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Article

New Applications of Differential Subordination Theory Concerning Fractional Integrals of Gaussian Hypergeometric Functions

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Abstract

In the present research, necessary and sufficient conditions for the univalence of the fractional integral of the Gaussian hypergeometric function are obtained by employing specific approaches from differential subordination theory. The first theorem provides the best dominant for a particular differential subordination, which is proved to be a Carathéodory function. These results facilitate the establishment of several corollaries that give the necessary and sufficient conditions for this fractional operator to be a function with a positive real part of the first derivative. The univalence properties obtained through this research enhance the potential for new applications in geometric function theory and related research domains.

Keywords: fractional integral; Gaussian hypergeometric function; differential subordination; best dominant; Carathéodory function

MSC: 30C45; 30C80

1. Introduction and Preliminaries

Over the last few decades, fractional calculus has expanded significantly within mathematical analysis, offering reliable methods for modeling and analysis across multiple scientific and engineering fields. Comprehensive reviews of the subject [1,2] outline these developments and detail its wide range of applications. In the context of complex analysis, Srivastava [3] demonstrated the advantages of incorporating fractional operators into geometric function theory. This work has prompted further research on the connection between fractional calculus and univalent functions.

Fractional integral and differential operators are standard tools for modeling processes that involve memory and hereditary effects. In recent decades, these operators have been applied to complex analysis, especially within geometric function theory. In this field, the theory of differential subordination is used to establish geometric properties of analytic functions, such as univalence, starlikeness, convexity, and close-to-convexity. Additionally, investigating the real-part properties of these fractional operators through their connection to the Carathéodory class provides a fundamental criterion to ensure their injectivity and univalence.

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In the research presented in this paper, specific approaches from differential subordination theory are used to obtain necessary and sufficient conditions for the univalence of the fractional integral of the Gaussian hypergeometric function. This fractional operator was introduced in [4] and was previously investigated within the framework of third-order differential subordination theory (see, for example [5]). The present approach to finding the univalence conditions for this fractional operator is motivated by the idea that in-depth knowledge of its geometric properties enhances the possibility of new applications in geometric function theory or related research domains. Recent studies confirm that this approach is valid and provides a powerful framework for establishing the structural characteristics of fractional hypergeometric operators. Geometric properties for the fractional integral of the Bessel function of the first kind were obtained in [6], motivating the subsequent use of this fractional operator to define a new integral operator [7]. The fractional integral of the confluent hypergeometric function was investigated for univalence conditions in [8], and geometric properties were highlighted for the fractional integral of the four-parameter Mittag-Leffler function in [9]. The geometric properties of a fractional integral operator associated with the generalized Marcum Q-function were recently analyzed in [10]. Third-order differential subordination results were established for a modified Tremblay fractional derivative operator in [11]. A series representation for this operator is established to investigate its analytic and geometric properties. Furthermore, certain geometric properties through differential subordination and superordination results for p-valent analytic functions were determined, involving the (r,k)-Srivastava fractional integral calculus in [12].

The motivation for this study arises from a remaining gap in the literature regarding the necessary and sufficient conditions for the univalence of the fractional integral of the Gaussian hypergeometric function. To address this, the methods of second-order differential subordination theory are applied to obtain certain geometric properties of this specific operator.

To establish the geometric function theory context of this research, the basic definitions and lemmas used throughout this study are presented.

We denote by $\mathcal{H}(\mathcal{U})$ the class of holomorphic functions in the open unit disk:

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$, we denote

$$H[a,n] = \{f \in \mathcal{H}(\mathcal{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, z \in \mathcal{U}\}.$$

Let

$$\mathcal{A} = \left\{ f \in \mathcal{H}(\mathcal{U}) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ f(0) = 0,\ f'(0) = 1 \right\}$$

$$\mathcal{S} = \{f \in \mathcal{H}(\mathcal{U}) : f \text{ is univalent in } \mathcal{U}\},$$

$$\mathcal{R} = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > 0, z \in \mathcal{U}\},$$

$$\mathcal{P} = \{p \in \mathcal{H}(\mathcal{U}) : p(0) = 1, \operatorname{Re} p(z) > 0, z \in \mathcal{U}\},$$

which represents the Carathéodory class of functions in the unit disk $\mathcal{U}$, and let

$$\mathcal{K} = \left\{ f \in \mathcal{H}(\mathcal{U}) : \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, f'(0) \neq 0, z \in \mathcal{U} \right\},$$

be the class of convex functions in $\mathcal{U}$.

Definition 1 ([13]). *A function $f \in \mathcal{H}(\mathcal{U})$ is called convex in $\mathcal{U}$ if f is univalent in $\mathcal{U}$ and $f(\mathcal{U})$ is a convex domain.*

The concept of a convex function was introduced by E. Study in the year 1913 [13].

Theorem 1 ([14]). *(Alexander's Duality Theorem). A function $f \in \mathcal{H}(\mathcal{U})$ is convex in $\mathcal{U}$ if and only if $F(z) = zf'(z)$ is starlike in $\mathcal{U}$.*

We denote by

$$\mathcal{S}^* = \left\{ f \in \mathcal{H}(\mathcal{U}) : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, f'(0) \neq 0, z \in \mathcal{U} \right\},$$

the class of starlike functions in $\mathcal{U}$.

The concept of a starlike function was introduced by J. Alexander in the year 1915 [14].

Definition 2 ([15–17]). *A function $f \in \mathcal{H}(\mathcal{U})$ is called close-to-convex in $\mathcal{U}$ if there exists a convex function ϕ in $\mathcal{U}$ such that*

$$\operatorname{Re} \frac{f'(z)}{\phi'(z)} > 0, \quad z \in \mathcal{U}.$$

Based on Alexander's Duality Theorem (Theorem 1 [14]), a function f is a close-to-convex function in $\mathcal{U}$ if and only if there exists a starlike function g in $\mathcal{U}$ such that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in \mathcal{U}.$$

For close-to-convex functions in $\mathcal{U}$, we have the following univalency criterion of Ozaki and Kaplan:

Lemma 1 ([15,16]). *Let $D \subset \mathbb{C}$ be a simply connected domain, and let $f \in \mathcal{H}(D)$. Suppose that there exists a univalent function ϕ in domain D such that $f(D) = \Delta$ is a convex domain and $\Delta \subset \mathbb{C}$ is an open and connected subset with the property that the line segment connecting any two points in the set is entirely contained within Δ. Then,*

$$\operatorname{Re} \frac{f'(z)}{\phi'(z)} > 0, \quad z \in D,$$

meaning that f is close-to-convex with respect to ϕ, which implies that f is univalent in D.

Definition 3 ([17–19]). *We denote by $\mathcal{R}$ the class of functions for which their derivative has a positive real part in the unit disk:*

$$\mathcal{R} = \{ f \in \mathcal{H}(\mathcal{U}) : \operatorname{Re} f'(z) > 0, z \in \mathcal{U} \}.$$

Lemma 2 ([17–19]). *(Noshiro–Warschawski–Wolff Univalency Criterion). If the function f is holomorphic in a convex domain $D \subset \mathbb{C}$ and if there exists a number $\gamma \in \mathbb{R}$ such that*

$$\operatorname{Re}[e^{i\gamma} f'(z)] > 0, \quad z \in D,$$

then the function f is univalent in D.

Definition 4 ([20])**.** *Let $f, g \in \mathcal{H}(\mathcal{U})$. We say that the function f is subordinate to the function g, and we write $f \prec g$ or $f(z) \prec g(z)$ if there exists a function $w \in \mathcal{H}(\mathcal{U})$ with $w(0) = 0$ and $|w(z)| < 1, z \in \mathcal{U}$, such that*

$$f(z) = g(w(z)), \quad z \in \mathcal{U}.$$

If the function g is univalent in $\mathcal{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.

Definition 5 ([21])**.** *Let $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ and let h be a univalent function in $\mathcal{U}$. If the function $p \in \mathcal{H}(\mathcal{U})$ satisfies the differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad z \in \mathcal{U}, \tag{1}$$

then the function p is called a solution of subordination (1). Subordination (1) is called a second-order differential subordination, and the univalent function q in $\mathcal{U}$ is called a dominant of the differential solutions if

$$p(z) \prec q(z)$$

for any function p that satisfies (1). A dominant $\tilde{q}$ is called the best dominant of the differential subordination (1) if $\tilde{q}(z) \prec q(z)$ for all dominants, q.

Definition 6 ([21])**.** *We denote by $\mathcal{Q}$ the set of all functions, q, that are holomorphic and injective on $\overline{\mathcal{U}} \setminus E(q)$, where*

$$E(q) = \left\{ \zeta \in \partial\mathcal{U} : \lim_{z \to \zeta} q(z) = \infty \right\},$$

and that satisfy $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} \setminus E(q)$, where $\partial\mathcal{U} = \{z \in \mathbb{C} : |z| = 1\}$ is the boundary of the open unit disk $\mathcal{U}$. The set $E(q)$ is called the exception set.

Definition 7 ([21])**.** *Let $\Omega \subseteq \mathbb{C}$, let $q \in \mathcal{Q}$, and let $n \in \mathbb{N}$, $n \geq 1$. We denote by $\Psi_n[\Omega, q]$ the class of functions $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\psi(r, s, t; z) \notin \Omega, \tag{2}$$

when

$$r = q(\zeta), \quad s = m\zeta q'(\zeta), \quad \mathrm{Re}\left(\frac{t}{s} + 1\right) \geq m \ \mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

where $z \in \mathcal{U}$, $\zeta \in \partial\mathcal{U} \setminus E(q)$, and $m \geq n \geq 1$. The set $\Psi_n[\Omega, q]$ is called the class of admissible functions, and condition (2) is called the admissibility condition.

Lemma 3 ([21])**.** *(S.S. Miller, P.T. Mocanu Lemma). Let $q \in \mathcal{Q}$ with $q(0) = a$, and let $p \in \mathcal{H}(\mathcal{U})$ satisfy $p(z) \neq a$ and $n \geq 1$. If $p(z)$ is not subordinate to the function $q(z)$ ($p(z) \not\prec q(z)$), then there exist points $z_0 = r_0e^{i\theta_0} \in \mathcal{U}$ and $\zeta_0 \in \partial\mathcal{U} \setminus E(q)$ and a real number $m \geq n \geq 1$ such that $p(\mathcal{U}(0, r_0)) \subset q(\mathcal{U})$ and*

(i) $p(z_0) = q(\zeta_0)$;
(ii) $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$;
(iii) $\mathrm{Re}\left(\frac{z_0p''(z_0)}{p'(z_0)} + 1\right) \geq m \ \mathrm{Re}\left(\frac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)} + 1\right)$.

In [4], the authors defined the fractional integral of the Gaussian hypergeometric function, $D_z^{-\lambda}F(a,b,c;z): U \to \mathbb{C},\ a,b,c \in \mathbb{C}, c \neq 0,-1,-2,\ldots, \lambda > 0$, which is given by

$$\begin{aligned} D_z^{-\lambda}F(a,b,c;z) &= \frac{1}{\Gamma(\lambda+1)} \cdot z^{\lambda} + \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=1}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^{\lambda+n}}{\Gamma(\lambda+n+1)} \\ &= \frac{1}{\Gamma(\lambda+1)} z^{\lambda} + \frac{ab}{c} \frac{1}{\Gamma(\lambda+2)} z^{\lambda+1} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^{\lambda+2}}{\Gamma(\lambda+3)} + \ldots, \end{aligned} \tag{3}$$

$z \in \mathcal{U}$.

For $\lambda = 1$, we have

$$D_z^{-1}F(a,b,c;z) = z + \frac{ab}{c} \frac{z^2}{2!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^3}{3!} + \ldots, \quad z \in \mathcal{U},$$

and

$$\left(D_z^{-1}F(a,b,c;z)\right)' = 1 + \frac{ab}{c} \cdot z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \ldots, \quad z \in \mathcal{U}, \tag{4}$$

with

$$\left[D_z^{-1}F(a,b,c;z)\right]'_{z=0} = 1.$$

2. Main Results

The first new result is a theorem that is further used to establish univalence criteria for the fractional integral of the Gaussian hypergeometric function. In this theorem, we obtain the best dominant for a certain differential subordination.

In previous works, the best dominant was determined using tools such as the admissibility condition for the class of admissible functions, the determination of univalent solutions for differential equations corresponding to the studied differential subordinations, and the determination of univalent solutions for second-order differential equations taking values in a certain fixed domain, as well as the method of subordination chains (Löwner chains). To prove the following theorem, we will use the Miller–Mocanu lemma (Lemma 3) and the properties of certain classes of functions, such as close-to-convex functions (Definition 2); the class of functions with a positive real part derivative in $\mathcal{U}$, denoted by $\mathcal{R}$; and the class of Carathéodory functions, denoted by $\mathcal{P}$.

Theorem 2. *Let $q \in \mathcal{K}$, with $\operatorname{Re} q(z) > 0$, $z \in \mathcal{U}$, $q'(0) \neq 0$, and let $\alpha,\ \phi \in \mathcal{H}(D)$, $D \supset q(\mathcal{U})$. Suppose the following:*

(i) $\operatorname{Re} \dfrac{\alpha'[q(z)] + z\phi'[q(z) + z^2q''(z)]}{\phi[q(z) + z^2q''(z)]} \geq 0, \quad z \in \mathcal{U};$

(ii) $G(z) = zq'(z)\phi[q(z) + z^2q''(z)]$ *is starlike in* $\mathcal{U}$.

If $p \in \mathcal{H}(\mathcal{U})$, with $p(0) = q(0) = 1$, $p(\mathcal{U}) \subset q(\mathcal{U})$, satisfies the subordination

$$\alpha[p(z)] + zp'(z)\phi[p(z) + z^2p''(z)] \prec \alpha[q(z)] + zq'(z)\phi[q(z) + z^2q''(z)], \tag{5}$$

then $p(z) \prec q(z)$, $z \in \mathcal{U}$, q is the best dominant of (5), and $p \in \mathcal{P}$.

Proof. We define the function $h(z)$ given by the following relation:

$$h(z) = \alpha[q(z)] + zq'(z)\phi[q(z) + z^2q''(z)], \quad z \in \mathcal{U}. \tag{6}$$

Differentiating relation (6), after a simple calculation, we obtain

$$\operatorname{Re}\frac{zh'(z)}{G(z)} = \operatorname{Re}\frac{\alpha'[q(z)] + z\phi'[q(z) + z^2q''(z)]}{\phi[q(z) + z^2q''(z)]} + \operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0. \tag{7}$$

Since G is starlike, relation (7) implies that h is a close-to-convex function and thus univalent according to Lemma 1.

Subordination (5) becomes $\alpha[p(z)] + zp'(z)\phi[p(z) + z^2p''(z)] \prec h(z)$, which is equivalent to the following mapping inclusion:

$$\left\{\alpha[p(z)] + zp'(z)\phi[p(z) + z^2p''(z)]\right\} \subset h(\mathcal{U}). \tag{8}$$

For some $z_0 \in \mathcal{U}$, relation (8) becomes

$$\alpha[p(z_0)] + z_0p'(z_0) \cdot \phi[p(z_0) + z_0^2p''(z_0)] \in h(\mathcal{U}).$$

To prove the theorem, we will use admissibility condition (2) and Lemma 3. For this purpose, we define the function $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ given by the following relation:

$$\psi(r, s, t; z) = \alpha(r) + s \cdot \phi(r + t), \quad r, s, t \in \mathbb{C}. \tag{9}$$

For $r = p(z)$, $s = zp'(z)$, and $t = z^2p''(z)$, relation (9) becomes

$$\psi(p(z), zp'(z), z^2p''(z)) = \alpha[p(z)] + zp'(z) \cdot \phi[p(z) + z^2p''(z)]. \tag{10}$$

Using relation (10), relation (8) becomes

$$\left\{\psi(p(z), zp'(z), z^2p''(z))\right\} \subset h(\mathcal{U}). \tag{11}$$

For $z = z_0 \in \mathcal{U}$, relation (11) becomes

$$\psi(p(z_0), z_0p'(z_0), z_0^2p''(z_0)) \in h(\mathcal{U}). \tag{12}$$

Suppose, by contradiction, that p is not subordinate to q ($p \not\prec q$). Then, by Lemma 3, it follows that there exist points $z_0 \in \mathcal{U}$, $z_0 = r_0e^{i\theta_0}$, $r_0 < 1$, $\zeta_0 \in \partial\mathcal{U} \setminus E(q)$, and $m \geq n \geq 1$, such that $p(\mathcal{U}(0, r_0)) \subset q(\mathcal{U})$ and

$$p(z_0) = q(\zeta_0), \quad z_0p'(z_0) = m\zeta_0q'(\zeta_0), \quad \operatorname{Re}\left(\frac{z_0p''(z_0)}{p'(z_0)} + 1\right) \geq m\left(\operatorname{Re}\frac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)} + 1\right). \tag{13}$$

Substituting $r = q(\zeta_0)$, $s = m\zeta_0q'(\zeta_0)$, and $t = z_0^2p''(z_0)$ into admissibility condition (2) from Definition 7, we have

$$\psi(q(\zeta_0), m\zeta_0q'(\zeta_0), z_0^2p''(z_0)) \notin h(\mathcal{U}). \tag{14}$$

Using the equalities given in (13) in relation (14), we obtain

$$\psi(p(z_0), z_0p'(z_0), z_0^2p''(z_0)) \notin h(\mathcal{U}). \tag{15}$$

Since relation (15) contradicts relation (12), it follows that the initial assumption is false, and therefore,

$$p(z) \prec q(z), \quad z \in \mathcal{U}.$$

The function q is a convex function in $\mathcal{U}$ and hence a univalent function in $\mathcal{U}$; since it satisfies Equation (6), it follows that it is the best dominant of differential subordination (5).

Next, we show that the function $p \in \mathcal{H}(\mathcal{U})$ is a Carathéodory function in $\mathcal{U}$. Since function q is a convex function in $\mathcal{U}$ and $\operatorname{Re} q(z) > 0$, $z \in \mathcal{U}$, with $q'(0) \neq 0$, the function q realizes a conformal mapping of the unit disk $\mathcal{U}$ onto $q(\mathcal{U}) = \{q(z) \in \mathbb{C} : \operatorname{Re} q(z) > 0\}$, a convex domain. Since the function q is univalent in $\mathcal{U}$, subordination (5) is equivalent to

$$\operatorname{Re} p(z) > 0, \quad z \in \mathcal{U}. \tag{16}$$

From the hypothesis of the theorem, we have $p(0) = 1$; then, relation (16) implies $p \in \mathcal{P}$, meaning that it is a Carathéodory function. □

The results obtained in this theorem allow us to determine the necessary and sufficient conditions under which the fractional integral of the Gaussian hypergeometric function is a function for which its derivative has a positive real part and hence is univalent in $\mathcal{U}$ according to Lemma 2.

If $p(z) = \left(D_z^{-\lambda} F(a,b,c;z)\right)'$ with $\lambda = 1$, $a,b,c \in \mathbb{C}$, $c \neq 0,-1,-2,\dots$, as given by relation (4), and $ab \neq 0$, from Theorem 2, we have the following corollary:

Corollary 1. *Let q be a convex function in $\mathcal{U}$, with $\operatorname{Re} q(z) > 0$, $z \in \mathcal{U}$, and $q'(0) \neq 0$, and let the functions α, $\phi \in \mathcal{H}(D)$, $D \supset q(\mathcal{U})$. Suppose that the following conditions are satisfied:*

(i) $\operatorname{Re} \dfrac{\alpha'[q(z)] + z\phi'[q(z) + z^2 q''(z)]}{\phi[q(z) + z^2 q''(z)]} \geq 0, \quad z \in \mathcal{U};$

(ii) $G(z) = zq'(z) \cdot \phi[q(z) + z^2 q''(z)]$ *is a starlike function in $\mathcal{U}$.*

If the functions $p(z) = \left(D_z^{-1} F(a,b,c;z)\right)' \in \mathcal{H}(\mathcal{U})$, with $p(0) = q(0) = 1$, $p(\mathcal{U}) \subset q(\mathcal{U})$, and

$$\begin{aligned}&\alpha\left[\left(D_z^{-1} F(a,b,c;z)\right)'\right]\\&\quad + z\left(D_z^{-1} F(a,b,c;z)\right)'' \cdot \phi\left[\left(D_z^{-1} F(a,b,c;z)\right)' + z^2\left(D_z^{-1} F(a,b,c;z)\right)'''\right]\end{aligned}$$

are holomorphic in $\mathcal{U}$ and satisfy the differential subordination

$$\begin{aligned}&\alpha\left[\left(D_z^{-1} F(a,b,c;z)\right)'\right] + z\left(D_z^{-1} F(a,b,c;z)\right)''\\&\quad \cdot \phi\left[\left(D_z^{-1} F(a,b,c;z)\right)' + z^2\left(D_z^{-1} F(a,b,c;z)\right)'''\right]\\&\quad \prec \alpha[q(z)] + zq'(z) \cdot \phi[q(z) + z^2 q''(z)], \quad z \in \mathcal{U},\end{aligned} \tag{17}$$

then, we have the following:

(a) $\left(D_z^{-1} F(a,b,c;z)\right)' \prec q(z)$, $z \in \mathcal{U}$, *and the function q is the best dominant of the subordination (17).*

(b) $\operatorname{Re}\left(D_z^{-1} F(a,b,c;z)\right)' > 0$, $z \in \mathcal{U}$, *i.e.,* $D_z^{-1} F(a,b,c;z) \in \mathcal{R}$, *and by Lemma 2,* $D_z^{-1} F(a,b,c;z) \in \mathcal{S}$, $z \in \mathcal{U}$.

For $\lambda \in (0,1)$, we define $\Omega^\lambda F(a,b,c;z) = \Gamma(\lambda+1) z^{-\lambda} D_z^{-\lambda} F(a,b,c;z)$, where the Γ function is the Euler integral of the second kind, and $D_z^{-\lambda} F(a,b,c;z)$ is given by relation (3).

Let

$$p(z) = \Omega^\lambda F(a,b,c;z) = \Gamma(\lambda+1) z^{-\lambda} D_z^{-\lambda} F(a,b,c;z). \tag{18}$$

By doing simple calculations using (18), we obtain

$$p(z) = 1 + \frac{ab}{c}\frac{1}{\lambda+1} z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{1}{(\lambda+1)(\lambda+2)} z^2 + \cdots. \tag{19}$$

For $z = 0$, we have $p(0) = 1$.

If $p(z)$ is given by (19), from Theorem 2, we have the following corollary:

Corollary 2. *Let $q \in \mathcal{K}$, with $\operatorname{Re} q(z) > 0$, $z \in \mathcal{U}$, $q'(0) \neq 0$, and let the functions α, $\phi \in \mathcal{H}(D), D \supset q(\mathcal{U})$. Suppose that we have the following:*

(i) $\operatorname{Re} \dfrac{\alpha'[q(z)] + z\phi'[q(z) + z^2 q''(z)]}{\phi[q(z) + z^2 q''(z)]} \geq 0, \quad z \in \mathcal{U};$

(ii) $G(z) = zq'(z)\phi[q(z) + z^2 q''(z)]$ *is starlike in* $\mathcal{U}$.

If the functions $p(z) = \Omega^\lambda F(a,b,c;z)$, $p(0) = q(0) = 1$, $p(\mathcal{U}) \subset q(\mathcal{U})$, and $\alpha(p(z)) + zp'(z) \cdot \phi\left[p(z) + z^2 p''(z)\right]$ are holomorphic in $\mathcal{U}$ and satisfy the differential subordination

$$\begin{aligned} &\alpha\left[\Omega^\lambda F(a,b,c;z)\right] + z\left(\Omega^\lambda F(a,b,c;z)\right)' \cdot \phi\left[\Omega^\lambda F(a,b,c;z) + z^2\left(\Omega^\lambda F(a,b,c;z)\right)''\right] \\ &\prec \alpha[q(z)] + zq'(z) \cdot \phi[q(z) + z^2 q''(z)], \quad z \in \mathcal{U}, \end{aligned} \tag{20}$$

then we have the following:

(a) $\Omega^\lambda F(a,b,c;z) \prec q(z)$, $z \in \mathcal{U}$, *and q is the best dominant of subordination* (20).

(b) $\operatorname{Re}\, \Omega^\lambda F(a,b,c;z) > 0$, $z \in \mathcal{U}$, *i.e.*, $\Omega^\lambda F(a,b,c;z) \in \mathcal{P}$.

Example 1. *For $\lambda = 1$, $a = -1$, $b = 2 - i$, and $c = 3 + i$, the fractional integral of the Gaussian hypergeometric function is given by*

$$D_z^{-1} F(-1, 2-i, 3+i; z) = z - \frac{1-i}{4} z^2.$$

We denote $p(z) = \left[D_z^{-1}F(-1,2-i,3+i;z)\right]' = \left(z - \frac{1-i}{4}z^2\right)' = 1 - \frac{1-i}{2}z$, which is a holomorphic function in $\mathcal{U}$. Let α, $\phi \in \mathcal{H}(D), D \supset q(\mathcal{U})$. Let the function $q(z) = \frac{1+z}{1-z}$ be a convex function in $\mathcal{U}$, with $q(0) = 1$, $\operatorname{Re} q(z) > 0$, $z \in \mathcal{U}$.

Using Corollary 1, we have the following.

Let $q(z) = \frac{1+z}{1-z}$ be a convex function in $\mathcal{U}$, with $\operatorname{Re} q(z) > 0$, $z \in \mathcal{U}$, and $q'(0) \neq 0$, and let α, $\phi \in \mathcal{H}(D), D \supset q(\mathcal{U})$.

Suppose the following conditions are satisfied:

(i′) $\operatorname{Re} \dfrac{\alpha'\left(\frac{1+z}{1-z}\right) + z \cdot \phi'\left[\frac{(1+z)(1-z)^2 + 4z^2}{(1-z)^3}\right]}{\phi\left[\frac{(1+z)(1-z)^2 + 4z^2}{(1-z)^3}\right]} \geq 0, \quad z \in \mathcal{U}.$

(ii′) $G(z) = \dfrac{2z}{(1-z)^2} \cdot \phi\left[\dfrac{(1+z)(1-z)^2 + 4z^2}{(1-z)^3}\right]$ *is a starlike function in* $\mathcal{U}$.

If the function $p(z) = 1 - \frac{1-i}{2}z$, with $p(0) = 1$, is a holomorphic function such that $p(\mathcal{U}) \subset q(\mathcal{U})$, and $\alpha\left(1 - \frac{1-i}{2}z\right) + z\left(-\frac{1-i}{2}\right) \cdot \phi\left(1 - \frac{1-i}{2}z\right)$ is holomorphic in $\mathcal{U}$ and satisfies the differential subordination

$$\begin{aligned} &\alpha\left(1 - \frac{1-i}{2}z\right) + z\left(-\frac{1-i}{2}\right) \cdot \phi\left(1 - \frac{1-i}{2}z\right) \\ &\prec \alpha\left(\frac{1+z}{1-z}\right) + \frac{2z}{(1-z)^2} \cdot \phi\left[\frac{(1+z)(1-z)^2 + 4z^2}{(1-z)^3}\right], \end{aligned}$$

then we have the following:

(a) $1 - \dfrac{1-i}{2}z \prec q(z) = \dfrac{1+z}{1-z}, \quad z \in \mathcal{U}$, *and $q(z) = \frac{1+z}{1-z}$ is the best dominant of the subordination.*

(b) $\operatorname{Re}\left(1 - \dfrac{1-i}{2}z\right) > 0, \quad z \in \mathcal{U}$, *i.e.*, $z - \dfrac{1-i}{4}z^2 \in \mathcal{R} \subset \mathcal{S}$.

Example 2. *For $\lambda > 1, \lambda \in \mathbb{N}$, for any $a, b, c \in \mathbb{C}, c \neq 0, -1, -2, \ldots$, set $\lambda = 3$ as an example, and we obtain*

$$D_z^{-3}F(a,b,c;z) = \frac{1}{\Gamma(4)}z^3 + \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=1}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^{3+n}}{\Gamma(4+n)} =$$

$$\frac{1}{\Gamma(4)}z^3 + \frac{ab}{c}\frac{1}{\Gamma(5)}z^4 + \cdots, z \in \mathcal{U}.$$

By differentiating, we obtain

$$\left[D_z^{-3}F(a,b,c;z)\right]' = \frac{3}{3!}z^2 + \frac{4}{4!}\frac{ab}{c}z^3 + \cdots$$

and

$$\left[D_z^{-3}F(a,b,c;z)\right]'_{z=0} = 0.$$

Since the necessary condition for univalency in $\mathcal{U}$ is not satisfied, $D_z^{-3}F(a,b,c;z)$ is not a univalent function in $\mathcal{U}$.

Example 3. *For $\lambda = \frac{1}{2}$, $a = -i$, $b = i$, and $c = 2i$, we have*

$$D_z^{-\frac{1}{2}}F(-i,i,2i;z) = \frac{2}{\sqrt{\pi}}z^{\frac{1}{2}} - \frac{2i}{3\sqrt{\pi}}z^{\frac{3}{2}} + \frac{-8(2+i)}{75\sqrt{\pi}}z^{\frac{5}{2}} + \ldots$$

We define

$$\begin{aligned}\Omega_{\frac{1}{2}}F(-i,i,2i;z) &= \Gamma\left(\frac{3}{2}\right)z^{-\frac{1}{2}} \cdot D_z^{-\frac{1}{2}}F(-i,i,2i;z) \\ &= 1 - \frac{i}{3}z + \frac{-8-4i}{75}z^2 + \ldots, \quad z \in \mathcal{U}.\end{aligned}$$

Applying Corollary 2, we have

Let $q \in \mathcal{K}$, $q(0) = 1$, $q'(0) \neq 0$, $\mathrm{Re}\, q(z) > 0$, $z \in U$, and $\alpha, \phi \in \mathcal{H}(D), D \supset q(\mathcal{U})$. We assume the following

(i) $\mathrm{Re}\left(\alpha'[q(z)] + \dfrac{z\phi'[q(z)+z^2q''(z)]}{\phi[q(z)+z^2q''(z)]}\right) > 0, \quad z \in \mathcal{U}.$

(ii) $G(z) = z \cdot q'(z) \cdot \phi[q(z) + z^2q''(z)]$ *is a starlike function in $\mathcal{U}$.*

If functions

$$p(z) = 1 - \frac{i}{3}z + \frac{-8-4i}{75}z^2 + \ldots,$$

and

$$\alpha\left(1 - \frac{i}{3}z + \frac{-8-4i}{75}z^2 + \ldots\right) + z\left(-\frac{i}{3} + \frac{-16-8i}{75}z + \ldots\right) \cdot \phi\left[1 - \frac{i}{3}z + \left(\frac{-24-12i}{75}\right)z^2 + \ldots\right],$$

are holomorphic functions in $\mathcal{U}$, then

$$\begin{aligned}&\alpha\left(1 - \frac{i}{3}z + \frac{-8-4i}{75}z^2 + \ldots\right) + z\left(-\frac{i}{3} + \frac{-16-8i}{75}z + \ldots\right) \cdot \phi\left[1 - \frac{i}{3}z + \left(\frac{-24-12i}{75}\right)z^2 + \ldots\right] \\ &\prec \alpha[q(z)] + z \cdot q'(z) \cdot \phi[q(z) + z^2q''(z)],\end{aligned}$$

implies the following:

(a) $1 - \frac{i}{3}z + \frac{-8-4i}{75}z^2 + \cdots \prec q(z)$, $z \in \mathcal{U}$, *and q is the best dominant.*

(b) $\mathrm{Re}\left(1 - \frac{i}{3}z + \frac{-8-4i}{75}z^2 + \ldots\right) > 0$, *i.e.*, $1 - \frac{i}{3}z + \frac{-8-4i}{75}z^2 + \cdots \in \mathcal{P}$.

3. Conclusions

The necessary and sufficient conditions for the univalence of the fractional integral of the Gaussian hypergeometric function are established in the corollaries derived from Theorem 2. The geometric criteria are obtained using the methods of differential subordination theory. The results obtained can be applied to define new subclasses of analytic and univalent functions associated with this operator. Consequently, the definition of these classes provides the necessary framework for investigating coefficient estimates, Fekete–Szegö inequalities, and higher-order Hankel or Toeplitz determinants. Furthermore, the established univalence properties have potential applications in operator theory and complex integral equations, where the injectivity of fractional operators is essential to ensure the uniqueness of solutions.

These geometric advancements are based on broader scientific frameworks. Fractional operators and differential subordination theory extend traditional mathematical analysis to non-integer or complex orders. These results are closely connected to special Gamma and Mittag-Leffler functions, while also relying on Laplace and Fourier integral transforms in the complex plane. Through these integral representations, the Gaussian hypergeometric function forms the structural basis of various fractional calculus operators. Consequently, the geometric investigation of their fractional integrals determines the conditions under which these operators preserve univalency.

Beyond the purely theoretical interest within geometric function theory, the analytical behavior of these fractional operators possesses potential relevance in related applied disciplines. In theoretical continuum mechanics and viscoelasticity, fractional-order frameworks are utilized to model memory effects and rheological properties. By establishing the geometric constraints under which these underlying hypergeometric representations remain univalent, our results provide structural insights into the analytic consistency of such non-integer models. Furthermore, in two-dimensional potential fluid dynamics, ensuring the univalency of these fractional integral mappings is critical for constructing complex velocity potentials and stream functions, ensuring that flow lines do not intersect in physical subdomains.

The findings presented in this paper complement the existing literature on analytic functions. They provide concrete mathematical references for further analysis of this specific fractional operator.

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