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Fekete–Szegő and Second Hankel Determinant Problems for a Bounded-Second-Derivative Class of Univalent Functions, and Its Fractional-Derivative Extension

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Abstract

Let $\mathcal{A}$ denote the class of functions $g(z) = z + \sum_{k=2}^{\infty} a_k z^k$ analytic and normalized in $\mathbb{U} = \{z : |z| < 1\}$, and let $D = \{g \in \mathcal{A} : |g''(z)| \leq 1, z \in \mathbb{U}\}$. We give a short, self-contained proof that D consists entirely of univalent functions by a direct line-integral estimate combined with the classical Noshiro–Warschawski theorem. We then solve, in closed form and with explicit extremal functions, two classical extremal coefficient problems for D : the Fekete–Szegő problem and the second Hankel determinant, using the exact Schur parametrization of bounded analytic functions applied to g'' . We also establish a sharp bound on a single coefficient $|a_n|$ for every $n \geq 2$, obtained by an elementary Parseval argument that does not require the full joint Schur parametrization needed for the Fekete–Szegő and Hankel problems. We then compose D with the Owa–Srivastava fractional derivative operator Ω_z^δ ($\delta \in [0, 1)$), used to build fractional bi-univalent subclasses, to define the one-parameter family $D(\delta) = \{g \in \mathcal{A} : |(\Omega_z^\delta g)''(z)| \leq 1\}$, and extend all three results to $D(\delta)$, again with explicit extremal functions. Because the underlying weights grow at different rates in the fractional parameter δ , the Fekete–Szegő problem exhibits a genuine, monotone bifurcation as δ ranges over $[0, 1)$, while the Hankel and general-coefficient bounds decrease monotonically—effects that are structural consequences of the fractional operator rather than a simple rescaling of the classical case. All results are verified in two independent numerical ways: the Schur parametrization lemma itself is checked against an explicit, independently constructed rational Schur function whose Taylor coefficients are extracted numerically via the Cauchy integral formula, and the sharp bounds are checked by Monte Carlo maximization over the admissible Schur parameters. Both checks are explained in detail and agree with the closed forms to at least four decimal places. Unlike the Carathéodory class of functions with positive real part, whose coefficient body is fixed at the origin (value 1) and is classically parametrized without an extra recursive step, the Schur class $\mathcal{B}$ used here does not pin down $\varphi(0)$, so its exact coefficient body genuinely requires the two-step recursive Schur parametrization applied below to g'' ; identifying and exploiting this distinction is part of the technical contribution of the paper.



Academic Editors: Gheorghe Oros and Georgia Irina Oros

Received: 20 July 2026

Revised: 10 August 2026

Accepted: 11 August 2026

Published: 17 August 2026

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Keywords: univalent functions; bounded derivatives; Noshiro–Warschawski theorem; Fekete–Szegő problem; second Hankel determinant; Schur functions; fractional derivative; Owa–Srivastava operator

MSC: 30C45; 30C50; 30C80; 26A33

1. Introduction

Let $\mathcal{A}$ denote the class of functions

$$g(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

analytic and normalized ($g(0) = 0$, $g'(0) = 1$) in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{S} \subset \mathcal{A}$ denote the univalent (injective) functions. Producing verifiable analytic conditions that force $g \in \mathcal{A}$ into $\mathcal{S}$ is a classical theme; recent work in this direction includes fractional-derivative-based coefficient bounds for bi-univalent functions [1], and, from a different and more classical angle, a recurring, natural strategy is to bound a derivative of g : MacGregor [2,3] and Singh [4] studied $|g'(z) - 1| < 1$; Obradović and Ponnusamy [5,6] and related authors studied various bounded-derivative and bounded-Schwarzian criteria; Al-Refai and Darus [7] established a general univalence criterion associated with the n th derivative; Al-Refai and Ali [8] introduced general coefficient estimates for bi-univalent functions. In this spirit, we study the class

$$D = \{g \in \mathcal{A} : |g''(z)| \leq 1, z \in \mathbb{U}\}, \quad (1)$$

i.e., functions whose second derivative is a Schur function (bounded by 1 in $\mathbb{U}$). This is a natural and easily stated bounded-derivative condition, and our first observation is a short, elementary, self-contained proof that $D \subset \mathcal{S}$, given in Proposition 1.

Having established $D \subset \mathcal{S}$, the next natural questions—and the main content of this paper—are the two classical extremal coefficient problems that typically follow a new univalence criterion: the Fekete–Szegő problem $\sup_{g \in D} |a_3 - \mu a_2^2|$ [9,10] and the second Hankel determinant $\sup_{g \in D} |a_2 a_4 - a_3^2|$ [11]. Both problems remain very active: recent work includes sharp Hankel and Fekete–Szegő bounds for classes associated with modified sigmoid and alpha-convex functions [12], with the sine function [13,14], with symmetric points [15], with logarithmic coefficients of strongly starlike and strongly convex functions [16], and general subclasses defined via subordination [17]. A closely related line of work defines subclasses of analytic and bi-univalent functions directly through the generating functions of classical special-polynomial sequences and derives their initial coefficient and Fekete–Szegő bounds by the same Faber-polynomial/subordination machinery, e.g., via Euler polynomials [18]; the underlying special-polynomial toolkit itself (generating functions, monomiality principle, connection formula) is developed in detail for Bell-based Apostol–Bernoulli-type polynomials by Ramírez et al. [19]. These works are complementary to, rather than overlapping with, the present paper: they use a fixed special-polynomial generating function to define a coefficient sequence and then bound the resulting $|a_2|$, $|a_3|$ by subordination, whereas here the class $D(\delta)$ is defined by a derivative bound and the coefficients are recovered via the Schur parametrization of g'' itself. Both problems are tractable here precisely because g'' is a genuine Schur function, which has an exact, sharp coefficient parametrization due to Schur [20]; this is a stronger structural fact than a triangle-inequality coefficient sufficient condition, and it is the key tool of the paper (Lemma 3). This case requires more care than the more familiar Carathéodory-class setting:

for a Carathéodory function $p(z) = 1 + p_1z + p_2z^2 + \cdots$ with $\operatorname{Re} p > 0$, the value $p(0) = 1$ is fixed, and the classical parametrization of (p_1, p_2) follows directly from one application of the Schwarz lemma. For $\varphi \in \mathcal{B}$, by contrast, $\varphi(0) = c_0$ is itself a free parameter of modulus at most 1, so a single Schwarz-lemma step only controls c_1 in terms of c_0 ; a second, nested application (recorded precisely in Lemma 3) is needed to control c_2 as well. This two-step recursive structure is what lets us solve the joint Fekete–Szegő and Hankel problems for D and $D(\delta)$ exactly, rather than only up to a coefficient-inequality bound.

Separately, motivated by the Owa–Srivastava fractional derivative operator [21,22], used by Al-Refai [1] to build fractional subclasses of bi-univalent functions and by Al-Refai and Darus [23] to extend the operator with applications to parabolic starlike and uniformly convex functions, we compose D with the normalized operator Ω_z^δ ($\delta \in [0, 1)$) to define a one-parameter family

$$D(\delta) := \{g \in \mathcal{A} : |(\Omega_z^\delta g)''(z)| \leq 1, z \in \mathbb{U}\}, \quad (2)$$

which reduces to D at $\delta = 0$. Since Ω_z^δ acts diagonally on Taylor coefficients, $g \in D(\delta)$ if and only if $G := \Omega_z^\delta g$ lies in the undecorated class D ; consequently every linear statement about D (in particular, univalence, via Proposition 1 applied to G) transfers to $D(\delta)$ by relabeling the coefficients of g with explicit weights $\lambda_k(\delta)$, with no new proof idea required. That relabeling observation is not, by itself, a strong result. Our actual contribution is that the nonlinear extremal problems (Fekete–Szegő, Hankel) do not merely relabel: because the functional $a_3 - \mu a_2^2$ is quadratic in a_2 and Ω_z^δ rescales a_2, a_3 by different weights, the location of the extremal problem’s bifurcation genuinely moves with δ . Fractional-order operators of this general type are also studied numerically in other settings, e.g., via a Bernstein-shifted Legendre spectral method for the Atangana–Baleanu–Caputo derivative applied to a perturbed Korteweg–de Vries equation [24]; our own use of the fractional operator here is purely analytic, with numerical verification limited to checking the resulting coefficient bounds in Section 7.

2. Preliminaries

2.1. The Class D and Its Univalence

Proposition 1 (Univalence criterion for D). *If $g \in D$, then $|g'(z) - 1| < 1$ for all $z \in \mathbb{U}$; consequently $\operatorname{Re} g'(z) > 0$ in $\mathbb{U}$, and g is univalent in $\mathbb{U}$. The bound is sharp: $g(z) = z + \frac{1}{2}z^2$ satisfies $g'' \equiv 1 \in D$ and $|g'(z) - 1| = |z| \rightarrow 1$ as $|z| \rightarrow 1$.*

Proof. Since g'' is analytic in the simply connected domain $\mathbb{U}$, the integral of g'' from 0 to z is path-independent; take the straight segment $w = tz$, $t \in [0, 1]$, $dw = z dt$. By the fundamental theorem of calculus for line integrals of analytic functions and $g'(0) = 1$,

$$g'(z) - 1 = \int_0^z g''(w) dw = \int_0^1 g''(tz) z dt.$$

Hence, by the ML-inequality (with $|g''(tz)| \leq 1$ for all $t \in [0, 1]$, $z \in \mathbb{U}$),

$$|g'(z) - 1| \leq |z| \int_0^1 |g''(tz)| dt \leq |z| \int_0^1 1 dt = |z| < 1, \quad z \in \mathbb{U}.$$

This proves the first claim. Now, $|g'(z) - 1| < 1$ means $g'(z)$ lies in the open disk of radius 1 centered at 1; every point $w = x + iy$ of this disk satisfies $(x - 1)^2 + y^2 < 1$, i.e., $x^2 + y^2 < 2x$, which forces $x > 0$ (if $x \leq 0$ then $x^2 + y^2 \leq 0$, forcing $x = y = 0$, but the origin lies on the boundary of the disk, not inside it—contradiction). Hence, $\operatorname{Re} g'(z) > 0$ throughout $\mathbb{U}$. Since $\mathbb{U}$ is convex, the classical Noshiro–Warschawski theorem [25,26] (if h

is analytic on a convex domain Ω and $\operatorname{Re} h'(z) > 0$ throughout Ω , then h is univalent on Ω ; see also [27], Section 2.5) gives that g is univalent in $\mathbb{U}$. Sharpness is immediate from the stated example, for which $g'' \equiv 1$ (so $|g''| \equiv 1 \leq 1$, i.e., $g \in D$) and $g'(z) - 1 = z$, so $|g'(z) - 1| = |z|$ approaches 1 as z approaches the boundary. $\square$

2.2. The Fractional Derivative Operator and $D(\delta)$

For $0 \leq \delta < 1$, the Owa–Srivastava fractional derivative [21,22] of f is

$$(D_z^\delta f)(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z f(\eta)(z-\eta)^{-\delta} d\eta,$$

and the normalized operator, used by Al-Refai [1] to define fractional bi-univalent subclasses, is $(\Omega_z^\delta g)(z) = \Gamma(2-\delta)z^\delta (D_z^\delta g)(z)$. For $g(z) = z + \sum_{k \geq 2} a_k z^k \in \mathcal{A}$,

$$G(z) := (\Omega_z^\delta g)(z) = z + \sum_{k=2}^{\infty} \lambda_k(\delta) a_k z^k \in \mathcal{A}, \quad \lambda_k(\delta) := \frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k+1-\delta)} = \frac{k!}{\prod_{j=2}^k (j-\delta)}. \quad (3)$$

Lemma 1 (Weights). For an integer $k \geq 2$ and $\delta \in [0, 1)$, $\lambda_k(\delta) > 0$, $\lambda_k(0) = 1$, and λ_k is strictly increasing in δ with $\lim_{\delta \rightarrow 1^-} \lambda_k(\delta) = k$; hence, $1 \leq \lambda_k(\delta) < k$. In particular

$$\lambda_2(\delta) = \frac{2}{2-\delta}, \quad \lambda_3(\delta) = \frac{6}{(2-\delta)(3-\delta)}, \quad \lambda_4(\delta) = \frac{24}{(2-\delta)(3-\delta)(4-\delta)}.$$

Proof. From $\lambda_k(\delta) = k! / \prod_{j=2}^k (j-\delta)$, positivity is clear since each factor $j-\delta > 0$ for $2 \leq j \leq k$, and $\delta < 1$; $\ln \lambda_k(\delta) = \ln k! - \sum_{j=2}^k \ln(j-\delta)$ has derivative $\sum_{j=2}^k (j-\delta)^{-1} > 0$, giving strict monotonicity; and $\prod_{j=2}^k (j-\delta) \rightarrow (k-1)!$ as $\delta \rightarrow 1^-$ gives the limit $k! / (k-1)! = k$. $\square$

Figure 1 plots these weights for $k = 2, \dots, 6$; their differing growth rates are what later drive the Fekete–Szegő bifurcation shift and the general coefficient-bound decay of Theorem 1.

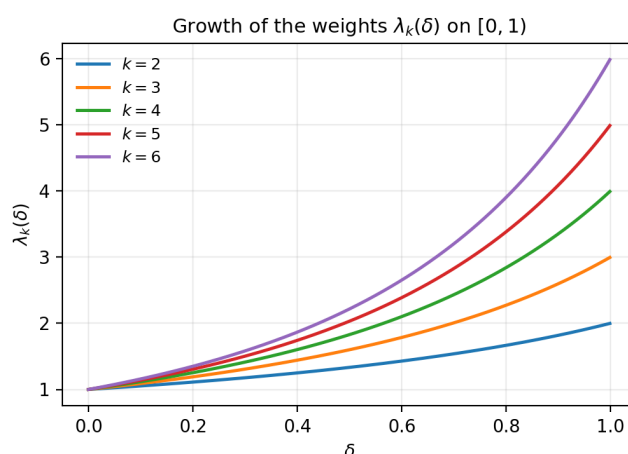


Figure 1. The weights $\lambda_k(\delta)$ of Lemma 1 for $k = 2, \dots, 6$, increasing from $\lambda_k(0) = 1$ to $\lambda_k(1^-) = k$. Their differing growth rates are what later drive the Fekete–Szegő bifurcation shift and the general coefficient-bound decay (Theorem 1).

Definition 1. For $\delta \in [0, 1)$, $g \in \mathcal{A}$ belongs to $D(\delta)$ if $G = \Omega_z^\delta g$ satisfies $|G''(z)| \leq 1$ in $\mathbb{U}$; equivalently, with $A_k := \lambda_k(\delta)a_k$, $|2A_2 + 6A_3z + 12A_4z^2 + \dots| \leq 1$ throughout $\mathbb{U}$. At $\delta = 0$, $D(\delta) = D$.

Corollary 1 (δ -univalence). *If $g \in D(\delta)$, then $G = \Omega_z^\delta g$ satisfies $|G'(z) - 1| < 1$ in $\mathbb{U}$ and is univalent, with sharp extremal function $G(z) = z + \frac{1}{2}z^2$, i.e., $g(z) = z + \frac{1}{2\lambda_2(\delta)}z^2 = z + \frac{2-\delta}{4}z^2$.*

Proof. $g \in D(\delta)$ means precisely $G \in D$ (Definition 1); apply Proposition 1 to G . Explicitly, since $G(z) = z + \frac{1}{2}z^2$ has z^2 -coefficient $\lambda_2(\delta)a_2 = \frac{1}{2}$, we get $a_2 = \frac{1}{2\lambda_2(\delta)}$; substituting the closed form $\lambda_2(\delta) = 2/(2-\delta)$ from Lemma 1 gives $a_2 = \frac{1}{2 \cdot 2/(2-\delta)} = \frac{2-\delta}{4}$, which confirms $\frac{1}{2\lambda_2(\delta)} = \frac{2-\delta}{4}$ as displayed. $\square$

Corollary 1 confirms $D(\delta)$ is a well-posed, non-trivial, univalence-guaranteeing family before we study coefficient functionals on it; as explained above, it is an immediate relabeling of Proposition 1 and is included only for context, not as a main result.

Definition 2 (The bound-parametrized family $D(\delta, \alpha)$). *For $\delta \in [0, 1)$ and $\alpha \in (0, 1]$, let*

$$D(\delta, \alpha) := \{g \in \mathcal{A} : |(\Omega_z^\delta g)''(z)| \leq \alpha, z \in \mathbb{U}\}.$$

By Definition 1, $D(\delta, 1) = D(\delta)$, and $D(0, \alpha) = \{g \in \mathcal{A} : |g''(z)| \leq \alpha\}$.

Proposition 2 (Monotonicity in the bound α). *Fix $\delta \in [0, 1)$. If $0 < \alpha_1 \leq \alpha_2 \leq 1$, then $D(\delta, \alpha_1) \subseteq D(\delta, \alpha_2)$. In particular, at $\delta = 0$, $D(0, \alpha_1) \subseteq D(0, \alpha_2)$ for $\alpha_1 \leq \alpha_2$, i.e., the classical family $\{g \in \mathcal{A} : |g''(z)| \leq \alpha\}_{\alpha \in (0, 1]}$ is nested and increasing in α , with $D(0, 1) = D$.*

Proof. Let $g \in D(\delta, \alpha_1)$, so $|G''(z)| \leq \alpha_1$ for all $z \in \mathbb{U}$, where $G = \Omega_z^\delta g$. Since $\alpha_1 \leq \alpha_2$, this gives $|G''(z)| \leq \alpha_1 \leq \alpha_2$ for all $z \in \mathbb{U}$, i.e., $g \in D(\delta, \alpha_2)$. As $g \in D(\delta, \alpha_1)$ was arbitrary, $D(\delta, \alpha_1) \subseteq D(\delta, \alpha_2)$. Setting $\delta = 0$ specializes $\Omega_z^0 g = g$ (Lemma 1, $\lambda_k(0) = 1$), giving the stated classical case. $\square$

This is a genuine pointwise containment of function classes (unlike monotonicity in the fractional order δ itself with the bound fixed at 1, which is a separate and harder question we do not resolve here): α enters only through the right-hand side of the defining inequality, so the containment $D(\delta, \alpha_1) \subseteq D(\delta, \alpha_2)$ for $\alpha_1 \leq \alpha_2$ is immediate from the definition and requires no multiplier argument.

Proposition 3 (Rescaled coefficient, Fekete–Szegő, and Hankel bounds). *Let $\delta \in [0, 1)$ and $\alpha \in (0, 1]$. If $g \in D(\delta, \alpha)$, then, for every $n \geq 2$,*

$$|a_n| \leq \frac{\alpha}{n(n-1)\lambda_n(\delta)},$$

sharp, with equality for $g(z) = z + \frac{\alpha}{n(n-1)\lambda_n(\delta)}e^{i\theta}z^n$; this bound is visibly increasing in α , consistent with Proposition 2. Moreover, the Fekete–Szegő and second Hankel determinant bounds of Theorems 2 and 3 (stated there for $D(\delta, 1) = D(\delta)$) rescale the same way for $D(\delta, \alpha)$, by α and α^2 respectively, since $a_3 - \mu a_2^2$ is homogeneous of degree 2 in (a_2, a_3) and $a_2 a_4 - a_3^2$ is homogeneous of degree 2 in (a_2, a_3, a_4) under $g \mapsto \alpha g$ (which is what replacing G'' by G''/α amounts to at the coefficient level).

Proof. Rescaling Theorem 1 (Section 3 below) by α —apply Lemma 2 to $G''/\alpha \in \mathcal{B}$ instead of G'' —gives the stated bound on $|a_n|$ directly, with sharpness inherited from the extremal function of Theorem 1 rescaled by α . The Fekete–Szegő and Hankel rescaling follows by the same substitution $G'' \mapsto G''/\alpha$ applied throughout the proofs of Theorems 2 and 3; we omit the routine restatement. $\square$

3. Sharp Coefficient Bound $|a_n|$ for General n

Before turning to the joint (Fekete–Szegő/Hankel) extremal problems, which require the full recursive Schur parametrization of Section 4, we record a simpler, general- n bound on a single coefficient $|a_n|$, valid for every $n \geq 2$. It follows from a classical, elementary fact about bounded analytic functions: Parseval's identity applied to a Schur function.

Lemma 2 (Parseval's inequality for bounded analytic functions; classical). *Let $\varphi(z) = \sum_{k=0}^{\infty} c_k z^k$ be analytic in $\mathbb{U}$ with $|\varphi(z)| \leq 1$ for $z \in \mathbb{U}$. Then*

$$\sum_{k=0}^{\infty} |c_k|^2 \leq 1,$$

and in particular $|c_k| \leq 1$ for every $k \geq 0$, with equality (for a given k) attained by $\varphi(z) = z^k$.

Proof. For $0 < r < 1$, expanding $|\varphi(re^{i\theta})|^2 = \varphi(re^{i\theta})\overline{\varphi(re^{i\theta})}$ as a double series and integrating term by term against $\frac{1}{2\pi} \int_0^{2\pi} e^{i(j-k)\theta} d\theta = \delta_{jk}$ (orthogonality of $\{e^{ik\theta}\}$) gives the classical Parseval identity

$$\frac{1}{2\pi} \int_0^{2\pi} |\varphi(re^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} |c_k|^2 r^{2k}.$$

Since $|\varphi(z)| \leq 1$ pointwise in $\mathbb{U}$, the left side is ≤ 1 for every $r < 1$, so $\sum_{k=0}^{\infty} |c_k|^2 r^{2k} \leq 1$ for every $r < 1$; letting $r \rightarrow 1^-$ (each partial sum is non-decreasing in r , so the limit equals the supremum) gives $\sum_{k=0}^{\infty} |c_k|^2 \leq 1$. Since all terms are non-negative, $|c_k|^2 \leq 1$ for each k . For $\varphi(z) = z^k$, $|\varphi(z)| = |z|^k < 1$ in $\mathbb{U}$ (so $\varphi \in \mathcal{B}$) and the only nonzero coefficient is $c_k = 1$, attaining equality. $\square$

Theorem 1 (Sharp bound on $|a_n|$, general n). *Let $\delta \in [0, 1)$ and $n \geq 2$. If $g(z) = z + \sum_{k=2}^{\infty} a_k z^k \in D(\delta)$, then*

$$|a_n| \leq \frac{1}{n(n-1)\lambda_n(\delta)} = \frac{\prod_{j=2}^n (j-\delta)}{n(n-1)n!}, \quad (4)$$

sharp, with equality only for

$$g(z) = z + \frac{1}{n(n-1)\lambda_n(\delta)} e^{i\theta} z^n = z + \frac{\prod_{j=2}^n (j-\delta)}{n(n-1)n!} e^{i\theta} z^n \quad (\theta \in \mathbb{R}).$$

In particular, at $\delta = 0$ (the class D), $|a_n| \leq 1/[n(n-1)]$, and as $\delta \rightarrow 1^-$, the bound decreases monotonically to $1/[n^2(n-1)]$.

Proof. Write $G = \Omega_z^\delta g = z + \sum_{k \geq 2} A_k z^k$, $A_k = \lambda_k(\delta) a_k$, so $g \in D(\delta)$ means $G'' \in \mathcal{B}$ (Definition 1). Expanding $G''(z) = \sum_{k=2}^{\infty} k(k-1) A_k z^{k-2} = \sum_{m=0}^{\infty} c_m z^m$ with $c_m = (m+2)(m+1) A_{m+2}$, Lemma 2 gives $|c_m| \leq 1$ for every $m \geq 0$; taking $m = n-2$,

$$n(n-1)|A_n| = n(n-1)\lambda_n(\delta)|a_n| \leq 1,$$

which is (4). Equality forces $G''(z) = e^{i\theta} z^{n-2}$ exactly (the unique Schur function with $|c_{n-2}| = 1$, since Lemma 2's ℓ^2 -bound forces every other c_m , $m \neq n-2$, to vanish); integrating twice, $G(z) = z + \frac{1}{n(n-1)} e^{i\theta} z^n$, and dividing by $\lambda_n(\delta)$ gives the stated extremal g . The closed form and its monotonicity in δ follow from Lemma 1: $\lambda_n(\delta) = n! / \prod_{j=2}^n (j-\delta)$ is increasing from $\lambda_n(0) = 1$ to $\lambda_n(1^-) = n$, so $1/[n(n-1)\lambda_n(\delta)]$ decreases from $1/[n(n-1)]$ to $1/[n(n-1) \cdot n] = 1/[n^2(n-1)]$. $\square$

Remark 1. Theorem 1 is consistent with, and independently corroborates, Theorem 3: setting $n = 2$ recovers the extremal function and bound of Corollary 1, and setting $n = 3, \mu = 0$ recovers exactly the small- μ branch of (10) and its extremal function. Unlike Theorems 2 and 3, Theorem 1 constrains only a single coefficient at a time and therefore needs only the elementary Parseval bound (Lemma 2), not the full joint Schur parametrization of Lemma 3; the latter becomes necessary precisely when two or more coefficients are coupled in one functional, as in the Fekete–Szegő and Hankel problems.

Figure 2 visualizes the extremal functions of Theorem 1 themselves, i.e., the images

$$g_{n,\delta}(\mathbb{U}), \quad g_{n,\delta}(z) = z + a_n(\delta)z^n, \quad a_n(\delta) = \frac{\prod_{j=2}^n (j - \delta)}{n(n-1)n!},$$

for $n = 2, 3, 4, 5$ (rows) and $\delta = 0, 0.5, 0.9$ (columns), obtained by mapping a polar grid of $\mathbb{U}$ (concentric circles and radial spokes) through $g_{n,\delta}$ and shading the image of the full disk. Two effects are visible simultaneously:

- Along a row (fixed n , δ increasing left to right): the extremal coefficient $a_n(\delta)$ strictly decreases (Lemma 1), so the image domain visibly flattens toward the identity map $g(z) = z$ (the undistorted disk) as $\delta \rightarrow 1^-$ —a direct geometric picture of the coefficient-bound improvement quantified in Theorem 1.
- Down a column (fixed δ , n increasing top to bottom): since $g'_{n,\delta}(z) - 1 = na_n(\delta)z^{n-1}$ has $n - 1$ zeros evenly spaced on a circle about the origin, the boundary of the image domain develops $n - 1$ symmetric lobes (cusps in the limiting sharp case); this is the extremal shape that saturates the single-coefficient bound of Theorem 1 at that value of n , exactly as the classical cardioid-type domain does for $n = 2$.

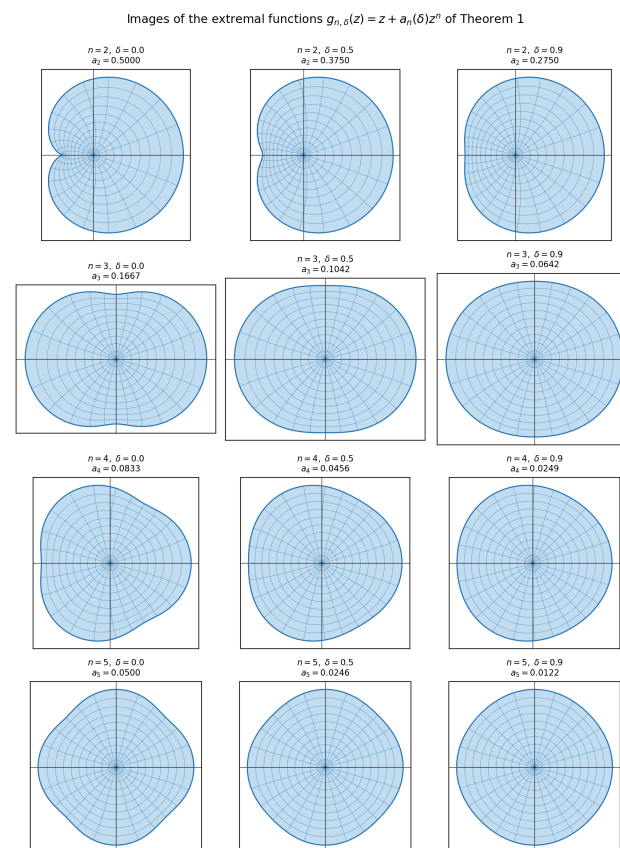


Figure 2. Images of the polar grid of $\mathbb{U}$ under the extremal functions $g_{n,\delta}(z) = z + a_n(\delta)z^n$ of Theorem 1, for $n = 2, 3, 4, 5$ (rows) and $\delta = 0, 0.5, 0.9$ (columns).

Because the coefficient bound in Theorem 1 is attained by a single, exactly known monomial for every (n, δ) , Figure 2 is not a numerical approximation of an extremal shape (as would typically be needed for a general extremal problem) but an exact plot of the actual sharp boundary case.

4. A Schur-Parametrization Lemma

The condition $|g''(z)| \leq 1$ means g'' belongs to the Schur class $\mathcal{B} = \{\varphi \text{ analytic in } \mathbb{U} : |\varphi(z)| \leq 1\}$. Unlike the Carathéodory class (positive real part, value at 0 fixed to be 1), the Schur class does not fix $\varphi(0)$, so its coefficient body requires the recursive Schur parametrization.

Lemma 3 (Schur [20]). *Let $\varphi(z) = c_0 + c_1z + c_2z^2 + \cdots \in \mathcal{B}$. Then there exist x, y with $|x| \leq 1, |y| \leq 1$ such that*

$$c_1 = (1 - |c_0|^2)x, \quad c_2 = (1 - |c_0|^2)[(1 - |x|^2)y - \bar{c}_0x^2]. \quad (5)$$

Conversely, every choice of c_0, x, y in the closed unit disk arises from some $\varphi \in \mathcal{B}$, so (5) is the full coefficient body.

Proof. We first dispose of the boundary case $|c_0| = 1$. If $|c_0| = |\varphi(0)| = 1$ for some $\varphi \in \mathcal{B}$, then φ attains its maximum modulus 1 at the interior point $z = 0$, so by the maximum modulus principle φ is constant, $\varphi \equiv c_0$; then $c_1 = c_2 = 0$, and (5) holds trivially for every choice of x, y with $|x|, |y| \leq 1$, since the prefactor $1 - |c_0|^2$ vanishes. This degenerate case therefore requires no further argument, and we may assume $|c_0| < 1$ for the rest of the proof.

Now suppose $|c_0| < 1$. Since $|c_0| = |\varphi(0)| \leq 1$, $\omega(z) := (\varphi(z) - c_0)/(1 - \bar{c}_0\varphi(z))$ is a self-map of $\mathbb{U}$ with $\omega(0) = 0$, so by the Schwarz lemma $\omega(z) = z\Omega(z)$ for some $\Omega \in \mathcal{B}$, giving

$$\varphi(z) = \frac{c_0 + z\Omega(z)}{1 + \bar{c}_0z\Omega(z)}.$$

Writing $\Omega(z) = x + d_1z + \cdots$ and applying the same argument to Ω gives $d_1 = (1 - |x|^2)y$ for some $|y| \leq 1$. Expanding φ by series division and collecting the coefficients of z^1, z^2 gives (5) (the z^1 -coefficient is $x - c_0\bar{c}_0x = (1 - |c_0|^2)x$; the z^2 -coefficient simplifies to $(1 - |c_0|^2)[(1 - |x|^2)y - \bar{c}_0x^2]$). The converse follows by reversing the construction, e.g., taking $\Omega(z) = (x + yz)/(1 + \bar{x}yz)$ explicitly realizes any admissible (c_0, x, y) . $\square$

Applying Lemma 3 to $h(z) := g''(z) = 2a_2 + 6a_3z + 12a_4z^2 + \cdots$ (so $c_0 = 2a_2, c_1 = 6a_3, c_2 = 12a_4$) gives, for $g \in D$,

$$a_2 = \frac{c_0}{2}, \quad a_3 = \frac{(1 - |c_0|^2)x}{6}, \quad a_4 = \frac{(1 - |c_0|^2)[(1 - |x|^2)y - \bar{c}_0x^2]}{12}, \quad (6)$$

with c_0, x, y ranging independently over the closed unit disk.

5. Sharp Fekete–Szegő and Hankel Bounds for D

Theorem 2. *Let $g(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots \in D$. Then for every $\mu \in \mathbb{C}$,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{6}, & |\mu| \leq \frac{2}{3}, \\ \frac{|\mu|}{4}, & |\mu| \geq \frac{2}{3}, \end{cases} \quad (7)$$

and

$$|a_2 a_4 - a_3^2| \leq \frac{1}{36}, \quad (8)$$

both sharp. For $|\mu| \neq 2/3$, equality in (7) is attained by a unique (up to the rotation θ) extremal function, as follows. Equality in the regime $|\mu| > 2/3$ of (7) holds only for

$$g(z) = z + \frac{1}{2}e^{i\theta}z^2 \quad (\theta \in \mathbb{R}),$$

the classical extremal function of D ($a_2 = \frac{1}{2}e^{i\theta}$, $a_3 = a_4 = 0$). Equality in the regime $|\mu| < 2/3$ of (7), and equality in (8), hold only for

$$g(z) = z + \frac{1}{6}e^{i\theta}z^3 \quad (\theta \in \mathbb{R}),$$

i.e., $a_2 = a_4 = 0$, $a_3 = \frac{1}{6}e^{i\theta}$. At the boundary value $|\mu| = 2/3$ the two regimes meet and (7) still holds with equality possible (the common value is $1/6 = |\mu|/4$ there), but uniqueness of the extremal function fails: as shown in the proof, $F(t) = 1/6$ for every $t \in [0, 1]$ when $|\mu| = 2/3$, so equality in (7) is attained by the entire one-parameter-in- t family $g(z) = z + \frac{t}{2}e^{i\theta_1}z^2 + \frac{1-t^2}{6}e^{i\theta_2}z^3$ for any $t \in [0, 1]$ and suitably aligned phases θ_1, θ_2 (which interpolates continuously between the two extremal functions above as t ranges over $[0, 1]$), rather than by a single function; we discuss this boundary case separately from the two open, uniqueness-bearing regimes $|\mu| < 2/3$ and $|\mu| > 2/3$.

Proof. Fekete–Szegő bound. By (6), with $t := |c_0| \in [0, 1]$,

$$a_3 - \mu a_2^2 = \frac{1-t^2}{6}x - \frac{\mu}{4}c_0^2.$$

For fixed c_0 , this is $Ax + B$ with $A = (1-t^2)/6 \geq 0$ real and $B = -\mu c_0^2/4$; since $|x| \leq 1$ ranges freely,

$$\max_{|x| \leq 1} |Ax + B| = A + |B|$$

(take $x = e^{i \arg B}$). Since $|B| = |\mu|t^2/4$ regardless of $\arg c_0$, we maximize

$$F(t) = \frac{1-t^2}{6} + \frac{|\mu|t^2}{4} = \frac{1}{6} + t^2 \left(\frac{|\mu|}{4} - \frac{1}{6} \right), \quad t \in [0, 1],$$

which is affine in t^2 , so its maximum on $[0, 1]$ is attained at an endpoint: $t = 0$ ($F = 1/6$) when $|\mu| \leq 2/3$, and $t = 1$ ($F = |\mu|/4$) when $|\mu| \geq 2/3$. This proves (7). When $|\mu| \neq 2/3$, F is strictly monotone in t^2 , so the maximizer $t \in \{0, 1\}$ is unique, which (after fixing the alignment phase) forces the stated extremal functions.

Hankel bound. Substituting (6) and simplifying,

$$a_2 a_4 - a_3^2 = (1-t^2) \left[\frac{c_0(1-s^2)y}{24} - \frac{(2+t^2)}{72}x^2 \right], \quad s := |x| \in [0, 1].$$

Maximizing over the free phases of y (term magnitude $t(1-s^2)/24$) and x^2 (term magnitude $(2+t^2)s^2/72$), independently alignable, gives

$$\Phi(t, s) = (1-t^2) \left[\frac{t(1-s^2)}{24} + \frac{(2+t^2)s^2}{72} \right].$$

The s^2 -coefficient in the bracket is $\frac{1}{72}(2+t^2-3t) = \frac{1}{72}(1-t)(2-t) \geq 0$, so Φ is non-decreasing in s^2 , maximized at $s = 1$:

$$\Phi(t, 1) = \frac{(1-t^2)(t^2+2)}{72}.$$

With $u = t^2$, $\psi(u) = (1 - u)(u + 2)/72$ has $\psi'(u) = -(2u + 1)/72 < 0$, and is strictly decreasing, so the unique maximum is at $u = 0$: $\Phi_{\max} = 1/36$. The unique extremizer $t = 0, s = 1$ gives $a_2 = 0, a_3 = \frac{1}{6}e^{i\theta}, a_4 = 0$. $\square$

6. The Fractional-Derivative Generalization

Since $g \in D(\delta)$ means $G'' = 2A_2 + 6A_3z + 12A_4z^2 + \dots$ ($A_k = \lambda_k(\delta)a_k$) is Schur, Lemma 3 gives, after dividing by $\lambda_k(\delta)$,

$$a_2 = \frac{c_0}{2\lambda_2(\delta)}, \quad a_3 = \frac{(1 - |c_0|^2)x}{6\lambda_3(\delta)}, \quad a_4 = \frac{(1 - |c_0|^2)[(1 - |x|^2)y - \bar{c}_0x^2]}{12\lambda_4(\delta)}. \quad (9)$$

Theorem 3. Let $\delta \in [0, 1)$ and $g(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \dots \in D(\delta)$. Then for every $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{6\lambda_3(\delta)} = \frac{(2 - \delta)(3 - \delta)}{36}, & |\mu| \leq \mu_0(\delta), \\ \frac{|\mu|}{4\lambda_2(\delta)^2} = \frac{|\mu|(2 - \delta)^2}{16}, & |\mu| \geq \mu_0(\delta), \end{cases} \quad \mu_0(\delta) := \frac{2\lambda_2(\delta)^2}{3\lambda_3(\delta)} = \frac{4(3 - \delta)}{9(2 - \delta)}, \quad (10)$$

and

$$|a_2a_4 - a_3^2| \leq \frac{1}{36\lambda_3(\delta)^2} = \frac{(2 - \delta)^2(3 - \delta)^2}{1296}, \quad (11)$$

both sharp for every $\delta \in [0, 1)$. For $|\mu| \neq \mu_0(\delta)$, equality in (10) is attained by a unique (up to rotation) extremal function, as follows. Equality in the regime $|\mu| > \mu_0(\delta)$ of (10) holds only for

$$g(z) = z + \frac{2 - \delta}{4}e^{i\theta}z^2.$$

Equality in the regime $|\mu| < \mu_0(\delta)$ of (10), and in (11), hold only for

$$g(z) = z + \frac{(2 - \delta)(3 - \delta)}{36}e^{i\theta}z^3.$$

As in Theorem 2, at the boundary value $|\mu| = \mu_0(\delta)$ the bound (10) still holds with equality possible, but the extremal function is not unique: the whole one-parameter family interpolating between the two functions above (via the free Schur parameter $t = |c_0| \in [0, 1]$, cf. the proof) attains equality, so we again restrict the uniqueness statement to the two open regimes $|\mu| < \mu_0(\delta)$ and $|\mu| > \mu_0(\delta)$. At $\delta = 0$, (10) and (11), and both extremal functions reduce exactly to Theorem 2 ($\mu_0(0) = 2/3$).

Proof. Fekete–Szegő bound. From (9), with $t = |c_0|$,

$$a_3 - \mu a_2^2 = \frac{1}{\lambda_3(\delta)} \left[\frac{1 - t^2}{6}x - \frac{\nu(\delta)}{4}c_0^2 \right], \quad \nu(\delta) := \mu \frac{\lambda_3(\delta)}{\lambda_2(\delta)^2}.$$

The bracket is exactly the expression maximized in Theorem 2's proof with μ replaced by $\nu(\delta)$, giving

$$\max |a_3 - \mu a_2^2| = \frac{1}{\lambda_3(\delta)} \times \begin{cases} 1/6, & |\nu(\delta)| \leq 2/3, \\ |\nu(\delta)|/4, & |\nu(\delta)| \geq 2/3. \end{cases}$$

The threshold $|\nu(\delta)| \leq 2/3$ is equivalent to $|\mu| \leq \frac{2}{3}\lambda_2(\delta)^2/\lambda_3(\delta) = \mu_0(\delta)$; substituting back gives (10), and the extremal Schur parameters $(t, x) = (1, \cdot)$, $(t, x) = (0, e^{i\theta})$ translate via (9)

into the stated functions. The closed forms follow from Lemma 1: $\lambda_2(\delta)^2 = 4/(2-\delta)^2$, $\lambda_3(\delta) = 6/[(2-\delta)(3-\delta)]$ give

$$\mu_0(\delta) = \frac{8(3-\delta)}{18(2-\delta)} = \frac{4(3-\delta)}{9(2-\delta)},$$

$$\frac{1}{6\lambda_3(\delta)} = \frac{(2-\delta)(3-\delta)}{36}, \quad \frac{1}{4\lambda_2(\delta)^2} = \frac{(2-\delta)^2}{16}.$$

Hankel bound. From (9),

$$a_2a_4 - a_3^2 = (1-t^2) \left[\frac{c_0(1-s^2)y}{24\lambda_2(\delta)\lambda_4(\delta)} - x^2 \left(\frac{t^2}{24\lambda_2(\delta)\lambda_4(\delta)} + \frac{1-t^2}{36\lambda_3(\delta)^2} \right) \right].$$

Let $P(\delta) = 1/(24\lambda_2(\delta)\lambda_4(\delta))$, $Q(\delta) = 1/(36\lambda_3(\delta)^2)$. Maximizing over the free phases of y, x^2 as before,

$$\Phi(t, s; \delta) = (1-t^2)[tP(\delta)(1-s^2) + s^2(t^2P(\delta) + (1-t^2)Q(\delta))].$$

The s^2 -coefficient is $(1-t)[(1+t)Q(\delta) - tP(\delta)]$. We verify explicitly that $Q(\delta)/P(\delta) \in (16/27, 2/3]$ for $\delta \in [0, 1)$. Using the closed forms of Lemma 1, $\lambda_2(\delta) = 2/(2-\delta)$, $\lambda_3(\delta) = 6/[(2-\delta)(3-\delta)]$, $\lambda_4(\delta) = 24/[(2-\delta)(3-\delta)(4-\delta)]$, so

$$\frac{\lambda_2(\delta)\lambda_4(\delta)}{\lambda_3(\delta)^2} = \frac{2}{2-\delta} \cdot \frac{24}{(2-\delta)(3-\delta)(4-\delta)} = \frac{48(3-\delta)}{36(4-\delta)} = \frac{4(3-\delta)}{3(4-\delta)},$$

$$\frac{1}{(2-\delta)^2(3-\delta)^2}$$

and hence,

$$\frac{Q(\delta)}{P(\delta)} = \frac{2}{3} \cdot \frac{\lambda_2(\delta)\lambda_4(\delta)}{\lambda_3(\delta)^2} = \frac{8}{9} \cdot \frac{3-\delta}{4-\delta} =: R(\delta).$$

Differentiating, $R'(\delta) = \frac{8}{9} \cdot \frac{-(4-\delta) + (3-\delta)}{(4-\delta)^2} = \frac{8}{9} \cdot \frac{-1}{(4-\delta)^2} < 0$, so R is strictly decreasing on $[0, 1)$; thus, $R(\delta) \leq R(0) = \frac{8}{9} \cdot \frac{3}{4} = \frac{2}{3}$ (attained at $\delta = 0$) and $R(\delta) > \lim_{\delta \rightarrow 1^-} R(\delta) = \frac{8}{9} \cdot \frac{2}{3} = \frac{16}{27}$ (approached, not attained, as $\delta \rightarrow 1^-$), confirming $Q(\delta)/P(\delta) \in (16/27, 2/3]$ as claimed. Since $t/(1+t) \leq 1/2 < 16/27$ for $t \in [0, 1]$, $(1+t)Q(\delta) - tP(\delta) > 0$ throughout, so Φ is increasing in s^2 , maximized at $s = 1$:

$$\Phi(t, 1; \delta) = (1-t^2)[t^2P(\delta) + (1-t^2)Q(\delta)].$$

With $u = t^2$, $\psi(u) = uP(\delta)(1-u) + Q(\delta)(1-u)^2$ has $\psi''(u) = -2(P(\delta) - Q(\delta))$; since $Q(\delta)/P(\delta) > 1/2$, $P(\delta) < 2Q(\delta)$, placing the critical point $u^* = \frac{P(\delta)-2Q(\delta)}{2(P(\delta)-Q(\delta))}$ strictly left of 0 ($P - 2Q < 0$, $P - Q > 0$), so ψ is concave with vertex left of $[0, 1]$, and is strictly decreasing there; the unique maximum is at $u = 0$: $\Phi_{\max} = Q(\delta) = 1/(36\lambda_3(\delta)^2)$, proving (11), with unique extremizer translating to $g(z) = z + \frac{(2-\delta)(3-\delta)}{36} e^{i\theta} z^3$.

At $\delta = 0$, $\lambda_k(0) = 1$ for all k (Lemma 1), so (10) and (11) and both extremal functions reduce to Theorem 2, and $\mu_0(0) = 2/3$. $\square$

Remark 2 (Why this is structural, not cosmetic). Were $a_3 - \mu a_2^2$ linear in the coefficients, composing with Ω_z^δ would just rescale the bound by a constant. It is quadratic in a_2 , and Ω_z^δ rescales a_2, a_3 by different weights $\lambda_2(\delta) \neq \lambda_3(\delta)$, so the extremal problem feels the effective parameter $\nu(\delta) = \mu\lambda_3(\delta)/\lambda_2(\delta)^2$, and the bifurcation point $\mu_0(\delta)$ genuinely moves (Figure 3), with the extremal function switching between the quadratic and cubic monomials of Theorem 3 at that value. This nonlinear effect has no counterpart in the purely linear univalence theory of Corollary 1.

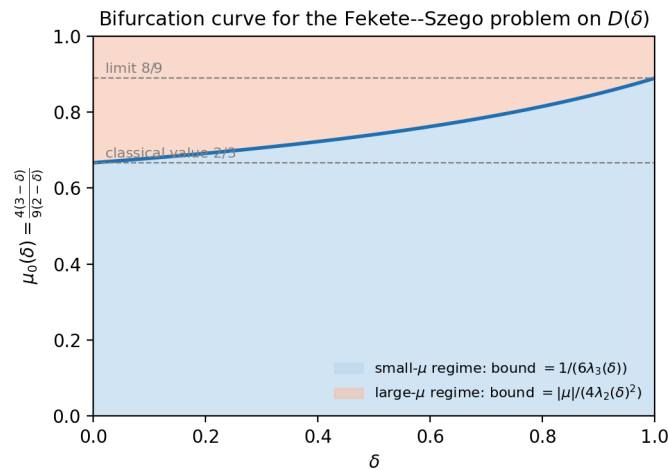


Figure 3. The bifurcation curve $\mu_0(\delta) = 4(3 - \delta) / [9(2 - \delta)]$ of Theorem 3, separating the two sharp Fekete–Szegő regimes (and their two distinct extremal functions). It moves monotonically from the classical value $2/3$ ($\delta = 0$) to the limit $8/9$ ($\delta \rightarrow 1^-$).

The second Hankel determinant bound of Theorem 3 also decreases monotonically with δ , as shown in Figure 4.

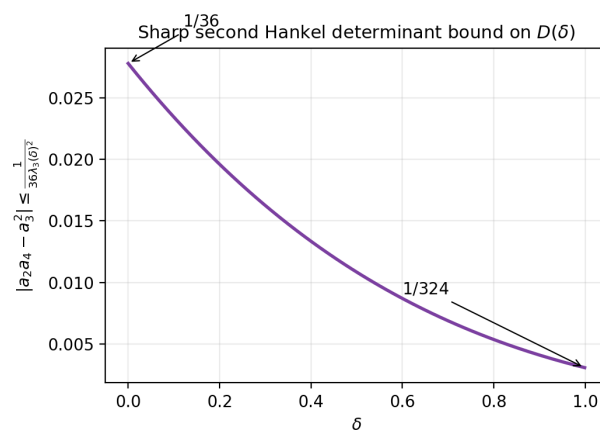


Figure 4. The sharp second Hankel determinant bound $1/(36\lambda_3(\delta)^2)$ of Theorem 3, decreasing monotonically from $1/36$ to $1/324$, attained throughout by $g(z) = z + \frac{(2-\delta)(3-\delta)}{36}z^3$.

Figure 5 visualizes the switch described in Remark 2 directly, plotting the sharp Fekete–Szegő bound as a function of μ and showing how its kink at $\mu_0(\delta)$ shifts as δ increases.

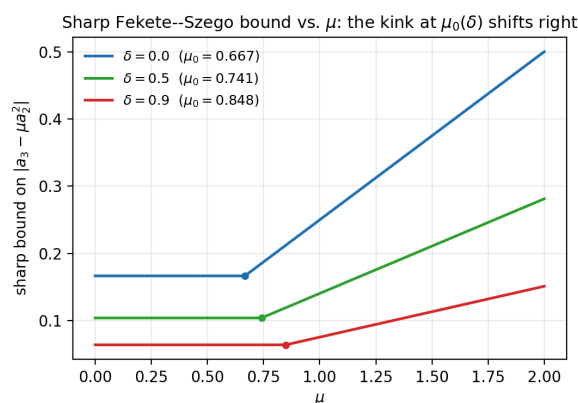


Figure 5. The sharp Fekete–Szegő bound of Theorem 3 as a function of μ , for $\delta = 0, 0.5, 0.9$. Each curve has a kink at its own $\mu_0(\delta)$ (marked); the kink shifts right as δ increases, visualizing Remark 2.

Table 1 collects, for reference, the sharp extremal function established above for each of the four coefficient problems treated in this paper.

Table 1. Summary of the sharp extremal functions established above, for $g \in D(\delta)$.

Extremal Problem	Regime	Sharp Extremal Function
$ a_3 - \mu a_2^2 $	$ \mu \geq \mu_0(\delta)$	$g(z) = z + \frac{2-\delta}{4} e^{i\theta} z^2$
$ a_3 - \mu a_2^2 $	$ \mu \leq \mu_0(\delta)$	$g(z) = z + \frac{(2-\delta)(3-\delta)}{36} e^{i\theta} z^3$
$ a_2 a_4 - a_3^2 $	(all δ)	$g(z) = z + \frac{(2-\delta)(3-\delta)}{36} e^{i\theta} z^3$
$ a_n $, general $n \geq 2$	(all δ)	$g(z) = z + \frac{\prod_{j=2}^n (j-\delta)}{n(n-1)n!} e^{i\theta} z^n$

7. Numerical Verification

We verify the results in two independent ways: first, the parametrization lemma itself (Lemma 3), by an explicit construction that does not assume the lemma; then the sharp bounds of Theorems 2 and 3, by Monte Carlo maximization using the lemma. All computations use IEEE double precision.

7.1. Verification of the Schur Parametrization (Lemma 3)

The lemma's proof is constructive: $\Omega(z) = (x + yz)/(1 + \bar{x}yz)$ is an explicit Schur function realizing the parameter y at the second recursive step, and

$$\varphi(z) = \frac{c_0 + z\Omega(z)}{1 + \bar{c}_0 z\Omega(z)} \quad (12)$$

is then an explicit rational Schur function for any (c_0, x, y) in the closed unit disk. To check (5) independently of the algebra used to derive it, we do not use the claimed formula: instead, for 400 independent random triples (c_0, x, y) (modulus uniform on $[0, 1]$, phase uniform on $[0, 2\pi)$, all six real degrees of freedom independent), we build φ from (12) and extract its Taylor coefficients c_1, c_2 purely numerically via the Cauchy integral formula,

$$c_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{\varphi(z)}{z^{n+1}} dz \approx \frac{1}{N} \sum_{j=0}^{N-1} \varphi(re^{i\theta_j}) e^{-in\theta_j} r^{-n}, \quad \theta_j = \frac{2\pi j}{N},$$

with $r = 0.5$ and $N = 4096$ (the trapezoidal rule on a circle is spectrally accurate for functions analytic in a neighborhood of the contour [28], so $N = 4096$ gives coefficients accurate to machine precision). We then compare these numerically extracted coefficients to the closed-form right-hand sides of (5). Over the 400 trials,

$$\max |c_{1,\text{numeric}} - c_{1,\text{formula}}| = 2.2 \times 10^{-16}, \quad \max |c_{2,\text{numeric}} - c_{2,\text{formula}}| = 3.3 \times 10^{-16},$$

i.e., agreement to machine precision, confirming (5) independently of its derivation. As a further sanity check, we evaluated $|\varphi(z)|$ at 50 random points per trial inside $\mathbb{U}$; the maximum observed value across all trials was $0.999948 < 1$, consistent with $\varphi \in \mathcal{B}$.

7.2. Monte Carlo Verification of the Sharp Bounds (Theorems 2 and 3)

Method. By Lemma 3, every admissible coefficient triple (a_2, a_3, a_4) of a function $g \in D(\delta)$ arises from some choice of (c_0, x, y) in the closed unit disk via (9), and every such choice is admissible. Hence,

$$\sup_{g \in D(\delta)} |a_3 - \mu a_2^2| = \sup_{|c_0|, |x| \leq 1} \left| \frac{(1 - |c_0|^2)x}{6\lambda_3(\delta)} - \mu \frac{c_0^2}{4\lambda_2(\delta)^2} \right|,$$

and similarly for the Hankel functional (which also involves y). We estimate the right-hand side by drawing $N = 2 \times 10^6$ independent samples of (c_0, x, y) —each with modulus drawn uniformly from $[0, 1]$ and phase drawn uniformly from $[0, 2\pi)$, all mutually independent—computing (a_2, a_3, a_4) for each sample via (9), evaluating the target functional, and recording the sample maximum. Because every sample gives a value that is genuinely attained by some $g \in D(\delta)$, the empirical maximum is always a valid lower bound on the true supremum, so it can never legitimately exceed the proved upper bound of Theorems 2 and 3; observing close agreement is therefore a strong, independent, algebra-error-catching check on both the upper-bound proof and the claimed location of the extremal point. (We do not sample moduli in an area-corrected way, e.g., via $t = \sqrt{U}$; since the goal is to search the parameter box for a maximum rather than to model a physical uniform distribution on the disk, a plain uniform grid in (t, θ) is an adequate and slightly simpler search measure, and in fact oversamples small and moderate t relative to area-uniform sampling, which if anything makes it a more conservative check of the interior of the domain.).

Results. Table 2 shows representative cases; the empirical maximum matches the closed form to at least four decimal places in every case, and never exceeds it, consistent with the closed form being both an upper bound and (up to Monte Carlo sampling resolution) sharp.

Table 2. Monte Carlo verification of Theorem 3 ($N = 2 \times 10^6$ samples per row).

δ	μ	Empirical $ a_3 - \mu a_2^2 _{\max}$	Closed Form	Empirical $ a_2 a_4 - a_3^2 _{\max}$	Closed Form
0.0	0.000	0.16666	0.16667	0.027775	0.027778
0.0	0.667	0.16667	0.16667	—	—
0.5	0.000	0.10416	0.10417	0.010850	0.010851
0.5	0.741	0.10417	0.10417	—	—
0.9	0.000	0.06416	0.06417	0.004117	0.004117
0.9	0.848	0.06417	0.06417	—	—

Remark 3. This numerical procedure demonstrates consistency of the closed forms with direct sampling; it is not itself a proof of sharpness. Sharpness is established analytically in Theorems 2 and 3 by exhibiting the exact extremal function that attains equality; the Monte Carlo experiment is an independent check that (a) no sampled configuration violates the proved upper bound, and (b) the empirical maximum approaches the bound as closely as sampling resolution allows, which is exactly the behavior expected near a true, attained supremum.

8. Discussion

8.1. Applications of Bounded-Derivative Univalence

Bounded-derivative univalence criteria of the type studied here are not purely internal to geometric function theory. Al-Refai [29] used higher-order derivative-constrained univalent conformal maps in complex potential flow simulation, where a bound on a higher derivative of the mapping function directly controls the numerical stiffness and stability of the resulting flow computation. The results of this paper—sharp, explicit extremal functions for the second-derivative-bounded class $D(\delta)$ —give closed-form, verifiable coefficient envelopes that could serve as design constraints in that kind of application, in place of a numerically searched admissible region.

8.2. Open Problem: Higher-Order Bounded-Derivative Analog

Open Problem 1. Determine the sharp Fekete–Szegő-type and Hankel-determinant-type bounds for the fractional-derivative extension of the higher-order class $\{g \in \mathcal{A} : |g'''(z)| \leq \beta(a_2)\}$ (bounded third derivative, with the right-hand side allowed to depend on a_2), with explicit extremal functions.

This is structurally different from the problem solved here: once the bound on g''' is allowed to depend on a_2 , the coefficient a_2 becomes a free disk-valued parameter decoupled from the Schur triple governing a_3, a_4, a_5 , so the natural extremal problem (e.g., $a_4 - \mu a_3^2$) requires a joint optimization over a free parameter and an independent Schur triple, rather than the single-Schur-triple optimization used in Theorems 2 and 3. We have not resolved this and state it as an open direction.

8.3. Speculative Outlook: Coefficient Bounds as Engineering Diagnostics

We flag one further, purely speculative direction, distinct from the open problem above. Several applied studies fit smooth empirical response curves to automotive thermal and emissions data—for instance, battery/powertrain thermal response under standard driving cycles [30] and emissions-based predictive-maintenance curves for hybrid vehicles [31]. Such a fitted curve, suitably normalized and re-expressed as a power series in a dimensionless variable (e.g., normalized time or state of charge), would in principle have its own coefficients $a_2, a_3, a_4, \dots$, and one could ask whether, under a physically motivated normalization, that response satisfies a bound of the type $|g''| \leq \beta$ or its fractional analog studied here—in which case the sharp Fekete–Szegő and Hankel bounds of Theorems 2 and 3 would translate into a priori envelopes on the curve’s second- and third-order coefficients, potentially readable as indices of nonlinear sensitivity and of higher-order structural consistency, respectively. We want to be explicit that this is only a possible avenue for future work, not a claim made or established in the present paper: neither ref. [30] nor ref. [31] uses bounded-derivative or univalent-function techniques, and nothing here shows that the response curves they study actually satisfy a bound of this form. Confirming or refuting that would require deriving such a bound from the underlying physical model (e.g., a thermal or electrochemical governing equation) or from the specific curve-fitting procedure used, rather than inferring it from the formal resemblance between the two settings; we leave this to future work.

8.4. Limitations

Theorems 2 and 3 concern D and $D(\delta)$ as defined by a bound on the second derivative only; we do not claim results for higher-order bounded-derivative classes. As emphasized in Section 2, $D(\delta)$ for $\delta > 0$ constrains $G = \Omega_z^\delta g$, and Theorem 3 describes the Taylor coefficients of g recovered through that constraint, not a condition imposed on g directly.

9. Conclusions

We introduced the bounded-second-derivative class $D = \{g \in \mathcal{A} : |g''(z)| \leq 1\}$, gave a short self-contained proof of its univalence via a direct line-integral estimate and the classical Noshiro–Warschawski theorem, and established a sharp bound on a single coefficient $|a_n|$ for every $n \geq 2$ via an elementary Parseval argument (Theorem 1). We then solved the Fekete–Szegő and second Hankel determinant problems for D in closed form with unique extremal functions (Theorem 2), using the exact Schur parametrization of bounded analytic functions. We then extended both results to the fractional-derivative family $D(\delta)$ built from the Owa–Srivastava operator (Theorem 3), and showed that the extremal problem’s bifurcation point moves monotonically and non-trivially with δ —a

structural consequence of composing a nonlinear coefficient functional with a fractional operator that rescales coefficients non-uniformly. All results were verified using two independent numerical approaches: direct construction and Cauchy-integral coefficient extraction for the underlying Schur lemma, and Monte Carlo maximization for the sharp bounds themselves.

Author Contributions: Conceptualization, O.A.-R.; methodology, O.A.-R. and S.J.A.A.; formal analysis, O.A.-R., S.J.A.A. and M.A.; investigation, O.A.-R., S.J.A.A., M.A. and A.A.; validation (numerical verification), S.J.A.A.; writing—original draft preparation, O.A.-R.; writing—review and editing, O.A.-R., S.J.A.A., M.A. and A.A.; visualization, S.J.A.A.; supervision, O.A.-R. and A.A.; project administration, O.A.-R. and A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The numerical verification data generated in this study (Monte Carlo maximization outputs underlying Table 2, and the Cauchy-integral coefficient-extraction results of Section 7.1), together with the figure-generation scripts, are available from the corresponding author upon reasonable request.

Acknowledgments: We thank the referees for their time and for valuable comments that improved the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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