fractal and fractional

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Numerical and Analytical Investigations of Hadamard Variable Order Fractional Differential Equations via Cumulative Distribution Functions

Mohammed Said Souid [1,2,3], Zoubida Bouazza [4], Souhila Sabit [5] and Kanokwan Sitthithakerngkiet [6,7,*]

1 Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences , Saveetha University, Chennai 02105, Tamil Nadu, India; souimed2008@yahoo.com
2 Department of Allied Sciences, Faculty of Arts and Science, Al-Ahliyya Amman University, Amman 19111, Jordan
3 Department of Economic Sciences, Ibn Khaldoun University of Tiaret, Tiaret 14000, Algeria
4 Department of Computer Sciences, Ibn Khaldoun University of Tiaret, Tiaret 14000, Algeria; bouazza3@yahoo.fr
5 Departement of Mathematics, Ibn Khaldoun University of Tiaret, Tiaret 14000, Algeria; sabitsouhila@yahoo.fr
6 Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok (KMUTNB), Bangkok 10800, Thailand
7 Intelligent and Nonlinear Dynamic Innovations Research Center, King Mongkut's University of Technology North Bangkok (KMUTNB), Bangkok 10800, Thailand
* Correspondence: kanokwan.s@sci.kmutnb.ac.th

Abstract

This paper investigates a class of Hadamard variable-order fractional differential equations in which the fractional order is determined by the cumulative distribution function (CDF) of a continuous random variable. The proposed framework establishes a novel connection between probability theory and variable-order fractional calculus by allowing the memory index of the fractional operator to evolve according to a prescribed distribution law. To facilitate the analysis, the CDF-based variable order is considered through a piecewise-constant representation on a finite partition of the interval, which transforms the original problem into a family of Hadamard fractional differential equations of constant order on successive subintervals. Existence and uniqueness results are established by converting the differential problem into an equivalent fractional integral equation and applying the Banach contraction principle in suitable Banach spaces. Sufficient conditions ensuring the well-posedness of the problem are derived in terms of explicit bounds involving the fractional order and the nonlinear term. In addition, the Ulam–Hyers stability of the proposed model is investigated, and stability criteria are obtained under the same analytical framework. To illustrate the applicability of the theoretical results, a numerical example involving a CDF-generated variable-order function is presented. The example verifies the assumptions of the existence, uniqueness, and stability theorems and demonstrates the effect of piecewise-constant approximations of the cumulative distribution function on the resulting numerical solutions. The obtained results show that the proposed CDF-based Hadamard variable-order framework provides a mathematically consistent setting for studying fractional differential equations whose memory characteristics depend on probabilistic distributions.

Keywords: Hadamard variable-order fractional derivative; cumulative distribution function; fractional differential equations; Banach contraction principle; existence and uniqueness; Ulam–Hyers stability; numerical approximation

MSC: 26A33; 34A08; 34D20

Check for updates

Academic Editor: Haci Mehmet Baskonus

Received: 3 May 2026
Revised: 28 June 2026
Accepted: 4 July 2026
Published: 6 July 2026

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1. Introduction

Fractional calculus has emerged as a powerful extension of classical calculus due to its ability to describe nonlocal phenomena and systems possessing memory and hereditary properties. In contrast to integer-order models, fractional operators incorporate the influence of previous states of a system, which makes them particularly suitable for modeling complex processes with long-range dependence. In recent decades, fractional differential equations have received considerable attention and have been successfully applied in numerous fields, including viscoelastic materials, anomalous diffusion, control systems, fluid mechanics, biological models, signal processing, finance, and probability theory; see, for instance, [1–4] and the references therein.

Among the different fractional operators available in the literature, the Hadamard fractional derivative occupies an important position because of its logarithmic kernel. Unlike the Riemann–Liouville and Caputo fractional derivatives, the Hadamard operator is naturally associated with multiplicative processes and logarithmic scaling phenomena. This feature makes it particularly appropriate for describing systems in which the evolution depends on relative changes rather than absolute differences. Consequently, Hadamard fractional differential equations have attracted increasing interest, and various results concerning existence, uniqueness, stability, qualitative properties, and numerical approximations have been developed; see [5–7] and references therein.

In many real-world applications, the memory characteristics of a system are not constant during its evolution. This observation has motivated the development of variable-order fractional calculus, where the fractional order is allowed to depend on time, space, or other state variables. Variable-order fractional operators provide additional flexibility compared with constant-order operators and have been successfully employed in the modeling of heterogeneous materials, anomalous transport processes, biological systems, adaptive control problems, and complex dynamical systems. Several theoretical investigations concerning existence results, stability properties, controllability, and numerical methods for variable-order fractional differential equations have recently been reported in [8–12].

A fundamental issue in variable-order fractional calculus concerns the determination of the order function. In most existing works, the variable order is introduced as a prescribed deterministic function selected according to the considered application or mathematical requirements. However, in many physical and biological systems, the evolution of memory effects is influenced by uncertainty, randomness, or statistical information. Therefore, it is natural to investigate new mechanisms that allow the fractional order to be generated from probabilistic structures.

Probability theory offers a suitable framework for describing uncertainty and random behavior. In particular, cumulative distribution functions (CDFs) possess several important properties, such as monotonicity, boundedness, and continuity, which make them natural candidates for constructing variable fractional orders. Since a cumulative distribution function takes values in the interval $[0, 1]$, it provides an admissible fractional order function without imposing additional restrictions. From a modeling point of view, a CDF-based fractional order enables the memory parameter to evolve according to the statistical distribution of an underlying random variable, creating a new interaction between probability theory and fractional calculus.

The relationship between fractional calculus and probability theory has been explored from different perspectives. Several fractional generalizations of classical probability distributions, including exponential, gamma, Rayleigh, Lévy, and other distributions, have been introduced and investigated in recent studies [13–15]. Furthermore, fractional and variable-order techniques have been incorporated into the construction of new probabilistic

models [16,17]. Nevertheless, the analysis of fractional differential equations in which the fractional order itself is directly determined by a cumulative distribution function remains largely unexplored.

Motivated by these observations, this paper investigates the following Hadamard variable-order fractional initial value problem:

$$\begin{cases} {}^{H}D_{1^+}^{F_X(\eta)} y(\eta) = f(\eta, y(\eta)), \quad \eta \in \Omega := [1, \tau], \\ y(1) = 0, \end{cases} \tag{1}$$

where $1 < \tau < +\infty$, ${}^{H}D_{1^+}^{F_X(\eta)}$ denotes the Hadamard fractional derivative of variable order, while F_X represents the cumulative distribution function associated with a continuous random variable X.

The main challenge in the analysis of problem (1) is the dependence of the fractional order on the cumulative distribution function. To overcome this difficulty, we consider a piecewise-constant approximation of the CDF over a finite partition of the considered interval. This assumption provides a suitable analytical framework in which the variable-order problem can be decomposed into a sequence of constant-order Hadamard fractional differential equations. Consequently, classical fixed-point techniques can be applied on each interval, allowing us to establish existence, uniqueness, and stability results.

The main contributions of this work are summarized as follows:

- A new class of Hadamard variable-order fractional differential equations is introduced, where the fractional order is generated by a cumulative distribution function.
- An equivalent integral formulation is established for the proposed problem on each subinterval induced by the piecewise representation of the cumulative distribution function.
- Using the Banach contraction mapping principle, sufficient conditions for the existence and uniqueness of solutions are obtained.
- The Ulam–Hyers stability property of the considered problem is investigated, and explicit stability conditions are derived.
- A numerical example is provided to demonstrate the applicability of the theoretical results and to illustrate the effect of the CDF-generated variable order on the obtained solutions.

The organization of the paper is as follows. Section 2 presents the necessary preliminaries concerning cumulative distribution functions and Hadamard fractional operators. Section 3 is devoted to the existence and uniqueness analysis of the proposed initial value problem. In Section 4, the Ulam–Hyers stability property is established. Section 5 provides a numerical example together with graphical illustrations and error analysis. Finally, Section 6 concludes the paper and discusses possible directions for future research.

2. Preliminaries

In this section, we recall the basic concepts, notation, and auxiliary results that will be used throughout the paper. We first introduce the cumulative distribution function and explain its role in generating a variable fractional order. Then, we recall the variable-order Hadamard fractional operators and some properties required in the analysis.

Let

$$\Omega := [1, \tau], \qquad 1 < \tau < +\infty.$$

We denote as $C(\Omega, \mathbb{R})$ the Banach space of continuous functions from Ω into $\mathbb{R}$ equipped with the norm

$$\|y\|_\infty = \sup_{\eta \in \Omega} |y(\eta)|.$$

For $0 \leq r \leq 1$, we also consider the weighted space

$$C_r(\Omega, \mathbb{R}) = \{y \in C(\Omega, \mathbb{R}) : (\ln \eta)^r y(\eta) \in C(\Omega, \mathbb{R})\},$$

endowed with the norm

$$\|y\|_{C_r} = \sup_{\eta \in \Omega} |(\ln \eta)^r y(\eta)|.$$

Throughout this work, X denotes a continuous random variable with probability density function f_X and cumulative distribution function F_X.

Definition 1 ([18]). *The cumulative distribution function associated with X is defined by*

$$F_X(\kappa) = \mathbb{P}(X \leq \kappa) = \int_{-\infty}^{\kappa} f_X(s)\, ds, \qquad \kappa \in \mathbb{R}.$$

Proposition 1 ([18]). *Let F_X be the cumulative distribution function of a continuous random variable X. Then, the following statements are true:*

1. $0 \leq F_X(\kappa) \leq 1$ *for every* $\kappa \in \mathbb{R}$;
2. F_X *is non-decreasing on* $\mathbb{R}$;
3. $$\lim_{\kappa \to -\infty} F_X(\kappa) = 0, \qquad \lim_{\kappa \to +\infty} F_X(\kappa) = 1;$$
4. F_X *is continuous on* $\mathbb{R}$.

Remark 1. *Since a cumulative distribution function takes values in $[0, 1]$, it provides a natural bounded function that can be used to generate an admissible fractional order. In this paper, the variable order is constructed by setting the fractional order according to F_X.*

Let

$$\alpha : \Omega \to (0, +\infty), \qquad \beta : \Omega \to (0, 1]$$

be continuous functions.

Definition 2 ([6]). *The left-sided Hadamard fractional integral of variable order is defined by*

$${}^{H}I_{1^+}^{\alpha(\eta)} f(\eta) = \frac{1}{\Gamma(\alpha(\eta))} \int_1^{\eta} \left(\ln \frac{\eta}{\rho}\right)^{\alpha(\eta)-1} \frac{f(\rho)}{\rho}\, d\rho, \qquad \eta > 1. \tag{2}$$

Definition 3 ([6]). *The left-sided Hadamard fractional derivative of variable order is defined as*

$${}^{H}D_{1^+}^{\beta(\eta)} f(\eta) = \frac{1}{\Gamma(n - \beta(\eta))} \left(\eta \frac{d}{d\eta}\right)^n \int_1^{\eta} \left(\ln \frac{\eta}{\rho}\right)^{n-\beta(\eta)-1} \frac{f(\rho)}{\rho}\, d\rho, \tag{3}$$

where

$$n = [\beta(\eta)] + 1.$$

Notice that, for a fixed evaluation point η, the value $\beta(\eta)$ is kept constant throughout the integration interval $[1, \eta]$. Therefore, the variable-order operator preserves the complete fractional memory associated with the history of the solution.

Remark 2. *When $\alpha(\eta)$ and $\beta(\eta)$ are constants, Definitions 2 and 3 reduce to the classical Hadamard fractional integral and derivative, respectively.*

The following lemmas summarize some properties of Hadamard fractional operators that will be used later.

Lemma 1 ([6]). *Let $\alpha > 0$ be constant and assume that $f \in L^1(1,\tau)$ with ${}^H D^{\alpha}_{1^+} f \in L^1(1,\tau)$. Then,*

$$ {}^H D^{\alpha}_{1^+} f(\eta) = 0$$

has the general solution

$$f(\eta) = \sum_{k=1}^{n} c_k (\ln \eta)^{\alpha-k},$$

where $n = [\alpha] + 1$.

Moreover,

$${}^H I^{\alpha}_{1^+} ({}^H D^{\alpha}_{1^+} f)(\eta) = f(\eta) + \sum_{k=1}^{n} c_k (\ln \eta)^{\alpha-k}.$$

Lemma 2 ([6]). *Let $\alpha > 0$ and $f \in L^1(1,\tau)$. Then,*

$${}^H D^{\alpha}_{1^+} \left({}^H I^{\alpha}_{1^+} f \right)(\eta) = f(\eta).$$

Lemma 3 ([6]). *Let $\alpha, \beta > 0$. Then,*

$${}^H I^{\alpha}_{1^+} ({}^H I^{\beta}_{1^+} f)(\eta) = {}^H I^{\beta}_{1^+} ({}^H I^{\alpha}_{1^+} f)(\eta) = {}^H I^{\alpha+\beta}_{1^+} f(\eta).$$

Lemma 4 ([9]). *Let $\alpha : \Omega \to (0,1]$ be continuous and $f \in C_r(\Omega, \mathbb{R})$ with $0 \le r \le 1$. Then, the variable-order Hadamard fractional integral ${}^H I^{\alpha(\eta)}_{1^+} f(\eta)$ exists for every $\eta \in \Omega$.*

To construct numerical approximations of the variable-order function generated by the cumulative distribution function, we introduce the following notion.

Definition 4 ([19]). *Let $I \subset \mathbb{R}$.*

A finite family $\wp$ of generalized intervals is called a partition of I if every point of I belongs to exactly one element of $\wp$.

A function $g : I \to \mathbb{R}$ is called piecewise constant with respect to $\wp$ if it is constant on every element of the partition.

In the numerical part of this work, the cumulative distribution function is approximated by a piecewise-constant function on the partition

$$\wp = \{[\tau_0, \tau_1), [\tau_1, \tau_2), \ldots, [\tau_{m-1}, \tau_m]\},$$

where

$$1 = \tau_0 < \tau_1 < \cdots < \tau_m = \tau.$$

The approximation is defined by

$$F_X^{(m)}(\eta) = F_X(\tau_i), \qquad \eta \in [\tau_{i-1}, \tau_i).$$

This piecewise-constant representation is introduced only as an approximation of the variable-order function F_X and does not modify the definition of the Hadamard variable-order operator. The fractional memory integral remains evaluated over the complete interval $[1, \eta]$ with the order corresponding to the evaluation point.

3. Existence and Uniqueness of Solutions

In this section, we establish the existence and uniqueness of solutions for the considered initial value problem. The main idea is to preserve the memory structure of the Hadamard variable-order operator while approximating the order function generated by the cumulative distribution function.

Let

$$\Omega = [1, \tau], \qquad 1 < \tau < +\infty,$$

and consider the partition

$$\wp = \{[\tau_0, \tau_1), [\tau_1, \tau_2), \ldots, [\tau_{n-1}, \tau_n]\},$$

where

$$1 = \tau_0 < \tau_1 < \cdots < \tau_n = \tau.$$

We assume that the cumulative distribution function F_X is approximated by the piecewise-constant function

$$F_X^{(n)}(\eta) = F_X(\tau_i), \qquad \eta \in [\tau_{i-1}, \tau_i),$$

for $i = 1, \ldots, n$. This approximation concerns only the variable-order function. The Hadamard fractional operator is still defined according to Definition 3; in particular, for a fixed evaluation point η, the order $F_X^{(n)}(\eta)$ is used over the whole memory interval $[1, \eta]$.

$\mathbf{H_1}$. *The function $F_X^{(n)} : \Omega \to (0, 1]$ is piecewise constant with respect to the partition $\wp$, namely,*

$$F_X^{(n)}(\eta) = \sum_{i=1}^{n} F_X(\tau_i)\chi_i(\eta),$$

where χ_i denotes the characteristic function of $[\tau_{i-1}, \tau_i)$.

For each $\eta \in \Omega_i = [\tau_{i-1}, \tau_i)$, the corresponding fractional order is therefore

$$\beta_i = F_X(\tau_i).$$

The variable-order Hadamard derivative at point η becomes

$${}^{H}D_{1^+}^{F_X^{(n)}(\eta)} y(\eta) = \frac{1}{\Gamma(1-\beta_i)} \left(\eta \frac{d}{d\eta}\right) \int_1^{\eta} \left(\ln \frac{\eta}{\rho}\right)^{-\beta_i} \frac{y(\rho)}{\rho}\, d\rho, \tag{4}$$

where the order β_i is constant ins the complete integration interval $[1, \eta]$.

Hence, the considered initial value problem

$$\begin{cases} {}^{H}D_{1^+}^{F_X^{(n)}(\eta)} y(\eta) = f(\eta, y(\eta)), & \eta \in \Omega, \\ y(1) = 0, \end{cases} \tag{5}$$

is equivalently written in its integral form as

$$y(\eta) = \frac{1}{\Gamma(F_X^{(n)}(\eta))} \int_1^{\eta} \left(\ln \frac{\eta}{\rho}\right)^{F_X^{(n)}(\eta)-1} \frac{f(\rho, y(\rho))}{\rho}\, d\rho. \tag{6}$$

The formulation above preserves the whole history of the solution and does not introduce artificial restarting conditions at the partition points.

Definition 5. *A function*

$$y \in C(\Omega, \mathbb{R})$$

is called a solution of the initial value problem (5) *if it satisfies the integral Equation* (6) *for every* $\eta \in \Omega$.

We now define the operator

$$(Ty)(\eta) = \frac{1}{\Gamma(F_X^{(n)}(\eta))} \int_1^\eta \left(\ln \frac{\eta}{\rho}\right)^{F_X^{(n)}(\eta)-1} \frac{f(\rho, y(\rho))}{\rho}\, d\rho.$$

A solution of the problem is therefore a fixed point of the operator T.

Let $y, z \in C(\Omega, \mathbb{R})$. Assuming that the nonlinear function satisfies the Lipschitz condition

$$|f(\eta, y) - f(\eta, z)| \le L|y - z|,$$

we obtain

$$\begin{aligned} |(Ty)(\eta) - (Tz)(\eta)| &\le \frac{L}{\Gamma(F_X^{(n)}(\eta))} \int_1^\eta \left(\ln \frac{\eta}{\rho}\right)^{F_X^{(n)}(\eta)-1} \frac{|y(\rho) - z(\rho)|}{\rho}\, d\rho \\ &\le \frac{L}{\Gamma(F_X^{(n)}(\eta)+1)} (\ln \eta)^{F_X^{(n)}(\eta)} \|y - z\|_\infty. \end{aligned} \tag{7}$$

Consequently,

$$\|Ty - Tz\|_\infty \le \Lambda \|y - z\|_\infty,$$

where

$$\Lambda = \sup_{\eta \in \Omega} \frac{L(\ln \eta)^{F_X^{(n)}(\eta)}}{\Gamma(F_X^{(n)}(\eta)+1)}.$$

If

$$\Lambda < 1,$$

then T is a contraction on $C(\Omega, \mathbb{R})$. Therefore, by the Banach fixed-point theorem, the initial value problem admits a unique solution.

To analyze the integral formulation of the variable-order problem, we establish the following lemma.

Lemma 5. *Assume that*

$$f \in C(\Omega \times \mathbb{R}, \mathbb{R}),$$

and that for some $r \in (0, 1)$, *the function*

$$(\ln \eta)^r f(\eta, y)$$

is continuous on $\Omega \times \mathbb{R}$.

Let $\eta \in \Omega_i = [\tau_{i-1}, \tau_i)$ *and denote*

$$\beta_i = F_X(\tau_i).$$

Then, a function

$$y \in C(\Omega, \mathbb{R})$$

is a solution of the variable-order Hadamard fractional initial value problem

$$\begin{cases} {}^{H}D_{1^+}^{\beta_i} y(\eta) = f(\eta, y(\eta)), & \eta \in \Omega_i, \\ y(1) = 0, \end{cases} \tag{8}$$

if and only if it satisfies the integral equation

$$y(\eta) = \frac{1}{\Gamma(\beta_i)} \int_1^{\eta} \left(\ln \frac{\eta}{\varrho} \right)^{\beta_i - 1} \frac{f(\varrho, y(\varrho))}{\varrho} \, d\varrho, \qquad \eta \in \Omega_i. \tag{9}$$

Proof. Assume first that

$$y \in C(\Omega, \mathbb{R})$$

is a solution of (8). Applying the Hadamard fractional integral of order β_i on both sides gives

$${}^{H}I_{1^+}^{\beta_i} \left({}^{H}D_{1^+}^{\beta_i} y \right)(\eta) = {}^{H}I_{1^+}^{\beta_i} f(\eta, y(\eta)).$$

Using Lemma 1, we obtain

$$y(\eta) = c(\ln \eta)^{\beta_i - 1} + \frac{1}{\Gamma(\beta_i)} \int_1^{\eta} \left(\ln \frac{\eta}{\varrho} \right)^{\beta_i - 1} \frac{f(\varrho, y(\varrho))}{\varrho} \, d\varrho.$$

Since the initial condition is

$$y(1) = 0,$$

and the admissible solution belongs to the weighted space associated with the Hadamard operator, the homogeneous term must vanish. Hence,

$$c = 0.$$

Therefore,

$$y(\eta) = \frac{1}{\Gamma(\beta_i)} \int_1^{\eta} \left(\ln \frac{\eta}{\varrho} \right)^{\beta_i - 1} \frac{f(\varrho, y(\varrho))}{\varrho} \, d\varrho,$$

which proves that every solution of the differential problem satisfies the integral equation.

Conversely, suppose that

$$y \in C(\Omega, \mathbb{R})$$

satisfies (9). Applying the Hadamard fractional derivative ${}^{H}D_{1^+}^{\beta_i}$ to both sides and using Lemma 2, we obtain

$${}^{H}D_{1^+}^{\beta_i} y(\eta) = f(\eta, y(\eta)),$$

for every

$$\eta \in \Omega_i.$$

Moreover, the integral representation directly gives

$$y(1) = 0.$$

Hence, y satisfies the initial value problem (8).

Therefore, the differential formulation and the integral formulation are equivalent. □

The following result is obtained by applying the Banach contraction mapping theorem.

Theorem 1. *Assume that the hypotheses of Lemma 5 hold. Moreover, suppose that there exists a constant $L > 0$ such that*

$$(\ln \eta)^r |f(\eta, y_1) - f(\eta, y_2)| \leq L|y_1 - y_2|,$$

for every

$$\eta \in \Omega, \qquad y_1, y_2 \in \mathbb{R}.$$

Let

$$\beta_i = F_X(\tau_i), \qquad \eta \in \Omega_i = [\tau_{i-1}, \tau_i).$$

Assume that

$$\frac{L(\ln \tau)^{\beta_i - r}}{\Gamma(\beta_i + 1)} < 1. \tag{10}$$

Then, the initial value problem (5) admits a unique solution in $C(\Omega, \mathbb{R})$.

Proof. We consider the operator

$$S : C(\Omega, \mathbb{R}) \longrightarrow C(\Omega, \mathbb{R})$$

defined by

$$(Sy)(\eta) = \frac{1}{\Gamma(\beta_i)} \int_1^\eta \left(\ln \frac{\eta}{\varrho}\right)^{\beta_i - 1} \frac{f(\varrho, y(\varrho))}{\varrho} \, d\varrho, \qquad \eta \in \Omega_i. \tag{11}$$

Since the kernel of the Hadamard fractional integral is integrable and $(\ln \eta)^r f$ is continuous, the operator S is well defined on $C(\Omega, \mathbb{R})$.

Let

$$u, v \in C(\Omega, \mathbb{R}).$$

Then, for $\eta \in \Omega_i$, we have

$$\begin{aligned} |Su(\eta) - Sv(\eta)| &\leq \frac{1}{\Gamma(\beta_i)} \int_1^\eta \left(\ln \frac{\eta}{\varrho}\right)^{\beta_i - 1} \frac{|f(\varrho, u(\varrho)) - f(\varrho, v(\varrho))|}{\varrho} \, d\varrho \\ &\leq \frac{L}{\Gamma(\beta_i)} \int_1^\eta \left(\ln \frac{\eta}{\varrho}\right)^{\beta_i - 1} (\ln \varrho)^{-r} |u(\varrho) - v(\varrho)| \frac{d\varrho}{\varrho}. \end{aligned}$$

Using

$$|u(\varrho) - v(\varrho)| \leq \|u - v\|_\infty,$$

we obtain

$$|Su(\eta) - Sv(\eta)| \leq \frac{L\|u - v\|_\infty}{\Gamma(\beta_i)} \int_1^\eta \left(\ln \frac{\eta}{\varrho}\right)^{\beta_i - 1} (\ln \varrho)^{-r} \frac{d\varrho}{\varrho}.$$

By the change of variables

$$s = \ln \varrho,$$

we get

$$\int_1^{\eta} \left(\ln \frac{\eta}{\varrho}\right)^{\beta_i - 1} (\ln \varrho)^{-r} \frac{d\varrho}{\varrho} = \frac{\Gamma(\beta_i)\Gamma(1-r)}{\Gamma(\beta_i + 1 - r)} (\ln \eta)^{\beta_i - r}.$$

Therefore,

$$|Su(\eta) - Sv(\eta)| \leq \frac{L(\ln \eta)^{\beta_i - r}}{\Gamma(\beta_i + 1 - r)} \|u - v\|_{\infty}.$$

Taking the supremum over $\eta \in \Omega$ gives

$$\|Su - Sv\|_{\infty} \leq \Lambda \|u - v\|_{\infty},$$

where

$$\Lambda = \sup_{\eta \in \Omega_i} \frac{L(\ln \eta)^{\beta_i - r}}{\Gamma(\beta_i + 1 - r)}.$$

Using assumption (10), we have

$$\Lambda < 1.$$

Hence, S is a strict contraction on $C(\Omega, \mathbb{R})$.

By the Banach fixed-point theorem, the operator S possesses a unique fixed point. This fixed point is the unique solution of the considered variable-order Hadamard fractional initial value problem. □

Next, we establish the existence and uniqueness of solutions for the initial value problem (1). We consider the following assumption.

H_2. *Assume that*

$$f \in C(\Omega \times \mathbb{R}, \mathbb{R}),$$

and that there exists $r \in (0, 1)$ *such that*

$$(\ln \eta)^r f(\eta, y) \in C(\Omega \times \mathbb{R}, \mathbb{R}).$$

Moreover, assume that there exists a constant $L > 0$ *satisfying*

$$(\ln \eta)^r |f(\eta, y_1) - f(\eta, y_2)| \leq L|y_1 - y_2|,$$

for every

$$\eta \in \Omega, \qquad y_1, y_2 \in \mathbb{R}.$$

Theorem 2. *Assume that* (H_1) *and* (H_2) *hold. If*

$$\Lambda = \sup_{\eta \in \Omega_i} \frac{L(\ln \eta)^{F_X(\tau_i) - r}}{\Gamma(F_X(\tau_i) + 1 - r)} < 1,$$

for every subinterval

$$\Omega_i = [\tau_{i-1}, \tau_i),$$

then the initial value problem (1) *admits a unique solution*

$$y \in C(\Omega, \mathbb{R}).$$

Proof. By assumption (H_1), the order function generated by the cumulative distribution function is approximated by the piecewise constant function

$$F_X^{(n)}(\eta) = F_X(\tau_i), \qquad \eta \in \Omega_i.$$

It is important to note that this approximation concerns only the value of the fractional order. The Hadamard operator keeps its complete memory over the interval $[1, \eta]$.

According to Lemma 5, the initial value problem is equivalent to the integral equation

$$y(\eta) = \frac{1}{\Gamma(F_X^{(n)}(\eta))} \int_1^\eta \left(\ln \frac{\eta}{\varrho}\right)^{F_X^{(n)}(\eta)-1} \frac{f(\varrho, y(\varrho))}{\varrho}\, d\varrho.$$

Define the operator

$$(Sy)(\eta) = \frac{1}{\Gamma(F_X^{(n)}(\eta))} \int_1^\eta \left(\ln \frac{\eta}{\varrho}\right)^{F_X^{(n)}(\eta)-1} \frac{f(\varrho, y(\varrho))}{\varrho}\, d\varrho.$$

From the continuity assumption in (H_2), the operator

$$S : C(\Omega, \mathbb{R}) \to C(\Omega, \mathbb{R})$$

is well defined.

Let

$$u, v \in C(\Omega, \mathbb{R}).$$

For $\eta \in \Omega_i$, we have

$$|Su(\eta) - Sv(\eta)| \le \frac{1}{\Gamma(F_X(\tau_i))} \int_1^\eta \left(\ln \frac{\eta}{\varrho}\right)^{F_X(\tau_i)-1} \times \frac{|f(\varrho, u(\varrho)) - f(\varrho, v(\varrho))|}{\varrho}\, d\varrho.$$

Using assumption (H_2), we obtain

$$|Su(\eta) - Sv(\eta)| \le \frac{L}{\Gamma(F_X(\tau_i))} \|u - v\|_\infty \times \int_1^\eta \left(\ln \frac{\eta}{\varrho}\right)^{F_X(\tau_i)-1} (\ln \varrho)^{-r} \frac{d\varrho}{\varrho}.$$

Applying the standard estimate for the Hadamard fractional integral yields

$$|Su(\eta) - Sv(\eta)| \le \frac{L(\ln \eta)^{F_X(\tau_i)-r}}{\Gamma(F_X(\tau_i) + 1 - r)} \|u - v\|_\infty.$$

Hence,

$$\|Su - Sv\|_\infty \le \Lambda \|u - v\|_\infty.$$

By the assumption

$$\Lambda < 1,$$

the operator S is a contraction on the Banach space $C(\Omega, \mathbb{R})$.

Therefore, the Banach fixed-point theorem guarantees the existence of a unique fixed point

$$y = Sy.$$

This fixed point satisfies the integral equation and, by Lemma 5, is a solution of the initial value problem (1).

Finally, if u and v are two solutions of (1), then both are fixed points of S. Hence,

$$\|u - v\|_\infty = \|Su - Sv\|_\infty \leq \Lambda \|u - v\|_\infty.$$

Since $\Lambda < 1$, we obtain

$$\|u - v\|_\infty = 0,$$

and consequently,

$$u(\eta) = v(\eta), \qquad \forall \eta \in \Omega.$$

Therefore, the solution is unique. □

4. Ulam–Hyers Stability

Definition 6 ([19]). *The initial value problem* (1) *is said to be Ulam–Hyers (UH) stable if there exists a constant* $c_f > 0$ *such that, for every* $\epsilon > 0$ *and every function* $z \in C(\Omega, \mathbb{R})$ *satisfying*

$$\left| {}^{H}D_{1^+}^{F_X^{(n)}(\eta)} z(\eta) - f(\eta, z(\eta)) \right| \leq \epsilon, \qquad \eta \in \Omega, \tag{12}$$

there exists an exact solution $y \in C(\Omega, \mathbb{R})$ *of* (1) *such that*

$$|z(\eta) - y(\eta)| \leq c_f \epsilon, \qquad \eta \in \Omega.$$

Theorem 3. *Assume that conditions* (H_1) *and* (H_2) *are satisfied and that*

$$\Lambda = \sup_{\eta \in \Omega_i} \frac{L(\ln \eta)^{F_X(\tau_i) - r}}{\Gamma(F_X(\tau_i) + 1 - r)} < 1.$$

Then the initial value problem (1) *is Ulam–Hyers stable.*

Proof. Let $\epsilon > 0$ and let

$$z \in C(\Omega, \mathbb{R})$$

be an approximate solution of the initial value problem (1) satisfying

$$\left| {}^{H}D_{1^+}^{F_X^{(n)}(\eta)} z(\eta) - f(\eta, z(\eta)) \right| \leq \epsilon, \qquad \eta \in \Omega. \tag{13}$$

Applying the variable-order Hadamard fractional integral ${}^{H}I_{1^+}^{F_X^{(n)}(\eta)}$ to both sides of (12) and using the properties of the Hadamard fractional operators, we obtain

$$\left| z(\eta) - \frac{1}{\Gamma(F_X^{(n)}(\eta))} \int_1^\eta \left(\ln \frac{\eta}{\varrho} \right)^{F_X^{(n)}(\eta)-1} \frac{f(\varrho, z(\varrho))}{\varrho} \, d\varrho \right|$$
$$\leq \frac{\epsilon}{\Gamma(F_X^{(n)}(\eta))} \int_1^\eta \left(\ln \frac{\eta}{\varrho} \right)^{F_X^{(n)}(\eta)-1} \frac{d\varrho}{\varrho}.$$

Using the change of variables

$$s = \ln \frac{\eta}{\varrho},$$

we obtain

$$\int_1^\eta \left(\ln \frac{\eta}{\varrho} \right)^{F_X^{(n)}(\eta)-1} \frac{d\varrho}{\varrho} = \frac{(\ln \eta)^{F_X^{(n)}(\eta)}}{F_X^{(n)}(\eta)}.$$

Hence,

$$|z(\eta) - (Sz)(\eta)| \leq \epsilon \frac{(\ln \eta)^{F_X^{(n)}(\eta)}}{\Gamma(F_X^{(n)}(\eta) + 1)}, \tag{14}$$

where the operator S is defined by

$$(Sy)(\eta) = \frac{1}{\Gamma(F_X^{(n)}(\eta))} \int_1^\eta \left(\ln \frac{\eta}{\varrho} \right)^{F_X^{(n)}(\eta)-1} \frac{f(\varrho, y(\varrho))}{\varrho} \, d\varrho.$$

Define

$$M = \sup_{\eta \in \Omega} \frac{(\ln \eta)^{F_X^{(n)}(\eta)}}{\Gamma(F_X^{(n)}(\eta) + 1)}.$$

Then, inequality (14) gives

$$\|z - Sz\|_\infty \leq M\epsilon. \tag{15}$$

By Theorem 2, the operator S is a contraction on $C(\Omega, \mathbb{R})$. Therefore, there exists a unique function

$$y \in C(\Omega, \mathbb{R})$$

such that

$$Sy = y.$$

This function is the unique exact solution of the initial value problem (1).
Now, using the fixed-point property of y, we have

$$|z(\eta) - y(\eta)| = |z(\eta) - Sy(\eta)|$$
$$\leq |z(\eta) - Sz(\eta)| + |Sz(\eta) - Sy(\eta)|.$$

Taking the supremum over $\eta \in \Omega$, we obtain

$$\|z - y\|_\infty \leq \|z - Sz\|_\infty + \|Sz - Sy\|_\infty.$$

From (15),

$$\|z - y\|_\infty \le M\epsilon + \|Sz - Sy\|_\infty.$$

Furthermore, for any $u, v \in C(\Omega, \mathbb{R})$ and $\eta \in \Omega_i$, we have

$$|Su(\eta) - Sv(\eta)| \le \frac{1}{\Gamma(F_X(\tau_i))} \int_1^\eta \left(\ln \frac{\eta}{\varrho}\right)^{F_X(\tau_i)-1} \times |f(\varrho, u(\varrho)) - f(\varrho, v(\varrho))| \frac{d\varrho}{\varrho}.$$

Using assumption (H_2), we obtain

$$|Su(\eta) - Sv(\eta)| \le \frac{L}{\Gamma(F_X(\tau_i))} \|u - v\|_\infty \times \int_1^\eta \left(\ln \frac{\eta}{\varrho}\right)^{F_X(\tau_i)-1} (\ln \varrho)^{-r} \frac{d\varrho}{\varrho}.$$

Applying the estimate used in Theorem 2, we obtain

$$|Su(\eta) - Sv(\eta)| \le \Lambda \|u - v\|_\infty,$$

where

$$\Lambda = \sup_{\eta \in \Omega_i} \frac{L(\ln \eta)^{F_X(\tau_i)-r}}{\Gamma(F_X(\tau_i) + 1 - r)}.$$

Since $\Lambda < 1$, we have

$$\|Sz - Sy\|_\infty \le \Lambda \|z - y\|_\infty.$$

Consequently,

$$\|z - y\|_\infty \le M\epsilon + \Lambda \|z - y\|_\infty.$$

Therefore,

$$(1 - \Lambda)\|z - y\|_\infty \le M\epsilon,$$

and hence

$$\|z - y\|_\infty \le \frac{M}{1 - \Lambda}\epsilon.$$

Finally, for every $\eta \in \Omega$,

$$|z(\eta) - y(\eta)| \le \|z - y\|_\infty \le \frac{M}{1 - \Lambda}\epsilon.$$

Setting

$$c_f = \frac{M}{1 - \Lambda},$$

we obtain

$$|z(\eta) - y(\eta)| \le c_f \epsilon, \qquad \eta \in \Omega.$$

Therefore, the initial value problem (1) is Ulam–Hyers stable. □

5. Numerical Results

In this section, we present a numerical example to illustrate the theoretical results obtained in the previous sections. Consider the following variable-order fractional initial value problem:

$$\begin{cases} D_{1^+}^{F_X(\eta)} y(\eta) = \dfrac{|2+y(\eta)|}{(\ln \eta)^{\frac{1}{9}}(1+e^{2\eta})(1+|y(\eta)|)}, & \eta \in \Omega = [1,3], \\ y(1) = 0. \end{cases} \tag{16}$$

The nonlinear function appearing in the problem is given by

$$f(\eta, y) = \frac{|2+y|}{(\ln \eta)^{\frac{1}{9}}(1+e^{2\eta})(1+|y|)}, \qquad (\eta, y) \in [1,3] \times \mathbb{R}.$$

The variable fractional order is generated from the probability density function

$$\rho(\eta) = \begin{cases} \dfrac{\eta+1}{6}, & \eta \in [1,3], \\ 0, & \eta \notin [1,3]. \end{cases}$$

Therefore, the associated cumulative distribution function is

$$F_X(\eta) = \int_1^{\eta} \rho(x)\, dx = \frac{1}{12}(\eta^2 + 2\eta - 3), \qquad \eta \in [1,3]. \tag{17}$$

It is clear that F_X satisfies the required properties of a cumulative distribution function. Indeed, the following statements hold true:

1. $$0 \le F_X(\eta) \le 1, \qquad \eta \in [1,3];$$
2. F_X is increasing since
$$F_X'(\eta) = \frac{\eta+1}{6} > 0;$$
3. $$\lim_{\eta \to -\infty} F_X(\eta) = 0, \qquad \lim_{\eta \to +\infty} F_X(\eta) = 1;$$
4. F_X is continuous and therefore right continuous.

We approximate the variable order $F_X(\eta)$ as a piecewise-constant function. Let

$$1 = \tau_0 < \tau_1 < \cdots < \tau_8 = 3$$

be the partition of the interval $[1,3]$, where

$$\Omega_i = [\tau_{i-1}, \tau_i], \qquad i = 1, \ldots, 8.$$

The corresponding constant values of the fractional order are

i	Ω_i	$F_X(\tau_i)$
1	$[1, \frac{5}{4}]$	$\frac{17}{192} = 0.0885$
2	$[\frac{5}{4}, \frac{3}{2}]$	$\frac{19}{192} = 0.0989$
3	$[\frac{3}{2}, \frac{7}{4}]$	$\frac{21}{192} = 0.1093$
4	$[\frac{7}{4}, 2]$	$\frac{23}{192} = 0.1197$

i	Ω_i	$F_X(\tau_i)$
5	$[2, \frac{9}{4}]$	$\frac{25}{192} = 0.1302$
6	$[\frac{9}{4}, \frac{5}{2}]$	$\frac{27}{192} = 0.1406$
7	$[\frac{5}{2}, \frac{11}{4}]$	$\frac{29}{192} = 0.1510$
8	$[\frac{11}{4}, 3]$	$\frac{31}{192} = 0.1614$

We first verify assumption (H_2). For arbitrary $y, z \in \mathbb{R}$, we have

$$\begin{aligned}
&(\ln \eta)^{\frac{1}{9}} |f(\eta, y) - f(\eta, z)| \\
&= \frac{1}{1 + e^{2\eta}} \left| \frac{|2 + y|}{1 + |y|} - \frac{|2 + z|}{1 + |z|} \right| \\
&\leq \frac{1}{1 + e^{2\eta}} ||2 + y| - |2 + z|| \\
&\leq \frac{1}{2} |y - z|.
\end{aligned}$$

Hence, assumption (H_2) is satisfied with

$$r = \frac{1}{9}, \qquad L = \frac{1}{2}.$$

Now we verify the contraction condition given by Equation (10). For each subinterval Ω_i, define

$$\Lambda_i = \frac{L \left(\ln \frac{\tau_i}{\tau_{i-1}} \right)^{F_X(\tau_i) - 1} \left((\ln \tau_i)^{1-r} - (\ln \tau_{i-1})^{1-r} \right)}{(1 - r)\Gamma(F_X(\tau_i))}.$$

Using $r = \frac{1}{9}$ and $L = \frac{1}{2}$, we obtain

i	1	2	3	4	5	6	7	8
Λ_i	0.05491	0.05056	0.05110	0.05216	0.00592	0.05910	0.05514	0.05583

Consequently,

$$\max_{1 \leq i \leq 8} \Lambda_i < 1.$$

Therefore, all assumptions of Theorem 1 are fulfilled, and the existence and uniqueness of the solution on every subinterval Ω_i are guaranteed.

Numerical Approximation

The numerical solution is obtained from the equivalent integral formulation. The interval $[1, 3]$ is discretized into a uniform grid

$$1 = \eta_0 < \eta_1 < \cdots < \eta_N = 3.$$

The fractional integral term is approximated by the trapezoidal quadrature formula, leading to the discrete approximation

$$y(\eta_j) = \frac{1}{\Gamma(F_X(\eta_j))} \sum_{k=0}^{j} \omega_k \left(\ln \frac{\eta_j}{\eta_k} \right)^{F_X(\eta_j) - 1} \frac{f(\eta_k, y(\eta_k))}{\eta_k},$$

where ω_k denotes the corresponding trapezoidal weights.

The resulting nonlinear algebraic system is solved iteratively to obtain the approximate values of the solution.

Figure 1 shows the numerical solution associated with the continuous fractional order

$$F_X(\eta) = \frac{1}{12}(\eta^2 + 2\eta - 3).$$

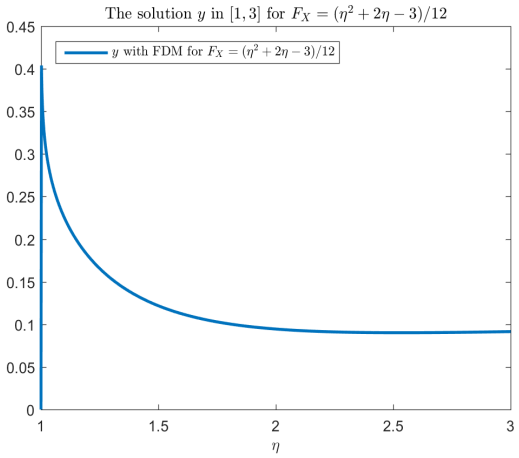


Figure 1. Numerical solution $y(\eta)$ corresponding to $F_X(\eta) = \frac{1}{12}(\eta^2 + 2\eta - 3)$.

To investigate the influence of the fractional order approximation, we compare the numerical solution $y(\eta)$ with the solutions $y_i(\eta)$ obtained by fixing the fractional order on each subinterval,

$$F_X(\eta) = F_X(\tau_i), \qquad \eta \in \Omega_i.$$

The obtained comparison is displayed in Figure 2.

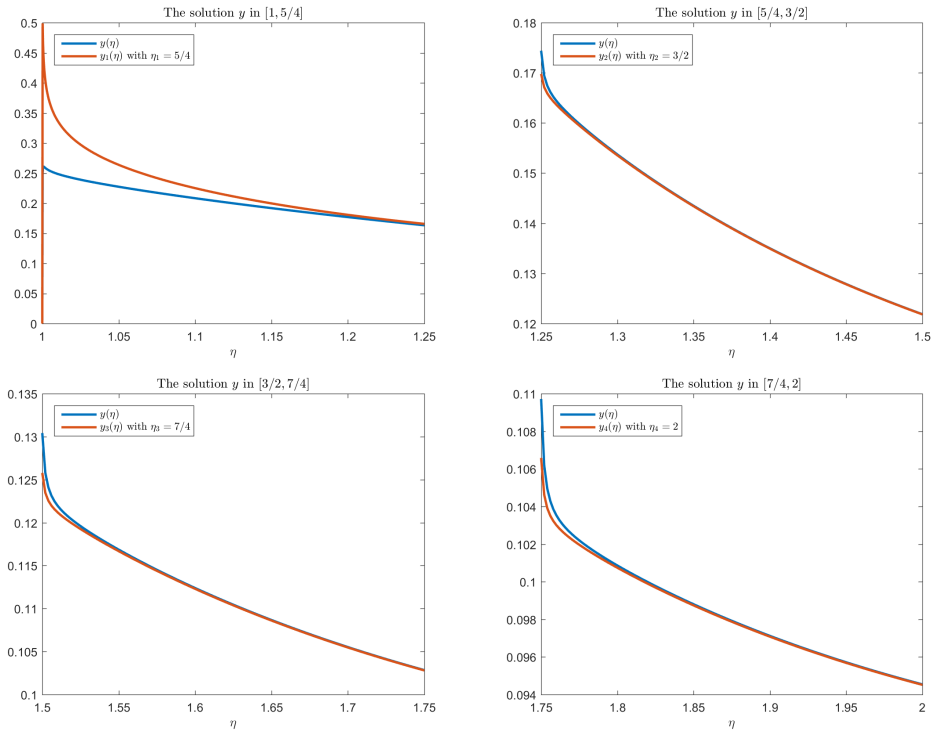


Figure 2. *Cont.*

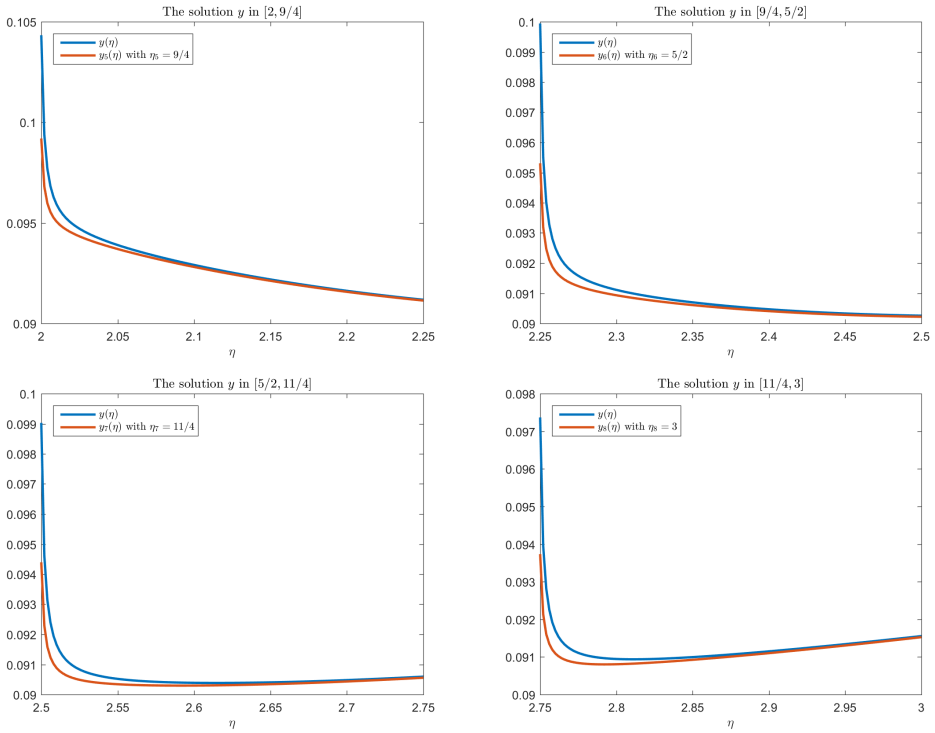


Figure 2. Comparison between the variable-order numerical solution $y(\eta)$ and the piecewise-constant order solutions $y_i(\eta)$.

The approximation error is measured by

$$Norm_i = \max_{\eta \in [\tau_{i-1}, \tau_i]} |y(\eta) - y_i(\eta)|.$$

The numerical results are summarized in Table 1.

Table 1. Errors between the variable-order solution and the piecewise-constant approximations.

τ_i	$F_X(\tau_i)$	$\lvert y_i(\tau_i) - y(\tau_i)\rvert$	$Norm_i$
$\frac{5}{4}$	0.0885	2.364×10^{-3}	0.2490
$\frac{3}{2}$	0.0989	3.678×10^{-5}	4.634×10^{-2}
$\frac{7}{4}$	0.1093	3.678×10^{-5}	4.634×10^{-3}
2	0.1197	2.487×10^{-5}	3.134×10^{-3}
$\frac{9}{4}$	0.1302	4.074×10^{-5}	5.134×10^{-3}
$\frac{5}{2}$	0.1406	3.678×10^{-5}	4.634×10^{-3}
$\frac{11}{4}$	0.1510	3.678×10^{-5}	4.634×10^{-3}
3	0.1614	2.884×10^{-5}	3.634×10^{-3}

The obtained numerical results show that the piecewise constant approximation of the variable fractional order provides a reliable approximation of the original model. Moreover, the numerical experiments confirm the theoretical results concerning existence, uniqueness, and Ulam–Hyers stability.

6. Discussion

The numerical example presented in the previous section serves to demonstrate the applicability of the theoretical framework developed in this work. In particular, the considered problem satisfies the assumptions required for the existence, uniqueness, and Ulam–Hyers

stability results established in Theorems 1 and 3. The example provides a practical illustration of the proposed approach, where the variable fractional order is generated through a cumulative distribution function and approximated by a piecewise-constant representation.

The numerical results show that the proposed approximation procedure is able to capture the behavior of the considered Hadamard variable-order fractional system. The comparison between the reference solution associated with the continuous order function

$$F_X(\eta) = \frac{1}{12}(\eta^2 + 2\eta - 3)$$

and the solutions obtained from the piecewise constant approximations $F_X(\eta) = F_X(\tau_i)$ demonstrates that the approximate solutions remain close to the reference solution on the corresponding subintervals. Moreover, the computed values of the error indicator

$$\text{Norm}_i = \max_{\eta \in [\tau_{i-1}, \tau_i]} |y(\eta) - y_i(\eta)|$$

show a decreasing tendency for most subintervals, indicating that the piecewise-constant representation provides a reasonable approximation of the continuous variable-order model in the considered example.

The obtained behavior is consistent with the theoretical analysis. Indeed, the piecewise-constant approximation transforms the original variable-order problem into a sequence of constant-order Hadamard fractional problems, for which the existence and uniqueness are ensured by the Banach fixed-point argument. The numerical results confirm that the influence of the cumulative distribution function on the memory parameter is reflected in the evolution of the solutions. Different values of $F_X(\tau_i)$ lead to slightly different fractional dynamics, which highlights the role of the distribution function in controlling the memory behavior of the system.

It should be emphasized that the numerical experiment presented here is not intended to provide a complete convergence analysis of the proposed numerical method. Rather, it is included to verify the applicability of the theoretical results and to illustrate the effect of the distribution-generated variable order on the solution behavior. A rigorous numerical analysis, including global convergence rates, stability estimates, and error bounds for general Hadamard variable-order fractional equations, remains an important topic for future research.

A main contribution of this work is the introduction of a probabilistic mechanism for constructing variable fractional orders through cumulative distribution functions. In contrast to classical variable-order fractional models, where the order is usually prescribed by a deterministic function, the present framework allows the fractional order to incorporate distributional information. This provides a flexible modeling perspective for systems in which memory effects may depend on uncertain or statistical characteristics.

The piecewise-constant representation adopted in this study plays a fundamental role in the mathematical analysis. It allows the variable-order operator to be decomposed into a finite collection of constant-order Hadamard fractional operators, making it possible to apply standard analytical tools. Although this representation provides an effective framework for the current investigation, its extension to more general cumulative distribution functions remains a challenging problem.

Several directions can be considered for future developments:

- Continuous distribution-generated orders. An important extension will be the investigation of continuously varying cumulative distribution functions without using piecewise approximations. This will require the development of new analytical techniques adapted to genuinely variable fractional orders.

- Influence of different probability distributions. Different choices of probability distributions may generate different memory structures. Studying the qualitative and quantitative influence of these choices on the solution behavior represents an interesting research direction.
- Numerical analysis. Further studies are needed to establish rigorous convergence results, error estimates, and stability properties of numerical schemes designed for Hadamard variable-order fractional differential equations.
- Generalizations to other fractional models. The proposed approach can potentially be extended to boundary value problems; impulsive fractional systems; coupled systems, fractional integral equations; and models involving other fractional operators, including Caputo–Hadamard, Hilfer, and Katugampola derivatives.

The results obtained in this paper indicate that cumulative-distribution-based variable fractional orders provide a promising framework for describing distribution-dependent memory phenomena. The combination of fractional calculus and probability theory may offer new mathematical tools for modeling complex systems where memory effects are influenced by uncertainty or statistical information.

7. Conclusions

In this paper, we studied a class of Hadamard variable-order fractional differential equations in which the fractional order is generated by the cumulative distribution function of a continuous random variable. This approach establishes a connection between probability theory and variable-order fractional calculus by allowing the memory parameter of the fractional operator to be determined through distributional information.

To overcome the analytical difficulties caused by the variable fractional order, the cumulative distribution function was represented by a piecewise-constant approximation over a finite partition of the considered interval. This formulation enabled the original problem to be decomposed into a family of constant-order Hadamard fractional differential equations. Using this representation, the Banach contraction principle was applied to establish sufficient conditions guaranteeing the existence and uniqueness of solutions. Furthermore, the Ulam–Hyers stability property was obtained, showing the robustness of the solutions with respect to small perturbations of the fractional differential equation.

A numerical example was provided to illustrate the theoretical results. The example verified the assumptions of the proposed theorems and showed how the choice of the cumulative distribution function affects the corresponding solutions. The comparison between the continuous variable-order model and its piecewise-constant approximations demonstrated the effectiveness of the adopted representation. However, the numerical results should be regarded as an illustration of the theoretical framework rather than a complete convergence study of the numerical method.

The main novelty of the proposed approach lies in introducing a probabilistic interpretation of the fractional order through cumulative distribution functions. Unlike many existing variable-order fractional models based on deterministic order functions, the present framework allows the memory index to be influenced by distributional information. This provides additional flexibility for modeling systems with uncertain or heterogeneous memory effects.

Several problems remain open for future investigation. These include the analysis of general continuous cumulative-distribution-function orders, the development of rigorous numerical convergence theories, and the extension of the results to more general fractional operators and mathematical models. Possible directions include boundary value problems; impulsive and integro-differential systems; coupled fractional equations; and models based on Caputo, Caputo–Hadamard, Hilfer, and Katugampola fractional derivatives.

The results presented in this work provide a first step toward the development of distribution-dependent variable-order fractional models and highlight the potential interaction between probability theory and fractional calculus in the description of systems with complex memory effects.

Author Contributions: Conceptualization, M.S.S. and K.S.; methodology, Z.B.; software, S.S.; validation, M.S.S., K.S. and Z.B.; formal analysis, S.S.; investigation, Z.B.; resources, K.S.; data curation, S.S.; writing—original draft preparation, M.S.S.; writing—review and editing, M.S.S.; visualization, K.S.; supervision, M.S.S.; project administration, M.S.S.; funding acquisition, K.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Science,Research and Innovation Fund (NSRF), King Mongkut's University of Technology North Bangkok, under Contract no. KMUTNB-FF-69-B-10.

Informed Consent Statement: Not applicable.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: The authors are grateful to the reviewers for useful suggestions that improved the content of this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

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