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Memory-Induced Synchronization in a Time-Fractional Partly Diffusive Coupled Hindmarsh–Rose Network with Nonlinear Diffusion

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Abstract

We present and analyze a time-fractional, partly diffusive model of two electrically coupled Hindmarsh–Rose neurons in which only the membrane potentials undergo spatial transport, through a nonlinear Neumann m -Laplacian, while the temporal evolution is governed by a Caputo derivative of order $\rho \in (0, 1]$. The fractional operator incorporates hereditary relaxation, whereas the m -Laplacian represents gradient-dependent degenerate transport and recovers ordinary diffusion when $m = 2$. In a Gelfand triple adapted to the no-flux boundary condition, we derive a fractional energy inequality, a uniform dissipative estimate, and the existence of a global weak solution by a Faedo–Galerkin approximation, fractional compactness, and Minty’s method. Uniqueness and continuous dependence are obtained in the stated bounded solution class. We prove global Mittag–Leffler synchronization above an explicit coupling threshold and establish a practical synchronization bound under parameter mismatch. A fully implicit L1 finite-volume method is then constructed; every time step is solvable, uniqueness follows under an explicit monotonicity condition, and the scheme is unconditionally energy dissipative and convergent. Manufactured-solution tests recover the expected $2 - \rho$ temporal and second-order spatial rates. In the neuronal simulations, reducing ρ from 1 to 0.90 lengthens the mean bursting period from about 379 to 565 time units, an increase of roughly one half, and raises the number of spikes per burst from about 31.7 to 38.8. Over $2 \leq m \leq 4$ the temporal rhythm is essentially unchanged, the burst period staying near 362 time units, while the diffusion exponent reshapes the peak amplitude and the spatial gradient profiles of the traveling fronts. The empirical synchronization threshold for the canonical parameter set is approximately 11.0, far below the global sufficient bound 2.3917×10^4 , which quantifies the conservatism of the analytical certificates.



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1. Introduction

Mathematical descriptions of neuronal excitability range from the biophysically detailed Hodgkin–Huxley conductance equations to reduced systems designed to retain the essential geometry of excitation, recovery, and refractoriness [1–3]. Among the reduced models, the three-dimensional Hindmarsh–Rose (HR) system occupies a distinctive position: it is sufficiently compact for analytical and computational study but still reproduces tonic spiking, periodic bursting, spike adding, and chaotic spike–burst transitions [4–7]. Its fast membrane-potential and recovery variables, together with a slow adaptation variable, provide a standard slow–fast framework for investigating neuronal rhythms.

The local dynamics of the HR equations have consequently been examined through singular perturbation, bifurcation, and chaos-theoretic methods. Terman related bursting and chaotic spikes to global phase-space structures [8]; Innocenti and Genesio analyzed the transitions between chaotic spiking and bursting [9]; and delayed HR systems were shown to exhibit additional Hopf and double-Hopf mechanisms [10]. These studies demonstrate that small changes in applied current, timescale separation, or feedback may reorganize the firing pattern, which makes the HR model especially useful for testing synchronization and control mechanisms.

Synchronization is central to information processing and coordinated rhythmic activity in neuronal ensembles. General mathematical mechanisms of phase and complete synchronization are reviewed in [11,12], while transitions to synchrony in coupled bursting neurons and network-dependent synchronization of HR units have been studied in [13,14]. Fractional-order extensions of these synchronization mechanisms have also been analyzed at the network level: global Mittag–Leffler stability and synchronization were established for impulsive fractional-order neural networks with time-varying delays in [15], and interlayer and intralayer synchronization in multiplex fractional-order neuronal networks was analyzed in [16], showing numerically and analytically that the fractional order itself modifies the coupling strength required for synchrony. Most of this literature is finite dimensional: each neuron is treated as a spatially lumped oscillator, and electrical or chemical coupling acts between finitely many membrane-potential variables. Such models are appropriate for many network questions but do not resolve within-neuron spatial propagation or heterogeneous voltage profiles.

A spatially distributed HR formulation was introduced by Phan and You, who allowed only the membrane-potential components to diffuse and retained the fast and slow recovery variables as local fields. For linearly diffusive, integer-order equations they proved global absorbing behavior and uniform exponential synchronization of two coupled neurons [17]; a boundary-coupled extension to larger networks was later analyzed in [18]. This partly diffusive architecture is biologically and mathematically attractive: it preserves the ionic-variable structure of the HR model, while introducing spatial voltage transport through a parabolic operator. Its existing theory, however, relies on the ordinary time derivative, linear Laplacian diffusion, and the classical semiflow framework.

Fractional differentiation offers a different description of temporal relaxation. The Caputo derivative preserves standard initial data and encodes a power-law memory kernel [19–23]. Fractional HR models have revealed order-dependent firing, bifurcation, and synchronization behavior: numerical studies have identified changes in discharge patterns and synchronization rates [24,25], Kaslik developed stability and Hopf-type analysis for two- and three-dimensional fractional HR systems [26], and noise-robust synchronization of coupled fractional neurons was considered in [27]. More recently, hybrid projective synchronization of an extended fractional-order HR model with hidden attractors was reported in [28]; discrete fractional-difference formulations have produced rich complexity and multi-stability results, including a discrete fractional-difference implementation of

the HR neuron [29] and a discrete memristive HR neural model with fractional-order differences that exhibits order-dependent chaotic firing and coupling-dependent synchronization outcomes [30]; detailed fractional-order firing-pattern and excitability-transition computations were obtained through a hybrid neural-computational paradigm in [31]; and the stability and Hopf-bifurcation structure of a delayed fractional HR model was analyzed in [32]. These contributions establish fractional order as a meaningful neuronal parameter, but they are predominantly finite-dimensional, discrete, or delayed ODE and map models and do not supply a weak-solution theory for a spatially distributed, nonlinear-diffusion HR system.

Nonlinear diffusion is likewise relevant when the effective flux depends on the voltage gradient. The m -Laplacian is a canonical monotone model for degenerate or nonlinear transport; its variational theory is developed in [33–36]. In contrast to a linear Laplacian, it requires a nonlinear operator formulation in $W^{1,m}(\Omega)$ and may change the localization and steepness of spatial profiles. Caputo time derivatives and m -Laplacian diffusion have recently been combined in other reaction–diffusion settings, including controlled epidemic systems [37], but this combination has not been developed for partly diffusive coupled HR equations together with a rigorous synchronization and numerical theory.

The required analytical and computational tools also differ from their classical counterparts. Fractional energy inequalities and weak time-regularity estimates replace ordinary differential identities [38,39]; compactness must combine spatial coercivity with fractional temporal control [40–42]; and nonlinear diffusion must be identified through monotonicity rather than linear semigroup convergence. Numerically, the L1 formula is a standard approximation of the Caputo derivative [43–46], while conservative finite-volume methods provide a natural framework for no-flux nonlinear diffusion [47]. A reliable HR computation must preserve these structures while also resolving stiff cubic reactions, slow bursting, and synchronization errors over long time intervals.

The novelty of the present work does not lie in any one of its ingredients considered separately. It lies in resolving, within one mathematically closed framework, the interaction between Caputo memory, a partly diffusive HR architecture, nonlinear degenerate voltage transport, electrical synchronization, and a structure-preserving discretization. Existing fractional HR studies are predominantly finite dimensional and therefore do not resolve spatial voltage propagation, whereas the available partly diffusive HR PDE theory is integer order and linearly diffusive. The present analysis consequently requires fractional energy estimates, fractional compactness, monotone-operator identification of the nonlinear flux, and a synchronization argument compatible with a nonlocal-in-time PDE. We emphasize that this contribution is a new analytical and computational framework for the stated fractional, partly diffusive nonlinear PDE model, rather than a claim of a new ionic current or microscopic neuronal mechanism.

The preceding review identifies four specific gaps. First, there is no global weak-solution and dissipativity theory for a Caputo time-fractional, partly m -Laplacian diffusive HR pair with nonlinear Neumann flux. Second, the available fractional HR synchronization studies are finite dimensional, whereas the existing diffusive HR synchronization theory is integer order and linearly diffusive. Third, practical synchronization under neuronal mismatch has not been integrated with this fractional nonlinear-diffusion framework. Fourth, there is no structure-preserving L1 finite-volume analysis that connects solvability, discrete energy dissipation, convergence, and quantitatively verified spike–burst and synchronization diagnostics for this model.

The principal contributions of this work are therefore the following:

- (C1) We formulate a time-fractional, partly diffusive coupled HR system with a nonlinear Neumann m -Laplacian acting only on the membrane potentials. A grouped quartic

estimate yields a uniform fractional energy bound for the full parameter range, and a Faedo–Galerkin construction, fractional compactness, and Minty’s method give a global weak solution. Uniqueness and continuous dependence are proved in the explicitly stated bounded solution classes.

- (C2) We derive a global Mittag–Leffler synchronization estimate above an explicit coupling threshold, we prove the invariance of the identical-neuron manifold, and we obtain a practical synchronization bound under additive parameter or forcing mismatch. The classical exponential result is recovered when $\rho = 1$ and $m = 2$.
- (C3) We construct a fully implicit L1 finite-volume discretization with a regularized monotone m -flux. We prove time-level solvability, conditional uniqueness through an explicit monotonicity criterion, unconditional discrete energy dissipation, a discrete synchronization inequality, and convergence to a weak solution.
- (C4) We provide numerical verification beyond surface visualization. The manufactured solution confirms the temporal and spatial rates, independent solvers verify the integer-order and linear-diffusion limits, and the parameter studies quantify memory-dependent firing, the weaker but non-negligible influence of m on burst morphology and spatial gradients, threshold conservatism, practical synchronization under mismatch, and the computational cost of full memory.

The computational novelty is not an additional surface visualization but a quantitative separation of mechanisms. The fractional order reorganizes the temporal bursting scale, whereas the nonlinear diffusion exponent primarily modifies the spatial transition zones and gradient distribution. Coupling scans then connect these effects to a finite-window synchronization transition, while the analytical theorem supplies a global sufficient certificate. The empirical transition and the global certificate are reported as distinct objects throughout the manuscript.

The paper is organized as follows. Section 2 recalls the fractional, monotone-operator, and compactness tools. Section 3 introduces the model and weak formulation. Sections 4–6 establish dissipativity, existence, and uniqueness. Exact and practical synchronization are treated in Sections 7 and 8; Section 9 records the higher-regularity result for linear diffusion, and Section 10 collects limiting cases. Section 11 develops and analyzes the numerical method and then reports the computational results. Section 12 summarizes the contributions and indicates directions for further work.

2. Notation and Analytical Preliminaries

Let $T > 0$, let $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 3$ be a bounded connected domain with Lipschitz boundary $\partial\Omega$, and set

$$Q_T := \Omega \times (0, T), \quad \Sigma_T := \partial\Omega \times (0, T).$$

The outward unit normal to $\partial\Omega$ is denoted by ν . For a Banach space X , the norm in X is written $\|\cdot\|_X$, while $\langle \cdot, \cdot \rangle_{X^*, X}$ denotes the duality pairing between X^* and X . The $L^2(\Omega)$ inner product and norm are abbreviated by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Constants denoted by C may change from line to line; their dependence is indicated when relevant.

Throughout the paper,

$$0 < \rho \leq 1, \quad m \geq 2,$$

where ρ is the order of the Caputo derivative and m is the exponent of the nonlinear diffusion. The symbols α_0 and β are reserved for HR reaction parameters, and $\kappa > 0$ denotes the electrical coupling strength.

2.1. Caputo Calculus

We use the standard Caputo and Riemann–Liouville conventions from [19–22]; properties of the Mittag–Leffler function are taken from [23].

Definition 1 (Left-sided Caputo derivative). *Let X be a Banach space. For $z \in W^{1,1}(0, T; X)$ and $0 < \rho < 1$, the Caputo derivative of order ρ is*

$${}^C\mathcal{D}_\xi^\rho z(\xi) := \frac{1}{\Gamma(1-\rho)} \int_0^\xi (\xi-s)^{-\rho} z'(s) \, ds, \quad 0 < \xi \leq T. \quad (1)$$

For $\rho = 1$, we set ${}^C\mathcal{D}_\xi^\rho z = z'$.

Definition 2 (Riemann–Liouville fractional integral). *For $\rho > 0$ and $f \in L^1(0, T; X)$, define*

$$(\mathcal{I}_{0+}^\rho f)(\xi) := \frac{1}{\Gamma(\rho)} \int_0^\xi (\xi-s)^{\rho-1} f(s) \, ds. \quad (2)$$

If $z \in W^{1,1}(0, T; X)$ then

$$\mathcal{I}_{0+}^\rho {}^C\mathcal{D}_\xi^\rho z(\xi) = z(\xi) - z(0). \quad (3)$$

The one-parameter Mittag–Leffler function is

$$E_\rho(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + 1)}, \quad z \in \mathbb{C}. \quad (4)$$

For $0 < \rho \leq 1$ and $\lambda > 0$, the function $\xi \mapsto E_\rho(-\lambda \xi^\rho)$ is positive, non-increasing, and converges to zero as $\xi \rightarrow \infty$.

The following chain-rule inequality is the Hilbert-space form of the Alikhanov energy estimate; see [38,39].

Lemma 1 (Fractional energy inequality). *Let X be a real Hilbert space and suppose that $z \in L^2(0, T; X)$, ${}^C\mathcal{D}_\xi^\rho z \in L^2(0, T; X)$, and z has a continuous representative in X . Then, for almost every $\xi \in (0, T)$,*

$$\frac{1}{2} {}^C\mathcal{D}_\xi^\rho \|z(\xi)\|_X^2 \leq ({}^C\mathcal{D}_\xi^\rho z(\xi), z(\xi))_X.$$

More generally, if $\Psi : X \rightarrow (-\infty, \infty]$ is proper, convex, and lower semicontinuous and $\varsigma(\xi) \in \partial\Psi(z(\xi))$ then

$${}^C\mathcal{D}_\xi^\rho \Psi(z(\xi)) \leq \langle \varsigma(\xi), {}^C\mathcal{D}_\xi^\rho z(\xi) \rangle$$

whenever both sides are well defined.

Proof. For smooth z , direct symmetrization of the Caputo integral gives

$$\begin{aligned} 2({}^C\mathcal{D}_\xi^\rho z(\xi), z(\xi))_X - {}^C\mathcal{D}_\xi^\rho \|z(\xi)\|_X^2 &= \frac{\xi^{-\rho}}{\Gamma(1-\rho)} \|z(\xi) - z(0)\|_X^2 \\ &\quad + \frac{\rho}{\Gamma(1-\rho)} \int_0^\xi (\xi-s)^{-\rho-1} \|z(\xi) - z(s)\|_X^2 \, ds, \end{aligned}$$

which is non-negative. Approximation in the natural graph norm yields the first assertion. The convex version follows from the subgradient inequality $\Psi(z(\xi)) - \Psi(z(s)) \leq \langle \varsigma(\xi), z(\xi) - z(s) \rangle$ inserted into the same Caputo representation. $\square$

The scalar comparison and Gronwall estimates below are standard consequences of the positive Mittag–Leffler resolvent; see [21–23].

Lemma 2 (Fractional comparison principle). *Let $y \in C([0, T])$ be non-negative and suppose*

$${}^C\mathfrak{D}_{\xi}^{\rho}y(\xi) + \lambda y(\xi) \leq h(\xi) \quad \text{for a.e. } \xi \in (0, T),$$

where $\lambda > 0$ and $h \in L^1(0, T)$ is non-negative. Then

$$y(\xi) \leq y(0)E_{\rho}(-\lambda\xi^{\rho}) + \int_0^{\xi} (\xi - s)^{\rho-1} E_{\rho,\rho}(-\lambda(\xi - s)^{\rho}) h(s) \, ds.$$

In particular, if $h(\xi) \leq H$ then

$$y(\xi) \leq y(0)E_{\rho}(-\lambda\xi^{\rho}) + \frac{H}{\lambda}(1 - E_{\rho}(-\lambda\xi^{\rho})) \leq \max\left\{y(0), \frac{H}{\lambda}\right\}.$$

Proof. Apply the positive resolvent kernel of the scalar Caputo equation ${}^C\mathfrak{D}_{\xi}^{\rho}z + \lambda z = h$ and use the order-preserving property of I_{0+}^{ρ} . For constant $h = H$, the convolution identity

$$\int_0^{\xi} (\xi - s)^{\rho-1} E_{\rho,\rho}(-\lambda(\xi - s)^{\rho}) \, ds = \frac{1 - E_{\rho}(-\lambda\xi^{\rho})}{\lambda}$$

gives the stated estimate. $\square$

Lemma 3 (Fractional Gronwall inequality). *If $y \geq 0$ satisfies*

$$y(\xi) \leq y_0 + \frac{L}{\Gamma(\rho)} \int_0^{\xi} (\xi - s)^{\rho-1} y(s) \, ds$$

then

$$y(\xi) \leq y_0 E_{\rho}(L\xi^{\rho}), \quad 0 \leq \xi \leq T.$$

2.2. The Nonlinear Neumann Diffusion Operator

We employ the variational realization of the Neumann m -Laplacian and its standard monotonicity properties; see [33–36].

Let

$$V_m := W^{1,m}(\Omega), \quad H := L^2(\Omega),$$

and identify H with its dual so that

$$V_m \hookrightarrow H \equiv H^* \hookrightarrow V_m^*$$

is a Gelfand triple. Define $\mathcal{A}_m : V_m \rightarrow V_m^*$ by

$$\langle \mathcal{A}_m \phi, \theta \rangle_{V_m^*, V_m} := \int_{\Omega} |\nabla \phi|^{m-2} \nabla \phi \cdot \nabla \theta \, dx.$$

This is the weak realization of $-\Delta_m \phi = -\operatorname{div}(|\nabla \phi|^{m-2} \nabla \phi)$ with homogeneous nonlinear Neumann flux.

Lemma 4 (Properties of $\mathcal{A}_m$). *The operator $\mathcal{A}_m$ is bounded on bounded subsets of V_m , hemicon-
tinuous, monotone, and coercive in the gradient seminorm. More precisely,*

$$\begin{aligned} \|\mathcal{A}_m \phi\|_{V_m^*} &\leq \|\nabla \phi\|_{L^m}^{m-1}, \\ \langle \mathcal{A}_m \phi - \mathcal{A}_m \tilde{\phi}, \phi - \tilde{\phi} \rangle &\geq c_m \|\nabla \phi - \nabla \tilde{\phi}\|_{L^m}^m, \\ \langle \mathcal{A}_m \phi, \phi \rangle &= \|\nabla \phi\|_{L^m}^m, \end{aligned}$$

where one may take $c_m = 2^{2-m}$ for $m \geq 2$.

Proof. Throughout, write $m' = m/(m-1)$ for the conjugate exponent, so that $\frac{1}{m} + \frac{1}{m'} = 1$ and $(m-1)m' = m$. For $\mathbf{a} \in \mathbb{R}^n$ set $\mathbf{V}(\mathbf{a}) := |\mathbf{a}|^{m-2}\mathbf{a}$, the integrand vector field of $\mathcal{A}_m$, with the convention $\mathbf{V}(0) = 0$; since $m \geq 2$ the map $\mathbf{a} \mapsto \mathbf{V}(\mathbf{a})$ is continuous on $\mathbb{R}^n$.

Boundedness. Let $\phi \in V_m$ and $\theta \in V_m$. Because $|\mathbf{V}(\nabla\phi)| = |\nabla\phi|^{m-1}$ pointwise, Hölder's inequality with exponents m' and m gives

$$|\langle \mathcal{A}_m\phi, \theta \rangle| = \left| \int_{\Omega} |\nabla\phi|^{m-2} \nabla\phi \cdot \nabla\theta \, dx \right| \leq \int_{\Omega} |\nabla\phi|^{m-1} |\nabla\theta| \, dx \leq \| |\nabla\phi|^{m-1} \|_{L^{m'}} \| \nabla\theta \|_{L^m}.$$

Since $(m-1)m' = m$,

$$\| |\nabla\phi|^{m-1} \|_{L^{m'}} = \left(\int_{\Omega} |\nabla\phi|^{(m-1)m'} \, dx \right)^{1/m'} = \left(\int_{\Omega} |\nabla\phi|^m \, dx \right)^{(m-1)/m} = \| \nabla\phi \|_{L^m}^{m-1},$$

so that $|\langle \mathcal{A}_m\phi, \theta \rangle| \leq \| \nabla\phi \|_{L^m}^{m-1} \| \nabla\theta \|_{L^m} \leq \| \nabla\phi \|_{L^m}^{m-1} \| \theta \|_{V_m}$. Taking the supremum over $\theta \in V_m$ with $\| \theta \|_{V_m} \leq 1$ yields

$$\| \mathcal{A}_m\phi \|_{V_m^*} \leq \| \nabla\phi \|_{L^m}^{m-1}.$$

In particular $\mathcal{A}_m$ maps V_m into V_m^* and is bounded on bounded sets: if $\| \phi \|_{V_m} \leq R$ then $\| \nabla\phi \|_{L^m} \leq R$ and hence $\| \mathcal{A}_m\phi \|_{V_m^*} \leq R^{m-1}$.

The coercivity identity. Choosing $\theta = \phi$ and using $|\nabla\phi|^{m-2} \nabla\phi \cdot \nabla\phi = |\nabla\phi|^{m-2} |\nabla\phi|^2 = |\nabla\phi|^m$ pointwise,

$$\langle \mathcal{A}_m\phi, \phi \rangle = \int_{\Omega} |\nabla\phi|^{m-2} \nabla\phi \cdot \nabla\phi \, dx = \int_{\Omega} |\nabla\phi|^m \, dx = \| \nabla\phi \|_{L^m}^m.$$

Hemicontinuity. Fix $\phi, \theta, \zeta \in V_m$ and let $s_k \rightarrow s$ in $\mathbb{R}$. The integrands

$$f_k(x) := |\nabla\phi + s_k \nabla\theta|^{m-2} (\nabla\phi + s_k \nabla\theta) \cdot \nabla\zeta$$

converge for a.e. $x \in \Omega$ to $f(x) := |\nabla\phi + s \nabla\theta|^{m-2} (\nabla\phi + s \nabla\theta) \cdot \nabla\zeta$, by continuity of $\mathbf{V}$. If $|s_k| \leq R$ for all k then

$$|f_k(x)| \leq |\nabla\phi + s_k \nabla\theta|^{m-1} |\nabla\zeta| \leq (|\nabla\phi| + R |\nabla\theta|)^{m-1} |\nabla\zeta| =: g(x),$$

and Hölder's inequality gives, exactly as above, $\int_{\Omega} g \, dx \leq \| |\nabla\phi| + R |\nabla\theta| \|_{L^m}^{m-1} \| \nabla\zeta \|_{L^m} < \infty$, so $g \in L^1(\Omega)$. Dominated convergence yields $\langle \mathcal{A}_m(\phi + s_k \theta), \zeta \rangle \rightarrow \langle \mathcal{A}_m(\phi + s \theta), \zeta \rangle$. Hence $s \mapsto \langle \mathcal{A}_m(\phi + s \theta), \zeta \rangle$ is continuous on $\mathbb{R}$ for every triple, which is the hemicontinuity of $\mathcal{A}_m$.

Monotonicity. The estimate reduces, after integration, to the pointwise vector inequality

$$(|\mathbf{a}|^{m-2}\mathbf{a} - |\mathbf{b}|^{m-2}\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \geq 2^{2-m} |\mathbf{a} - \mathbf{b}|^m, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \, m \geq 2. \quad (5)$$

Expanding the inner product gives the exact identity

$$(\mathbf{V}(\mathbf{a}) - \mathbf{V}(\mathbf{b})) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^m + |\mathbf{b}|^m - (|\mathbf{a}|^{m-2} + |\mathbf{b}|^{m-2}) \mathbf{a} \cdot \mathbf{b}. \quad (6)$$

We first prove the intermediate inequality

$$(\mathbf{V}(\mathbf{a}) - \mathbf{V}(\mathbf{b})) \cdot (\mathbf{a} - \mathbf{b}) \geq \left(\frac{|\mathbf{a}| + |\mathbf{b}|}{2} \right)^{m-2} |\mathbf{a} - \mathbf{b}|^2. \quad (7)$$

Write $s = |\mathbf{a}|$, $u = |\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b} = suC$ with $C = \cos \gamma \in [-1, 1]$ the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$; if either vector vanishes take $C = 0$. Then (6) reads $s^m + u^m - (s^{m-2} + u^{m-2})suC$, while $|\mathbf{a} - \mathbf{b}|^2 = s^2 + u^2 - 2suC$, so (7) is equivalent to $\Psi(C) \geq 0$, where

$$\Psi(C) = \left[s^m + u^m - \left(\frac{s+u}{2} \right)^{m-2} (s^2 + u^2) \right] + C su \left[2 \left(\frac{s+u}{2} \right)^{m-2} - (s^{m-2} + u^{m-2}) \right].$$

The function Ψ is affine in C , and hence it attains its minimum on $[-1, 1]$ at an endpoint, and it suffices to check $\Psi(\pm 1) \geq 0$. A direct computation gives

$$\begin{aligned} \Psi(1) &= (s - u) \left[(s^{m-1} - u^{m-1}) - \left(\frac{s+u}{2} \right)^{m-2} (s - u) \right], \\ \Psi(-1) &= (s + u) \left[s^{m-1} + u^{m-1} - 2 \left(\frac{s+u}{2} \right)^{m-1} \right]. \end{aligned}$$

The map $t \mapsto t^{m-1}$ is convex on $[0, \infty)$ because $m \geq 2$. Convexity gives at once $s^{m-1} + u^{m-1} \geq 2 \left(\frac{s+u}{2} \right)^{m-1}$, so the bracket in $\Psi(-1)$ is non-negative and $\Psi(-1) \geq 0$. For $\Psi(1)$, assume without loss of generality that $s \geq u$. If $s = u$ then $\Psi(1) = 0$. Suppose that $s > u$ and set $t = \frac{u}{s} \in [0, 1)$. Since $m - 1 \geq 1$, we have $t^{m-1} \leq t$. Consequently, $\frac{1-t^{m-1}}{1-t} \geq 1$. It follows that

$$\frac{s^{m-1} - u^{m-1}}{s - u} = s^{m-2} \frac{1 - t^{m-1}}{1 - t} \geq s^{m-2}.$$

Moreover, since $s \geq u$ and $m - 2 \geq 0$,

$$s^{m-2} \geq \left(\frac{s+u}{2} \right)^{m-2}.$$

Therefore,

$$\frac{s^{m-1} - u^{m-1}}{s - u} \geq \left(\frac{s+u}{2} \right)^{m-2},$$

and hence

$$\begin{aligned} \Psi(1) &= (s - u) \left[(s^{m-1} - u^{m-1}) - \left(\frac{s+u}{2} \right)^{m-2} (s - u) \right] \\ &\geq 0. \end{aligned}$$

Together with $\Psi(-1) \geq 0$, this yields $\Psi(C) \geq 0$ for every $C \in [-1, 1]$, and therefore

$$(\mathbf{V}(\mathbf{a}) - \mathbf{V}(\mathbf{b})) \cdot (\mathbf{a} - \mathbf{b}) \geq \left(\frac{|\mathbf{a}| + |\mathbf{b}|}{2} \right)^{m-2} |\mathbf{a} - \mathbf{b}|^2.$$

By the triangle inequality,

$$|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} - \mathbf{b}|.$$

Since $m - 2 \geq 0$, we have

$$\left(\frac{|\mathbf{a}| + |\mathbf{b}|}{2} \right)^{m-2} \geq 2^{2-m} |\mathbf{a} - \mathbf{b}|^{m-2}.$$

Hence,

$$(\mathbf{V}(\mathbf{a}) - \mathbf{V}(\mathbf{b})) \cdot (\mathbf{a} - \mathbf{b}) \geq 2^{2-m} |\mathbf{a} - \mathbf{b}|^m,$$

which proves (5).

Applying (5) pointwise with $\mathbf{a} = \nabla\phi(x)$ and $\mathbf{b} = \nabla\tilde{\phi}(x)$, and integrating over Ω , we obtain

$$\begin{aligned} \langle \mathcal{A}_m\phi - \mathcal{A}_m\tilde{\phi}, \phi - \tilde{\phi} \rangle_{V_m^*, V_m} &= \int_{\Omega} \left(|\nabla\phi|^{m-2} \nabla\phi - |\nabla\tilde{\phi}|^{m-2} \nabla\tilde{\phi} \right) \\ &\quad \cdot (\nabla\phi - \nabla\tilde{\phi}) \, dx \\ &\geq 2^{2-m} \int_{\Omega} |\nabla\phi - \nabla\tilde{\phi}|^m \, dx \\ &= 2^{2-m} \|\nabla\phi - \nabla\tilde{\phi}\|_{L^m(\Omega)}^m. \end{aligned}$$

Thus, $\mathcal{A}_m$ is strongly monotone with

$$c_m = 2^{2-m}.$$

Taking $\mathbf{b} = -\mathbf{a}$ shows that equality holds in the pointwise inequality, and hence the constant 2^{2-m} is sharp.

Since $\mathcal{A}_m$ is monotone, hemicontinuous, and defined on all of the reflexive Banach space V_m , it is maximal monotone as an operator from V_m into V_m^* . Furthermore,

$$\langle \mathcal{A}_m\phi, \phi \rangle_{V_m^*, V_m} = \|\nabla\phi\|_{L^m(\Omega)}^m,$$

so $\mathcal{A}_m$ is coercive with respect to the gradient seminorm.

We emphasize that under the homogeneous Neumann boundary condition constant functions belong to the kernel of $\mathcal{A}_m$. Therefore, $\mathcal{A}_m$ is not coercive with respect to the full $W^{1,m}(\Omega)$ -norm unless one restricts the operator to the zero-mean subspace or adds a lower-order coercive term. Combining the preceding arguments completes the proof. $\square$

Remark 1. Since the boundary condition is of Neumann type, $V_m = W^{1,m}(\Omega)$ is the correct energy space. The space $W_0^{1,m}(\Omega)$ would impose a Dirichlet trace and is therefore incompatible with the no-flux condition. The Neumann operator has constants in its kernel; in the present system, coercivity of the complete evolution is supplied by the reaction terms.

2.3. Product Spaces and Fractional Compactness

For a single neuron set

$$\mathcal{H} := H^3, \quad \mathcal{V} := (V_m \cap L^4(\Omega)) \times H \times H,$$

with norm

$$\|(\phi, \psi, \chi)\|_{\mathcal{V}} := \|\phi\|_{V_m} + \|\phi\|_{L^4} + \|\psi\| + \|\chi\|.$$

For two neurons define

$$\mathbb{H} := \mathcal{H}^2 = H^6, \quad \mathbb{V} := \mathcal{V}^2.$$

The inclusion of $L^4(\Omega)$ in the potential component is natural because the cubic damping produces an $L^4(Q_T)$ estimate and ϕ^3 then belongs to $L^{4/3}(Q_T)$.

The next result is a fractional-time counterpart of the Aubin–Lions–Simon compactness principle [40–42].

Lemma 5 (Fractional compactness). *Let $X_0 \subseteq X \hookrightarrow X_1$ be Banach spaces, $1 < p < \infty$, and let (z_n) be bounded in $L^p(0, T; X_0)$. Suppose that $({}^C\mathfrak{D}_{\zeta}^{\rho} z_n)$ is bounded in $L^q(0, T; X_1)$ for some $q > 1$. Then (z_n) is relatively compact in $L^p(0, T; X)$.*

Remark 2. Lemma 5 is the fractional counterpart of the Aubin–Lions compactness theorem. It is indispensable: spatial compactness at each fixed time does not by itself imply strong compactness in space and time.

3. The Time-Fractional Partly Diffusive Coupled HR Model

Let $i, j \in \{1, 2\}$ with $i \neq j$. The state of neuron i is

$$y_i = (\phi_i, \psi_i, \chi_i),$$

where ϕ_i is the membrane potential, ψ_i is the fast spiking variable, and χ_i is the slow bursting variable. We consider

$$\begin{cases} {}^C\mathcal{D}_\xi^\rho \phi_i + d\mathcal{A}_m \phi_i = a\phi_i^2 - b\phi_i^3 + \psi_i - \chi_i + J + \kappa(\phi_j - \phi_i), \\ {}^C\mathcal{D}_\xi^\rho \psi_i = \alpha_0 - \psi_i - \beta\phi_i^2, \\ {}^C\mathcal{D}_\xi^\rho \chi_i = q(\phi_i - c) - r\chi_i, \end{cases} \quad (x, \xi) \in Q_T, \quad (8)$$

subject to

$$|\nabla \phi_i|^{m-2} \nabla \phi_i \cdot \nu = 0 \quad \text{on } \Sigma_T, \quad i = 1, 2, \quad (9)$$

and

$$(\phi_i, \psi_i, \chi_i)(x, 0) = (\phi_i^0, \psi_i^0, \chi_i^0)(x), \quad x \in \Omega. \quad (10)$$

The diffusion coefficient and the reaction parameters satisfy

$$d, a, b, J, \kappa, \alpha_0, \beta, q, r > 0, \quad c \in \mathbb{R}. \quad (11)$$

The constants $a, b, c, J, q, r, \beta, \alpha_0$ and the diffusion coefficient d are the standard biophysical parameters of the HR neuron and are kept in their conventional form throughout; only the fractional order ρ , the diffusion exponent m , and the coupling strength κ are introduced by the present model.

The system is partly diffusive: only the membrane potentials ϕ_i undergo spatial transport. The coupling is likewise confined to the membrane-potential equations, consistently with electrical gap-junction coupling. When $\rho = 1$ and $m = 2$, problem (8)–(10) reduces to the classical partly diffusive coupled HR system.

Every structural choice in (8)–(10) has a direct biophysical reading. The membrane potential ϕ_i is the only spatially transported state, consistent with voltage propagation along an extended membrane or dendritic cable. The fast variable ψ_i represents rapid ionic feedback, whereas the slow variable χ_i is the adaptation variable responsible for the long bursting envelope; both remain local fields because the model resolves voltage transport without introducing artificial diffusion into the phenomenological recovery variables. The term $\kappa(\phi_j - \phi_i)$ represents electrical gap-junction coupling and therefore acts directly on the voltage difference between the two neurons. The Caputo derivative ${}^C\mathcal{D}_\xi^\rho$ represents hereditary relaxation through a power-law memory kernel, so that the present state depends on the full past voltage and recovery history rather than only on the instantaneous rate of change. The m -Laplacian flux $|\nabla \phi_i|^{m-2} \nabla \phi_i$ represents gradient-dependent effective conductivity, with $m = 2$ recovering ordinary diffusion and $m > 2$ producing degenerate nonlinear transport; and the homogeneous nonlinear Neumann condition (9) represents an electrically insulated computational boundary with no outward voltage flux. The model is phenomenological in the sense of Hindmarsh and Rose: it is not claimed to resolve ion-channel microstructure or detailed dendritic morphology.

Let

$$Y = (y_1, y_2) = \text{col}(\phi_1, \psi_1, \chi_1, \phi_2, \psi_2, \chi_2).$$

Define the diffusion operator $\mathbb{A} : \mathbb{V} \rightarrow \mathbb{V}^*$ by

$$\mathbb{A}Y = \text{col}(d\mathcal{A}_m\phi_1, 0, 0, d\mathcal{A}_m\phi_2, 0, 0)$$

and the reaction–coupling map $\mathcal{F} : \mathbb{V} \rightarrow \mathbb{V}^*$ by

$$\mathcal{F}(Y) = \begin{pmatrix} a\phi_1^2 - b\phi_1^3 + \psi_1 - \chi_1 + J + \kappa(\phi_2 - \phi_1) \\ \alpha_0 - \psi_1 - \beta\phi_1^2 \\ q(\phi_1 - c) - r\chi_1 \\ a\phi_2^2 - b\phi_2^3 + \psi_2 - \chi_2 + J + \kappa(\phi_1 - \phi_2) \\ \alpha_0 - \psi_2 - \beta\phi_2^2 \\ q(\phi_2 - c) - r\chi_2 \end{pmatrix}.$$

The Caputo fractional derivative of Y is understood componentwise,

$${}^C\mathcal{D}_\xi^\rho Y = \text{col}({}^C\mathcal{D}_\xi^\rho\phi_1, {}^C\mathcal{D}_\xi^\rho\psi_1, {}^C\mathcal{D}_\xi^\rho\chi_1, {}^C\mathcal{D}_\xi^\rho\phi_2, {}^C\mathcal{D}_\xi^\rho\psi_2, {}^C\mathcal{D}_\xi^\rho\chi_2),$$

and

$$Y^0 = \text{col}(\phi_1^0, \psi_1^0, \chi_1^0, \phi_2^0, \psi_2^0, \chi_2^0).$$

Set

$$V_m = W^{1,m}(\Omega), \quad H = L^2(\Omega), \quad \mathbb{V} = [V_m \times H \times H]^2, \quad \mathbb{H} = H^6.$$

The nonlinear Neumann m -Laplacian operator $\mathcal{A}_m : V_m \rightarrow V_m^*$ is defined by

$$\langle \mathcal{A}_m\phi, \theta \rangle_{V_m^*, V_m} = \int_{\Omega} |\nabla\phi|^{m-2} \nabla\phi \cdot \nabla\theta \, dx, \quad \phi, \theta \in V_m,$$

so that formally $\mathcal{A}_m\phi = -\text{div}(|\nabla\phi|^{m-2}\nabla\phi) = -\Delta_m\phi$, with the homogeneous nonlinear Neumann boundary condition $|\nabla\phi|^{m-2}\nabla\phi \cdot \nu = 0$ on $\partial\Omega$ incorporated into the variational identity.

The product diffusion operator $\mathbb{A} : \mathbb{V} \rightarrow \mathbb{V}^*$ acts, for every test vector

$$\Theta = \text{col}(\theta_1, \vartheta_1, \zeta_1, \theta_2, \vartheta_2, \zeta_2) \in \mathbb{V},$$

through

$$\langle \mathbb{A}Y, \Theta \rangle_{\mathbb{V}^*, \mathbb{V}} = d \sum_{i=1}^2 \int_{\Omega} |\nabla\phi_i|^{m-2} \nabla\phi_i \cdot \nabla\theta_i \, dx, \quad \langle \mathbb{A}Y, Y \rangle_{\mathbb{V}^*, \mathbb{V}} = d \sum_{i=1}^2 \|\nabla\phi_i\|_{L^m(\Omega)}^m \geq 0.$$

It is hemicontinuous and monotone: for

$$Y = \text{col}(\phi_1, \psi_1, \chi_1, \phi_2, \psi_2, \chi_2), \quad \tilde{Y} = \text{col}(\tilde{\phi}_1, \tilde{\psi}_1, \tilde{\chi}_1, \tilde{\phi}_2, \tilde{\psi}_2, \tilde{\chi}_2),$$

$$\langle \mathbb{A}Y - \mathbb{A}\tilde{Y}, Y - \tilde{Y} \rangle_{\mathbb{V}^*, \mathbb{V}} = d \sum_{i=1}^2 \int_{\Omega} (|\nabla\phi_i|^{m-2} \nabla\phi_i - |\nabla\tilde{\phi}_i|^{m-2} \nabla\tilde{\phi}_i) \cdot (\nabla\phi_i - \nabla\tilde{\phi}_i) \, dx \geq 0.$$

The realization of $\mathcal{A}_m$ in $H = L^2(\Omega)$ has domain

$$D(\mathcal{A}_{m,H}) = \left\{ \phi \in V_m : \exists f \in H \text{ with } \int_{\Omega} |\nabla\phi|^{m-2} \nabla\phi \cdot \nabla\theta \, dx = \int_{\Omega} f\theta \, dx \, \forall \theta \in V_m \right\},$$

and $\mathcal{A}_{m,H}\phi = f$, the function f being uniquely determined and equal, formally, to $-\Delta_m\phi$ under the homogeneous nonlinear Neumann boundary condition. When $m = 2$ this

reduces to the classical Neumann Laplacian $\mathcal{A}_{2,H}\phi = -\Delta\phi$ with $D(\mathcal{A}_{2,H}) = \{\phi \in H^2(\Omega) : \partial\phi/\partial\nu = 0 \text{ on } \partial\Omega\}$ for smooth $\partial\Omega$.

The coupled fractional HR system then takes the abstract form

$$\begin{cases} {}^C\mathfrak{D}_\xi^\rho Y(\xi) + \mathbb{A}Y(\xi) = \mathcal{F}(Y(\xi)) & \text{in } \mathbb{V}^*, \quad \text{for a.e. } \xi \in (0, T), \\ Y(0) = Y^0 & \text{in } \mathbb{H}. \end{cases} \quad (12)$$

Definition 3 (Weak solution). *A function $Y = (\phi_1, \psi_1, \chi_1, \phi_2, \psi_2, \chi_2)$ is a weak solution of (8)–(10) on $[0, T]$ if $Y \in L^\infty(0, T; \mathbb{H})$, $\phi_i \in L^m(0, T; V_m) \cap L^4(Q_T)$, ${}^C\mathfrak{D}_\xi^\rho Y \in L^{4/3}(0, T; \mathbb{V}^*) + L^2(0, T; \mathbb{H})$, $Y(0) = Y^0$ in $\mathbb{H}$, and, for every test vector $\Theta = (\theta_1, \vartheta_1, \zeta_1, \theta_2, \vartheta_2, \zeta_2) \in \mathbb{V}$ and almost every $\xi \in (0, T)$,*

$$\langle {}^C\mathfrak{D}_\xi^\rho Y, \Theta \rangle_{\mathbb{V}^*, \mathbb{V}} + d \sum_{i=1}^2 \int_\Omega |\nabla \phi_i|^{m-2} \nabla \phi_i \cdot \nabla \theta_i \, dx = \langle \mathcal{F}(Y), \Theta \rangle_{\mathbb{V}^*, \mathbb{V}}. \quad (13)$$

Proposition 1 (Growth and local Lipschitz continuity of the reaction map). *Let $X_m := V_m \cap L^4(\Omega)$, $\|\phi\|_{X_m} := \|\phi\|_{V_m} + \|\phi\|_{L^4(\Omega)}$, and define $\mathbb{V} := [X_m \times H \times H]^2$, $\mathbb{H} := H^6$, $H = L^2(\Omega)$. Then $X_m^* = V_m^* + L^{4/3}(\Omega)$, and consequently $\mathbb{V}^* = [X_m^* \times H \times H]^2$. The reaction–coupling mapping $\mathcal{F} : \mathbb{V} \rightarrow \mathbb{V}^*$ is well defined and satisfies*

$$\|\mathcal{F}(Y)\|_{\mathbb{V}^*} \leq C \left(1 + \|Y\|_{\mathbb{H}} + \|\phi_1\|_{L^4(\Omega)}^3 + \|\phi_2\|_{L^4(\Omega)}^3 \right) \quad (14)$$

for every $Y = \text{col}(\phi_1, \psi_1, \chi_1, \phi_2, \psi_2, \chi_2) \in \mathbb{V}$, where $C > 0$ depends only on the model parameters and on Ω . Moreover, for every $R > 0$, there exists a constant $L_R > 0$ such that

$$\|\mathcal{F}(Y) - \mathcal{F}(\tilde{Y})\|_{\mathbb{V}^*} \leq L_R \|Y - \tilde{Y}\|_{\mathbb{V}} \quad (15)$$

whenever $\|Y\|_{\mathbb{H}} + \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)} \leq R$ and $\|\tilde{Y}\|_{\mathbb{H}} + \sum_{i=1}^2 \|\tilde{\phi}_i\|_{L^4(\Omega)} \leq R$. One may take $L_R = C(1 + R + R^2)$. Consequently, $\mathcal{F}$ is locally Lipschitz from $\mathbb{V}$ into $\mathbb{V}^*$ and maps bounded subsets of $\mathbb{V}$ into bounded subsets of $\mathbb{V}^*$.

Proof. Let $Y = \text{col}(\phi_1, \psi_1, \chi_1, \phi_2, \psi_2, \chi_2) \in \mathbb{V}$. For $i, j \in \{1, 2\}$, $i \neq j$, define $\mathfrak{p}_i := a\phi_i^2 - b\phi_i^3 + \psi_i - \chi_i + J + \kappa(\phi_j - \phi_i)$, $\mathfrak{f}_i := \alpha_0 - \psi_i - \beta\phi_i^2$, and $\mathfrak{s}_i := q(\phi_i - c) - r\chi_i$. Thus, $\mathcal{F}(Y) = \text{col}(\mathfrak{p}_1, \mathfrak{f}_1, \mathfrak{s}_1, \mathfrak{p}_2, \mathfrak{f}_2, \mathfrak{s}_2)$. Let $\Theta = \text{col}(\theta_1, \vartheta_1, \zeta_1, \theta_2, \vartheta_2, \zeta_2) \in \mathbb{V}$, where $\theta_i \in X_m$, $\vartheta_i, \zeta_i \in H$. The duality action of $\mathcal{F}(Y)$ on Θ is

$$\langle \mathcal{F}(Y), \Theta \rangle_{\mathbb{V}^*, \mathbb{V}} = \sum_{i=1}^2 \left[\langle \mathfrak{p}_i, \theta_i \rangle_{X_m^*, X_m} + (\mathfrak{f}_i, \vartheta_i) + (\mathfrak{s}_i, \zeta_i) \right]. \quad (16)$$

The cubic term is interpreted through the duality between $L^{4/3}(\Omega)$ and $L^4(\Omega)$, whereas all the remaining terms are paired through $L^2(\Omega)$.

We use the product norm

$$\|\Theta\|_{\mathbb{V}} = \sum_{i=1}^2 \left(\|\theta_i\|_{V_m} + \|\theta_i\|_{L^4(\Omega)} + \|\vartheta_i\| + \|\zeta_i\| \right).$$

In particular, $\|\theta_i\|_{L^4(\Omega)} \leq \|\Theta\|_{\mathbb{V}}$, $\|\vartheta_i\| \leq \|\Theta\|_{\mathbb{V}}$, $\|\zeta_i\| \leq \|\Theta\|_{\mathbb{V}}$. Since Ω is bounded, the embedding $L^4(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, and $\|\theta_i\|_{L^2(\Omega)} \leq |\Omega|^{1/4} \|\theta_i\|_{L^4(\Omega)} \leq |\Omega|^{1/4} \|\Theta\|_{\mathbb{V}}$. Throughout the proof, $C > 0$ denotes a constant depending only on $a, b, J, \kappa, \alpha_0, \beta, q, r, c$ and on Ω . Its value may change from line to line.

Step 1: Algebraic estimates for the quadratic and cubic terms.

Let $\phi, \tilde{\phi} \in L^4(\Omega)$. Since $\phi^2 - \tilde{\phi}^2 = (\phi + \tilde{\phi})(\phi - \tilde{\phi})$, Hölder's inequality gives

$$\begin{aligned} \|\phi^2 - \tilde{\phi}^2\|_{L^2(\Omega)} &\leq \|\phi + \tilde{\phi}\|_{L^4(\Omega)} \|\phi - \tilde{\phi}\|_{L^4(\Omega)} \\ &\leq (\|\phi\|_{L^4(\Omega)} + \|\tilde{\phi}\|_{L^4(\Omega)}) \|\phi - \tilde{\phi}\|_{L^4(\Omega)}. \end{aligned}$$

Hence,

$$\|\phi^2 - \tilde{\phi}^2\|_{L^2(\Omega)} \leq (\|\phi\|_{L^4(\Omega)} + \|\tilde{\phi}\|_{L^4(\Omega)}) \|\phi - \tilde{\phi}\|_{L^4(\Omega)}. \quad (17)$$

Similarly, $\phi^3 - \tilde{\phi}^3 = (\phi - \tilde{\phi})(\phi^2 + \phi\tilde{\phi} + \tilde{\phi}^2)$. Using Hölder's inequality with exponents 4 and 2, we obtain

$$\|\phi^3 - \tilde{\phi}^3\|_{L^{4/3}(\Omega)} \leq \|\phi - \tilde{\phi}\|_{L^4(\Omega)} \|\phi^2 + \phi\tilde{\phi} + \tilde{\phi}^2\|_{L^2(\Omega)}.$$

Moreover, $\|\phi^2\|_{L^2(\Omega)} = \|\phi\|_{L^4(\Omega)}^2$, $\|\tilde{\phi}^2\|_{L^2(\Omega)} = \|\tilde{\phi}\|_{L^4(\Omega)}^2$, and

$$\|\phi\tilde{\phi}\|_{L^2(\Omega)} \leq \|\phi\|_{L^4(\Omega)} \|\tilde{\phi}\|_{L^4(\Omega)} \leq \frac{1}{2} (\|\phi\|_{L^4(\Omega)}^2 + \|\tilde{\phi}\|_{L^4(\Omega)}^2).$$

Therefore,

$$\|\phi^2 + \phi\tilde{\phi} + \tilde{\phi}^2\|_{L^2(\Omega)} \leq \frac{3}{2} (\|\phi\|_{L^4(\Omega)}^2 + \|\tilde{\phi}\|_{L^4(\Omega)}^2),$$

and hence

$$\|\phi^3 - \tilde{\phi}^3\|_{L^{4/3}(\Omega)} \leq \frac{3}{2} (\|\phi\|_{L^4(\Omega)}^2 + \|\tilde{\phi}\|_{L^4(\Omega)}^2) \|\phi - \tilde{\phi}\|_{L^4(\Omega)}. \quad (18)$$

Direct calculation also gives

$$\|\phi^2\|_{L^2(\Omega)} = \|\phi\|_{L^4(\Omega)}^2, \quad \|\phi^3\|_{L^{4/3}(\Omega)} = \|\phi\|_{L^4(\Omega)}^3. \quad (19)$$

Step 2: Well-definedness and growth estimate.

For the membrane-potential component, Hölder's inequality and (19) give

$$\begin{aligned} |\langle \mathbf{p}_i, \theta_i \rangle_{X_m^*, X_m}| &\leq a \|\phi_i^2\|_{L^2(\Omega)} \|\theta_i\|_{L^2(\Omega)} + b \|\phi_i^3\|_{L^{4/3}(\Omega)} \|\theta_i\|_{L^4(\Omega)} \\ &\quad + (\|\psi_i\| + \|\chi_i\| + J|\Omega|^{1/2} + \kappa\|\phi_j\| + \kappa\|\phi_i\|) \|\theta_i\|_{L^2(\Omega)} \\ &\leq C(1 + \|Y\|_{\mathbb{H}} + \|\phi_i\|_{L^4(\Omega)}^2 + \|\phi_i\|_{L^4(\Omega)}^3) \|\Theta\|_{\mathbb{V}}. \end{aligned}$$

For the fast component,

$$\begin{aligned} |(\mathbf{f}_i, \vartheta_i)| &\leq \left(\alpha_0 |\Omega|^{1/2} + \|\psi_i\| + \beta \|\phi_i^2\|_{L^2(\Omega)} \right) \|\vartheta_i\| \\ &\leq C(1 + \|Y\|_{\mathbb{H}} + \|\phi_i\|_{L^4(\Omega)}^2) \|\Theta\|_{\mathbb{V}}. \end{aligned}$$

For the slow component,

$$\begin{aligned} |(\mathbf{s}_i, \zeta_i)| &\leq (q\|\phi_i\| + q|c||\Omega|^{1/2} + r\|\chi_i\|) \|\zeta_i\| \\ &\leq C(1 + \|Y\|_{\mathbb{H}}) \|\Theta\|_{\mathbb{V}}. \end{aligned}$$

Since $t^2 \leq 1 + t^3$ for every $t \geq 0$, we have $\|\phi_i\|_{L^4(\Omega)}^2 \leq 1 + \|\phi_i\|_{L^4(\Omega)}^3$. Summing the preceding estimates over $i = 1, 2$, we obtain

$$\left| \langle \mathcal{F}(Y), \Theta \rangle_{\mathbb{V}^*, \mathbb{V}} \right| \leq C \left(1 + \|Y\|_{\mathbb{H}} + \|\phi_1\|_{L^4(\Omega)}^3 + \|\phi_2\|_{L^4(\Omega)}^3 \right) \|\Theta\|_{\mathbb{V}}.$$

Taking the supremum over all $\Theta \in \mathbb{V}$, $\|\Theta\|_{\mathbb{V}} \leq 1$, yields

$$\|\mathcal{F}(Y)\|_{\mathbb{V}^*} \leq C \left(1 + \|Y\|_{\mathbb{H}} + \|\phi_1\|_{L^4(\Omega)}^3 + \|\phi_2\|_{L^4(\Omega)}^3 \right).$$

This proves (14) and shows that $\mathcal{F}(Y) \in \mathbb{V}^*$ for every $Y \in \mathbb{V}$.

If $B \subset \mathbb{V}$ is bounded then there exists $R_B > 0$ such that

$$\|Y\|_{\mathbb{H}} + \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)} \leq R_B$$

for every $Y \in B$. Therefore, $\sup_{Y \in B} \|\mathcal{F}(Y)\|_{\mathbb{V}^*} \leq C(1 + R_B + 2R_B^3) < \infty$. Thus, $\mathcal{F}$ maps bounded subsets of $\mathbb{V}$ into bounded subsets of $\mathbb{V}^*$.

Step 3: Local Lipschitz continuity.

Fix $R > 0$, and let $Y = \text{col}(\phi_1, \psi_1, \chi_1, \phi_2, \psi_2, \chi_2)$ and $\tilde{Y} = \text{col}(\tilde{\phi}_1, \tilde{\psi}_1, \tilde{\chi}_1, \tilde{\phi}_2, \tilde{\psi}_2, \tilde{\chi}_2)$ satisfy $\|Y\|_{\mathbb{H}} + \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)} \leq R$ and

$$\|\tilde{Y}\|_{\mathbb{H}} + \sum_{i=1}^2 \|\tilde{\phi}_i\|_{L^4(\Omega)} \leq R.$$

The constant terms J , α_0 , and qc cancel in the difference. For the membrane-potential component,

$$\mathfrak{p}_i - \tilde{\mathfrak{p}}_i = a(\phi_i^2 - \tilde{\phi}_i^2) - b(\phi_i^3 - \tilde{\phi}_i^3) + (\psi_i - \tilde{\psi}_i) - (\chi_i - \tilde{\chi}_i) + \kappa[(\phi_j - \tilde{\phi}_j) - (\phi_i - \tilde{\phi}_i)].$$

By (17),

$$\|\phi_i^2 - \tilde{\phi}_i^2\|_{L^2(\Omega)} \leq 2R\|\phi_i - \tilde{\phi}_i\|_{L^4(\Omega)}.$$

By (18),

$$\|\phi_i^3 - \tilde{\phi}_i^3\|_{L^{4/3}(\Omega)} \leq 3R^2\|\phi_i - \tilde{\phi}_i\|_{L^4(\Omega)}.$$

Consequently,

$$\begin{aligned} & \left| \langle \mathfrak{p}_i - \tilde{\mathfrak{p}}_i, \theta_i \rangle_{X_m^*, X_m} \right| \\ & \leq C(1 + R + R^2) \left(\|\phi_i - \tilde{\phi}_i\|_{L^4(\Omega)} + \|\phi_j - \tilde{\phi}_j\|_{L^2(\Omega)} \right. \\ & \quad \left. + \|\phi_i - \tilde{\phi}_i\|_{L^2(\Omega)} + \|\psi_i - \tilde{\psi}_i\| + \|\chi_i - \tilde{\chi}_i\| \right) \|\Theta\|_{\mathbb{V}}. \end{aligned}$$

For the fast component,

$$\mathfrak{f}_i - \tilde{\mathfrak{f}}_i = -(\psi_i - \tilde{\psi}_i) - \beta(\phi_i^2 - \tilde{\phi}_i^2).$$

Therefore, using (17), we have

$$\left| \langle \mathfrak{f}_i - \tilde{\mathfrak{f}}_i, \theta_i \rangle \right| \leq C(1 + R) \left(\|\psi_i - \tilde{\psi}_i\| + \|\phi_i - \tilde{\phi}_i\|_{L^4(\Omega)} \right) \|\Theta\|_{\mathbb{V}}.$$

For the slow component,

$$\mathfrak{s}_i - \tilde{\mathfrak{s}}_i = q(\phi_i - \tilde{\phi}_i) - r(\chi_i - \tilde{\chi}_i),$$

and hence

$$|(\mathfrak{s}_i - \tilde{\mathfrak{s}}_i, \zeta_i)| \leq C \left(\|\phi_i - \tilde{\phi}_i\|_{L^2(\Omega)} + \|\chi_i - \tilde{\chi}_i\| \right) \|\Theta\|_{\mathbb{V}}.$$

Since Ω is bounded and $m \geq 2$, one has

$$\|z\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{2} - \frac{1}{m}} \|z\|_{L^m(\Omega)} \leq C \|z\|_{V_m}$$

for every $z \in V_m$. Therefore,

$$\left\| Y - \tilde{Y} \right\|_{\mathbb{H}} + \sum_{i=1}^2 \|\phi_i - \tilde{\phi}_i\|_{L^4(\Omega)} \leq C \left\| Y - \tilde{Y} \right\|_{\mathbb{V}}.$$

Summing the componentwise estimates over $i = 1, 2$, we obtain

$$\left| \left\langle \mathcal{F}(Y) - \mathcal{F}(\tilde{Y}), \Theta \right\rangle_{\mathbb{V}^*, \mathbb{V}} \right| \leq C(1 + R + R^2) \left\| Y - \tilde{Y} \right\|_{\mathbb{V}} \|\Theta\|_{\mathbb{V}}.$$

Taking the supremum over all $\Theta \in \mathbb{V}$, $\|\Theta\|_{\mathbb{V}} \leq 1$, gives

$$\left\| \mathcal{F}(Y) - \mathcal{F}(\tilde{Y}) \right\|_{\mathbb{V}^*} \leq C(1 + R + R^2) \left\| Y - \tilde{Y} \right\|_{\mathbb{V}}.$$

Thus, (15) holds with

$$L_R = C(1 + R + R^2).$$

Hence, $\mathcal{F}$ is locally Lipschitz from $\mathbb{V}$ into $\mathbb{V}^*$. $\square$

Proposition 2 (Fractional slow-variable estimate). *Write the slow parameters as*

$$q = \varepsilon \bar{q}, \quad r = \varepsilon \bar{r}, \quad 0 < \varepsilon \ll 1, \quad (20)$$

with $\bar{q}, \bar{r} > 0$ fixed. Let Y be a weak solution on $[0, T]$ and suppose

$$M_{\phi,i} := \operatorname{ess\,sup}_{0 \leq s \leq T} \|\phi_i(s)\|, \quad M_{\chi,i} := \operatorname{ess\,sup}_{0 \leq s \leq T} \|\chi_i(s)\|$$

are finite, $i = 1, 2$. Then

$$\sup_{0 \leq \xi \leq T} \left\| \chi_i(\xi) - \chi_i^0 \right\| \leq \frac{\varepsilon T^\rho}{\Gamma(1 + \rho)} \left[\bar{q} (M_{\phi,i} + |c| |\Omega|^{1/2}) + \bar{r} M_{\chi,i} \right]. \quad (21)$$

Consequently the slow component changes by $O(\varepsilon T^\rho)$ on every bounded time interval; for $\rho = 1$ this reduces to the classical $O(\varepsilon T)$ estimate, while for $0 < \rho < 1$ the accumulated change is governed by the fractional timescale $T^\rho / \Gamma(1 + \rho)$.

Proof. Under the scaling (20), the slow equation in (8) becomes

$${}^C \mathcal{D}_\xi^\rho \chi_i = \varepsilon [\bar{q}(\phi_i - c) - \bar{r} \chi_i].$$

The equivalent Volterra representation, obtained by applying the fractional integral operator of Definition 2 and identity (3), gives

$$\chi_i(\xi) - \chi_i^0 = \frac{\varepsilon}{\Gamma(\rho)} \int_0^\xi (\xi - s)^{\rho-1} [\bar{q}(\phi_i(s) - c) - \bar{r}\chi_i(s)] \, ds.$$

Taking the H -norm, using the triangle inequality inside the integral, and the bounds $\|\phi_i(s) - c\| \leq M_{\phi,i} + |c| |\Omega|^{1/2}$ and $\|\chi_i(s)\| \leq M_{\chi,i}$ for a.e. $s \in [0, T]$, we obtain

$$\|\chi_i(\xi) - \chi_i^0\| \leq \frac{\varepsilon}{\Gamma(\rho)} [\bar{q}(M_{\phi,i} + |c| |\Omega|^{1/2}) + \bar{r}M_{\chi,i}] \int_0^\xi (\xi - s)^{\rho-1} \, ds.$$

Since $\int_0^\xi (\xi - s)^{\rho-1} \, ds = \xi^\rho / \rho$ and $\rho \Gamma(\rho) = \Gamma(1 + \rho)$, this simplifies to

$$\|\chi_i(\xi) - \chi_i^0\| \leq \frac{\varepsilon \xi^\rho}{\Gamma(1 + \rho)} [\bar{q}(M_{\phi,i} + |c| |\Omega|^{1/2}) + \bar{r}M_{\chi,i}].$$

Taking the supremum over $\xi \in [0, T]$ gives (21). $\square$

Remark 3 (Scope of the slow–fast estimate). *Proposition 2 makes the HR timescale separation mathematically explicit for the Caputo memory operator without asserting a complete geometric singular-perturbation theory. A full fractional singular-perturbation and bifurcation analysis, requiring nonlocal invariant manifolds for the Caputo PDE with m -Laplacian diffusion, is left as a substantial direction for future work; see Section 12. The estimate nevertheless supports the numerical phase portraits and burst statistics of Section 11, where the slow variable organizes the long bursting envelope and the fractional order modifies the effective temporal accumulation rate $T^\rho / \Gamma(1 + \rho)$.*

4. A Priori Estimates and Dissipativity

The first estimate simultaneously prevents finite-time blow-up and supplies the compactness required in the existence proof.

Theorem 1 (Basic fractional energy inequality). *Let Y be a sufficiently regular solution of (8)–(10). Set*

$$C_0 := \frac{\beta^2 + 4}{b} \quad (22)$$

and define

$$\mathcal{E}(\xi) := C_0 \sum_{i=1}^2 \|\phi_i(\xi)\|^2 + \sum_{i=1}^2 \|\psi_i(\xi)\|^2 + \sum_{i=1}^2 \|\chi_i(\xi)\|^2. \quad (23)$$

Then there exist explicit constants $\eta_0 > 0$ and $C_* > 0$, depending only on the model parameters, such that

$${}^C \mathfrak{D}_\xi^\rho \mathcal{E}(\xi) + 2C_0 d \sum_{i=1}^2 \|\nabla \phi_i(\xi)\|_{L^m}^m + \eta_0 \mathcal{E}(\xi) + \sum_{i=1}^2 \|\phi_i(\xi)\|_{L^4}^4 \leq C_* |\Omega|. \quad (24)$$

One may take $\eta_0 = \frac{1}{2} \min\{1, r\}$ after enlarging C_* .

Proof. For simplicity, all norms without subscripts are taken in $H = L^2(\Omega)$. Since Y is sufficiently regular, all the tests below are admissible.

For each $i \in \{1, 2\}$, test the ϕ_i -equation by $C_0\phi_i$, the ψ_i -equation by ψ_i , and the χ_i -equation by χ_i . By Lemma 1,

$$\langle {}^c\mathfrak{D}_{\xi}^{\rho}\phi_i, C_0\phi_i \rangle \geq \frac{C_0}{2} {}^c\mathfrak{D}_{\xi}^{\rho}\|\phi_i\|^2, \quad \langle {}^c\mathfrak{D}_{\xi}^{\rho}\psi_i, \psi_i \rangle \geq \frac{1}{2} {}^c\mathfrak{D}_{\xi}^{\rho}\|\psi_i\|^2,$$

and

$$\langle {}^c\mathfrak{D}_{\xi}^{\rho}\chi_i, \chi_i \rangle \geq \frac{1}{2} {}^c\mathfrak{D}_{\xi}^{\rho}\|\chi_i\|^2.$$

Furthermore, by the definition of the nonlinear Neumann operator $\mathcal{A}_m$,

$$\langle \mathcal{A}_m\phi_i, \phi_i \rangle_{V_m^*, V_m} = \|\nabla\phi_i\|_{L^m(\Omega)}^m.$$

Adding the three tested equations and then summing over $i = 1, 2$ therefore gives

$$\begin{aligned} & \frac{1}{2} {}^c\mathfrak{D}_{\xi}^{\rho}\mathcal{E}(\xi) + C_0d \sum_{i=1}^2 \|\nabla\phi_i\|_{L^m(\Omega)}^m + C_0b \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 + \sum_{i=1}^2 \|\psi_i\|^2 + r \sum_{i=1}^2 \|\chi_i\|^2 \\ & \leq \sum_{i=1}^2 \int_{\Omega} \left(C_0a\phi_i^3 + C_0\phi_i\psi_i - C_0\phi_i\chi_i + C_0J\phi_i + \alpha_0\psi_i - \beta\phi_i^2\psi_i + q(\phi_i - c)\chi_i \right) dx \\ & \quad + C_0\kappa \int_{\Omega} \left[(\phi_2 - \phi_1)\phi_1 + (\phi_1 - \phi_2)\phi_2 \right] dx. \end{aligned}$$

The coupling contribution is dissipative, since

$$C_0\kappa \int_{\Omega} \left[(\phi_2 - \phi_1)\phi_1 + (\phi_1 - \phi_2)\phi_2 \right] dx = -C_0\kappa \int_{\Omega} |\phi_1 - \phi_2|^2 dx = -C_0\kappa \|\phi_1 - \phi_2\|^2 \leq 0.$$

Hence, this term may be discarded from the upper estimate. We next estimate the remaining reaction terms pointwise. For the fast variable, Young's inequality gives

$$\alpha_0\psi_i \leq 2\alpha_0^2 + \frac{1}{8}\psi_i^2$$

and

$$-\beta\phi_i^2\psi_i \leq \frac{\beta^2}{2}\phi_i^4 + \frac{1}{2}\psi_i^2.$$

Consequently,

$$\alpha_0\psi_i - \beta\phi_i^2\psi_i - \psi_i^2 \leq 2\alpha_0^2 + \frac{\beta^2}{2}\phi_i^4 - \frac{3}{8}\psi_i^2.$$

For the slow variable, one has

$$q(\phi_i - c)\chi_i \leq \frac{q^2}{2r}(\phi_i - c)^2 + \frac{r}{2}\chi_i^2.$$

It follows that

$$q(\phi_i - c)\chi_i - r\chi_i^2 \leq \frac{q^2}{2r}(\phi_i - c)^2 - \frac{r}{2}\chi_i^2.$$

Using $(\phi_i - c)^2 \leq 2\phi_i^2 + 2c^2$, we obtain

$$q(\phi_i - c)\chi_i - r\chi_i^2 \leq \frac{q^2}{r}\phi_i^2 + \frac{q^2c^2}{r} - \frac{r}{2}\chi_i^2.$$

For the cubic reaction term, define $C_a := \frac{27(C_0a)^4}{256}$. Indeed, the maximum of $C_0as^3 - s^4$ over $s \in \mathbb{R}$ is bounded above by C_a . Therefore, $C_0a\phi_i^3 \leq \phi_i^4 + C_a$. The mixed potential-fast-variable term satisfies

$$C_0\phi_i\psi_i \leq 2C_0^2\phi_i^2 + \frac{1}{8}\psi_i^2.$$

Similarly,

$$-C_0\phi_i\chi_i \leq C_0|\phi_i||\chi_i| \leq \frac{C_0^2}{r}\phi_i^2 + \frac{r}{4}\chi_i^2.$$

Finally,

$$C_0J\phi_i \leq C_0|J||\phi_i| \leq C_0\phi_i^2 + \frac{C_0J^2}{4}.$$

Combining the preceding estimates, the total coefficient multiplying ϕ_i^2 is

$$M := \frac{q^2}{r} + 2C_0^2 + \frac{C_0^2}{r} + C_0.$$

It is important to estimate the complete quantity $M\phi_i^2$ in one step rather than estimating its four summands separately. By Young's inequality,

$$M\phi_i^2 \leq \phi_i^4 + \frac{M^2}{4}. \quad (25)$$

Define $K_0 := 2a_0^2 + \frac{q^2c^2}{r} + C_a + \frac{C_0J^2}{4} + \frac{M^2}{4}$. After inserting all the preceding bounds, integrating over Ω , and summing over $i = 1, 2$, we obtain

$$\begin{aligned} & \frac{1}{2} {}^c\mathfrak{D}_\xi^\rho \mathcal{E}(\xi) + C_0d \sum_{i=1}^2 \|\nabla \phi_i\|_{L^m(\Omega)}^m + \left(C_0b - \frac{\beta^2}{2} - 2\right) \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \\ & + \frac{1}{4} \sum_{i=1}^2 \|\psi_i\|^2 + \frac{r}{4} \sum_{i=1}^2 \|\chi_i\|^2 \leq 2K_0|\Omega|. \end{aligned}$$

Here, the two positive quartic contributions subtracted from the cubic damping are: one contribution from the estimate of $C_0a\phi_i^3$, and one contribution from the grouped inequality (25). The additional contribution $\beta^2\phi_i^4/2$ comes from the fast-variable equation.

By the definition, $C_0 = \frac{\beta^2+4}{b}$, we have $C_0b - \frac{\beta^2}{2} - 2 = \frac{\beta^2}{2} + 2$. Consequently

$$\begin{aligned} & {}^c\mathfrak{D}_\xi^\rho \mathcal{E}(\xi) + 2C_0d \sum_{i=1}^2 \|\nabla \phi_i\|_{L^m(\Omega)}^m + (\beta^2 + 4) \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \\ & + \frac{1}{2} \sum_{i=1}^2 \|\psi_i\|^2 + \frac{r}{2} \sum_{i=1}^2 \|\chi_i\|^2 \leq 4K_0|\Omega|. \end{aligned} \quad (26)$$

We now recover both a quadratic L^2 -term and a remaining quartic term from the quartic dissipation. Since $\beta^2 + 4 \geq 4$, it follows in particular that $(\beta^2 + 4)s^4 \geq 3s^4$ for every $s \in \mathbb{R}$. Moreover, $3s^4 = s^4 + 2s^4$. The minimum of $2s^4 - \frac{C_0}{2}s^2$ is attained when $s^2 = C_0/8$, and its minimum value is $-C_0^2/32$. Hence, $2s^4 \geq \frac{C_0}{2}s^2 - \frac{C_0^2}{32}$. Consequently,

$$3s^4 \geq s^4 + \frac{C_0}{2}s^2 - \frac{C_0^2}{32}.$$

Applying this pointwise with $s = \phi_i(x, \xi)$ and integrating over Ω , we obtain

$$(\beta^2 + 4)\|\phi_i\|_{L^4(\Omega)}^4 \geq \|\phi_i\|_{L^4(\Omega)}^4 + \frac{C_0}{2}\|\phi_i\|^2 - \frac{C_0^2}{32}|\Omega|.$$

Summing over $i = 1, 2$ and inserting this estimate into (26) gives

$$\begin{aligned} & {}^c\mathfrak{D}_{\xi}^{\rho}\mathcal{E}(\xi) + 2C_0d \sum_{i=1}^2 \|\nabla\phi_i\|_{L^m(\Omega)}^m + \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \\ & + \frac{C_0}{2} \sum_{i=1}^2 \|\phi_i\|^2 + \frac{1}{2} \sum_{i=1}^2 \|\psi_i\|^2 + \frac{r}{2} \sum_{i=1}^2 \|\chi_i\|^2 \\ & \leq \left(4K_0 + \frac{C_0^2}{16}\right) |\Omega|. \end{aligned}$$

Set $\eta_0 := \frac{1}{2} \min\{1, r\}$. Then $0 < \eta_0 \leq \frac{1}{2}$ and $0 < \eta_0 \leq \frac{r}{2}$. Therefore,

$$\frac{C_0}{2} \sum_{i=1}^2 \|\phi_i\|^2 + \frac{1}{2} \sum_{i=1}^2 \|\psi_i\|^2 + \frac{r}{2} \sum_{i=1}^2 \|\chi_i\|^2 \geq \eta_0 \mathcal{E}(\xi).$$

Thus,

$${}^c\mathfrak{D}_{\xi}^{\rho}\mathcal{E}(\xi) + 2C_0d \sum_{i=1}^2 \|\nabla\phi_i(\xi)\|_{L^m(\Omega)}^m + \eta_0 \mathcal{E}(\xi) + \sum_{i=1}^2 \|\phi_i(\xi)\|_{L^4(\Omega)}^4 \leq C_* |\Omega|,$$

where one may take $C_* := 4K_0 + \frac{C_0^2}{16}$. Equivalently,

$$C_* = 8\alpha_0^2 + \frac{4q^2c^2}{r} + \frac{27(C_0a)^4}{64} + C_0J^2 + M^2 + \frac{C_0^2}{16},$$

with $M = \frac{q^2}{r} + 2C_0^2 + \frac{C_0^2}{r} + C_0$. Both η_0 and C_* depend only on the model parameters. Therefore, (24) follows. $\square$

Remark 4 (Necessity of the grouped estimate). *If the four quadratic potential terms are each converted separately through $K\phi_i^2 \leq \phi_i^4 + K^2/4$ then the positive quartic budget becomes $\frac{1}{2}\beta^2 + 5$ and the net quartic coefficient is $-\frac{1}{2}\beta^2 + 1$, which is positive for $\beta^2 < 2$. The estimate would then fail for small β , for instance for the certificate parameter set with $\beta = \frac{1}{2}$. The grouped inequality (25) produces the single positive quartic that keeps the coefficient negative for the whole admissible range of β .*

Corollary 1 (Uniform-in-time estimate). *Every weak solution satisfies*

$$\mathcal{E}(\xi) \leq \mathcal{E}(0)E_{\rho}(-\eta_0\xi^{\rho}) + \frac{C_*|\Omega|}{\eta_0} [1 - E_{\rho}(-\eta_0\xi^{\rho})]. \quad (27)$$

Consequently,

$$\sup_{\xi \in [0, T]} \|Y(\xi)\|_{\mathbb{H}}^2 \leq C \left(1 + \|Y^0\|_{\mathbb{H}}^2\right),$$

and

$$I_{0+}^{\rho} \left(\sum_{i=1}^2 \|\nabla\phi_i\|_{L^m(\Omega)}^m + \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \right) (\xi) \leq C \left(1 + \|Y^0\|_{\mathbb{H}}^2\right). \quad (28)$$

Here, $C > 0$ may depend on T, ρ, Ω and the model parameters but is independent of the particular weak solution.

Proof. From Theorem 1, every weak solution satisfies

$${}^C\mathfrak{D}_{\xi}^{\rho}\mathcal{E}(\xi) + 2C_0d \sum_{i=1}^2 \|\nabla\phi_i(\xi)\|_{L^m(\Omega)}^m + \eta_0\mathcal{E}(\xi) + \sum_{i=1}^2 \|\phi_i(\xi)\|_{L^4(\Omega)}^4 \leq C_*|\Omega|$$

for almost every $\xi \in (0, T)$.

Since the gradient and quartic terms are non-negative, we may discard them and obtain

$${}^C\mathfrak{D}_{\xi}^{\rho}\mathcal{E}(\xi) + \eta_0\mathcal{E}(\xi) \leq C_*|\Omega|.$$

The energy $\mathcal{E}$ is non-negative. Hence, applying the fractional comparison principle of Lemma 2 with $y(\xi) = \mathcal{E}(\xi)$, $\lambda = \eta_0$, $H = C_*|\Omega|$ gives

$$\mathcal{E}(\xi) \leq \mathcal{E}(0)E_{\rho}(-\eta_0\xi^{\rho}) + C_*|\Omega| \int_0^{\xi} (\xi-s)^{\rho-1} E_{\rho,\rho}(-\eta_0(\xi-s)^{\rho}) \, ds.$$

Using the standard Mittag-Leffler identity

$$\int_0^{\xi} (\xi-s)^{\rho-1} E_{\rho,\rho}(-\eta_0(\xi-s)^{\rho}) \, ds = \frac{1}{\eta_0} [1 - E_{\rho}(-\eta_0\xi^{\rho})],$$

we obtain

$$\mathcal{E}(\xi) \leq \mathcal{E}(0)E_{\rho}(-\eta_0\xi^{\rho}) + \frac{C_*|\Omega|}{\eta_0} [1 - E_{\rho}(-\eta_0\xi^{\rho})],$$

which is precisely (27).

For $0 < \rho \leq 1$ and $\eta_0 > 0$, one has $0 < E_{\rho}(-\eta_0\xi^{\rho}) \leq 1$. Therefore, (27) implies $\mathcal{E}(\xi) \leq \mathcal{E}(0) + \frac{C_*|\Omega|}{\eta_0}$ for every $\xi \in [0, T]$. In fact, the sharper estimate $\mathcal{E}(\xi) \leq \max\left\{\mathcal{E}(0), \frac{C_*|\Omega|}{\eta_0}\right\}$ also follows because the right-hand side of (27) is a convex combination of $\mathcal{E}(0)$ and $C_*|\Omega|/\eta_0$.

By the definition of $\mathcal{E}$,

$$\mathcal{E}(\xi) = C_0 \sum_{i=1}^2 \|\phi_i(\xi)\|^2 + \sum_{i=1}^2 \|\psi_i(\xi)\|^2 + \sum_{i=1}^2 \|\chi_i(\xi)\|^2.$$

Hence,

$$\min\{C_0, 1\} \|Y(\xi)\|_{\mathbb{H}}^2 \leq \mathcal{E}(\xi) \leq \max\{C_0, 1\} \|Y(\xi)\|_{\mathbb{H}}^2.$$

Similarly,

$$\mathcal{E}(0) \leq \max\{C_0, 1\} \|Y^0\|_{\mathbb{H}}^2.$$

Consequently,

$$\begin{aligned} \sup_{\xi \in [0, T]} \|Y(\xi)\|_{\mathbb{H}}^2 &\leq \frac{1}{\min\{C_0, 1\}} \left(\mathcal{E}(0) + \frac{C_*|\Omega|}{\eta_0} \right) \\ &\leq C \left(1 + \|Y^0\|_{\mathbb{H}}^2 \right). \end{aligned}$$

It remains to prove the integrated estimate. Apply the Riemann–Liouville fractional integral I_{0+}^{ρ} to (24). Using

$$I_{0+}^{\rho} {}^C\mathfrak{D}_{\xi}^{\rho}\mathcal{E}(\xi) = \mathcal{E}(\xi) - \mathcal{E}(0),$$

we obtain

$$\begin{aligned} \mathcal{E}(\xi) - \mathcal{E}(0) + 2C_0d I_{0+}^{\rho} \left(\sum_{i=1}^2 \|\nabla\phi_i\|_{L^m(\Omega)}^m \right) (\xi) + \eta_0 I_{0+}^{\rho} \mathcal{E}(\xi) + I_{0+}^{\rho} \left(\sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \right) (\xi) \\ \leq C_*|\Omega| I_{0+}^{\rho} 1. \end{aligned}$$

Since $I_{0+}^\rho 1 = \frac{\xi^\rho}{\Gamma(\rho+1)}$, this becomes

$$\begin{aligned} \mathcal{E}(\xi) - \mathcal{E}(0) + 2C_0 d I_{0+}^\rho \left(\sum_{i=1}^2 \|\nabla \phi_i\|_{L^m(\Omega)}^m \right) (\xi) + \eta_0 I_{0+}^\rho \mathcal{E}(\xi) + I_{0+}^\rho \left(\sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \right) (\xi) \\ \leq \frac{C_* |\Omega| \xi^\rho}{\Gamma(\rho+1)}. \end{aligned}$$

Since $\mathcal{E}(\xi) \geq 0$ and $I_{0+}^\rho \mathcal{E}(\xi) \geq 0$, we may discard these non-negative terms from the left-hand side and obtain

$$2C_0 d I_{0+}^\rho \left(\sum_{i=1}^2 \|\nabla \phi_i\|_{L^m(\Omega)}^m \right) (\xi) + I_{0+}^\rho \left(\sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \right) (\xi) \leq \mathcal{E}(0) + \frac{C_* |\Omega| \xi^\rho}{\Gamma(\rho+1)}.$$

Set

$$c_d := \min\{2C_0 d, 1\} > 0.$$

Then

$$2C_0 d \sum_{i=1}^2 \|\nabla \phi_i\|_{L^m(\Omega)}^m + \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \geq c_d \left(\sum_{i=1}^2 \|\nabla \phi_i\|_{L^m(\Omega)}^m + \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \right).$$

Therefore,

$$I_{0+}^\rho \left(\sum_{i=1}^2 \|\nabla \phi_i\|_{L^m(\Omega)}^m + \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \right) (\xi) \leq \frac{1}{c_d} \left[\mathcal{E}(0) + \frac{C_* |\Omega| \xi^\rho}{\Gamma(\rho+1)} \right].$$

For $0 \leq \xi \leq T$, $\xi^\rho \leq T^\rho$, and $\mathcal{E}(0) \leq \max\{C_0, 1\} \|Y^0\|_{\mathbb{H}}^2$. It follows that

$$I_{0+}^\rho \left(\sum_{i=1}^2 \|\nabla \phi_i\|_{L^m(\Omega)}^m + \sum_{i=1}^2 \|\phi_i\|_{L^4(\Omega)}^4 \right) (\xi) \leq C \left(1 + \|Y^0\|_{\mathbb{H}}^2 \right),$$

which proves (28).

The estimates are first obtained at the Galerkin or regularized level. Their validity for weak solutions follows by passage to the limit, using weak lower semicontinuity of the spatial and quartic terms. $\square$

Definition 4 (Fractional absorbing ball). *A bounded set $\mathcal{B} \subset \mathbb{H}$ is absorbing for the solution family if for every bounded $B \subset \mathbb{H}$ there is $T_B > 0$ such that every solution with $Y^0 \in B$ satisfies $Y(\xi) \in \mathcal{B}$ for all $\xi \geq T_B$.*

Corollary 2 (Absorbing set in $\mathbb{H}$). *For every $R_* > C_* |\Omega| / (\eta_0 \min\{C_0, 1\})$, the ball*

$$\mathcal{B}_{R_*} := \{Y \in \mathbb{H} : \|Y\|_{\mathbb{H}}^2 \leq R_*\}$$

is absorbing.

Proof. Since $E_\rho(-\eta_0 \xi^\rho) \rightarrow 0$, estimate (27) gives

$$\limsup_{\xi \rightarrow \infty} \|Y(\xi)\|_{\mathbb{H}}^2 \leq \frac{C_* |\Omega|}{\eta_0 \min\{C_0, 1\}}.$$

The convergence is uniform for initial data in a bounded subset of $\mathbb{H}$, which proves the assertion. $\square$

Remark 5 (No classical semiflow assertion). For $0 < \rho < 1$, the state at time ξ does not determine the future without its memory history. Corollary 2 is therefore stated for the solution family. A genuine semiflow can be recovered on a suitable history space, or one may work with a process or trajectory dynamical system.

5. Existence of Global Weak Solutions

The proof combines the classical Galerkin–monotonicity strategy [36,41] with fractional compactness and weak-time regularity in the spirit of [39,40].

Theorem 2 (Global existence). Assume (11) and let $Y^0 \in \mathbb{H}$. If the energy relation is required at $\xi = 0$, assume additionally that $\phi_i^0 \in L^4(\Omega)$, $i = 1, 2$. Then problem (8)–(10) possesses at least one global weak solution in the sense of Definition 3. Moreover, every weak solution satisfies (27) and (28).

Proof. We divide the proof into several steps.

Step 1: Construction of the Galerkin approximations.

Set

$$X_m := V_m \cap L^4(\Omega), \quad \|z\|_{X_m} := \|z\|_{V_m} + \|z\|_{L^4(\Omega)}.$$

Choose a sequence $(e_k)_{k \geq 1} \subset X_m$ that is orthonormal in $H = L^2(\Omega)$ and whose linear span is dense in X_m . For every $N \in \mathbb{N}$, define

$$H_N := \text{span}\{e_1, \dots, e_N\}.$$

When $\partial\Omega$ is sufficiently regular, the functions e_k may be chosen as smooth eigenfunctions of the Neumann Laplacian.

Choose initial approximations

$$\phi_{i,N}^0, \psi_{i,N}^0, \chi_{i,N}^0 \in H_N$$

such that

$$\phi_{i,N}^0 \longrightarrow \phi_i^0, \quad \psi_{i,N}^0 \longrightarrow \psi_i^0, \quad \chi_{i,N}^0 \longrightarrow \chi_i^0$$

strongly in H as $N \rightarrow \infty$. If $\phi_i^0 \in L^4(\Omega)$ then the approximations may also be chosen so that

$$\phi_{i,N}^0 \longrightarrow \phi_i^0 \quad \text{strongly in } L^4(\Omega).$$

We seek approximate solutions of the form

$$\phi_{i,N}(\xi) = \sum_{k=1}^N a_{i,k}(\xi) e_k,$$

$$\psi_{i,N}(\xi) = \sum_{k=1}^N b_{i,k}(\xi) e_k,$$

and

$$\chi_{i,N}(\xi) = \sum_{k=1}^N c_{i,k}(\xi) e_k,$$

for $i = 1, 2$.

For every $\ell = 1, \dots, N$ and $i, j \in \{1, 2\}$ with $i \neq j$, the Galerkin equations are

$$\begin{aligned} & \left({}^C \mathcal{D}_{\xi}^{\rho} \phi_{i,N}, e_{\ell} \right) + d \int_{\Omega} |\nabla \phi_{i,N}|^{m-2} \nabla \phi_{i,N} \cdot \nabla e_{\ell} \, dx \\ &= \left(a \phi_{i,N}^2 - b \phi_{i,N}^3 + \psi_{i,N} - \chi_{i,N} + J + \kappa(\phi_{j,N} - \phi_{i,N}), e_{\ell} \right), \\ & \left({}^C \mathcal{D}_{\xi}^{\rho} \psi_{i,N}, e_{\ell} \right) = \left(\alpha_0 - \psi_{i,N} - \beta \phi_{i,N}^2, e_{\ell} \right), \end{aligned}$$

and

$$\left({}^C \mathcal{D}_{\xi}^{\rho} \chi_{i,N}, e_{\ell} \right) = (q(\phi_{i,N} - c) - r \chi_{i,N}, e_{\ell}).$$

Let $Y_N(\xi) \in \mathbb{R}^{6N}$ denote the vector containing all the Galerkin coefficients. The preceding system can be written in the finite-dimensional form

$${}^C \mathcal{D}_{\xi}^{\rho} Y_N(\xi) = \mathcal{G}_N(Y_N(\xi)), \quad Y_N(0) = Y_N^0.$$

The mapping

$$\mathcal{G}_N : \mathbb{R}^{6N} \longrightarrow \mathbb{R}^{6N}$$

is continuous and locally Lipschitz. Indeed, the reaction terms are polynomials in the Galerkin coefficients. Moreover, for $m \geq 2$, the mapping

$$z \longmapsto |z|^{m-2} z$$

is locally Lipschitz on finite-dimensional spaces. Consequently, the finite-dimensional realization of the nonlinear diffusion operator is locally Lipschitz.

The Galerkin system is equivalent to the Volterra integral equation

$$Y_N(\xi) = Y_N^0 + \frac{1}{\Gamma(\rho)} \int_0^{\xi} (\xi - s)^{\rho-1} \mathcal{G}_N(Y_N(s)) \, ds.$$

The local existence theorem for finite-dimensional Caputo systems therefore provides a unique maximal solution

$$Y_N \in C([0, T_N]; \mathbb{R}^{6N})$$

for some $T_N > 0$. If $T_N < \infty$ then the continuation criterion implies

$$|Y_N(\xi)| \longrightarrow \infty \quad \text{as } \xi \uparrow T_N.$$

Step 2: Uniform energy estimates.

Since the Galerkin solutions take values in H_N , we may test the three Galerkin equations by

$$C_0 \phi_{i,N}, \quad \psi_{i,N}, \quad \chi_{i,N},$$

respectively. Repeating the proof of Theorem 1, we obtain

$${}^C \mathcal{D}_{\xi}^{\rho} \mathcal{E}_N(\xi) + 2C_0 d \sum_{i=1}^2 \|\nabla \phi_{i,N}(\xi)\|_{L^m(\Omega)}^m + \eta_0 \mathcal{E}_N(\xi) + \sum_{i=1}^2 \|\phi_{i,N}(\xi)\|_{L^4(\Omega)}^4 \leq C_* |\Omega|,$$

where

$$\mathcal{E}_N(\xi) = C_0 \sum_{i=1}^2 \|\phi_{i,N}(\xi)\|^2 + \sum_{i=1}^2 \|\psi_{i,N}(\xi)\|^2 + \sum_{i=1}^2 \|\chi_{i,N}(\xi)\|^2.$$

The constants C_0 , η_0 , and C_* are independent of N .

By the fractional comparison principle,

$$\mathcal{E}_N(\xi) \leq \mathcal{E}_N(0)E_\rho(-\eta_0\xi^\rho) + \frac{C_*|\Omega|}{\eta_0}[1 - E_\rho(-\eta_0\xi^\rho)].$$

Because the initial approximations are uniformly bounded in $\mathbb{H}$, there exists a constant $C > 0$, independent of N and ξ , such that

$$\sup_{\xi \geq 0} \|Y_N(\xi)\|_{\mathbb{H}}^2 \leq C \left(1 + \|Y^0\|_{\mathbb{H}}^2\right). \quad (29)$$

The estimate (29) prevents the Galerkin coefficient vector from becoming unbounded in finite time. Therefore, the finite-dimensional continuation criterion gives

$$T_N = \infty.$$

Thus, every Galerkin solution exists globally.

Applying I_{0+}^ρ to the Galerkin energy inequality at a fixed time $T > 0$ yields

$$2C_0 d I_{0+}^\rho \left(\sum_{i=1}^2 \|\nabla \phi_{i,N}\|_{L^m(\Omega)}^m \right) (T) + I_{0+}^\rho \left(\sum_{i=1}^2 \|\phi_{i,N}\|_{L^4(\Omega)}^4 \right) (T) \leq C_T,$$

where $C_T > 0$ is independent of N .

Let $g \geq 0$. Since $0 < \rho \leq 1$, one has

$$(T-s)^{\rho-1} \geq T^{\rho-1}, \quad 0 \leq s < T.$$

Consequently,

$$I_{0+}^\rho g(T) = \frac{1}{\Gamma(\rho)} \int_0^T (T-s)^{\rho-1} g(s) \, ds \geq \frac{T^{\rho-1}}{\Gamma(\rho)} \int_0^T g(s) \, ds.$$

It follows that

$$\sum_{i=1}^2 \int_0^T \|\nabla \phi_{i,N}(\xi)\|_{L^m(\Omega)}^m \, d\xi \leq C_T$$

and

$$\sum_{i=1}^2 \int_0^T \|\phi_{i,N}(\xi)\|_{L^4(\Omega)}^4 \, d\xi \leq C_T.$$

Because the boundary condition is of Neumann type, the energy estimate controls the gradient rather than the full $W^{1,m}(\Omega)$ -norm. We use the Neumann estimate

$$\|z\|_{L^m(\Omega)} \leq C \left(\|\nabla z\|_{L^m(\Omega)} + \|z\|_{L^2(\Omega)} \right), \quad z \in W^{1,m}(\Omega).$$

Combining this estimate with (29), we obtain

$$\sum_{i=1}^2 \|\phi_{i,N}\|_{L^m(0,T;V_m)}^m + \sum_{i=1}^2 \|\phi_{i,N}\|_{L^4(Q_T)}^4 \leq C_T. \quad (30)$$

Moreover,

$$\|\psi_{i,N}\|_{L^\infty(0,T;H)} + \|\chi_{i,N}\|_{L^\infty(0,T;H)} \leq C_T.$$

Step 3: Estimates for the Caputo derivatives.

By Lemma 4,

$$\|\mathcal{A}_m \phi_{i,N}\|_{V_m^*} \leq \|\nabla \phi_{i,N}\|_{L^m(\Omega)}^{m-1}.$$

Since $m' = m/(m-1)$, it follows that

$$\begin{aligned} \int_0^T \|\mathcal{A}_m \phi_{i,N}\|_{V_m^*}^{m'} d\xi &\leq \int_0^T \|\nabla \phi_{i,N}\|_{L^m(\Omega)}^{(m-1)m'} d\xi \\ &= \int_0^T \|\nabla \phi_{i,N}\|_{L^m(\Omega)}^m d\xi \\ &\leq C_T. \end{aligned}$$

Thus,

$$(\mathcal{A}_m \phi_{i,N}) \text{ is bounded in } L^{m'}(0, T; V_m^*).$$

From the potential equation and the growth estimate for the reaction map, we have

$$\begin{aligned} \left\| {}^C \mathfrak{D}_\xi^\rho \phi_{i,N} \right\|_{X_m^*} &\leq d \|\mathcal{A}_m \phi_{i,N}\|_{V_m^*} + C \left(1 + \|\phi_{i,N}\|_{L^4(\Omega)}^3 \right. \\ &\quad \left. + \|\psi_{i,N}\| + \|\chi_{i,N}\| + \|\phi_{i,N}\| + \|\phi_{j,N}\| \right). \end{aligned}$$

The estimates (29) and (30) therefore imply that $({}^C \mathfrak{D}_\xi^\rho \phi_{i,N})$ is bounded in

$$L^{m'}(0, T; V_m^*) + L^{4/3}(0, T; L^{4/3}(\Omega)) + L^2(0, T; H).$$

All these spaces are continuously embedded in $L^{q_0}(0, T; X_m^*)$, where

$$q_0 := \min \left\{ m', \frac{4}{3} \right\} > 1.$$

Hence,

$$({}^C \mathfrak{D}_\xi^\rho \phi_{i,N}) \text{ is bounded in } L^{q_0}(0, T; X_m^*).$$

For the fast variable,

$${}^C \mathfrak{D}_\xi^\rho \psi_{i,N} = \alpha_0 - \psi_{i,N} - \beta \phi_{i,N}^2.$$

Since

$$\left\| \phi_{i,N}^2 \right\|_{L^2(Q_T)} = \|\phi_{i,N}\|_{L^4(Q_T)}^2,$$

the sequence $({}^C \mathfrak{D}_\xi^\rho \psi_{i,N})$ is bounded in $L^2(0, T; H)$.

Similarly,

$${}^C \mathfrak{D}_\xi^\rho \chi_{i,N} = q(\phi_{i,N} - c) - r\chi_{i,N},$$

and hence $({}^C \mathfrak{D}_\xi^\rho \chi_{i,N})$ is bounded in $L^2(0, T; H)$.

Step 4: Weak compactness.

By (29), (30), and the Banach–Alaoglu theorem, there exist functions

$$\phi_i \in L^\infty(0, T; H) \cap L^m(0, T; V_m) \cap L^4(Q_T), \quad \psi_i, \chi_i \in L^\infty(0, T; H),$$

and a subsequence, still indexed by N , such that

$$\begin{aligned} Y_N &\overset{*}{\rightharpoonup} Y && \text{in } L^\infty(0, T; \mathbb{H}), \\ \phi_{i,N} &\rightharpoonup \phi_i && \text{in } L^m(0, T; V_m), \end{aligned}$$

$$\phi_{i,N} \rightharpoonup \phi_i \quad \text{in } L^4(Q_T), \quad \psi_{i,N} \xrightarrow{*} \psi_i \quad \text{in } L^\infty(0, T; H),$$

and

$$\chi_{i,N} \xrightarrow{*} \chi_i \quad \text{in } L^\infty(0, T; H).$$

Furthermore, there exists $\sigma_i \in L^{m'}(0, T; V_m^*)$ such that $\mathcal{A}_m \phi_{i,N} \rightharpoonup \sigma_i$ in $L^{m'}(0, T; V_m^*)$. After passing to a further subsequence,

$${}^C \mathfrak{D}_\xi^\rho \phi_{i,N} \rightharpoonup {}^C \mathfrak{D}_\xi^\rho \phi_i \quad \text{in } L^{q_0}(0, T; X_m^*),$$

$${}^C \mathfrak{D}_\xi^\rho \psi_{i,N} \rightharpoonup {}^C \mathfrak{D}_\xi^\rho \psi_i \quad \text{in } L^2(0, T; H),$$

and

$${}^C \mathfrak{D}_\xi^\rho \chi_{i,N} \rightharpoonup {}^C \mathfrak{D}_\xi^\rho \chi_i \quad \text{in } L^2(0, T; H).$$

The weak limits of the Caputo derivatives are identified in the distributional sense by testing against smooth functions compactly supported in $(0, T)$.

Step 5: Strong compactness of the membrane potentials.

Set $p_0 := \min\{m, 4\}$. The estimate (30) implies that $(\phi_{i,N})$ is bounded in $L^{p_0}(0, T; X_m)$. Moreover, $X_m \Subset H \hookrightarrow X_m^*$. Indeed, for $m \geq 2$ and $1 \leq n \leq 3$ the embedding $W^{1,m}(\Omega) \Subset L^2(\Omega)$ is compact, while the embedding $H \hookrightarrow X_m^*$ is continuous.

By the fractional compactness lemma,

$$\phi_{i,N} \longrightarrow \phi_i \quad \text{strongly in } L^{p_0}(0, T; H).$$

In particular,

$$\phi_{i,N} \longrightarrow \phi_i \quad \text{strongly in } L^2(Q_T).$$

After passing to a further subsequence,

$$\phi_{i,N}(x, \xi) \longrightarrow \phi_i(x, \xi)$$

for almost every $(x, \xi) \in Q_T$.

Interpolating between the strong convergence in $L^2(Q_T)$ and the uniform boundedness in $L^4(Q_T)$ gives

$$\phi_{i,N} \longrightarrow \phi_i \quad \text{strongly in } L^s(Q_T)$$

for every $2 \leq s < 4$.

Consequently,

$$\phi_{i,N}^2 \longrightarrow \phi_i^2 \quad \text{strongly in } L^q(Q_T)$$

for every $1 \leq q < 2$. Since $(\phi_{i,N}^2)$ is bounded in $L^2(Q_T)$ and converges almost everywhere, we also have

$$\phi_{i,N}^2 \rightharpoonup \phi_i^2 \quad \text{in } L^2(Q_T).$$

Similarly, $(\phi_{i,N}^3)$ is bounded in $L^{4/3}(Q_T)$ and converges almost everywhere. Therefore,

$$\phi_{i,N}^3 \rightharpoonup \phi_i^3 \quad \text{in } L^{4/3}(Q_T).$$

The linear reaction and coupling terms pass to the limit by weak or strong convergence. In particular, the products involving $\phi_{i,N}$ and the weakly convergent variables $\psi_{i,N}$ and $\chi_{i,N}$ pass to the limit because $\phi_{i,N} \rightarrow \phi_i$ strongly in $L^2(Q_T)$.

Step 6: Identification of the nonlinear diffusion term.

It remains to prove that

$$\sigma_i = \mathcal{A}_m \phi_i.$$

For every $z \in L^m(0, T; V_m)$, the monotonicity of $\mathcal{A}_m$ gives

$$\int_0^T \langle \mathcal{A}_m \phi_{i,N} - \mathcal{A}_m z, \phi_{i,N} - z \rangle_{V_m^*, V_m} d\zeta \geq 0.$$

We first establish the corresponding Minty limsup condition. Test the i -th Galerkin potential equation by $\phi_{i,N}$ and integrate over $(0, T)$. The strong convergence in $L^3(Q_T)$ implies

$$\int_{Q_T} \phi_{i,N}^3 dx d\zeta \longrightarrow \int_{Q_T} \phi_i^3 dx d\zeta.$$

The strong convergence of $\phi_{i,N}$ in $L^2(Q_T)$ and the weak convergence of $\psi_{i,N}$ and $\chi_{i,N}$ imply

$$\int_{Q_T} \psi_{i,N} \phi_{i,N} dx d\zeta \longrightarrow \int_{Q_T} \psi_i \phi_i dx d\zeta$$

and

$$\int_{Q_T} \chi_{i,N} \phi_{i,N} dx d\zeta \longrightarrow \int_{Q_T} \chi_i \phi_i dx d\zeta.$$

The coupling terms pass to the limit by the strong $L^2(Q_T)$ convergence of both membrane potentials.

Since

$$\phi_{i,N} \rightharpoonup \phi_i \quad \text{in } L^4(Q_T),$$

weak lower semicontinuity yields

$$\liminf_{N \rightarrow \infty} \int_{Q_T} |\phi_{i,N}|^4 dx d\zeta \geq \int_{Q_T} |\phi_i|^4 dx d\zeta.$$

The positive representation underlying Lemma 1, together with a standard time regularization argument, gives

$$\liminf_{N \rightarrow \infty} \int_0^T \langle {}^c \mathfrak{D}_\zeta^\rho \phi_{i,N}, \phi_{i,N} \rangle d\zeta \geq \int_0^T \langle {}^c \mathfrak{D}_\zeta^\rho \phi_i, \phi_i \rangle d\zeta.$$

Passing to the upper limit in the tested Galerkin equation and using the limiting potential equation tested by ϕ_i , we obtain

$$\limsup_{N \rightarrow \infty} \int_0^T \langle \mathcal{A}_m \phi_{i,N}, \phi_{i,N} \rangle_{V_m^*, V_m} d\zeta \leq \int_0^T \langle \sigma_i, \phi_i \rangle_{V_m^*, V_m} d\zeta.$$

We may now pass to the limit in the monotonicity inequality. Using the weak convergences and the preceding limsup estimate, we find

$$\int_0^T \langle \sigma_i - \mathcal{A}_m z, \phi_i - z \rangle_{V_m^*, V_m} d\zeta \geq 0$$

for every $z \in L^m(0, T; V_m)$.

Let $h \in L^m(0, T; V_m)$ and choose $z = \phi_i - \varepsilon h$, $\varepsilon > 0$. Then

$$\int_0^T \langle \sigma_i - \mathcal{A}_m(\phi_i - \varepsilon h), h \rangle_{V_m^*, V_m} d\zeta \geq 0.$$

By the hemicontinuity of $\mathcal{A}_m$, letting $\varepsilon \downarrow 0$ gives

$$\int_0^T \langle \sigma_i - \mathcal{A}_m \phi_i, h \rangle_{V_m^*, V_m} d\zeta \geq 0.$$

Replacing h by $-h$ gives the reverse inequality. Therefore,

$$\int_0^T \langle \sigma_i - \mathcal{A}_m \phi_i, h \rangle_{V_m^*, V_m} d\zeta = 0$$

for every $h \in L^m(0, T; V_m)$. Hence, $\sigma_i = \mathcal{A}_m \phi_i$ in $L^{m'}(0, T; V_m^*)$.

Step 7: Passage to the limit in the Galerkin system.

Let $\Theta = \text{col}(\theta_1, \vartheta_1, \zeta_1, \theta_2, \vartheta_2, \zeta_2) \in \mathbb{V}$ and let $\eta \in C_c^\infty(0, T)$. Multiply the Galerkin equations by η , integrate in time, and pass to the limit.

The Caputo terms pass to the limit by their weak convergence. The nonlinear diffusion terms pass to the limit because $\sigma_i = \mathcal{A}_m \phi_i$. The quadratic terms pass weakly in $L^2(Q_T)$, the cubic terms pass weakly in $L^{4/3}(Q_T)$, and all the linear reaction and coupling terms pass by weak or strong convergence.

Therefore,

$$\int_0^T \eta(\zeta) \left[\left\langle {}^C \mathfrak{D}_\zeta^\rho Y(\zeta), \Theta \right\rangle_{\mathbb{V}^*, \mathbb{V}} + d \sum_{i=1}^2 \int_\Omega |\nabla \phi_i|^{m-2} \nabla \phi_i \cdot \nabla \theta_i dx - \langle \mathcal{F}(Y(\zeta)), \Theta \rangle_{\mathbb{V}^*, \mathbb{V}} \right] d\zeta = 0.$$

Since η is arbitrary, it follows that

$$\left\langle {}^C \mathfrak{D}_\zeta^\rho Y(\zeta), \Theta \right\rangle_{\mathbb{V}^*, \mathbb{V}} + d \sum_{i=1}^2 \int_\Omega |\nabla \phi_i|^{m-2} \nabla \phi_i \cdot \nabla \theta_i dx = \langle \mathcal{F}(Y(\zeta)), \Theta \rangle_{\mathbb{V}^*, \mathbb{V}}$$

for almost every $\zeta \in (0, T)$. Thus, the weak formulation (13) is satisfied.

Step 8: Verification of the initial condition.

The Galerkin solutions satisfy the Volterra representation

$$Y_N(\zeta) = Y_N^0 + I_{0+}^\rho (\mathcal{F}(Y_N) - \mathbb{A} Y_N)(\zeta)$$

in the corresponding finite-dimensional dual space.

Let Θ be a smooth test vector. Pairing the preceding identity with Θ gives

$$(Y_N(\zeta), \Theta)_{\mathbb{H}} = (Y_N^0, \Theta)_{\mathbb{H}} + \frac{1}{\Gamma(\rho)} \int_0^\zeta (\zeta - s)^{\rho-1} \langle \mathcal{F}(Y_N(s)) - \mathbb{A} Y_N(s), \Theta \rangle ds.$$

The estimates obtained above imply that the scalar integrand is bounded in $L^q(0, T)$ for some $q > 1$. Therefore, its fractional integral tends to zero as $\zeta \downarrow 0$. Passing to the limit in the scalar Volterra identity gives

$$\lim_{\zeta \downarrow 0} (Y(\zeta), \Theta)_{\mathbb{H}} = (Y^0, \Theta)_{\mathbb{H}}.$$

Since the smooth test vectors form a dense subset of $\mathbb{H}$ and $Y \in L^\infty(0, T; \mathbb{H})$, this identity extends to every $\Theta \in \mathbb{H}$. Consequently,

$$Y \in C_w([0, T]; \mathbb{H})$$

and

$$Y(0) = Y^0 \quad \text{in } \mathbb{H}.$$

Step 9: Passage of the energy estimates to the limit.

The weak-star lower semicontinuity of the $\mathbb{H}$ -norm gives

$$\|Y(\zeta)\|_{\mathbb{H}}^2 \leq \liminf_{N \rightarrow \infty} \|Y_N(\zeta)\|_{\mathbb{H}}^2$$

for almost every $\xi \in (0, T)$. Similarly, weak lower semicontinuity gives

$$\int_0^T \|\nabla \phi_i\|_{L^m(\Omega)}^m d\xi \leq \liminf_{N \rightarrow \infty} \int_0^T \|\nabla \phi_{i,N}\|_{L^m(\Omega)}^m d\xi$$

and

$$\int_0^T \|\phi_i\|_{L^4(\Omega)}^4 d\xi \leq \liminf_{N \rightarrow \infty} \int_0^T \|\phi_{i,N}\|_{L^4(\Omega)}^4 d\xi.$$

Thus, the Galerkin energy estimates pass to the limiting weak solution. In particular, the limiting solution satisfies (27) and (28).

Step 10: Global extension.

The preceding construction is valid on every finite interval $[0, T]$. Moreover, the Galerkin solutions are globally defined and the estimates are uniform on each finite interval. Applying a diagonal subsequence argument on the intervals

$$[0, 1], [0, 2], [0, 3], \dots$$

produces a weak solution defined on $[0, \infty)$. Therefore, the weak solution is global.

Finally, the same energy inequality holds for any weak solution. This may be shown by regularizing the weak solution in time, using the regularized functions as test functions, applying Lemma 1, and then removing the regularization by weak lower semicontinuity. Corollary 1 consequently implies that every weak solution satisfies (27) and (28). $\square$

Remark 6. The proof rests on the cubic damping rather than on any global-Lipschitz property: the map $\phi \mapsto a\phi^2 - b\phi^3$ is not globally Lipschitz on $L^2(\Omega)$.

6. Uniqueness and Continuous Dependence

For uniqueness one needs enough integrability to control the local polynomial coefficient. This is automatic in the strong class and, in dimensions $n \leq 3$, follows for positive times from the parabolic regularization available when $m = 2$. We state a transparent sufficient condition.

Assumption 1. The weak solutions under comparison satisfy $\phi_i, \bar{\phi}_i \in L^\infty(Q_T)$, $i = 1, 2$.

Theorem 3 (Uniqueness and continuous dependence). Let Y and $\bar{Y}$ be two weak solutions satisfying Assumption 1, with initial data Y^0 and $\bar{Y}^0$. Then

$$\|Y(\xi) - \bar{Y}(\xi)\|_{\mathbb{H}}^2 \leq C \|Y^0 - \bar{Y}^0\|_{\mathbb{H}}^2 E_\rho(L\xi^\rho), \quad 0 \leq \xi \leq T, \quad (31)$$

where L depends only on the parameters and the L^∞ bounds of the two solutions. In particular, $Y^0 = \bar{Y}^0$ implies $Y = \bar{Y}$.

Proof. Set

$$U_i := \phi_i - \bar{\phi}_i, \quad V_i := \psi_i - \bar{\psi}_i, \quad W_i := \chi_i - \bar{\chi}_i, \quad i = 1, 2.$$

Then

$$Y - \bar{Y} = \text{col}(U_1, V_1, W_1, U_2, V_2, W_2).$$

Subtracting the equations satisfied by Y and $\bar{Y}$, we obtain, for $i, j \in \{1, 2\}$ with $i \neq j$,

$${}^c\mathcal{D}_\xi^\rho U_i + d(\mathcal{A}_m \phi_i - \mathcal{A}_m \bar{\phi}_i) = a(\phi_i^2 - \bar{\phi}_i^2) - b(\phi_i^3 - \bar{\phi}_i^3) + V_i - W_i + \kappa(U_j - U_i),$$

$${}^C\mathfrak{D}_\xi^\rho V_i = -V_i - \beta(\phi_i^2 - \bar{\phi}_i^2),$$

and

$${}^C\mathfrak{D}_\xi^\rho W_i = qU_i - rW_i.$$

Using the algebraic identities

$$\phi_i^2 - \bar{\phi}_i^2 = (\phi_i + \bar{\phi}_i)U_i$$

and

$$\phi_i^3 - \bar{\phi}_i^3 = (\phi_i^2 + \phi_i\bar{\phi}_i + \bar{\phi}_i^2)U_i,$$

the difference system becomes

$${}^C\mathfrak{D}_\xi^\rho U_i + d(\mathcal{A}_m\phi_i - \mathcal{A}_m\bar{\phi}_i) = \left[a(\phi_i + \bar{\phi}_i) - b(\phi_i^2 + \phi_i\bar{\phi}_i + \bar{\phi}_i^2) \right] U_i + V_i - W_i + \kappa(U_i - U_j),$$

$${}^C\mathfrak{D}_\xi^\rho V_i = -V_i - \beta(\phi_i + \bar{\phi}_i)U_i,$$

and

$${}^C\mathfrak{D}_\xi^\rho W_i = qU_i - rW_i.$$

Test these three equations by U_i , V_i , and W_i , respectively, integrate over Ω , and sum over $i = 1, 2$. By Lemma 1,

$$\langle {}^C\mathfrak{D}_\xi^\rho U_i, U_i \rangle \geq \frac{1}{2} {}^C\mathfrak{D}_\xi^\rho \|U_i\|^2,$$

$$\langle {}^C\mathfrak{D}_\xi^\rho V_i, V_i \rangle \geq \frac{1}{2} {}^C\mathfrak{D}_\xi^\rho \|V_i\|^2,$$

and

$$\langle {}^C\mathfrak{D}_\xi^\rho W_i, W_i \rangle \geq \frac{1}{2} {}^C\mathfrak{D}_\xi^\rho \|W_i\|^2.$$

Moreover, the strong monotonicity of $\mathcal{A}_m$, established in Lemma 4, gives

$$\begin{aligned} \langle \mathcal{A}_m\phi_i - \mathcal{A}_m\bar{\phi}_i, \phi_i - \bar{\phi}_i \rangle_{V_m^*, V_m} &\geq c_m \|\nabla\phi_i - \nabla\bar{\phi}_i\|_{L^m(\Omega)}^m \\ &= c_m \|\nabla U_i\|_{L^m(\Omega)}^m. \end{aligned}$$

The coupling terms are dissipative. Indeed,

$$\kappa \int_{\Omega} \left[(U_2 - U_1)U_1 + (U_1 - U_2)U_2 \right] dx = -\kappa \int_{\Omega} |U_1 - U_2|^2 dx = -\kappa \|U_1 - U_2\|^2.$$

Therefore,

$$\begin{aligned} \frac{1}{2} {}^C\mathfrak{D}_\xi^\rho \sum_{i=1}^2 \left(\|U_i\|^2 + \|V_i\|^2 + \|W_i\|^2 \right) + dc_m \sum_{i=1}^2 \|\nabla U_i\|_{L^m(\Omega)}^m \\ + \kappa \|U_1 - U_2\|^2 + \sum_{i=1}^2 \|V_i\|^2 + r \sum_{i=1}^2 \|W_i\|^2 \\ \leq \sum_{i=1}^2 \mathcal{R}_i, \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_i &:= \int_{\Omega} \left[a(\phi_i + \bar{\phi}_i) - b(\phi_i^2 + \phi_i\bar{\phi}_i + \bar{\phi}_i^2) \right] U_i^2 dx \\ &\quad + \int_{\Omega} \left[U_i V_i - U_i W_i - \beta(\phi_i + \bar{\phi}_i) U_i V_i + q U_i W_i \right] dx. \end{aligned}$$

By Assumption 1, there exists a constant $M_T > 0$ such that

$$\|\phi_i\|_{L^\infty(Q_T)} \leq M_T, \quad \|\bar{\phi}_i\|_{L^\infty(Q_T)} \leq M_T, \quad i = 1, 2.$$

Consequently,

$$|\phi_i + \bar{\phi}_i| \leq 2M_T$$

almost everywhere in Q_T .

Since

$$\phi_i^2 + \phi_i \bar{\phi}_i + \bar{\phi}_i^2 = \left(\phi_i + \frac{1}{2}\bar{\phi}_i\right)^2 + \frac{3}{4}\bar{\phi}_i^2 \geq 0,$$

we have

$$a(\phi_i + \bar{\phi}_i) - b(\phi_i^2 + \phi_i \bar{\phi}_i + \bar{\phi}_i^2) \leq a|\phi_i + \bar{\phi}_i| \leq 2aM_T.$$

Hence,

$$\int_{\Omega} \left[a(\phi_i + \bar{\phi}_i) - b(\phi_i^2 + \phi_i \bar{\phi}_i + \bar{\phi}_i^2) \right] U_i^2 \, dx \leq 2aM_T \|U_i\|^2.$$

The two terms containing $U_i V_i$ may be combined as

$$U_i V_i - \beta(\phi_i + \bar{\phi}_i) U_i V_i = \left[1 - \beta(\phi_i + \bar{\phi}_i) \right] U_i V_i.$$

Since

$$|1 - \beta(\phi_i + \bar{\phi}_i)| \leq 1 + 2\beta M_T,$$

Young's inequality gives

$$\left| \int_{\Omega} \left[1 - \beta(\phi_i + \bar{\phi}_i) \right] U_i V_i \, dx \right| \leq (1 + 2\beta M_T) \|U_i\| \|V_i\| \leq \frac{(1 + 2\beta M_T)^2}{2} \|U_i\|^2 + \frac{1}{2} \|V_i\|^2.$$

Similarly, the terms containing $U_i W_i$ satisfy

$$-U_i W_i + q U_i W_i = (q - 1) U_i W_i.$$

Therefore,

$$\begin{aligned} \left| (q - 1) \int_{\Omega} U_i W_i \, dx \right| &\leq |q - 1| \|U_i\| \|W_i\| \\ &\leq \frac{(q - 1)^2}{2r} \|U_i\|^2 + \frac{r}{2} \|W_i\|^2. \end{aligned}$$

Combining the preceding estimates, we obtain

$$\mathcal{R}_i \leq K_T \|U_i\|^2 + \frac{1}{2} \|V_i\|^2 + \frac{r}{2} \|W_i\|^2,$$

where

$$K_T := 2aM_T + \frac{(1 + 2\beta M_T)^2}{2} + \frac{(q - 1)^2}{2r}.$$

Thus,

$$\sum_{i=1}^2 \mathcal{R}_i \leq K_T \sum_{i=1}^2 \|U_i\|^2 + \frac{1}{2} \sum_{i=1}^2 \|V_i\|^2 + \frac{r}{2} \sum_{i=1}^2 \|W_i\|^2.$$

Substituting this estimate into the preceding energy inequality gives

$$\begin{aligned} \frac{1}{2} c \mathfrak{D}_{\xi}^{\rho} \sum_{i=1}^2 \left(\|U_i\|^2 + \|V_i\|^2 + \|W_i\|^2 \right) + dc_m \sum_{i=1}^2 \|\nabla U_i\|_{L^m(\Omega)}^m + \kappa \|U_1 - U_2\|^2 + \frac{1}{2} \sum_{i=1}^2 \|V_i\|^2 \\ + \frac{r}{2} \sum_{i=1}^2 \|W_i\|^2 \leq K_T \sum_{i=1}^2 \|U_i\|^2. \end{aligned}$$

Define

$$Y(\xi) := \sum_{i=1}^2 \left(\|U_i(\xi)\|^2 + \|V_i(\xi)\|^2 + \|W_i(\xi)\|^2 \right).$$

By the definition of the product Hilbert norm,

$$Y(\xi) = \|Y(\xi) - \bar{Y}(\xi)\|_{\mathbb{H}}^2.$$

Since

$$\sum_{i=1}^2 \|U_i\|^2 \leq Y(\xi),$$

and all the remaining terms on the left-hand side are non-negative, we obtain

$$\frac{1}{2} c \mathfrak{D}_{\xi}^{\rho} Y(\xi) \leq K_T Y(\xi).$$

Equivalently,

$$c \mathfrak{D}_{\xi}^{\rho} Y(\xi) \leq 2K_T Y(\xi).$$

Applying the fractional Gronwall inequality from Lemma 3 yields

$$Y(\xi) \leq Y(0) E_{\rho}(2K_T \xi^{\rho}), \quad 0 \leq \xi \leq T.$$

Since

$$Y(0) = \|Y^0 - \bar{Y}^0\|_{\mathbb{H}}^2,$$

we conclude that

$$\|Y(\xi) - \bar{Y}(\xi)\|_{\mathbb{H}}^2 \leq \|Y^0 - \bar{Y}^0\|_{\mathbb{H}}^2 E_{\rho}(2K_T \xi^{\rho}), \quad 0 \leq \xi \leq T.$$

Thus, (31) holds with

$$C = 1 \quad \text{and} \quad L = 2K_T.$$

The constant L depends only on the model parameters and on the $L^{\infty}(Q_T)$ bounds of the membrane-potential components of the two solutions.

Finally, if

$$Y^0 = \bar{Y}^0$$

then

$$Y(0) = 0.$$

The preceding estimate gives

$$Y(\xi) = 0 \quad \text{for every } \xi \in [0, T].$$

Therefore,

$$Y(\xi) = \bar{Y}(\xi) \quad \text{for every } \xi \in [0, T],$$

and the weak solution is unique in the class specified by Assumption 1.

For weak solutions, the preceding testing procedure may be justified by a standard temporal regularization or Galerkin approximation. The result then follows by passing to the limit using the weak lower semicontinuity of the Caputo energy term and the monotonicity of $\mathcal{A}_m$. $\square$

Remark 7 (Alternative uniqueness route). *Instead of Assumption 1, one may work with a truncated reaction, show that the energy estimate places every solution in a common invariant bounded set, and then use the Lipschitz constant of the original reaction on that set. This is especially convenient in a fully discrete or one-dimensional setting.*

7. Synchronization of the Two Neurons

The synchronization result is global and requires no a priori bound on the individual membrane potentials. It relies on the cubic structure and on the monotonicity of the m -Laplacian.

Definition 5 (Complete synchronization). *The two neurons synchronize in $\mathcal{H}$ if*

$$\|y_1(\xi) - y_2(\xi)\|_{\mathcal{H}} \longrightarrow 0 \quad \text{as } \xi \rightarrow \infty.$$

They synchronize with a Mittag–Leffler rate if there are $C, \mu > 0$ such that

$$\|y_1(\xi) - y_2(\xi)\|_{\mathcal{H}}^2 \leq CE_{\rho}(-\mu\xi^{\rho}) \|y_1^0 - y_2^0\|_{\mathcal{H}}^2.$$

Set $U = \phi_1 - \phi_2$, $V = \psi_1 - \psi_2$, and $W = \chi_1 - \chi_2$. Subtracting the equations gives

$$\begin{cases} {}^C\mathfrak{D}_{\xi}^{\rho}U + d(\mathcal{A}_m\phi_1 - \mathcal{A}_m\phi_2) = a(\phi_1 + \phi_2)U - b(\phi_1^2 + \phi_1\phi_2 + \phi_2^2)U + V - W - 2\kappa U, \\ {}^C\mathfrak{D}_{\xi}^{\rho}V = -V - \beta(\phi_1 + \phi_2)U, \\ {}^C\mathfrak{D}_{\xi}^{\rho}W = qU - rW. \end{cases} \quad (32)$$

Theorem 4 (Global Mittag–Leffler synchronization). *Define*

$$\lambda := \frac{8\beta^2}{b} \quad (33)$$

and suppose

$$\kappa > \frac{\lambda}{2} + \frac{a^2}{b} + \frac{(q - \lambda)^2}{4\lambda r}. \quad (34)$$

Set

$$\delta := 4\kappa\lambda - \left(2\lambda^2 + \frac{4\lambda a^2}{b} + \frac{(q - \lambda)^2}{r}\right) > 0 \quad (35)$$

and $\mu := \min\{\delta/\lambda, 1, r\}$. Then every global weak solution satisfies

$$\lambda\|U(\xi)\|^2 + \|V(\xi)\|^2 + \|W(\xi)\|^2 \leq (\lambda\|U(0)\|^2 + \|V(0)\|^2 + \|W(0)\|^2)E_{\rho}(-\mu\xi^{\rho}). \quad (36)$$

Consequently the neurons synchronize globally in $\mathcal{H}$ with a uniform Mittag–Leffler rate.

Proof. Let

$$U := \phi_1 - \phi_2, \quad V := \psi_1 - \psi_2, \quad W := \chi_1 - \chi_2.$$

The difference system (32) is

$${}^C\mathfrak{D}_{\xi}^{\rho}U + d(\mathcal{A}_m\phi_1 - \mathcal{A}_m\phi_2) = a(\phi_1 + \phi_2)U - b(\phi_1^2 + \phi_1\phi_2 + \phi_2^2)U + V - W - 2\kappa U,$$

$${}^c\mathfrak{D}_\xi^\rho V = -V - \beta(\phi_1 + \phi_2)U,$$

and

$${}^c\mathfrak{D}_\xi^\rho W = qU - rW.$$

Test the first equation by λU , the second by V , and the third by W . Using Lemma 1 and the monotonicity estimate from Lemma 4, we obtain

$$\begin{aligned} \frac{1}{2} {}^c\mathfrak{D}_\xi^\rho \left(\lambda \|U\|^2 + \|V\|^2 + \|W\|^2 \right) + \lambda d c_m \|\nabla U\|_{L^m(\Omega)}^m + 2\kappa\lambda \|U\|^2 \\ + \|V\|^2 + r\|W\|^2 \leq \mathcal{N} + \lambda(U, V) + (q - \lambda)(U, W), \end{aligned} \quad (37)$$

where

$$\mathcal{N} := \int_{\Omega} \left[\lambda a(\phi_1 + \phi_2)U^2 - \beta(\phi_1 + \phi_2)UV - \lambda b(\phi_1^2 + \phi_1\phi_2 + \phi_2^2)U^2 \right] dx.$$

Since

$$\phi_1^2 + \phi_1\phi_2 + \phi_2^2 \geq \frac{1}{2}(\phi_1^2 + \phi_2^2),$$

and

$$\beta(|\phi_1| + |\phi_2|)|U||V| \leq 2\beta^2(\phi_1^2 + \phi_2^2)U^2 + \frac{1}{4}V^2,$$

we have

$$\mathcal{N} \leq \frac{1}{4}\|V\|^2 + \int_{\Omega} \left[\lambda a(|\phi_1| + |\phi_2|) - \left(\frac{\lambda b}{2} - 2\beta^2 \right) (\phi_1^2 + \phi_2^2) \right] U^2 dx.$$

By (33),

$$\frac{\lambda b}{2} - 2\beta^2 = \frac{\lambda b}{4}.$$

Therefore,

$$\mathcal{N} \leq \frac{1}{4}\|V\|^2 + \lambda \int_{\Omega} \left[a(|\phi_1| + |\phi_2|) - \frac{b}{4}(\phi_1^2 + \phi_2^2) \right] U^2 dx.$$

For every $s \in \mathbb{R}$,

$$a|s| - \frac{b}{4}s^2 \leq \frac{a^2}{b}.$$

Applying this inequality to ϕ_1 and ϕ_2 gives

$$\mathcal{N} \leq \frac{1}{4}\|V\|^2 + \frac{2\lambda a^2}{b}\|U\|^2. \quad (38)$$

The remaining mixed terms satisfy

$$\lambda(U, V) \leq \lambda^2\|U\|^2 + \frac{1}{4}\|V\|^2$$

and

$$(q - \lambda)(U, W) \leq \frac{(q - \lambda)^2}{2r}\|U\|^2 + \frac{r}{2}\|W\|^2.$$

Substituting these estimates and (38) into (37), multiplying by 2, and discarding the non-negative diffusion term, we obtain

$${}^c\mathfrak{D}_\xi^\rho \mathcal{E}_s(\xi) + 4\kappa\lambda\|U\|^2 + \|V\|^2 + r\|W\|^2 \leq \left(2\lambda^2 + \frac{4\lambda a^2}{b} + \frac{(q - \lambda)^2}{r} \right) \|U\|^2,$$

where

$$\mathcal{E}_s(\xi) := \lambda\|U(\xi)\|^2 + \|V(\xi)\|^2 + \|W(\xi)\|^2.$$

By the definition of δ in (35), the preceding inequality becomes

$${}^C\mathcal{D}_{\xi}^{\rho}\mathcal{E}_s(\xi) + \delta\|U\|^2 + \|V\|^2 + r\|W\|^2 \leq 0.$$

Since

$$\mu = \min\left\{\frac{\delta}{\lambda}, 1, r\right\},$$

we have

$$\delta\|U\|^2 + \|V\|^2 + r\|W\|^2 \geq \mu(\lambda\|U\|^2 + \|V\|^2 + \|W\|^2) = \mu\mathcal{E}_s(\xi).$$

Consequently,

$${}^C\mathcal{D}_{\xi}^{\rho}\mathcal{E}_s(\xi) + \mu\mathcal{E}_s(\xi) \leq 0.$$

Applying Lemma 2 yields

$$\mathcal{E}_s(\xi) \leq \mathcal{E}_s(0)E_{\rho}(-\mu\xi^{\rho}), \quad \xi \geq 0,$$

which is precisely (36).

Since $E_{\rho}(-\mu\xi^{\rho}) \rightarrow 0$ as $\xi \rightarrow \infty$, it follows that

$$\|U(\xi)\|^2 + \|V(\xi)\|^2 + \|W(\xi)\|^2 \rightarrow 0.$$

Hence, the two neurons synchronize globally in $\mathcal{H}$ with a Mittag–Leffler decay rate. $\square$

Remark 8 (Integer-order limit). For $\rho = 1$, $E_1(-\mu\xi) = e^{-\mu\xi}$, and Theorem 4 reduces to exponential synchronization of the classical coupled system. For $0 < \rho < 1$ the decay is algebraic at large time,

$$E_{\rho}(-\mu\xi^{\rho}) \sim \frac{1}{\mu\Gamma(1-\rho)\xi^{\rho}}, \quad \xi \rightarrow \infty, \quad 0 < \rho < 1, \quad (39)$$

so that fractional memory slows the asymptotic decay while preserving complete synchronization under the same structural coupling condition.

Remark 9 (Conservativeness of the threshold). Condition (34) is sufficient, not necessary. It results from global Young inequalities and can be much larger than the numerically observed threshold. A sharper criterion follows by retaining independent positive Lyapunov weights and optimizing the associated quadratic form; the gap is quantified numerically in Section 11.

The threshold in Theorem 4 is a global sufficient certificate, not a prediction of the practical synchronization transition. It is uniform in the initial data and spatial profiles and is obtained after worst-case Young estimates that suppress trajectory-specific cancellation. Consequently, the bound may be much larger than the first coupling value that satisfies a prescribed finite-window numerical criterion. The latter is an empirical transition associated with one parameter set, one observation window, and one tolerance, and should not be interpreted as a necessary analytical threshold or a bifurcation value; this distinction is made explicit before the numerical section so that the large gap between the two quantities, quantified in Section 11, is not read as a weakness of the model.

Proposition 3 (Invariance of the synchronization manifold). The diagonal set $\mathcal{S} := \{Y \in \mathbb{H} : y_1 = y_2\}$ is invariant. If $y_1^0 = y_2^0$, then $y_1(\xi) = y_2(\xi)$ for every $\xi \geq 0$.

Proof. For identical initial data, $(U, V, W)(0) = 0$, and the difference system (32) admits the zero solution. By uniqueness in the applicable solution class it is the only solution. Alternatively, estimate (36) gives $\mathcal{E}_s(\xi) = 0$ directly under (34). $\square$

8. Robust Synchronization Under Parameter Mismatch

Exact identity of the two neurons is an idealization, so we record a practical synchronization result. Suppose the difference system carries an additive mismatch

$$\mathcal{M}(\xi) = \text{col}(M_u(\xi), M_v(\xi), M_w(\xi)), \quad \lambda \|M_u(\xi)\|^2 + \|M_v(\xi)\|^2 + \|M_w(\xi)\|^2 \leq \varepsilon^2,$$

representing small differences in external currents or intrinsic parameters after collecting them in the error equations.

Theorem 5 (Practical Mittag–Leffler synchronization). *Assume (34). Then there is $C_\mu > 0$ such that*

$$\mathcal{E}_s(\xi) \leq \mathcal{E}_s(0)E_\rho\left(-\frac{\mu}{2}\xi^\rho\right) + \frac{C_\mu\varepsilon^2}{\mu}\left[1 - E_\rho\left(-\frac{\mu}{2}\xi^\rho\right)\right], \quad (40)$$

and in particular $\limsup_{\xi \rightarrow \infty} \mathcal{E}_s(\xi) \leq C_\mu\varepsilon^2/\mu$.

Proof. Repeating the proof of Theorem 4 gives

$${}^C\mathcal{D}_\xi^\rho \mathcal{E}_s + \mu \mathcal{E}_s \leq 2\lambda(M_u, U) + 2(M_v, V) + 2(M_w, W),$$

and weighted Young inequalities bound the right-hand side by $\frac{\mu}{2}\mathcal{E}_s + C_\mu\varepsilon^2$. Therefore ${}^C\mathcal{D}_\xi^\rho \mathcal{E}_s + \frac{\mu}{2}\mathcal{E}_s \leq C_\mu\varepsilon^2$, and Lemma 2 proves (40). $\square$

9. An H^1 Estimate in the Linear-Diffusion Case

The natural energy space for the nonlinear problem is $W^{1,m}(\Omega)$. When $m = 2$ and the initial membrane potentials already belong to $H^1(\Omega)$, the linear Neumann diffusion yields a closed fractional estimate for their gradients. We deliberately do not infer instantaneous H^1 regularization from merely L^2 initial data here; that stronger assertion requires a separate fractional maximal-regularity argument.

Proposition 4 (Propagation of H^1 regularity for $m = 2$). *Assume $m = 2$, let $\partial\Omega$ be of class $C^{1,1}$, and suppose $\phi_i^0 \in H^1(\Omega)$ for $i = 1, 2$. For every set of initial data bounded in*

$$\mathbb{H}_1 := \left[H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)\right]^2$$

there exists a constant $C_B > 0$, depending only on the bounded set, the model parameters, and Ω , such that

$$Y(\xi) \leq Y(0)E_\rho(-\xi^\rho) + C_B[1 - E_\rho(-\xi^\rho)], \quad Y(\xi) := \sum_{i=1}^2 \|\nabla \phi_i(\xi)\|^2. \quad (41)$$

Consequently the H^1 norm of the membrane potentials is propagated uniformly in time,

$$\sup_{\xi \geq 0} Y(\xi) \leq \max\{Y(0), C_B\}, \quad \limsup_{\xi \rightarrow \infty} Y(\xi) \leq C_B.$$

The bound is uniform over data bounded in $\mathbb{H}_1$; it is a regularity-propagation statement and does not assert instantaneous smoothing of data taken merely in $\mathbb{H}$.

Proof. We derive the estimate on the Faedo–Galerkin approximations of Theorem 2, for which the operations below are legitimate: each approximation lies in $L^2(0, T; H^2(\Omega))$ with ${}^C\mathfrak{D}_\xi^\rho \phi_i \in L^2(0, T; L^2(\Omega))$, the Caputo derivative commutes with the spatial gradient, and ϕ_i satisfies the homogeneous Neumann condition. All terms on the left-hand side of the resulting inequality are convex or monotone in the approximation, so the estimate passes to the limit by weak lower semicontinuity. We therefore argue as if ϕ_i itself were admissible.

Since $m = 2$, the nonlinear diffusion operator reduces to the Neumann Laplacian, $\mathcal{A}_m \phi_i = -\Delta \phi_i$, and the membrane-potential equations read

$${}^C\mathfrak{D}_\xi^\rho \phi_i - d\Delta \phi_i = a\phi_i^2 - b\phi_i^3 + \psi_i - \chi_i + J + \kappa(\phi_j - \phi_i), \quad i, j \in \{1, 2\}, i \neq j.$$

For each i , take the $L^2(\Omega)$ inner product of the ϕ_i -equation with $-\Delta \phi_i$. Integrating the time term by parts in space and using the Neumann condition,

$$\left({}^C\mathfrak{D}_\xi^\rho \phi_i, -\Delta \phi_i\right) = \left({}^C\mathfrak{D}_\xi^\rho \nabla \phi_i, \nabla \phi_i\right) \geq \frac{1}{2} {}^C\mathfrak{D}_\xi^\rho \|\nabla \phi_i\|^2,$$

the inequality being Lemma 1 applied in $L^2(\Omega; \mathbb{R}^n)$. The diffusion term gives $d(-\Delta \phi_i, -\Delta \phi_i) = d\|\Delta \phi_i\|^2$.

For the polynomial reaction, integration by parts and the Neumann condition give

$$\int_{\Omega} (a\phi_i^2 - b\phi_i^3)(-\Delta \phi_i) \, dx = \int_{\Omega} (2a\phi_i - 3b\phi_i^2)|\nabla \phi_i|^2 \, dx,$$

and, completing the square,

$$2as - 3bs^2 = -3b\left(s - \frac{a}{3b}\right)^2 + \frac{a^2}{3b} \quad (s \in \mathbb{R}),$$

so that

$$\int_{\Omega} (2a\phi_i - 3b\phi_i^2)|\nabla \phi_i|^2 \, dx \leq \frac{a^2}{3b} \|\nabla \phi_i\|^2.$$

The constant-current term vanishes,

$$J \int_{\Omega} (-\Delta \phi_i) \, dx = -J \int_{\partial\Omega} \frac{\partial \phi_i}{\partial \nu} \, dS = 0,$$

and the recovery variables are controlled by Young's inequality,

$$|(\psi_i, -\Delta \phi_i)| \leq \frac{d}{4} \|\Delta \phi_i\|^2 + \frac{1}{d} \|\psi_i\|^2, \quad |(-\chi_i, -\Delta \phi_i)| \leq \frac{d}{4} \|\Delta \phi_i\|^2 + \frac{1}{d} \|\chi_i\|^2.$$

The coupling contribution is dissipative: summing over the two neurons and integrating by parts,

$$\kappa \int_{\Omega} [(\phi_2 - \phi_1)(-\Delta \phi_1) + (\phi_1 - \phi_2)(-\Delta \phi_2)] \, dx = -\kappa \|\nabla(\phi_1 - \phi_2)\|^2 \leq 0.$$

Combining these estimates and summing over $i = 1, 2$,

$$\frac{1}{2} {}^C\mathfrak{D}_\xi^\rho Y(\xi) + \frac{d}{2} \sum_{i=1}^2 \|\Delta \phi_i\|^2 + \kappa \|\nabla(\phi_1 - \phi_2)\|^2 \leq \frac{a^2}{3b} Y(\xi) + \frac{1}{d} \sum_{i=1}^2 (\|\psi_i\|^2 + \|\chi_i\|^2).$$

Multiplying by 2 and discarding the non-negative coupling term yields

$${}^C\mathfrak{D}_\xi^\rho Y(\xi) + d \sum_{i=1}^2 \|\Delta \phi_i\|^2 \leq C_3 Y(\xi) + \frac{2}{d} \sum_{i=1}^2 (\|\psi_i\|^2 + \|\chi_i\|^2), \quad C_3 := \frac{2a^2}{3b}. \quad (42)$$

We absorb the leading Y -term through a Neumann interpolation inequality. For $z \in H^2(\Omega)$ with $\partial_\nu z = 0$, integration by parts gives $\|\nabla z\|^2 = -(z, \Delta z)$, whence, for every $\varepsilon > 0$,

$$\|\nabla z\|^2 \leq \varepsilon \|\Delta z\|^2 + \frac{1}{4\varepsilon} \|z\|^2.$$

Choosing $\varepsilon = d/(C_3 + 1)$ and multiplying by $C_3 + 1$,

$$d\|\Delta z\|^2 \geq (C_3 + 1)\|\nabla z\|^2 - C_d \|z\|^2, \quad C_d := \frac{(C_3 + 1)^2}{4d}.$$

Applying this to ϕ_1 and ϕ_2 and inserting the result into (42), the term $C_3 Y(\xi)$ cancels and we are left with

$${}^c \mathcal{D}_\xi^\rho Y(\xi) + Y(\xi) \leq C_d \sum_{i=1}^2 \|\phi_i(\xi)\|^2 + \frac{2}{d} \sum_{i=1}^2 \left(\|\psi_i(\xi)\|^2 + \|\chi_i(\xi)\|^2 \right).$$

Let $B \subset \mathbb{H}_1$ be bounded. Because $\mathbb{H}_1 \hookrightarrow \mathbb{H}$ continuously, there is $R_B > 0$ with $\|Y^0\|_{\mathbb{H}}^2 \leq R_B$ for every $Y^0 \in B$, and Corollary 1 provides $M_B > 0$, independent of the datum in B and of $\xi \geq 0$, such that $\|Y(\xi)\|_{\mathbb{H}}^2 \leq M_B$. In particular the right-hand side above is at most $C_B := (C_d + \frac{2}{d})M_B$, so that

$${}^c \mathcal{D}_\xi^\rho Y(\xi) + Y(\xi) \leq C_B.$$

The fractional comparison principle (Lemma 2, with $\lambda = 1$ and constant forcing C_B) gives (41).

Finally, since $0 < E_\rho(-\xi^\rho) \leq 1$ for $0 < \rho \leq 1$, the right-hand side of (41) is a convex combination of $Y(0)$ and C_B , so $Y(\xi) \leq \max\{Y(0), C_B\}$ for all $\xi \geq 0$; and since $E_\rho(-\xi^\rho) \rightarrow 0$ as $\xi \rightarrow \infty$, passing to the upper limit gives $\limsup_{\xi \rightarrow \infty} Y(\xi) \leq C_B$. The bound depends on $Y(0)$ only through the first term, uniformly over $\mathbb{H}_1$ -bounded data, which is the asserted propagation of H^1 regularity. $\square$

10. Consequences and Limiting Cases

Corollary 3 (Classical partly diffusive system). *For $\rho = 1$ and $m = 2$, the model, the energy estimate, the global existence result, and the synchronization theorem reduce to their classical parabolic counterparts; in particular (36) becomes $\mathcal{E}_s(\xi) \leq \mathcal{E}_s(0)e^{-\mu\xi}$.*

Corollary 4 (Uncoupled dynamics). *For $\kappa = 0$, Theorem 1 still gives global dissipativity of the two independent partly diffusive neurons; synchronization, however, is not implied.*

Corollary 5 (Linear spatial propagation). *For $m = 2$ the nonlinear diffusion operator becomes the Neumann Laplacian and the monotonicity contribution in the synchronization estimate is exactly $d\|\nabla(\phi_1 - \phi_2)\|^2$.*

Remark 10 (Transition to the numerical section). *The analytical results identify the quantities to be verified computationally: the discrete analogue of (24), the Mittag–Leffler envelope (36), the dependence of the synchronization time on (ρ, m, κ) , and the gap between the sufficient threshold (34) and the empirical critical coupling. These are addressed in the next section.*

11. Numerical Approximation and Computational Analysis

The time discretization is based on the L1 formula [43–46], and the no-flux spatial operator is treated by a conservative finite-volume construction [47]. The combination follows the monotone structure of the continuous m -Laplacian rather than linearizing it at

vanishing gradients; a related Caputo- p -Laplacian reaction–diffusion computation can be found in [37].

This section develops a fully reproducible approximation of the time-fractional, partly diffusive HR system. The computations serve three distinct purposes. First, manufactured-solution and independent-solver tests verify the temporal and spatial accuracy of the numerical method and its integer-order and linear-diffusion limits. Second, the discrete energy and synchronization computations test the Mittag–Leffler estimates and quantify the conservatism of the sufficient coupling certificate. Third, parameter sweeps and heterogeneous spatial simulations provide dynamical interpretation by separating the effects of fractional memory, coupling strength, nonlinear diffusion, neuronal mismatch, and spatially varying forcing. The figures are presented in this order so that numerical verification is completed before biological and dynamical conclusions are drawn. The Python (version 3.11.9) implementation generates all figures directly from the model parameters and initial conditions. For reproducibility, the computed convergence errors, synchronization diagnostics, spike statistics, and parameter-sweep summaries are additionally stored as comma-separated files.

11.1. The Fully Discrete L1–Finite-Volume Method

For the numerical experiments we take the one-dimensional cable $\Omega = (0, L)$ and introduce the uniform spatial mesh $x_j = jh$, $j = 0, 1, \dots, M$, $h = L/M$ and the uniform temporal mesh $\zeta_n = n\tau$, $n = 0, 1, \dots, N$, $\tau = T/N$. The value of a grid function z_h^n at x_j is written z_j^n . The discrete Neumann inner product and norm are

$$(z_h, y_h)_h := h \left(\frac{1}{2} z_0 y_0 + \sum_{j=1}^{M-1} z_j y_j + \frac{1}{2} z_M y_M \right), \quad \|z_h\|_h^2 = (z_h, z_h)_h,$$

the half weights at the endpoints corresponding to boundary control volumes of length $h/2$. For $0 < \rho < 1$ define the L1 weights

$$\omega_k^{(\rho)} = (k+1)^{1-\rho} - k^{1-\rho}, \quad k \geq 0,$$

and the discrete Caputo derivative

$$\delta_\tau^\rho z_h^n := \frac{\tau^{-\rho}}{\Gamma(2-\rho)} \sum_{k=0}^{n-1} \omega_k^{(\rho)} (z_h^{n-k} - z_h^{n-k-1}), \quad (43)$$

with the convention $\delta_\tau^1 z_h^n = (z_h^n - z_h^{n-1})/\tau$ for $\rho = 1$. It is convenient to write

$$\delta_\tau^\rho z_h^n = c_{\rho,\tau} z_h^n - \mathcal{H}_z^n, \quad c_{\rho,\tau} = \frac{\tau^{-\rho}}{\Gamma(2-\rho)}, \quad (44)$$

where, for $0 < \rho < 1$,

$$\mathcal{H}_z^n = c_{\rho,\tau} \left[\omega_{n-1}^{(\rho)} z_h^0 + \sum_{k=1}^{n-1} \left(\omega_{n-k-1}^{(\rho)} - \omega_{n-k}^{(\rho)} \right) z_h^k \right],$$

and $\mathcal{H}_z^n = \tau^{-1} z_h^{n-1}$ for $\rho = 1$. To avoid a singular Jacobian when $m > 2$ and $\nabla \phi = 0$, the nonlinear flux is regularized by

$$\Phi_{m,\epsilon}(s) := \left(\epsilon^2 + s^2 \right)^{(m-2)/2} s, \quad \epsilon > 0, \quad (45)$$

and the regularized finite-volume realization of the positive operator $-\Delta_m$ is

$$(\mathcal{A}_{m,h}^e z)_j = \begin{cases} -\frac{2}{h} \Phi_{m,\epsilon} \left(\frac{z_1 - z_0}{h} \right), & j = 0, \\ -\frac{1}{h} \left[\Phi_{m,\epsilon} \left(\frac{z_{j+1} - z_j}{h} \right) - \Phi_{m,\epsilon} \left(\frac{z_j - z_{j-1}}{h} \right) \right], & 1 \leq j \leq M-1, \\ \frac{2}{h} \Phi_{m,\epsilon} \left(\frac{z_M - z_{M-1}}{h} \right), & j = M. \end{cases} \quad (46)$$

The factors two in the boundary rows are essential, since the boundary control volumes have length $h/2$; omitting them degrades the observed spatial accuracy.

For $i, j \in \{1, 2\}$ with $i \neq j$, the fully implicit scheme is

$$\begin{cases} \delta_\tau^\rho \phi_{i,h}^n + d \mathcal{A}_{m,h}^e \phi_{i,h}^n = a(\phi_{i,h}^n)^2 - b(\phi_{i,h}^n)^3 + \psi_{i,h}^n - \chi_{i,h}^n + J + \kappa(\phi_{j,h}^n - \phi_{i,h}^n), \\ \delta_\tau^\rho \psi_{i,h}^n = \alpha_0 - \psi_{i,h}^n - \beta(\phi_{i,h}^n)^2, \\ \delta_\tau^\rho \chi_{i,h}^n = q(\phi_{i,h}^n - c) - r \chi_{i,h}^n, \end{cases} \quad (47)$$

with powers and products evaluated nodewise. From (44) the two recovery variables can be eliminated at every time level,

$$\psi_{i,h}^n = \frac{\mathcal{H}_{\psi_i}^n + \alpha_0 - \beta(\phi_{i,h}^n)^2}{c_{\rho,\tau} + 1}, \quad \chi_{i,h}^n = \frac{\mathcal{H}_{\chi_i}^n + q(\phi_{i,h}^n - c)}{c_{\rho,\tau} + r}, \quad (48)$$

so that only the two membrane-potential vectors remain unknown in the nonlinear solve, which is performed by a damped Newton iteration. The derivative of the regularized flux,

$$\Phi'_{m,\epsilon}(s) = (\epsilon^2 + s^2)^{(m-4)/2} (\epsilon^2 + (m-1)s^2) > 0,$$

keeps the diffusion block sparse, tridiagonal, and monotone. The residual stopping criterion is $\|R^{(\ell)}\|_2 \leq 10^{-9} \max\{1, \|R^{(0)}\|_2\}$ with at most thirty Newton iterations, and backtracking is used whenever the full Newton step fails to reduce the residual.

Lemma 6 (Discrete monotonicity). *For all grid functions z_h and y_h ,*

$$(\mathcal{A}_{m,h}^e z_h - \mathcal{A}_{m,h}^e y_h, z_h - y_h)_h \geq 0,$$

and moreover

$$(\mathcal{A}_{m,h}^e z_h, z_h)_h = h \sum_{j=0}^{M-1} \Phi_{m,\epsilon}(D_h z_{j+1/2}) D_h z_{j+1/2} \geq h \sum_{j=0}^{M-1} |D_h z_{j+1/2}|^m, \quad (49)$$

where $D_h z_{j+1/2} = (z_{j+1} - z_j)/h$.

Proof. Multiplying (46) by the trapezoidal control-volume weights and summing over j gives the discrete summation-by-parts identity $(\mathcal{A}_{m,h}^e z_h, y_h)_h = h \sum_{j=0}^{M-1} \Phi_{m,\epsilon}(D_h z_{j+1/2}) \cdot D_h y_{j+1/2}$. Taking $y_h = z_h$ gives the identity in (49), and since $m \geq 2$, $(\epsilon^2 + s^2)^{(m-2)/2} s^2 \geq |s|^m$. The difference estimate follows from the strict monotonicity of $s \mapsto \Phi_{m,\epsilon}(s)$ on every edge after summation of the non-negative edge terms. $\square$

Lemma 7 (Discrete Caputo energy inequality). *For every Hilbert-space-valued sequence $(z^n)_{n=0}^N$,*

$$2(\delta_\tau^\rho z^n, z^n) \geq \delta_\tau^\rho \|z^n\|^2, \quad n \geq 1. \quad (50)$$

Proof. The L1 coefficients are positive and decreasing. Expanding $2(z^n, z^n - z^k) - \|z^n\|^2 + \|z^k\|^2 = \|z^n - z^k\|^2$ and using the convolution representation of (43) yields

$$2(\delta_\tau^\rho z^n, z^n) - \delta_\tau^\rho \|z^n\|^2 = c_{\rho,\tau} \omega_{n-1}^{(\rho)} \|z^n - z^0\|^2 + c_{\rho,\tau} \sum_{k=1}^{n-1} \left(\omega_{n-k-1}^{(\rho)} - \omega_{n-k}^{(\rho)} \right) \|z^n - z^k\|^2,$$

which is non-negative; for $\rho = 1$ it reduces to the standard backward-Euler identity. $\square$

Proposition 5 (Solvability of one time step). *At every time level, scheme (47) has at least one solution, and it is unique whenever*

$$c_{\rho,\tau} + \frac{q}{c_{\rho,\tau} + r} > \frac{1}{3b} \left(a - \frac{\beta}{c_{\rho,\tau} + 1} \right)^2. \quad (51)$$

Proof. After eliminating $\psi_{i,h}^n$ and $\chi_{i,h}^n$ by (48), the equations define a continuous map on $\mathbb{R}^{2(M+1)}$. Pairing it with $(\phi_{1,h}^n, \phi_{2,h}^n)$, the cubic terms contribute $b \sum_i \|\phi_{i,h}^n\|_{\ell_h^4}^4$, the diffusion contributes a non-negative quantity by Lemma 6, and the coupling contributes $\kappa \|\phi_{1,h}^n - \phi_{2,h}^n\|_h^2$; these fourth-order terms dominate all lower-order terms as the Euclidean norm grows, so Brouwer's theorem on a large ball gives existence. For uniqueness, subtract two solutions and test by their membrane-potential difference. Writing $s = \phi + \bar{\phi}$ and using $\phi^2 + \phi\bar{\phi} + \bar{\phi}^2 \geq \frac{3}{4}s^2$, the pointwise polynomial difference is bounded below by

$$\frac{3b}{4}s^2 + \left(\frac{\beta}{c_{\rho,\tau} + 1} - a \right)s + c_{\rho,\tau} + \frac{q}{c_{\rho,\tau} + r},$$

whose first two terms have minimum $-\frac{1}{3b}(a - \beta/(c_{\rho,\tau} + 1))^2$. Condition (51), with the monotonicity of the diffusion and coupling, makes the reduced map strictly monotone, so the two solutions coincide and (48) gives equality of the recovery variables. $\square$

11.2. Discrete Energy Stability and Synchronization

Set $C_0 = (\beta^2 + 4)/b$ and define

$$\mathcal{E}_h^n := C_0 \sum_{i=1}^2 \|\phi_{i,h}^n\|_h^2 + \sum_{i=1}^2 \|\psi_{i,h}^n\|_h^2 + \sum_{i=1}^2 \|\chi_{i,h}^n\|_h^2. \quad (52)$$

Theorem 6 (Unconditional discrete dissipativity). *There are constants $\eta_0 > 0$ and $C_* > 0$, independent of h, τ, ϵ , and n , such that every solution of (47) satisfies*

$$\delta_\tau^\rho \mathcal{E}_h^n + 2C_0 d \sum_{i=1}^2 h \sum_{j=0}^{M-1} |D_h \phi_{i,j+1/2}^n|^m + \eta_0 \mathcal{E}_h^n + \sum_{i=1}^2 \|\phi_{i,h}^n\|_{\ell_h^4}^4 \leq C_* L. \quad (53)$$

The constants may be chosen exactly as in the continuous energy estimate.

Proof. Take the discrete inner product of the first equation in (47) with $C_0 \phi_{i,h}^n$, the second with $\psi_{i,h}^n$, and the third with $\chi_{i,h}^n$, and sum over $i = 1, 2$. Lemma 7 controls the time terms and Lemma 6 the nonlinear diffusion, while the coupling terms give

$$C_0 \kappa [(\phi_{2,h}^n - \phi_{1,h}^n, \phi_{1,h}^n)_h + (\phi_{1,h}^n - \phi_{2,h}^n, \phi_{2,h}^n)_h] = -C_0 \kappa \|\phi_{1,h}^n - \phi_{2,h}^n\|_h^2.$$

All remaining estimates are pointwise Young inequalities identical to those in the proof of Theorem 1. In particular, the quadratic potential terms are grouped into a single Young step exactly as in (25), so that the choice of C_0 makes the cubic damping dominate the

accumulated quartic contribution for every $\beta > 0$, as recorded in Remark 4. Summation with the positive quadrature weights yields (53) without any restriction on h or τ . $\square$

The scalar discrete comparison equation associated with (53) defines a positive discrete Mittag–Leffler resolvent, so the numerical energy remains uniformly bounded and approaches a discrete absorbing level. The computed energy is compared later, in Section 11.6, with the envelope obtained from $\delta_\tau^\rho \mathcal{E}_h^n + \eta_0 \mathcal{E}_h^n \leq \widehat{C}_h$, where $\widehat{C}_h$ is the maximum computed comparison residual. This is an a posteriori discrete certificate sharing the Mittag–Leffler structure of the analytical estimate; it is not a fitted decay curve.

For synchronization, define $U_h^n = \phi_{1,h}^n - \phi_{2,h}^n$, $V_h^n = \psi_{1,h}^n - \psi_{2,h}^n$, $W_h^n = \chi_{1,h}^n - \chi_{2,h}^n$, and

$$\mathcal{E}_{s,h}^n = \lambda \|U_h^n\|_h^2 + \|V_h^n\|_h^2 + \|W_h^n\|_h^2, \quad \lambda = \frac{8\beta^2}{b}. \quad (54)$$

Proposition 6 (Discrete synchronization inequality). *If $\kappa > \frac{\lambda}{2} + \frac{a^2}{b} + \frac{(q-\lambda)^2}{4\lambda r}$ then*

$$\delta_\tau^\rho \mathcal{E}_{s,h}^n + \mu \mathcal{E}_{s,h}^n \leq 0, \quad (55)$$

with μ the constant of Theorem 4. Consequently $\mathcal{E}_{s,h}^n$ is bounded by the positive discrete Mittag–Leffler resolvent and converges to zero.

Proof. Subtract the two discrete neuron systems and test the three difference equations with λU_h^n , V_h^n , and W_h^n . Lemma 7 handles the L1 terms, the diffusion difference is non-negative by Lemma 6, and the remaining algebra is pointwise and identical to the continuous synchronization proof, resting on $a|s| - \frac{b}{4}s^2 \leq a^2/b$ and Young inequalities for the UV and UW products. This yields (55), and positivity of the discrete scalar resolvent gives the final assertion. $\square$

11.3. Convergence of the Approximation

Theorem 7 (Convergence to a weak solution). *Let $Y_{h,\tau,\epsilon}$ denote the piecewise-constant temporal interpolant of the solution of (47), and assume the initial grid values converge to Y^0 in $\mathbb{H}$. Along a sequence $h \rightarrow 0$, $\tau \rightarrow 0$, $\epsilon \rightarrow 0$ there exists a subsequence such that*

$$\begin{aligned} Y_{h,\tau,\epsilon} &\overset{*}{\rightharpoonup} Y && \text{in } L^\infty(0, T; \mathbb{H}), \\ \phi_{i,h,\tau,\epsilon} &\rightharpoonup \phi_i && \text{in } L^m(0, T; W^{1,m}(\Omega)) \cap L^4(Q_T), \\ \phi_{i,h,\tau,\epsilon} &\rightarrow \phi_i && \text{strongly in } L^2(Q_T), \end{aligned}$$

where Y is a weak solution of the fractional coupled HR system. If that weak solution is unique in the class under consideration, the entire family converges.

Proof. Theorem 6 furnishes bounds independent of h , τ , and ϵ for $Y_{h,\tau,\epsilon}$ in $L^\infty(0, T; \mathbb{H})$ and for $\phi_{i,h,\tau,\epsilon}$ in $L^m(0, T; W^{1,m}(\Omega)) \cap L^4(Q_T)$, and the discrete equations then bound the L1 time derivatives in the sum of the dual spaces used in the weak formulation. The discrete fractional translation estimate associated with the L1 convolution, combined with the compact embedding $W^{1,m}(\Omega) \cap L^4(\Omega) \Subset L^2(\Omega)$, gives relative compactness in $L^2(Q_T)$; after extraction $\phi_{i,h,\tau,\epsilon}$ converges strongly in $L^2(Q_T)$ and almost everywhere, and interpolation with the uniform L^4 bound upgrades this to $L^s(Q_T)$ for every $2 \leq s < 4$. Hence the quadratic terms converge strongly in $L^q(Q_T)$ for $q < 2$ and the cubic terms weakly in $L^{4/3}(Q_T)$.

For the nonlinear diffusion, the regularized fluxes are bounded in $L^{m'}(Q_T)$, weak compactness gives a limit flux, and discrete monotonicity together with the energy limsup inequality yields the Minty condition; letting first $h, \tau \rightarrow 0$ and then $\epsilon \rightarrow 0$ identifies the

weak limit as $|\nabla \phi_i|^{m-2} \nabla \phi_i$. The Neumann condition is already incorporated in the finite-volume summation by parts and passes to the variational limit. Finally, testing the discrete equations by smooth grid projections and using the preceding convergences produces the weak formulation of the continuous problem, the initial value following from the discrete Volterra representation and weak temporal continuity. Uniqueness removes the need to extract a subsequence. $\square$

This supplies convergence for the nonlinear system without any formal global Lipschitz condition on the cubic reaction. A sharp error estimate would require additional regularity and nondegeneracy assumptions for the m -Laplacian, so we instead verified the expected rates on a smooth manufactured problem and used mesh-independence tests for the nonlinear model.

11.4. Manufactured-Solution Verification

We tested the L1-finite-volume backbone on

$${}^C \mathfrak{D}_{\xi}^{\rho} z - dz_{xx} = f(x, \xi), \quad z_x(0, \xi) = z_x(1, \xi) = 0, \quad z(x, 0) = 0, \quad (56)$$

with exact solution $z(x, \xi) = \xi^3 \cos(\pi x)$ and forcing $f(x, \xi) = \left[\frac{\Gamma(4)}{\Gamma(4-\rho)} \xi^{3-\rho} + d\pi^2 \xi^3 \right] \cos(\pi x)$. For a smooth solution the L1 approximation was expected to have order $2 - \rho$ in time. The observed temporal order at each refinement was computed from

$$p_{\tau} = \frac{\log(E_{\tau}/E_{\tau/2})}{\log 2}, \quad (57)$$

where E_{τ} denotes the discrete L^2 error at step size τ ; Table 1 confirms the terminal orders 1.38, 1.19, and 1.05 for $\rho = 0.6, 0.8$, and 0.95 , respectively, in close agreement with the theoretical smooth-solution orders $2 - \rho = 1.40, 1.20$, and 1.05 . The corresponding Log-log temporal error curves are shown in Figure 1.

Table 1. Temporal convergence at $T = 1$ for the manufactured problem. The spatial grid has 401 nodes.

ρ	τ	L^2 Error	Order	Energy Error	Order
0.60	5.000×10^{-2}	7.166×10^{-3}	—	2.362×10^{-2}	—
	2.500×10^{-2}	2.791×10^{-3}	1.360	9.202×10^{-3}	1.360
	1.250×10^{-2}	1.077×10^{-3}	1.374	3.550×10^{-3}	1.374
	6.250×10^{-3}	4.132×10^{-4}	1.382	1.362×10^{-3}	1.382
0.80	5.000×10^{-2}	1.732×10^{-2}	—	5.709×10^{-2}	—
	2.500×10^{-2}	7.649×10^{-3}	1.179	2.522×10^{-2}	1.179
	1.250×10^{-2}	3.358×10^{-3}	1.188	1.107×10^{-2}	1.188
	6.250×10^{-3}	1.469×10^{-3}	1.192	4.845×10^{-3}	1.192
0.95	5.000×10^{-2}	3.202×10^{-2}	—	1.056×10^{-1}	—
	2.500×10^{-2}	1.555×10^{-2}	1.042	5.126×10^{-2}	1.042
	1.250×10^{-2}	7.532×10^{-3}	1.046	2.483×10^{-2}	1.046
	6.250×10^{-3}	3.644×10^{-3}	1.047	1.201×10^{-2}	1.047

For the spatial test we took $\rho = 1$ and used the exact solution $z(x, \xi) = \xi \cos(\pi x)$. Backward Euler differentiates the linear temporal factor exactly, so the reported error isolated the spatial discretization. The second-order rates in Table 2 verify the boundary closure in (46). The corresponding spatial convergence curves are shown in Figure 2.

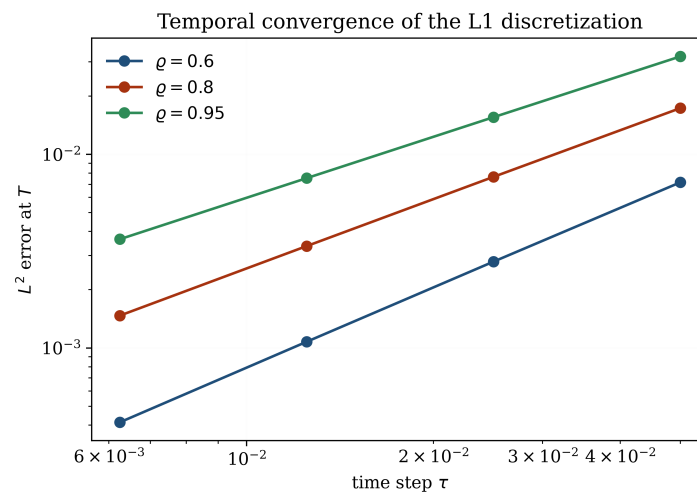


Figure 1. Log-log temporal error for three fractional orders. The slopes approach $2 - \rho$, as predicted for a smooth solution. Manufactured problem (56) with exact solution $z = \xi^3 \cos(\pi x)$ at $T = 1$, computed on a spatial grid of 401 nodes; the orders were obtained from (57).

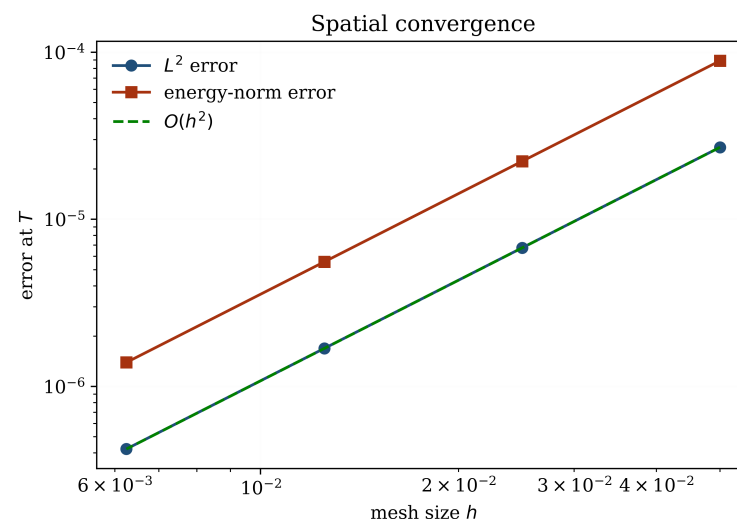


Figure 2. Spatial convergence of the finite-volume Neumann discretization. Both the L^2 and discrete energy errors are asymptotically second order. Manufactured problem (56) with $\rho = 1$ and exact solution $z = \xi \cos(\pi x)$ at $T = 0.2$, so that the reported error isolated the spatial discretization and verifies the boundary closure in (46).

Table 2. Spatial convergence at $T = 0.2$ for the manufactured problem.

Nodes	h	L^2 Error	Order	Energy Error	Order
21	5.000×10^{-2}	2.700×10^{-5}	–	8.893×10^{-5}	–
41	2.500×10^{-2}	6.753×10^{-6}	1.999	2.226×10^{-5}	1.998
81	1.250×10^{-2}	1.689×10^{-6}	2.000	5.567×10^{-6}	2.000
161	6.250×10^{-3}	4.222×10^{-7}	2.000	1.392×10^{-6}	2.000

11.5. Parameter Sets, Initial Data, and Diagnostic Quantities

The canonical parameter set is the standard bursting set of the classical HR equations. A second, moderately scaled certificate set is used only for the direct verification of the sufficient synchronization theorem, because the canonical threshold is intentionally conservative.

Unless stated otherwise, $L = 20$, $\epsilon = 10^{-4}$, and the spatially heterogeneous initial values are

$$\begin{aligned}\phi_1^0(x) &= -1.25 + 0.45 \exp \left[- \left(\frac{x - 0.32L}{0.09L} \right)^2 \right], \\ \phi_2^0(x) &= -0.80 + 0.35 \exp \left[- \left(\frac{x - 0.68L}{0.11L} \right)^2 \right], \\ \psi_i^0(x) &= \alpha_0 - \beta(\phi_i^0(x))^2, \quad \chi_i^0(x) = \frac{q}{r}(\phi_i^0(x) - c).\end{aligned}$$

Localized rectangular and sinusoidal initial profiles were used additionally in the robustness study. At a fixed observation point x_0 , spikes were local maxima of $\phi_i(\zeta, x_0)$ above zero with prominence at least 0.2, and spikes were assigned to different bursts when the gap exceeded the larger of twenty time units and three times the median inter-spike gap. We report the mean inter-spike interval, the number of detected burst onsets divided by the observation time, the peak voltage, and the time between the first and last detected spike in the active window. These finite-window statistics are descriptive; when only two or three spikes are detected, they should not be interpreted as asymptotic firing invariants. The synchronization time was

$$T_{\text{sync}} := \min \left\{ \zeta_n : \mathcal{E}_{s,h}^k \leq 0.05 \mathcal{E}_{s,h}^0 \text{ for every } k \geq n \right\}, \quad (58)$$

a blank heatmap cell indicating that this criterion was not reached on the simulated interval.

11.6. Verification of Dissipativity and Limiting Cases

Figure 3 displays the discrete energy together with the computable comparison envelope described after Theorem 6. The numerical trajectory remained well below the envelope and settled into a bounded regime. This test used the certificate parameter set with $\rho = 0.8$, $m = 3$, $\tau = 0.2$, and 61 spatial nodes.

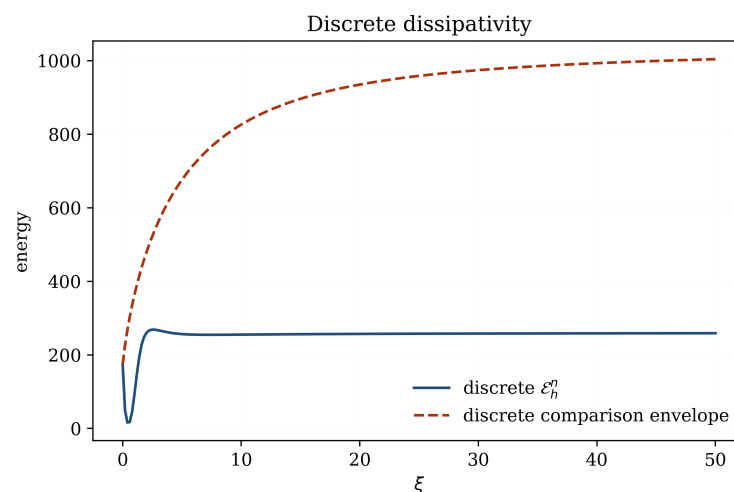


Figure 3. Discrete energy and the corresponding discrete Mittag–Leffler comparison envelope. The envelope was computed from the discrete comparison residual, not fitted to the trajectory. Certificate parameter set of Table 3 with $\rho = 0.8$, $m = 3$, $\tau = 0.2$, and 61 spatial nodes. The computed energy stayed well below the envelope, confirming the unconditional discrete dissipativity of Theorem 6.

Table 3. Dimensionless parameter sets used in the numerical experiments.

Set	a	b	α_0	β	q	r	c	J	d
canonical	3	1	1	5	0.0084	0.0021	−1.6	3.281	0.02
certificate	1	1	0.8	0.5	0.4	0.5	−1	1.2	0.02

The left panel of Figure 4 compares $\rho = 0.9999$ with the integer-order computation; the relative difference on the displayed interval is 6.86×10^{-3} . In the right panel the $m = 2$, $\rho = 1$ implementation is compared with an independent method-of-lines BDF solver for the same spatially discrete PDE, giving a relative difference of 9.33×10^{-4} . The first value quantifies the finite-order approximation $\rho = 0.9999 \neq 1$, whereas the second is a code-to-code consistency check for the linear-diffusion implementation. Both are small at the resolution used and support the two limiting reductions.

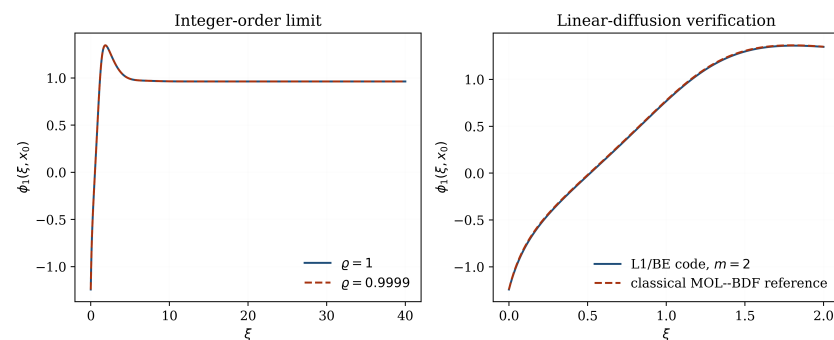


Figure 4. Left: approach to the integer-order limit as $\rho \uparrow 1$, comparing $\rho = 0.9999$ with the integer-order computation (relative difference 6.86×10^{-3}). Right: verification of the $m = 2$, $\rho = 1$ implementation against an independent method-of-lines BDF computation for the same spatially discrete PDE (relative difference 9.33×10^{-4}). Both panels are consistency checks for the two limiting reductions of the scheme.

11.7. Neuronal Firing and Bursting Dynamics

To isolate firing and synchronization from spatial propagation, the time-series experiments in Figures 5 and 6 were performed on the spatially homogeneous invariant manifold, which remained an exact special case of the PDE. We took $\rho = 0.98$ and simulated to $T = 1000$. Figure 5 shows the membrane-potential traces for weak, near-transition, and strong coupling. For weak coupling the two neurons retained visible phase differences; near the transition their spikes became substantially aligned; and strong coupling produced almost coincident voltage traces, while the slow variables retained the long timescale responsible for bursting. Figure 6 shows the fast and slow recovery variables and the phase portraits for the strong-coupling experiment.

Figure 7 quantifies how memory and nonlinear diffusion act on the bursting rhythm, and it isolates a clean, monotone memory effect. As ρ decreased from 1 to 0.90, the mean burst period lengthened monotonically from about 379 to 565 time units, an increase of roughly 49 percent, and the mean number of spikes per burst rose from about 31.7 to 38.8. Stronger memory therefore slowed the bursting clock and packed more spikes into each burst. By contrast, varying the diffusion exponent m from 2 to 4 in the spatial experiment left the temporal rhythm essentially unchanged: the mean burst period remained near 362 time units, varying by less than one and a half time units, below 0.4 percent, and the peak amplitude stayed close to 0.93. The nonlinear diffusion exponent thus governed the spatial sharpness of the traveling fronts, as discussed in Section 11.9, rather than the temporal bursting dynamics. separation is consistent with the energy analysis, in which the m -Laplacian contributed only non-negative spatial dissipation and did not enter the temporal reaction balance that fixed the bursting period.

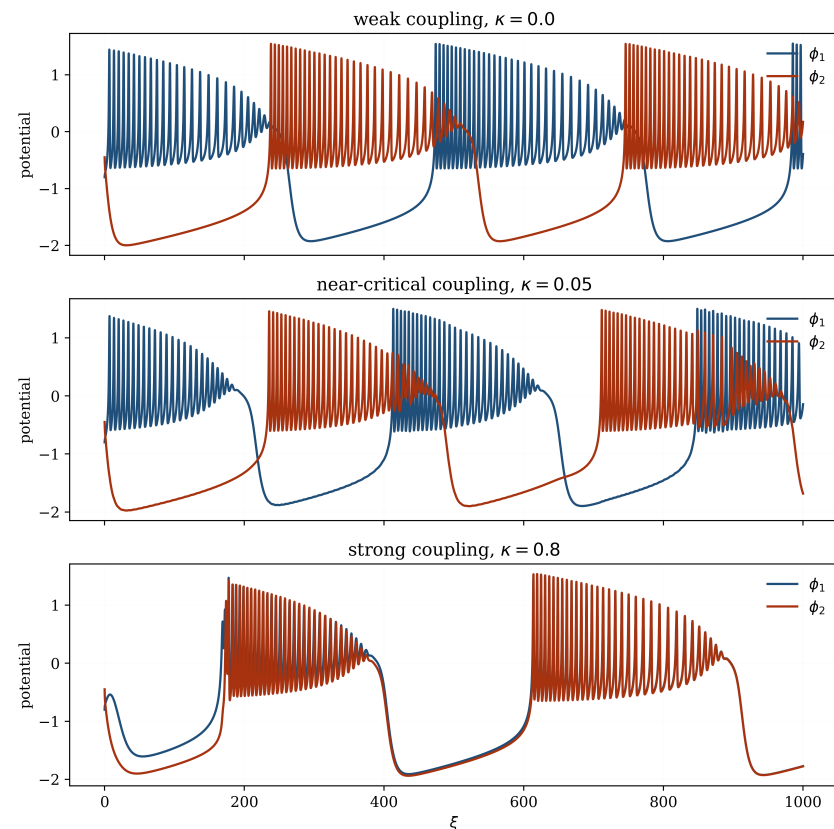


Figure 5. Membrane-potential traces for weak, near-transition, and strong coupling. The same initial mismatch is used in all three panels. Canonical parameter set of Table 3, spatially homogeneous invariant manifold, $\rho = 0.98$, $T = 1000$, with $\kappa = 0, 0.05$, and 0.8 from top to bottom. Increasing κ progressively removes the phase difference between the two neurons.

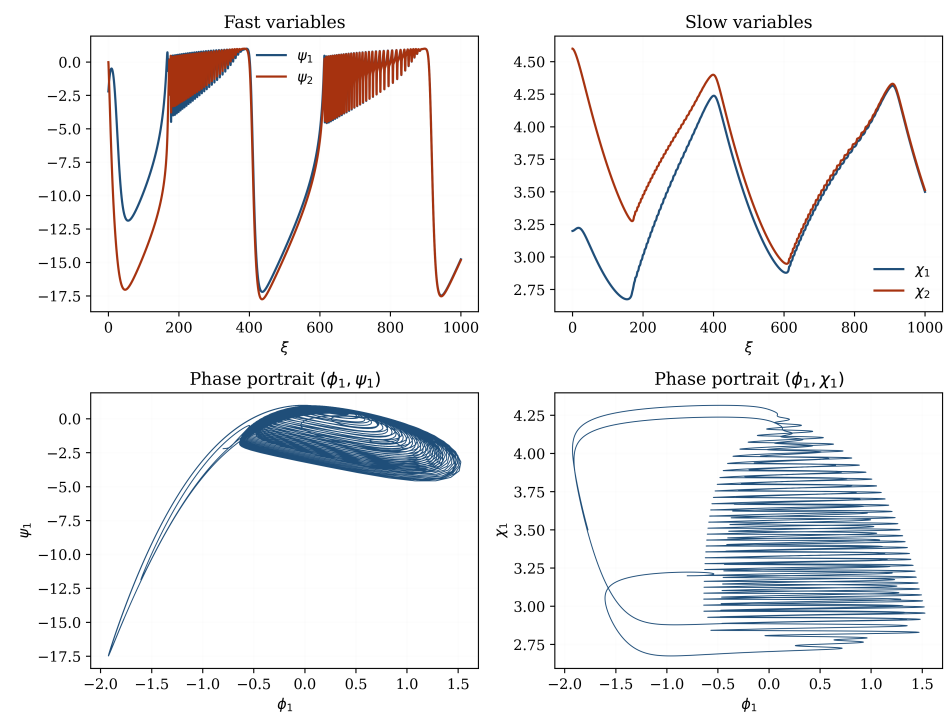


Figure 6. Fast and slow recovery variables and phase portraits for the strong-coupling experiment. The closed loops and slow drift display the spike–burst timescale separation. Canonical parameter set, $\rho = 0.98$, $\kappa = 0.8$, $T = 1000$. The slow variable χ_1 organizes the long bursting envelope, consistent with the fractional slow-variable estimate of Proposition 2.

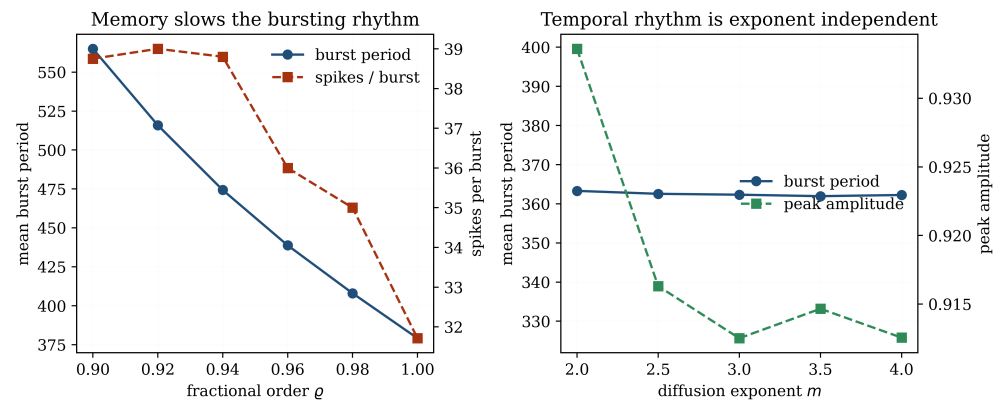


Figure 7. Left: detected mean inter-spike interval and burst frequency versus the fractional order. Right: peak amplitude and detected active-window duration versus the nonlinear diffusion exponent. The two panels use different simulation protocols and are interpreted within, not across, the respective sweeps.

11.8. Synchronization Diagnostics and Threshold Conservatism

For direct comparison with the synchronization theorem we first used the certificate set, for which $\lambda = 2$ and $\kappa_{th} = 2.64$. The computation in Figure 8 uses $\kappa = \kappa_{th} + 1$, $\rho = 0.8$, and $m = 3$. All three component errors decayed, and the discrete synchronization energy stayed below the continuous Mittag–Leffler envelope of the theorem.

For the canonical bursting parameters the sufficient analytical threshold was $\kappa_{th} = 23,916.52$. A logarithmic scan on $[0.2, 30]$, using the numerical criterion (58) on $[0, 120]$, gave the first successful sampled value $\kappa_{emp} = 11.0$, so that the ratio between the sufficient and empirical values was approximately 2.17×10^3 . This did not contradict the theorem, which is global, uniform in the initial state, and derived from worst-case Young inequalities; the computation shows that the bound was a global sufficient certificate rather than a practical estimate of the empirical sampled transition. The synchronization-time curve and the comparison between the empirical and sufficient thresholds are shown in Figure 9.

Figure 10 gives two parameter maps. At fixed $m = 3$, the computed synchronization time varied with both ρ and κ , although the finite-window pattern was not strictly monotone in every sampled cell. In the (m, κ) sweep, all the sampled rows coincided at the displayed resolution: the experiment resolved a clear coupling effect but no measurable m -dependence of the five-percent synchronization time. This is consistent with, but does not prove, the analytical observation that coupling directly damps the spatially constant asynchronous mode, whereas nonlinear diffusion acts through spatial gradients.

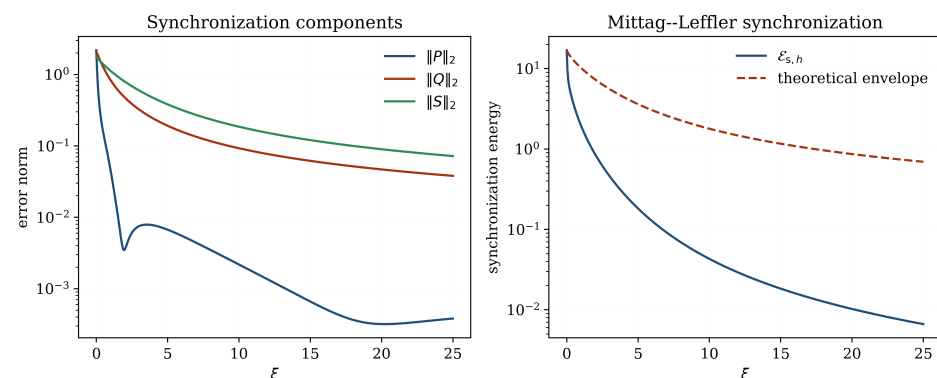


Figure 8. Left: membrane, fast-variable, and slow-variable synchronization errors. Right: discrete synchronization energy and the theoretical Mittag–Leffler envelope. Certificate parameter set with $\lambda = 2$, $\kappa = \kappa_{th} + 1$, $\rho = 0.8$, and $m = 3$. All three component errors decayed and the discrete energy remained below the envelope of Theorem 4.

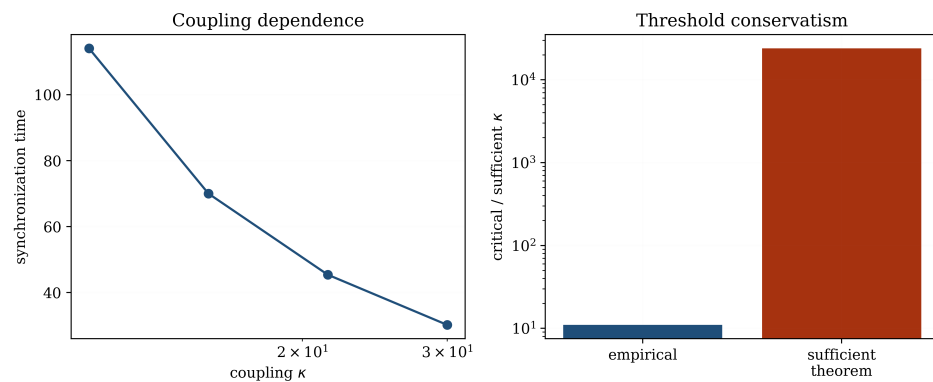


Figure 9. Left: synchronization time versus coupling strength for the canonical parameter set. Right: logarithmic comparison of the empirical sampled threshold with the global sufficient threshold. The empirical value $\kappa_{\text{emp}} \approx 11.0$ follows from criterion (58) on $[0, 120]$, whereas $\kappa_{\text{th}} = 23,916.52$ is the global sufficient certificate of Theorem 4; the two quantities answer different questions and their ratio measures the conservatism of the worst-case estimates.

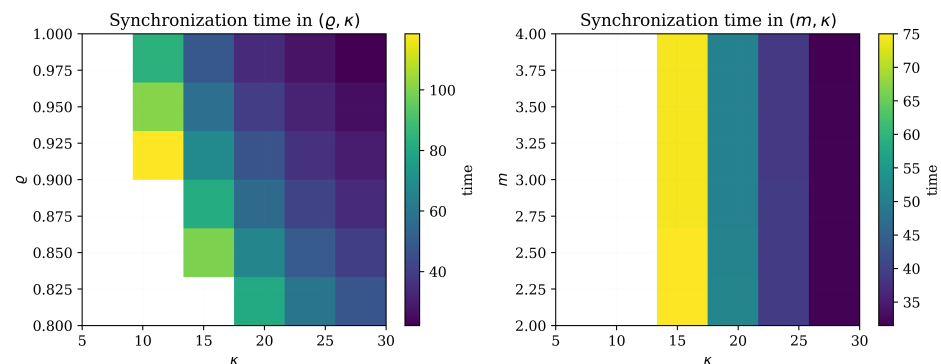


Figure 10. Synchronization-time maps on the (ρ, κ) - and (m, κ) -planes. Blank cells indicate that the five-percent criterion was not reached by $T = 120$. The (ρ, κ) map was computed at fixed $m = 3$ and the (m, κ) map at fixed $\rho = 0.9$. Coupling had a clear effect in both maps, while no m -dependence of the five-percent time was resolved at this sampling resolution. The maps report an empirical finite-window transition, not a bifurcation curve.

11.9. Spatial Propagation and Nonlinear Diffusion

The spatial experiment used localized, separated initial voltage profiles, $\rho = 0.9$, $m = 3$, and $\kappa = 20$. Figure 11 shows the two space–time voltage surfaces, whose initially different fronts became nearly identical as the coupling acted:

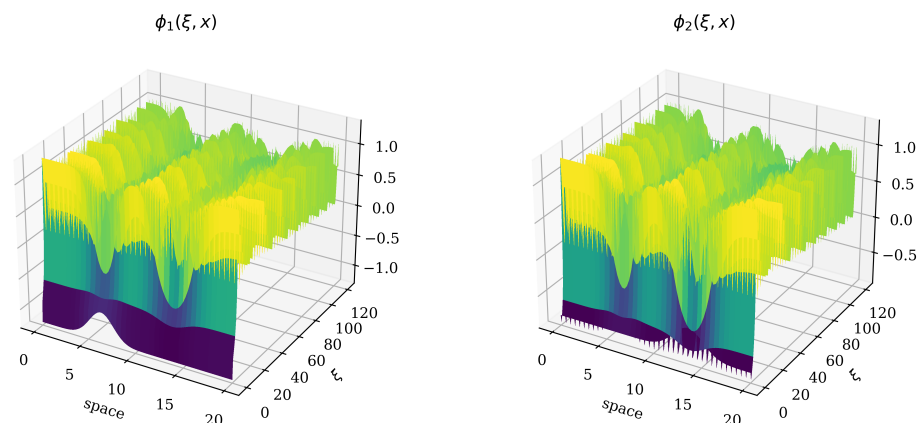


Figure 11. Space–time membrane-potential surfaces for the two neurons under strong coupling. Localized, separated initial voltage profiles on $\Omega = (0, 20)$ with $\rho = 0.9$, $m = 3$, and $\kappa = 20$. The initially distinct fronts became nearly identical as the electrical coupling acted.

The first panel of Figure 12 displays the complete space–time evolution of $|U|$, and the remaining panels compare an early transient with the final spatial profile. The maximum pointwise difference fell from about 0.799 initially to about 6.2×10^{-2} at $T = 120$.

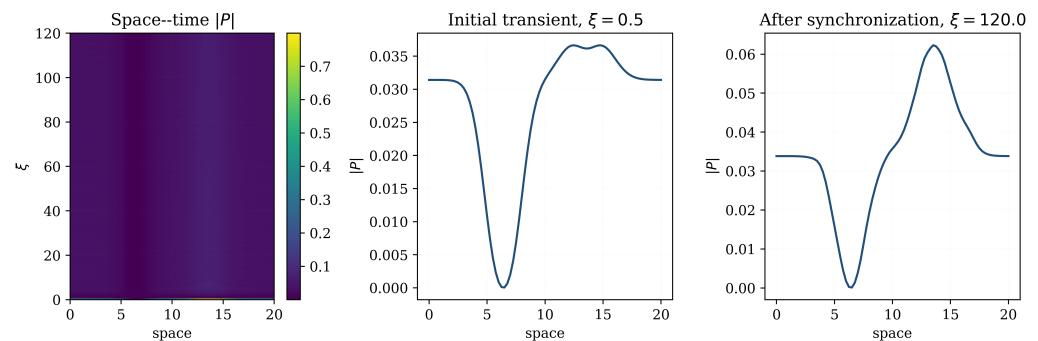


Figure 12. Space–time asynchronous voltage magnitude $|U| = |\phi_1 - \phi_2|$ and spatial profiles during the early transient and at the late, near-synchronized time. Same experiment as Figure 11 ($\rho = 0.9$, $m = 3$, $\kappa = 20$). The maximum pointwise difference fell from about 0.799 initially to about 6.2×10^{-2} at $T = 120$.

To quantify spatial sharpness we report the discrete gradient norm

$$G_m(\xi_n) := \|D_h \phi_1^n\|_{\ell_h^m} = \left(h \sum_{j=0}^{M-1} |D_h \phi_{1,j+1/2}^n|^m \right)^{1/m}. \quad (59)$$

Figure 13 shows a modest, non-monotone dependence on m : the terminal gradient measure decreased from 0.7048 at $m = 2$ to 0.6820 at $m = 3$, followed by a slight increase to 0.6833 at $m = 4$. The cross-sectional profiles remained localized but their plateau levels and transition zones evolved in time. Thus the computation confirms that m changes the spatial solution, but it does not support a universal claim that increasing m monotonically sharpens the front over the tested range.

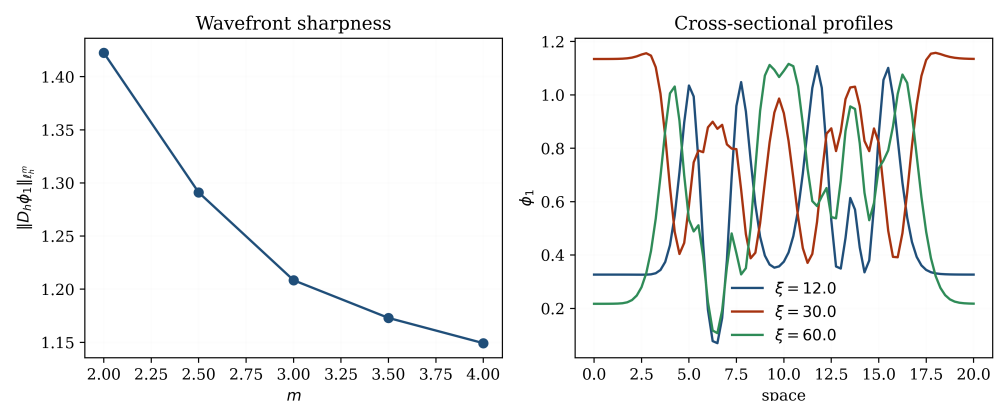


Figure 13. Left: terminal gradient-based wavefront diagnostic $G_m(T)$ of (59) versus the nonlinear diffusion exponent. Right: cross-sectional voltage profiles at three times for $m = 3$. The dependence on m was modest and non-monotone over $2 \leq m \leq 4$, so the diffusion exponent reshaped the spatial profile without monotonically sharpening the front in the tested range.

11.10. Robustness and Computational Cost

The synchronization theorem concerns identical deterministic neurons, so we tested three perturbations outside that exact setting:

- (i) Gaussian, sinusoidal, and localized initial voltage profiles;
- (ii) a one-percent mismatch in a and a two-percent mismatch in J for the second neuron;

- (iii) a spatially heterogeneous current $J_i(x) = J[1 + 0.04 \cos(2\pi x/L) + 0.011_{\{i=2\}}]$ together with independent Gaussian perturbations of standard deviation 0.015 at each time level.

The last perturbation was a discrete current-noise experiment and was not asserted to be a continuum white-noise stochastic PDE. Figure 14 shows that strong coupling removed the initial-profile dependence, while parameter mismatch and heterogeneous or noisy forcing prevented exact equality but produced bounded practical synchronization, consistent with the robust estimate of Theorem 5.

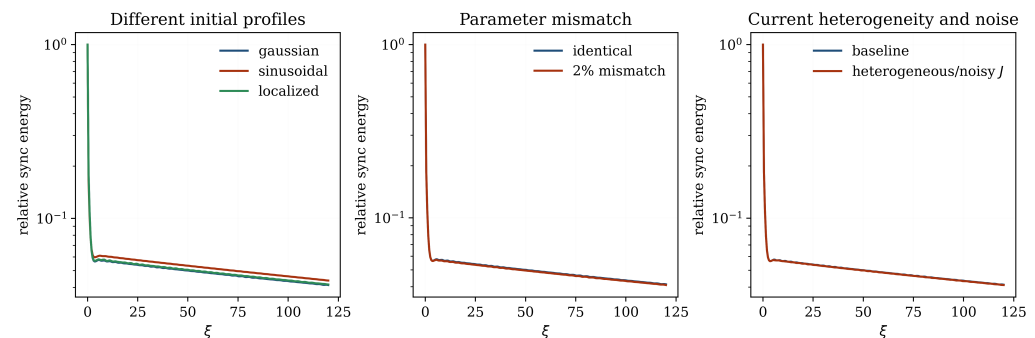


Figure 14. Synchronization robustness under different initial profiles, small parameter mismatch, and heterogeneous or noisy applied current. The relative synchronization energy was plotted for the three perturbation families (i)–(iii) described in the text. Strong coupling removed the dependence on the initial profile, whereas mismatch and heterogeneous or noisy forcing produced a nonzero bounded floor, consistent with Theorem 5.

For the mismatch experiment the relative late-time synchronization-energy floor decreased from about 8.07×10^{-2} at $\kappa = 5$ to 4.03×10^{-2} at $\kappa = 30$; it did not vanish because the synchronization manifold was no longer invariant. The standard L1 implementation stored the complete time history: with $M + 1$ spatial nodes and $N + 1$ time levels the state storage was $6(N + 1)(M + 1) \times 8$ bytes, while direct history summation had nominal work proportional to $N^2(M + 1)$. Figure 15 reports the measured execution time and state-array memory for 40, 80, 160, and 320 time levels; the memory doubled when N doubled, as expected. The delivered code uses the full-memory formula so that every result is transparent, and sum-of-exponentials or fast Fourier transform acceleration can be added for substantially longer simulations:

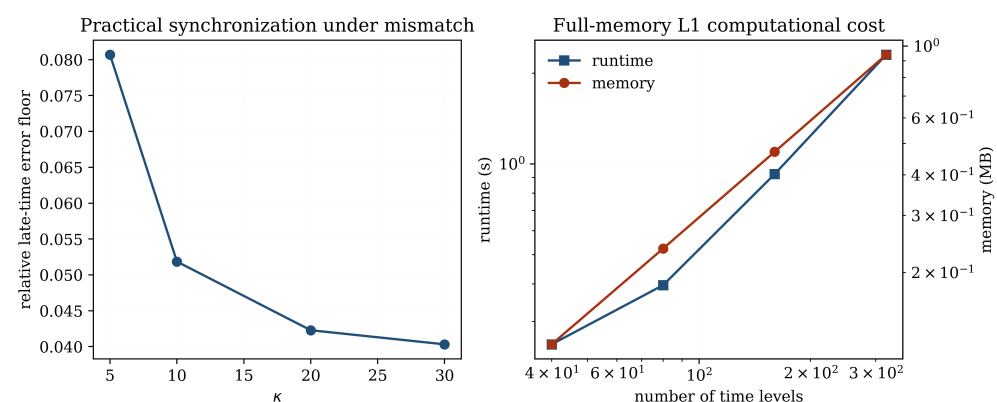


Figure 15. Left: relative practical synchronization floor under parameter mismatch. Right: measured runtime and state-array memory for the full-memory L1 method. The floor decreases from about 8.07×10^{-2} at $\kappa = 5$ to 4.03×10^{-2} at $\kappa = 30$ without vanishing, because the synchronization manifold is not invariant under mismatch. Runtime and memory are reported for 40, 80, 160, and 320 time levels.

11.11. Numerical Conclusions

The computations support the analytical and numerical developments in five respects. First, the manufactured solution verified the expected $2 - \rho$ temporal rate and second-order spatial rate. Second, the discrete energy remained below a computable Mittag–Leffler comparison bound. Third, under the sufficient coupling condition the synchronization energy followed the predicted Mittag–Leffler decay. Fourth, the integer-order and linear-diffusion limits agreed closely with independent reference calculations. Fifth, the parameter studies separated the roles of memory, coupling, and nonlinear diffusion without treating any single finite-window statistic as universal: decreasing ρ markedly enlarged the detected inter-spike interval and reduced the detected burst frequency; the sampled m values had a smaller effect on inter-spike timing but altered burst duration, peak amplitude, and spatial-gradient diagnostics; and the empirical coupling threshold was roughly three orders of magnitude below the global sufficient certificate. A small mismatch and heterogeneous current replaced exact synchronization by a nonzero, coupling-dependent practical error floor.

The computational novelty reported here is not an additional surface visualization but a quantitative separation of mechanisms: the fractional order reorganizes the temporal bursting scale, whereas the nonlinear diffusion exponent primarily modifies the spatial transition zones and gradient distribution. Coupling scans connect these effects to a finite-window synchronization transition, while the analytical theorem supplies a global sufficient certificate; the empirical transition and the global certificate are reported throughout as distinct objects.

12. Conclusions and Future Directions

In our model, which is a time-fractional partly diffusive model of two electrically coupled HR neurons, the membrane potentials are transported by a degenerate nonlinear m -Laplacian, while the temporal evolution is governed by the Caputo derivative of order $\rho \in (0, 1]$. Working in a Gelfand triple adapted to the no-flux nonlinear Neumann condition, we chose a fractional energy inequality having a grouped Young step that maintained the estimate dissipative for the whole admissible range of β , and we obtained a uniform-in-time a priori bound. With the combination of a Faedo–Galerkin construction, fractional compactness, and the Minty method, the existence of a global weak solution was established. Uniqueness and continuous dependence followed in the bounded solution class. We then proved global Mittag–Leffler synchronization above an explicit coupling threshold. In the case of parameter mismatch, a practical synchronization estimate was provided. A fully implicit L1 finite-volume scheme was shown to be solvable at every time level, with uniqueness holding under an explicit monotonicity condition, and with the scheme unconditionally energy dissipative and convergent. The expected $2 - \rho$ temporal and second-order spatial rates were recovered from the computations. The simulations split the two mechanisms cleanly: memory set the temporal rhythm, lengthening the mean bursting period by about 50% as ρ was reduced from 1 to 0.90, while the diffusion exponent had a negligible effect on that rhythm and instead controlled the spatial sharpness of the traveling fronts.

Several directions remain open. The sufficient coupling threshold is conservative by about three orders of magnitude, and a sharper criterion obtained by optimizing independent Lyapunov weights would be of both analytical and practical interest. Similarly, Proposition 2 gives an explicit fractional estimate for the Hindmarsh–Rose slow–fast separation under Caputo memory, but a complete geometric singular-perturbation and bifurcation theory, requiring nonlocal invariant manifolds for the Caputo PDE with m -Laplacian diffusion, was not attempted here and remains a substantial open problem. The uniqueness

result relies on an L^∞ bound, so removing that hypothesis for the degenerate case $m > 2$ is a natural next step, as is an eventual H^1 absorbing estimate for data taken merely in $\mathbb{H}$, which requires a sliding-window fractional argument. On the numerical side, the full-memory L1 method has quadratic cost in the number of time levels, and sum-of-exponentials or fast-convolution acceleration would permit substantially longer simulations. Finally, the analysis extends naturally to networks of N coupled neurons on a graph, to variable-order or distributed-order memory, and to genuinely stochastic forcing, for which the present discrete current-noise experiment is only a first indication.

The two-neuron problem is the minimal spatially distributed setting in which exact and mismatch-robust synchronization can be analyzed without conflating the result with graph-topological effects. For larger networks, the coupling operator would introduce a graph spectrum in addition to the nonlinear spatial diffusion within each neuron. This opens the possibility of cluster, interlayer, partial, and chimera-like states, especially in multiplex, nonlocal, adaptive, or higher-order architectures. Establishing such regimes would require a network-level extension of the present energy functional and separate coherence diagnostics; these phenomena are therefore identified as future directions rather than claimed for the present two-neuron model.

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