is deliberate: it models communication uncertainty that simultaneously distorts all opinions in a correlated manner, preserving the overall structure of the group while introducing randomness.

The leader's adaptive control input. The term $u(t)$ in the leader's equation is defined as

$$u(t) = \gamma \eta(t) \sum_{j=1}^N \psi_{0j}(t)(x_j(t) - x_0(t)), \quad (5)$$

where $\psi_{0j}(t) = \psi(\|x_j(t) - x_0(t)\|)$ characterizes the influence of follower j on the leader. The function ψ shares the same qualitative properties as ϕ , but since followers are typically less influential than the leader, the net effect is kept small by the adaptive coefficient $\eta(t)$:

$$\eta(t) = \frac{1}{2} \min \left\{ \frac{\psi_{0\iota(t)}}{N - \psi_{0\iota(t)}}, \frac{2C}{\gamma \sum_{j=1}^N \|x_j(t) - x_0(t)\|} \right\}, \quad (6)$$

where $\iota(t)$ denotes the smallest index at each time t , and $C > 0$ is a constant such that $\|u(t)\| \leq C$. The role of $\eta(t)$ is to ensure that the leader's opinion is only marginally affected by the followers: when the followers' opinions are far from the leader's, or when their connections are weak, $\eta(t)$ becomes small, and the leader's opinion remains nearly unchanged. The leader then acts as a stable anchor, gradually pulling the followers toward its own position through the $\gamma \phi_{i0}(t)$ terms, ultimately enabling the entire group to reach consensus.

Remark 4. The well-posedness of the proposed fractional stochastic H–K system follows from the general existence and uniqueness framework for fractional stochastic differential equations developed in Sakthivel et al. [21]. In particular, the proposed model corresponds to a finite-dimensional special case of the system considered there. Specifically, by taking the state space as $\mathbb{R}^{N+1}$ and setting

$$A = 0, \quad B_1 = 0, \quad B_2 = 0, \quad x_t = x(t), \quad I_k = 0,$$

the abstract fractional stochastic system in [20] is reduced to the present finite-dimensional stochastic system without delay, integral operators and impulsive effects. Here, $x_t = x(t)$ means that the history-dependent terms are restricted to the current state.

Let $\mathbf{x}(t) = (x_0(t), x_1(t), \dots, x_N(t))^T$, then the system can be expressed as

$${}^C \mathcal{D}_t^\alpha \mathbf{x}(t) = F(\mathbf{x}(t), t) + G(\mathbf{x}(t), t) \frac{dB_t}{dt},$$

where the drift coefficient is given by

$$F(\mathbf{x}(t), t) = \begin{pmatrix} u(t) \\ \frac{1}{N} \sum_{j=1}^N a_{1j}(t)(x_j(t) - x_1(t)) + \gamma \phi_{10}(t)(x_0(t) - x_1(t)) \\ \vdots \\ \frac{1}{N} \sum_{j=1}^N a_{Nj}(t)(x_j(t) - x_N(t)) + \gamma \phi_{N0}(t)(x_0(t) - x_N(t)) \end{pmatrix}$$

and the diffusion coefficient is

$$G(\mathbf{x}(t), t) = \sqrt{2\sigma} \operatorname{diag}(x_0(t), x_1(t), \dots, x_N(t)).$$

The confidence function $a(r)$ in (3) is constructed as a bounded smooth function. The influence functions $\phi(r)$ and $\psi(r)$ are chosen with the same regularity properties, namely they are continuous, bounded, positive and Lipschitz continuous on the considered bounded opinion domain. Consequently, the drift coefficient satisfies the required local Lipschitz condition. In particular, the follower interaction terms are locally Lipschitz due to the regularity of $a(r)$ and $\phi(r)$, while the adaptive leader feedback term $u(t)$ is also locally Lipschitz because it is constructed from the bounded Lipschitz influence function $\psi(r)$ and the bounded relative opinion differences. Moreover, the diffusion coefficient obviously satisfies the Lipschitz continuity and linear growth conditions required for the existence and uniqueness result in [20]. Therefore, the well-posedness framework established in [20] is applicable to the proposed fractional stochastic H–K model and the fractional stochastic system (1) admits a unique solution in the sense of the fractional stochastic integral formulation.

Remark 5. We emphasize two key distinctions between the present fractional-order model and its integer-order counterpart studied in [14]. First, the Caputo derivative introduces a memory kernel into the dynamics, so that the evolution of opinions at time t depends not only on the instantaneous configuration but also on the entire past trajectory. Second, as we shall see in the main theorem, this memory effect fundamentally alters the convergence rate: while integer-order systems with multiplicative noise exhibit exponential consensus, the fractional-order dynamics yield a power-law decay. These differences will be made precise in the following analysis.

3.2. Consensus Analysis

Definition 3. Consider the inhomogeneous fractional-order system (1) consisting of N followers with opinions $x_i(t)$ and one leader with opinion $x_0(t)$. We say that the system achieves almost sure consensus towards the leader if

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} \|x_i(t) - x_0(t)\| = 0\right) = 1$$

for every follower $i = 1, 2, \dots, N$, i.e., the opinion of each follower converges to the leader's opinion with probability one.

Lemma 4. Let $\mathbf{x}(t) = (x_0(t), x_1(t), x_2(t), \dots, x_N(t))^T$ be the solution to the fractional stochastic system (1) with bounded initial configurations, and let

$$V(\mathbf{x}(t)) := \sum_{i=1}^N (x_i(t) - x_0(t))^2$$

be the total squared deviation of the followers from the leader. Then, for $\frac{1}{2} < \alpha < 1$, the following pathwise fractional differential inequality holds:

$${}^C\mathcal{D}_t^\alpha V(\mathbf{x}(t)) \leq -\lambda V(\mathbf{x}(t)) + 2\sqrt{2\sigma} V(\mathbf{x}(t)) \frac{dB_t}{dt}, \quad (7)$$

where

$$\lambda := 2\gamma\phi_m - \frac{N}{N-1}\gamma - 4 > 0.$$

It should be noted that ϕ_m is the theoretical lower bound of the leader-follower influence function over the entire bounded opinion domain, rather than a value obtained from a particular numerical realization.

Proof. Applying the fractional Lyapunov inequality (Lemma 2) to the convex function $V(\mathbf{x}(t))$, we have

$${}^C\mathcal{D}_t^\alpha V(\mathbf{x}(t)) \leq 2 \sum_{i=1}^N (x_i(t) - x_0(t)) \cdot {}^C\mathcal{D}_t^\alpha (x_i(t) - x_0(t)). \quad (8)$$

Using the linearity of the Caputo derivative, we obtain

$${}^C\mathcal{D}_t^\alpha (x_i(t) - x_0(t)) = {}^C\mathcal{D}_t^\alpha x_i(t) - {}^C\mathcal{D}_t^\alpha x_0(t).$$

Substituting the model Equation (1) into the right-hand side and recalling (8) yields

$$\begin{aligned} {}^C\mathcal{D}_t^\alpha V(\mathbf{x}(t)) &\leq -\frac{2}{N} \sum_{i,j=1}^N a_{ij}(t) (x_i(t) - x_j(t)) \cdot (x_i(t) - x_0(t)) \\ &\quad - 2\gamma \sum_{i=1}^N \phi_{i0}(t) (x_i(t) - x_0(t))^2 \\ &\quad - 2u(t) \cdot \sum_{i=1}^N (x_i(t) - x_0(t)) \\ &\quad + 2\sqrt{2\sigma} \sum_{i=1}^N (x_i(t) - x_0(t))^2 \frac{dB_t}{dt}. \end{aligned} \quad (9)$$

Next, let us estimate the first three terms in the right-hand side of (9).

Estimate of the leader's feedback term. By the definition of $u(t)$ in (5) and the elementary inequality $2a \cdot b \leq \|a\|^2 + \|b\|^2$, we have

$$\begin{aligned} &-2u(t) \cdot \sum_{i=1}^N (x_i(t) - x_0(t)) \\ &= -2\gamma\eta(t) \sum_{i,j=1}^N \psi_{0j}(t) (x_j(t) - x_0(t)) \cdot (x_i(t) - x_0(t)) \\ &\leq \gamma\eta(t) \left(\sum_{i,j=1}^N \|x_j(t) - x_0(t)\|^2 + \sum_{i,j=1}^N \|x_i(t) - x_0(t)\|^2 \right) \\ &= 2N\gamma\eta(t) V(\mathbf{x}(t)). \end{aligned}$$

From the definition of $\eta(t)$ and the property of the minimum operator,

$$\eta(t) \leq \frac{1}{2} \frac{\psi_{0l(t)}}{N - \psi_{0l(t)}} \leq \frac{1}{2(N-1)},$$

then it follows that

$$-2u(t) \cdot \sum_{i=1}^N (x_i(t) - x_0(t)) \leq \frac{N}{N-1} \gamma V(\mathbf{x}(t)).$$

Estimate of the follower–follower interaction term. Using the identity

$$(x_j - x_i) \cdot (x_i - x_0) = (x_j - x_0) \cdot (x_i - x_0) - \|x_i - x_0\|^2,$$

and the fact that $0 \leq a_{ij}(t) \leq 1$, we can derive

$$\begin{aligned} & -\frac{2}{N} \sum_{i,j=1}^N a_{ij}(t) (x_i(t) - x_j(t)) \cdot (x_i(t) - x_0(t)) \\ &= -\frac{2}{N} \sum_{i,j=1}^N a_{ij}(t) (x_j(t) - x_0(t)) \cdot (x_i(t) - x_0(t)) + \frac{2}{N} \sum_{i,j=1}^N a_{ij}(t) \|x_i(t) - x_0(t)\|^2 \\ &\leq \frac{2}{N} \sum_{i,j=1}^N \|x_j(t) - x_0(t)\| \|x_i(t) - x_0(t)\| + \frac{2}{N} \sum_{i,j=1}^N \|x_i(t) - x_0(t)\|^2 \end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\frac{2}{N} \sum_{i,j=1}^N \|x_j - x_0\| \|x_i - x_0\| = \frac{2}{N} \left(\sum_{i=1}^N \|x_i - x_0\| \right)^2 \leq \frac{2}{N} \cdot NV(\mathbf{x}(t)) = 2V(\mathbf{x}(t))$$

and

$$\frac{2}{N} \sum_{i,j=1}^N \|x_i - x_0\|^2 = \frac{2}{N} \sum_{i=1}^N \sum_{j=1}^N \|x_i - x_0\|^2 = \frac{2}{N} \cdot N \cdot V(\mathbf{x}(t)) = 2V(\mathbf{x}(t)).$$

Therefore,

$$-\frac{2}{N} \sum_{i,j=1}^N a_{ij}(t) (x_i(t) - x_j(t)) \cdot (x_i(t) - x_0(t)) \leq 4V(\mathbf{x}(t)).$$

It should be noted that the above estimate is conservative. The follower–follower interaction term has an intrinsic dissipative structure with respect to pairwise disagreement among followers. However, since the Lyapunov function in this paper measures the deviation from the leader, this term cannot be directly converted into a negative-definite contribution to $V(\mathbf{x}(t))$. Therefore, the above bound is adopted to obtain a closed inequality solely in terms of the leader deviation function. The resulting condition is sufficient but not necessary.

Estimate of the leader’s attractive force. Since $\phi_{i0}(t) \geq \phi_m > 0$, we have

$$-2\gamma \sum_{i=1}^N \phi_{i0}(t) \|x_i(t) - x_0(t)\|^2 \leq -2\gamma \phi_m V(\mathbf{x}(t)).$$

Substituting the three estimates into (9), we obtain

$$\begin{aligned} {}^c\mathcal{D}_t^\alpha V(\mathbf{x}(t)) &\leq \left(4 - 2\gamma\phi_m + \frac{N}{N-1}\gamma \right) V(\mathbf{x}(t)) + 2\sqrt{2}\sigma \sum_{i=1}^N (x_i(t) - x_0(t))^2 \frac{dB_t}{dt} \\ &= -\lambda V(\mathbf{x}(t)) + 2\sqrt{2}\sigma V(\mathbf{x}(t)) \frac{dB_t}{dt}. \end{aligned}$$

This completes the proof. $\square$

Theorem 1. Let $\mathbf{x}(t) = (x_0(t), x_1(t), x_2(t), \dots, x_N(t))^T$ be the global solution of system (1) with bounded initial configurations. Suppose the communication functions $a(r)$, $\phi(r)$ and $\psi(r)$ satisfy the assumptions in Section 3.1 and the leadership strength γ fulfils

$$\gamma > \frac{4}{2\phi_m - \frac{N}{N-1}} > 0. \quad (10)$$

Then the system achieves almost sure global consensus towards the leader's opinion, i.e.,

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_0(t)\| = 0 \text{ a.s.}, \quad \forall i = 1, \dots, N.$$

Moreover, due to the hereditary nature of the Caputo fractional derivative, the convergence is power-law with rate

$$\mathbb{E}[V(\mathbf{x}(t))] = O(t^{-\alpha}), \quad t \rightarrow \infty. \quad (11)$$

Remark 6. The sufficient condition on γ requires

$$\phi_m > \frac{N}{2(N-1)}.$$

This requirement only concerns the minimum interaction strength within the bounded opinion domain. For the influence function

$$\phi(r) = \frac{1}{(1+r^2)^{1/4}},$$

the condition is equivalent to

$$D < \sqrt{\left(\frac{2(N-1)}{N}\right)^4 - 1}.$$

Hence, the condition provides a quantitative relationship between the allowable opinion range and the required leadership strength. It should be noted that this is only a sufficient condition resulting from the Lyapunov estimate and is not necessary for consensus. This sufficient condition is conservative because the follower–follower interaction term is bounded by a general upper estimate. A less conservative analysis may exploit the intrinsic dissipative structure of the interaction topology.

Proof. We divide the proof into four steps.

Step 1: In the following consensus analysis, we consider solutions satisfying the integrability condition

$$\mathbb{E}[V(\mathbf{x}(t))] < \infty.$$

Since the Caputo fractional derivative is defined through a fractional integral operator, the expectation operator can be interchanged with the fractional integral under this integrability condition by the Fubini theorem. Therefore,

$$\mathbb{E}\left[{}^C\mathcal{D}_t^\alpha V(\mathbf{x}(t))\right] = {}^C\mathcal{D}_t^\alpha \mathbb{E}[V(\mathbf{x}(t))].$$

Hence, taking expectation on both sides of the inequality (7) in Lemma 4 yields

$${}^C\mathcal{D}_t^\alpha \mathbb{E}[V(\mathbf{x}(t))] \leq -\lambda \mathbb{E}[V(\mathbf{x}(t))],$$

where the stochastic convolution term vanishes. Indeed, the stochastic term is interpreted as

$$M(t) = \frac{2\sqrt{2\sigma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} V(\mathbf{x}(s)) dB_s.$$

Under the square-integrability condition

$$\mathbb{E} \int_0^t (t-s)^{2\alpha-2} V^2(\mathbf{x}(s)) ds < \infty,$$

$M(t)$ is a well-defined Itô integral and a martingale. Therefore,

$$\mathbb{E}[M(t)] = 0.$$

Applying the fractional Grönwall inequality (Lemma 3) with $y(t) = \mathbb{E}[V(\mathbf{x}(t))]$ and $a = \lambda$, we get

$$\mathbb{E}[V(\mathbf{x}(t))] \leq \mathbb{E}[V(\mathbf{x}(0))] E_\alpha(-\lambda t^\alpha). \quad (12)$$

Using the asymptotic property of the Mittag–Leffler function (Lemma 1) yields

$$E_\alpha(-\lambda t^\alpha) \sim \frac{1}{\lambda \Gamma(1-\alpha)} t^{-\alpha}, \quad t \rightarrow \infty.$$

Combining this with (12), we conclude that there exists a constant $C_0 > 0$ (whose explicit form is immaterial for our purpose and thus omitted) such that

$$\mathbb{E}[V(\mathbf{x}(t))] \leq C_0 t^{-\alpha}, \quad \forall t > 0. \quad (13)$$

This establishes the power-law decay rate claimed in (11).

Step 2: Choose a time subsequence $t_n = n^\beta$ with $\beta > 1/\alpha$ (such a β exists because $\frac{1}{2} < \alpha < 1$). From (13) we have

$$\sum_{n=1}^{\infty} \mathbb{E}[V(\mathbf{x}(t_n))] \leq C_0 \sum_{n=1}^{\infty} n^{-\alpha\beta} < \infty,$$

since $\alpha\beta > 1$. For any $\varepsilon > 0$, Markov's inequality yields

$$\sum_{n=1}^{\infty} \mathbb{P}(V(\mathbf{x}(t_n)) > \varepsilon) \leq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \mathbb{E}[V(\mathbf{x}(t_n))] < \infty.$$

By the Borel–Cantelli lemma, only finitely many events $\{V(\mathbf{x}(t_n)) > \varepsilon\}$ occur almost surely. Letting $\varepsilon \rightarrow 0^+$ along a countable sequence, we obtain

$$\lim_{n \rightarrow \infty} V(\mathbf{x}(t_n)) = 0 \quad \text{a.s.}$$

Step 3: The result obtained in Step 2 only guarantees convergence along the discrete subsequence $\{t_n\}$. We next extend this convergence to the whole continuous-time trajectory. For this purpose, define

$$S_n := \sup_{t_n \leq t \leq t_{n+1}} V(\mathbf{x}(t)).$$

We first establish an estimate for $\mathbb{E}[S_n]$. According to the fractional stochastic integral formulation in Remark 1, the evolution of the disagreement variables on the interval $[t_n, t_{n+1}]$ can be expressed as the sum of a deterministic fractional convolution term and a stochastic convolution term. The stochastic convolution associated with the i -th follower is given by

$$M_i(t) = \frac{\sqrt{2\sigma}}{\Gamma(\alpha)} \int_{t_n}^t (t-s)^{\alpha-1} (x_i(s) - x_0(s)) dB_s, \quad t_n \leq t \leq t_{n+1}.$$

Using the Burkholder–Davis–Gundy inequality together with the standard maximal inequality for fractional stochastic convolutions, we obtain

$$\mathbb{E} \left[\sup_{t_n \leq t \leq t_{n+1}} |M_i(t)|^2 \right] \leq C \mathbb{E} \int_{t_n}^{t_{n+1}} (t_{n+1} - s)^{2\alpha-2} \|x_i(s) - x_0(s)\|^2 ds,$$

here and in the following, C denotes a positive constant, independent of the main parameters, whose exact value is irrelevant to the conclusion and may differ from line to line. Since

$$\|x_i(s) - x_0(s)\|^2 \leq V(\mathbf{x}(s)),$$

it follows from (13) in Step 1 that

$$\mathbb{E}[\|x_i(s) - x_0(s)\|^2] \leq C_0 s^{-\alpha}.$$

Moreover, for $s \in [t_n, t_{n+1}]$, we have $s^{-\alpha} \leq t_n^{-\alpha}$. Therefore,

$$\mathbb{E}\left[\sup_{t_n \leq t \leq t_{n+1}} |M_i(t)|^2\right] \leq C t_n^{-\alpha} \int_{t_n}^{t_{n+1}} (t_{n+1} - s)^{2\alpha-2} ds.$$

Due to $\alpha \in (1/2, 1)$,

$$\int_{t_n}^{t_{n+1}} (t_{n+1} - s)^{2\alpha-2} ds = \frac{(t_{n+1} - t_n)^{2\alpha-1}}{2\alpha - 1} < \infty.$$

Hence,

$$\mathbb{E}\left[\sup_{t_n \leq t \leq t_{n+1}} |M_i(t)|^2\right] \leq C t_n^{-\alpha}.$$

Summing the above estimate over all followers and combining it with the corresponding deterministic fractional convolution estimate, we obtain

$$\mathbb{E}[S_n] \leq C t_n^{-\alpha}.$$

Now choose the subsequence $t_n = n^\beta$, where $\beta > 1/\alpha$, as in Step 2. Then,

$$\sum_{n=1}^{\infty} \mathbb{E}[S_n] \leq C \sum_{n=1}^{\infty} t_n^{-\alpha} = C \sum_{n=1}^{\infty} n^{-\alpha\beta} < \infty.$$

For any $\varepsilon > 0$, Markov's inequality gives

$$\sum_{n=1}^{\infty} \mathbb{P}(S_n > \varepsilon) \leq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \mathbb{E}[S_n] < \infty.$$

Therefore, by the Borel–Cantelli lemma,

$$\lim_{n \rightarrow \infty} S_n = 0, \quad \text{a.s..}$$

Finally, for any $t \geq t_1$, there exists a unique index n such that $t \in [t_n, t_{n+1}]$. By the definition of S_n ,

$$0 \leq V(\mathbf{x}(t)) \leq S_n.$$

Since $S_n \rightarrow 0$ almost surely, we conclude that

$$\lim_{t \rightarrow \infty} V(\mathbf{x}(t)) = 0, \quad \text{a.s..} \quad (14)$$

This completes the proof of the continuous-time convergence.

Step 4: Since for each $i = 1, \dots, N$, we have

$$\|x_i(t) - x_0(t)\|^2 \leq V(\mathbf{x}(t)).$$

Hence the limit (14) implies

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_0(t)\| = 0 \quad \text{a.s.}$$

for every $i = 1, \dots, N$. This proof is completed. $\square$

4. Numerical Simulations

In this section, numerical simulations are conducted to verify the theoretical results and illustrate the consensus behavior of the proposed fractional stochastic inhomogeneous H-K model. The simulations demonstrate the influence of key system parameters, including the fractional order, multiplicative noise intensity and leadership strength, on the convergence dynamics. In particular, we examine the algebraic convergence behavior induced by the fractional memory effect and provide statistical verification based on multiple stochastic realizations. The numerical results confirm the theoretical consensus conditions and illustrate the robustness of the proposed model under stochastic perturbations.

Remark 7. Let $t_n = n\Delta t$ and $F(X(t_k))$ denote the deterministic drift of the complete leader–follower state. The corrected numerical update used in the revised simulations is

$$\begin{aligned} X(t_n) \approx & X(0) + \frac{(\Delta t)^\alpha}{\Gamma(\alpha + 1)} \sum_{k=0}^{n-1} [(n-k)^\alpha - (n-k-1)^\alpha] F(X(t_k)) \\ & + \frac{\sqrt{2\sigma}}{\Gamma(\alpha)} \sum_{k=0}^{n-1} (t_n - t_k)^{\alpha-1} X(t_k) \Delta B_k, \end{aligned} \quad (15)$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$ and the same scalar Brownian increment is applied to all agents and both opinion components. Thus, the complete history is retained in both the deterministic fractional convolution and the stochastic convolution.

For the short-time representative trajectories, we consider $N = 25$ followers and one leader in two-dimensional opinion space. The initial follower opinions are selected in the disk of radius 0.5 centered at the origin and the same fixed initial configuration is used in every Monte Carlo realization. The leader is initialized at $x_0(0) = (0.75, 0.75)^T$. These representative trajectories are computed over $0 \leq t \leq 1$ with $\Delta t = 0.004$. Unless otherwise stated, the baseline parameters are $\sigma = 0.02$, $\gamma = 10$, $\delta = 0.04$, $\epsilon = 0.01$ and $\alpha = 0.999$.

The leader–follower and follower–leader communication functions are chosen as

$$\phi(r) = \psi(r) = \frac{1}{(1 + r^2)^{1/4}}, \quad (16)$$

whereas the follower–follower interaction is described by

$$a(r) = \begin{cases} 1, & 0 \leq r \leq \delta, \\ \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{1}{r - \delta} + \frac{1}{r - (\delta + \epsilon)}\right), & \delta < r < \delta + \epsilon, \\ 0, & r \geq \delta + \epsilon. \end{cases} \quad (17)$$

Since $\|x_i(0)\| \leq 0.5$ and $\|x_0(0)\| = \sqrt{0.75^2 + 0.75^2}$, the maximum initial leader–follower distance satisfies

$$D_0 = \max_{1 \leq i \leq N} \|x_i(0) - x_0(0)\| \leq 0.5 + \sqrt{0.75^2 + 0.75^2} \approx 1.56. \quad (18)$$

Consequently, we may take

$$\phi_m = (1 + D_0^2)^{-1/4} \approx 0.735, \quad (19)$$

and the sufficient condition in Theorem 1 gives

$$\gamma > \frac{4}{2\phi_m - \frac{N}{N-1}} \approx 9.34. \quad (20)$$

Therefore, $\gamma = 10$, $\gamma = 14$ and $\gamma = 20$ satisfy this sufficient condition, whereas $\gamma = 9$ is included as a below-threshold comparison. Since the condition is sufficient rather than necessary, numerical behavior below the threshold does not contradict the theorem.

Both the representative trajectory plots and the ensemble-statistical comparisons are computed over $0 \leq t \leq 1$ with $\Delta t = 0.004$. This interval provides sufficient temporal resolution of the initial consensus process and permits a direct comparison between the representative paths and the ensemble results. For each parameter setting, we perform $N_{MC} = 100$ independent realizations using independently generated Brownian paths. Here N_{MC} denotes the number of Monte Carlo realizations and is distinct from the model constant $M = 2$. For the m th realization, define

$$V^{(m)}(t) = \sum_{i=1}^N \|x_i^{(m)}(t) - x_0^{(m)}(t)\|^2, \quad m = 1, \dots, N_{MC}. \quad (21)$$

The ensemble mean and sample standard deviation are

$$\bar{V}(t) = \frac{1}{N_{MC}} \sum_{m=1}^{N_{MC}} V^{(m)}(t) \quad (22)$$

and

$$s_V(t) = \left[\frac{1}{N_{MC} - 1} \sum_{m=1}^{N_{MC}} \left(V^{(m)}(t) - \bar{V}(t) \right)^2 \right]^{1/2}. \quad (23)$$

In each statistical figure, panel (a) presents $\bar{V}(t)$ with a shaded band $\bar{V}(t) \pm s_V(t)$, and panel (b) presents $s_V(t)$ separately. Because the statistical curves are displayed on logarithmic axes, nonpositive lower-band values are replaced only for plotting by a floor of 10^{-10} ; the statistics themselves are not altered. The trajectory figures should therefore be viewed as qualitative illustrations, while the quantitative comparisons below are based on the ensemble statistics.

4.1. Influence of Internal System Parameters

We first investigate the effects of the multiplicative noise intensity, the leadership strength and the bounded-confidence parameters. The same initial configuration and time discretization are used for every parameter setting.

4.1.1. Effect of Multiplicative Noise Intensity σ

We first examine the robustness of the consensus mechanism against communication uncertainty by varying the noise intensity σ . We fix $\gamma = 10$, $\delta = 0.04$, $\epsilon = 0.01$ and $\alpha = 0.999$, and simulate the system for three values: $\sigma = 0$ (deterministic case), $\sigma = 0.01$ and $\sigma = 0.05$.

As shown in Figure 1, both the deterministic and stochastic systems eventually achieve consensus. Under multiplicative noise, however, the followers' trajectories exhibit more irregular fluctuations before aligning with the leader's opinion. A larger σ introduces stronger stochastic perturbations and weakens the contraction effect among agents, yet the system still converges to the consensus state. This observation confirms the theoretical

prediction that the consensus condition (10) is independent of σ ; the noise affects only the transient dynamics, not the asymptotic convergence.

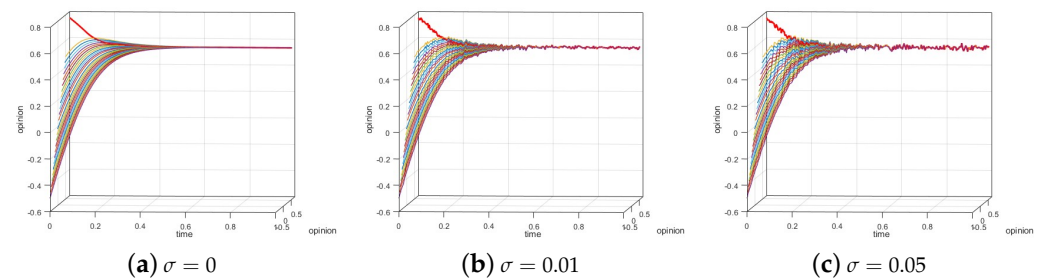


Figure 1. Representative sample paths illustrating the effect of the multiplicative noise intensity σ on the consensus dynamics. The red curve denotes the leader, and the multicolored curves denote the followers.

Figure 1 contains one representative realization for each noise intensity. The corresponding ensemble statistics are reported in Figure 2.

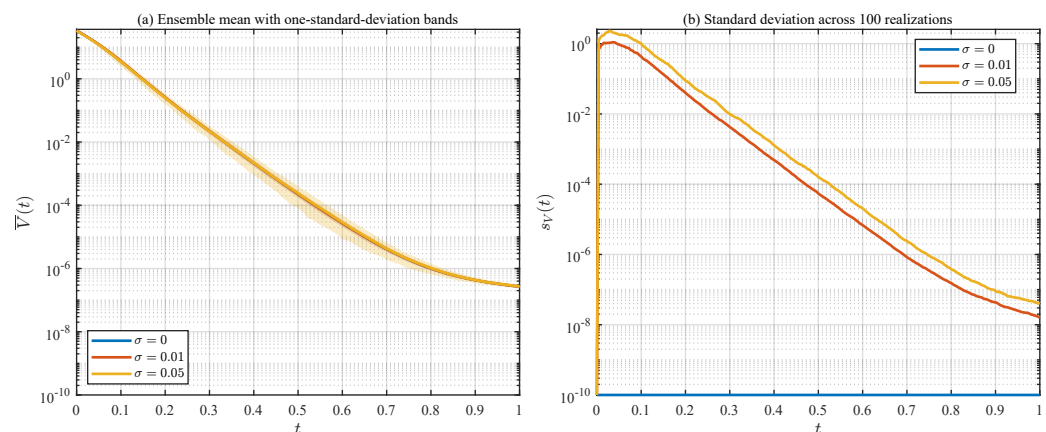


Figure 2. Ensemble statistics of the disagreement under different noise intensities. Panel (a) shows $\bar{V}(t)$ over $N_{MC} = 100$ independent realizations, with shaded one–standard–deviation bands. Panel (b) shows the corresponding sample standard deviation $s_V(t)$.

The ensemble means have similar decay profiles for the tested noise intensities, while the dispersion between independent realizations increases as σ increases. In the deterministic case $\sigma = 0$, the standard deviation vanishes because all realizations coincide. These results indicate that, in the tested parameter regime, the noise intensity mainly affects sample-to-sample variability rather than the overall ensemble-averaged convergence trend.

4.1.2. Effect of Leadership Strength γ

We now investigate how the leader's influence strength γ affects the convergence speed. We fix $\sigma = 0.02$, $\delta = 0.04$, $\epsilon = 0.01$ and $\alpha = 0.999$, and vary γ as 9, 10, 14 and 20. The value $\gamma = 9$ is included as a case clearly below the sufficient threshold in (20).

Figure 3 gives representative trajectories, while Figure 4 provides the ensemble comparison.

Figure 3 shows that as γ increases, the followers are attracted more strongly toward the leader's opinion, and the convergence time decreases significantly. Larger γ also leads to a steeper decay of the disagreement among agents. Although the asymptotic convergence remains power-law due to the fractional dynamics, increasing γ effectively reduces the transient stage and accelerates consensus formation. This numerical observation

is consistent with Theorem 1. The cases $\gamma = 10$, $\gamma = 14$ and $\gamma = 20$, which satisfy the sufficient condition (20), display faster decay than the below-threshold case $\gamma = 9$. Nevertheless, $\bar{V}(t)$ also decreases for $\gamma = 9$ in the $N_{MC} = 100$ realizations considered here. This does not extend the guarantee of Theorem 1 below the threshold; rather, it numerically illustrates that the theorem provides a conservative sufficient condition and not a necessary condition for consensus. These ensemble results also support the theoretical role of leadership strength in accelerating consensus.

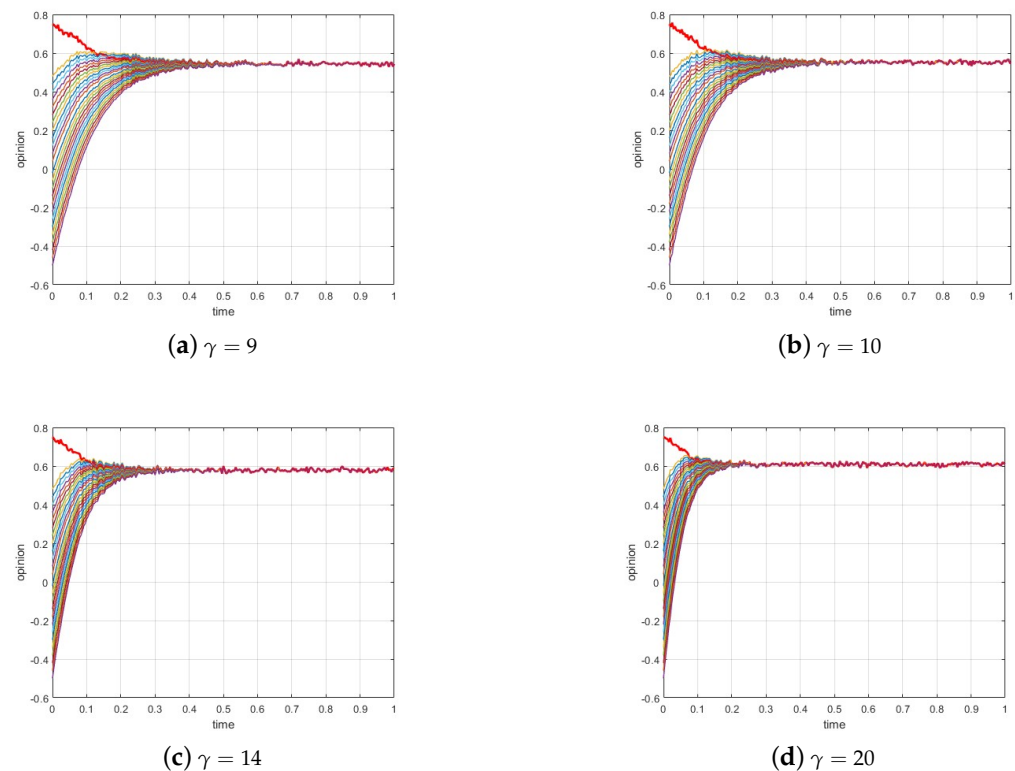


Figure 3. Representative sample paths under different leadership strengths $\gamma = 9, 10, 14$ and 20 . The red curve denotes the leader, and the multicolored curves denote the followers.

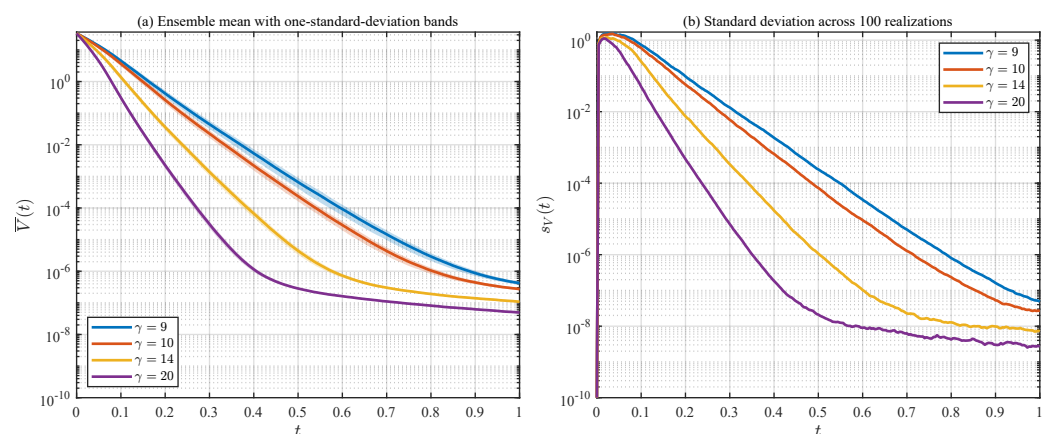


Figure 4. Ensemble statistics of the disagreement under leadership strengths $\gamma = 9, 10, 14$, and 20 . Panel (a) shows $\bar{V}(t)$ over $N_{MC} = 100$ independent realizations, with shaded one-standard-deviation bands. Panel (b) shows $s_V(t)$.

4.1.3. Effect of Bounded-Confidence Parameters δ and ϵ

The parameters δ and ϵ determine the effective interaction range among followers: δ defines the radius of full interaction, while ϵ controls the width of the transition region from full interaction to non-interaction. To investigate their influence, we fix $\sigma = 0.02$,

$\gamma = 10$ and $\alpha = 0.999$, and simulate the system under three representative configurations: $(\delta, \epsilon) = (0.005, 0.01)$, $(0.02, 0.1)$ and $(0.99, 1)$.

The corresponding statistical comparison is shown in Figure 5.

As shown in Figure 6, when δ and ϵ are small, the followers' trajectories converge slowly toward the leader's opinion, and the disagreement among agents persists for a relatively long time. This is because each agent interacts only with a limited number of neighbors, resulting in a sparse communication network and inefficient information propagation. As δ and ϵ increase, the effective interaction range becomes significantly larger, network connectivity is substantially enhanced, and the trajectories rapidly collapse toward the consensus state. The asymptotic consensus behavior remains unchanged, in agreement with the theoretical analysis. The ensemble means and standard deviations remain relatively close for the three tested confidence configurations. A broader interaction range slightly changes the transient decay, but its quantitative influence is weaker than that of the leadership strength for the parameter regime considered here.

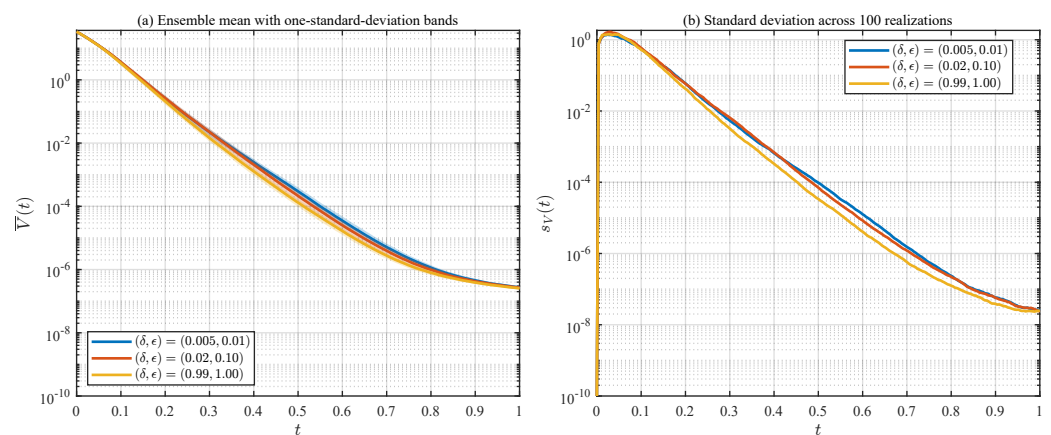


Figure 5. Ensemble statistics of the disagreement under different bounded-confidence parameters. Panel (a) shows $\bar{V}(t)$ over $N_{MC} = 100$ independent realizations, with shaded one-standard-deviation bands. Panel (b) shows $s_V(t)$.

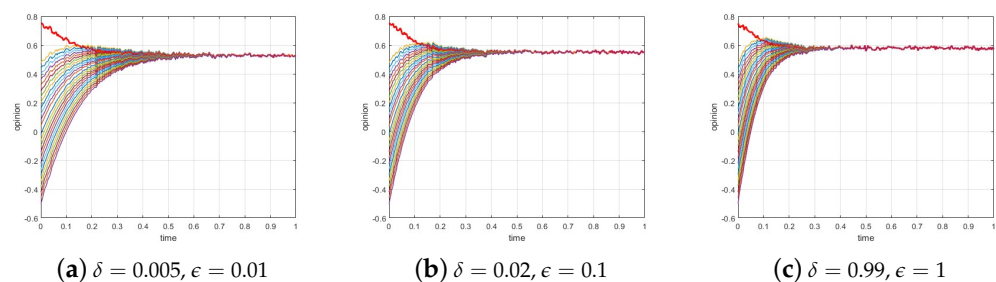


Figure 6. Representative sample paths under different bounded-confidence parameters (δ, ϵ) . The red curve denotes the leader, and the multicolored curves denote the followers.

4.2. Influence of Fractional Order

We now turn to the core feature of our model: the memory effect encoded by the Caputo fractional derivative. We fix $\sigma = 0.02$, $\gamma = 10$, $\delta = 0.04$ and $\epsilon = 0.01$, and vary the fractional order α over three values: 0.65, 0.85 and 0.999.

The ensemble statistics corresponding to the representative paths in Figure 7 are shown in Figure 8.

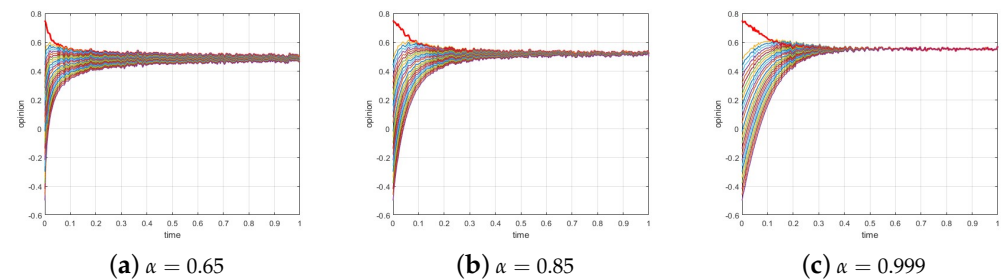


Figure 7. Representative sample paths under different fractional orders α . The red curve denotes the leader, and the multicolored curves denote the followers.

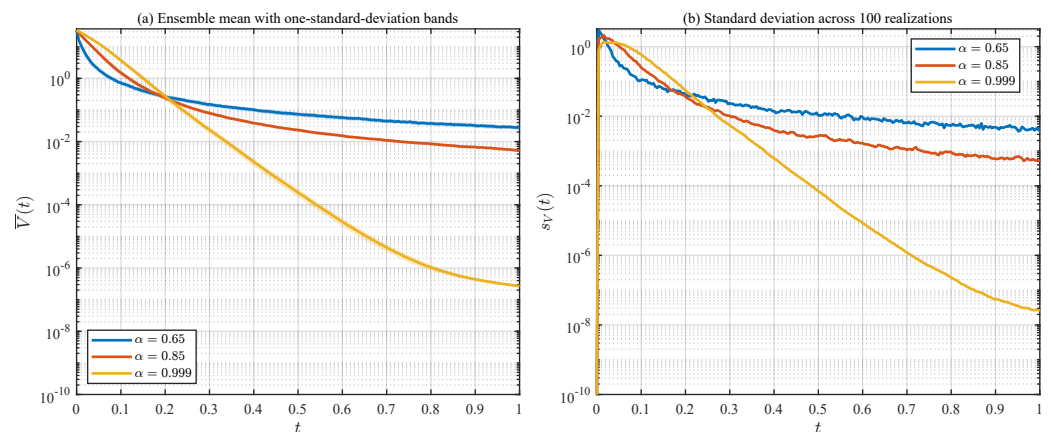


Figure 8. Ensemble statistics of the disagreement under different fractional orders. Panel (a) shows $\bar{V}(t)$ over $N_{MC} = 100$ independent realizations, with shaded one-standard-deviation bands. Panel (b) shows $s_V(t)$.

Figure 7 reveals a clear transition in the convergence behavior as α varies. When α is smaller (e.g., 0.65), the system exhibits a slow and prolonged convergence process, with noticeable persistence of oscillatory fluctuations before reaching consensus. This reflects the strong memory effect: past states significantly influence the current dynamics, resulting in a slower decay rate consistent with the theoretical power-law prediction $t^{-\alpha}$.

As α increases, the convergence speed becomes progressively faster. The trajectories show a more direct contraction toward the consensus region, and the oscillatory behavior weakens. When α is close to 1 (e.g., 0.999), the system may exhibit faster transient convergence within a finite simulation interval, while the long-time asymptotic behavior remains a power-law decay.

It should be pointed out that although the convergence becomes faster as α approaches one, the fractional system still exhibits algebraic decay rather than exponential convergence. This follows from the asymptotic property of the Mittag–Leffler function given in Lemma 1. Only in integer-order limit does the Mittag–Leffler function reduce to

$$E_1(-\lambda t) = e^{-\lambda t},$$

leading to exponential convergence. For every fixed $\alpha < 1$, including $\alpha = 0.999$, the system preserves the fractional algebraic asymptotic behavior. Therefore, a fractional order close to one represents faster fractional convergence instead of a true exponential decay.

Figure 8 shows that the fractional order affects both the initial transient and the subsequent decay. The short-time ordering of the curves need not coincide with their long-

time ordering; hence a conclusion based only on the earliest transient can be misleading. The long-time behavior is related to the Mittag–Leffler asymptotic formula

$$E_{\alpha}(-ct^{\alpha}) \sim \frac{1}{c\Gamma(1-\alpha)}t^{-\alpha}, \quad t \rightarrow \infty, \quad (24)$$

which characterizes the algebraic tail generated by fractional memory.

4.3. Representative Long-Time Dynamics

The preceding trajectory figures use $T = 1$ and $\Delta t = 0.004$ to resolve the short-time transient. To observe the evolution over a much longer interval, we additionally simulate the system up to $T = 100$ with $N = 10$, $\Delta t = 0.01$, $\alpha = 0.65$, $\gamma = 10$, and $\sigma = 0.02$. The reduced system size follows the reviewer’s suggestion and makes it computationally feasible to retain the complete fractional history. This figure is a representative realization; the quantitative assessment of stochastic variability is based on the ensemble statistics defined in (21)–(23).

Figure 9 shows that the followers approach the leader during the initial transient. Once the relative disagreement is small, the agents continue to fluctuate together under the common Brownian motion. Consensus therefore means that the relative disagreement tends to zero; it does not require the common opinion to converge to a deterministic constant.

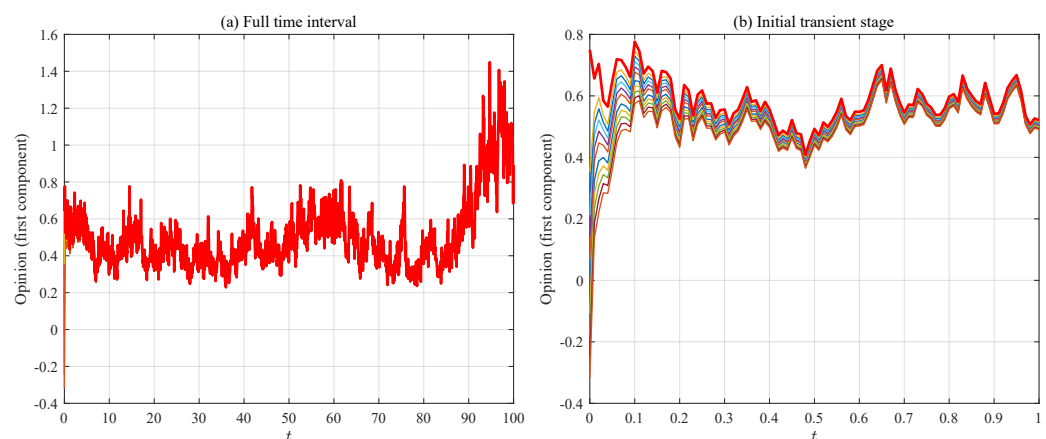


Figure 9. Representative long-time opinion trajectories for $T = 100$, $N = 10$, $\Delta t = 0.01$, $\alpha = 0.65$, $\gamma = 10$, and $\sigma = 0.02$. Panel (a) shows the complete interval, and panel (b) enlarges the initial transient $0 \leq t \leq 1$. The red curve denotes the leader, and the multicolored curves denote the followers.

The approximately linear tail in logarithmic coordinates indicates algebraic rather than exponential decay. Because Figure 10 represents only one realization, its fitted slope is used as qualitative long-time evidence and not as an ensemble estimate.

Remark 8. The long horizon $T = 100$ is required to examine the asymptotic behavior of the fractional system. However, this horizon is much larger than the short transient time scale $T = 1$. Consequently, displaying only the complete interval $0 \leq t \leq 100$ compresses the rapid initial motion into a narrow region and makes the transient consensus process difficult to observe. For this reason, Figure 9a reports the complete long-time evolution, whereas Figure 9b enlarges $0 \leq t \leq 1$ to show the initial convergence dynamics clearly. The two panels present the same simulation realization on different time scales.

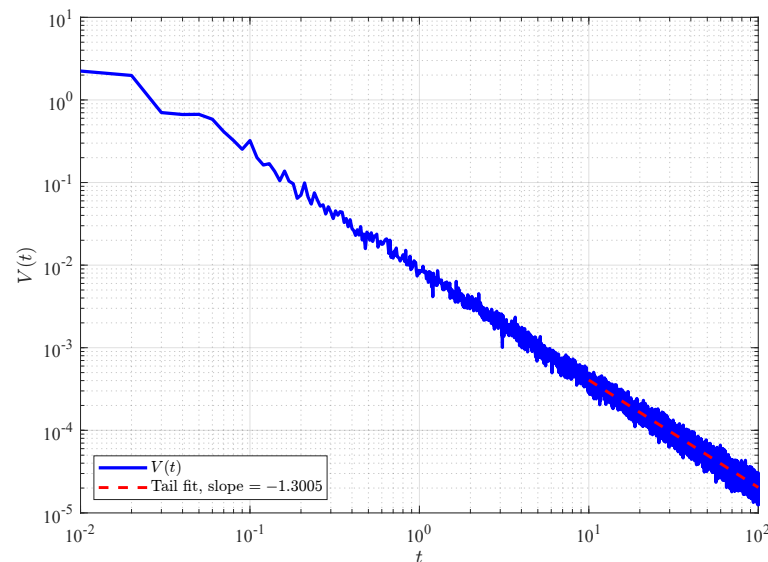


Figure 10. Log-log plot of $V(t)$ for the representative long-time realization in Figure 9. A least-squares fit over $10 \leq t \leq 100$ gives a slope of approximately -1.30 .

4.4. Numerical Step-Size Sensitivity

To examine the sensitivity of the Grünwald–Letnikov discretization to the time step, we compare the deterministic solutions obtained with $\Delta t = 0.02, 0.01$, and 0.005 over $0 \leq t \leq 20$. The deterministic case $\sigma = 0$ is used to separate discretization error from Monte Carlo sampling error. The solution with $\Delta t = 0.005$ is treated as the reference. For a time step h , the relative discrete L^2 error is defined by

$$E_h = \frac{\|V_h - V_{0.005}\|_2}{\|V_{0.005}\|_2}, \quad (25)$$

where the reference solution is interpolated onto the corresponding coarser time grid.

The three curves in Figure 11a have the same decay profile and become closer under time-step refinement. Quantitatively, the relative error decreases from 12.83% for $\Delta t = 0.02$ to 7.25% for $\Delta t = 0.01$. This monotone reduction indicates a clear numerical convergence trend. We therefore use $\Delta t = 0.01$ in the final long-time simulation as a compromise between temporal resolution and computational cost.

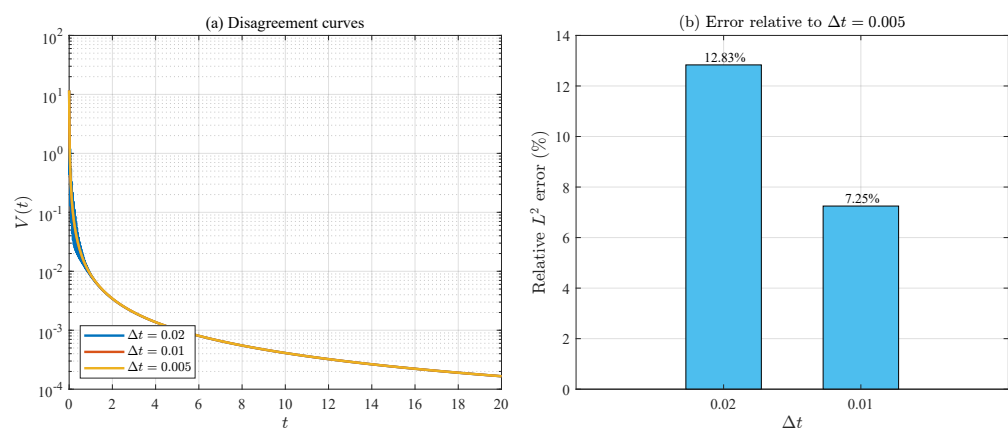


Figure 11. Step-size sensitivity of the Grünwald–Letnikov scheme. Panel (a) compares the disagreement curves for $\Delta t = 0.02, 0.01$, and 0.005 . Panel (b) reports the relative discrete L^2 errors with respect to the $\Delta t = 0.005$ reference solution.

4.5. Convergence Rate Analysis

To complement the representative long-time trajectory, we estimate the decay rate using the ensemble-averaged disagreement. The experiment uses $N_{MC} = 100$ independent Brownian paths with $T = 100$, $N = 10$, $\Delta t = 0.01$, $\gamma = 10$, $\sigma = 0.01$, $\alpha = 0.65$, $\delta = 0.04$ and $\epsilon = 0.01$. For each time point, $\bar{V}(t)$ and $s_V(t)$ are calculated from the 100 values $\{V^{(m)}(t)\}_{m=1}^{100}$ according to (22) and (23). The logarithmic fit is applied to the tail $10 \leq t \leq 100$ of $\bar{V}(t)$ rather than to a single sample path.

Figure 12 presents the log-log plot of the Lyapunov function and the corresponding linear fitting result. As shown in Figure 12a, the curve of $\bar{V}(t)$ exhibits a clear power-law decay with respect to time. The slope obtained from the linear regression in logarithmic coordinates is shown in Figure 12b.

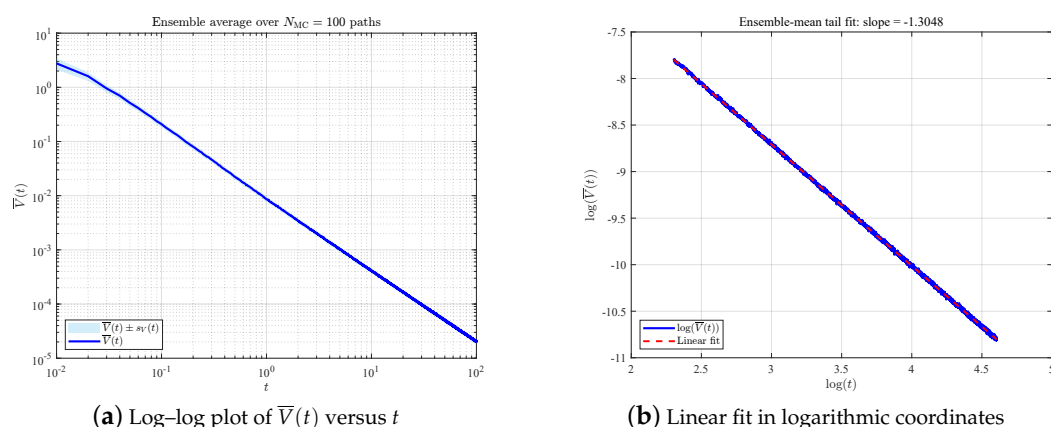


Figure 12. Power-law decay analysis of the ensemble-averaged disagreement obtained from $N_{MC} = 100$ independent realizations. Panel (a) shows $\bar{V}(t)$ with the shaded one-standard-deviation band $\bar{V}(t) \pm s_V(t)$. Panel (b) shows the least-squares fit of $\log(\bar{V}(t))$ against $\log(t)$ over $10 \leq t \leq 100$. The fitted tail exponent is reported in panel (b).

We point out that the theoretical result in Theorem 1 provides an upper bound for the convergence rate of the Lyapunov function, namely,

$$\mathbb{E}[V(\mathbf{x}(t))] \leq C_0 t^{-\alpha}. \quad (26)$$

This is a sufficient upper bound and is not asserted to be a sharp asymptotic equivalence and this estimate guarantees the algebraic convergence behavior but does not require the exact decay exponent to be exactly $-\alpha$. Therefore, an empirical decay exponent whose magnitude is larger than α remains consistent with (26). In particular, no identity between the measured exponent and 2α is assumed. The ensemble result supports the claimed power-law convergence while also showing that the theoretical estimate may be conservative for the tested parameter setting.

It should be noted that the theoretical estimate concerns the expected Lyapunov function $\mathbb{E}[V(\mathbf{x}(t))]$, whereas the numerical result shown here is obtained from a representative sample path. Therefore, the simulation aims to verify the algebraic decay behavior rather than provide a direct estimation of the expectation bound.

In summary, the numerical results presented in this section corroborate the theoretical findings of Theorem 1: the consensus condition is independent of the noise intensity and the fractional order, while the convergence rate is intrinsically determined by α through the power-law decay $t^{-\alpha}$. The ensemble statistics demonstrate that the numerical observations are not based on a single Brownian sample path. The noise intensity primarily changes stochastic dispersion in the tested regime, stronger leadership accelerates the reduction of disagreement, and the fractional order affects both transient and long-time

behavior. The bounded-confidence parameters have a comparatively weaker influence for the configurations considered here. The long-time and step-size studies provide additional evidence for algebraic decay and indicate a clear time-step convergence trend for the reported simulations.

Remark 9. *A complete mathematical treatment of the numerical solution and the rigorous convergence analysis of the stochastic fractional scheme is addressed in a separate study devoted specifically to numerical methods for fractional differential equations. Such an analysis requires additional solution-regularity assumptions, strong or mean-square error estimates, and a detailed treatment of the singular stochastic convolution. These issues constitute an independent numerical-analysis problem and lie beyond the consensus-theoretic scope of the present paper. Accordingly, the present work reports the implemented discretization and its step-size sensitivity, but does not claim a new rigorous convergence theorem for the numerical scheme.*

5. Conclusions

This paper has proposed a fractional-order inhomogeneous Hegselmann–Krause model with multiplicative noise, where the Caputo derivative captures memory effects in social interactions and the multiplicative noise models communication uncertainty. Using fractional Lyapunov and Gronwall inequalities, we have established a sufficient condition for almost sure consensus and revealed a power-law decay rate $t^{-\alpha}$, in contrast to the exponential convergence of integer-order models. The condition is independent of noise intensity, indicating that large groups with strong leadership are inherently robust against communication uncertainty. Numerical simulations validate the theoretical results and show that the fractional order α significantly modulates convergence speed, with smaller α yielding slower decay. Future work includes extending the model to time delays, switching topologies and multi-leader scenarios.

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