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# Multifractal Analysis of Refined Sets in Branching Random Walks on Galton–Watson Trees

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**Abstract**

In this paper, we investigate refined level sets associated with branching random walks on the boundary of a supercritical Galton–Watson tree. More precisely, for a prescribed deterministic sequence $\mathbf{s} := (r_n, \tilde{r}_n)$, we introduce refined deviation sets, denoted by $E_{\mathbf{s}}(\alpha, \beta)$, consisting of boundary points for which the deviations $S_n\mathsf{X}(t) - \alpha S_n\widetilde{\mathsf{X}}(t)$ and $S_n\mathsf{Y}(t) - \beta S_n\widetilde{\mathsf{Y}}(t)$ are asymptotically equivalent to $r_n$ and $\tilde{r}_n$, respectively, as $n \to \infty$. This setting extends the classical law-of-large-numbers level sets by allowing for controlled deviations around the typical linear behavior of the branching random walks. By constructing suitable inhomogeneous Mandelbrot measures, we establish sufficient conditions under which the sets $E_{\mathbf{s}}(\alpha, \beta)$ have maximal Hausdorff and packing dimensions.

**Keywords:** branching random walk; Hausdorff and packing dimensions; Galton–Watson tree; multifractal analysis

## 1. Introduction

Multifractal analysis provides a unified framework for the investigation of mathematical objects displaying pronounced variability across scales. Since the pioneering works of B. Mandelbrot in the early 1980s [1,2], this theory has evolved into a central tool in modern analysis, with significant impact in both theoretical and applied settings. In its classical formulation, multifractal analysis aims to characterize the fine local behavior of a function or signal through pointwise scaling exponents, which capture its local regularity.

Let $d \geq 1$ and endow $\mathbb{R}^d$ with its canonical inner product $\langle \cdot, \cdot \rangle$ and the associated Euclidean norm $\| \cdot \|$. Consider a measurable function $X : \mathbb{R}^d \to \mathbb{R}$. For each point $x \in \mathbb{R}^d$, the local regularity of $X$ is described by a pointwise exponent $\alpha_X(x)$. This naturally leads to the study of the level sets

$$E_\alpha = \{x \in \mathbb{R}^d : \alpha_X(x) = \alpha\},$$

whose geometric complexity is commonly quantified using Hausdorff and packing dimensions. The collection of these dimensions as $\alpha$ varies defines the multifractal spectrum of $X$ [3,4]. This formalism is particularly well adapted to the analysis of random signals and time series, which often exhibit long-range correlations and nontrivial power-law scaling behavior.

Let $\partial\mathsf{T}$ denote the boundary of a Galton–Watson tree $\mathsf{T}$ with offspring distribution $N$. Such a tree constitutes a natural probabilistic model for the evolution of a branching

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population, where individuals reproduce independently according to the same law. To each infinite ray $t = (t_1, t_2, \dots) \in \partial\mathsf{T}$, we associate branching random walks defined by

$$S_n Z(t) = \sum_{k=1}^{n} \mathsf{Z}_{t_1 \cdots t_k},$$

where $Z$ ranges over the family $\mathcal{R} = \{\mathsf{X}, \mathsf{Y}, \widetilde{\mathsf{X}}, \widetilde{\mathsf{Y}}\}$ and the random variables $\{Z_u\}_{u \in \mathsf{T}}$ are indexed by the vertices of the tree (see Section 2 for precise definitions). We define the associated branching random walks along each infinite ray $t = (t_1, t_2, \dots) \in \partial\mathsf{T}$ by

$$S_n Z(t) = \sum_{k=1}^{n} \mathsf{Z}_{t_1 \cdots t_k}, \qquad Z \in \mathcal{R}.$$

The main object of interest in this work is a refined description of the asymptotic behavior of the sequences of ratios

$$\left\{ \frac{S_n \mathsf{X}(t)}{S_n \widetilde{\mathsf{X}}(t)} \right\}_{n \geq 1} \quad \text{and} \quad \left\{ \frac{S_n \mathsf{Y}(t)}{S_n \widetilde{\mathsf{Y}}(t)} \right\}_{n \geq 1},$$

The motivation to introduce and study $\left\{ \frac{S_n \mathsf{X}(t)}{S_n \widetilde{\mathsf{X}}(t)} \right\}_{n \geq 1}$ comes from the idea of studying the dimension of level sets under the distance of the random metric $d_{\widetilde{\mathsf{X}}}$ defined as

$$d_{\widetilde{\mathsf{X}}} : (s, t) \mapsto \exp(-S_{|s \wedge t|} \widetilde{\mathsf{X}}(s \wedge t))$$

for all $s, t \in \partial\mathsf{T}$, where $s \wedge t$ stands for the longest common prefix of $s$ and $t$, and with the convention that $\exp(-\infty) = 0$. Indeed, $\lim_n S_n \mathsf{X}(t) / S_n \widetilde{\mathsf{X}}(t) = \alpha$ is equivalent to $\lim_n S_n(\mathsf{X}(t) - \alpha \widetilde{\mathsf{X}}(t))/n = 0$ under (4).

To introduce our work, we consider first the level sets

$$E(\alpha, \beta) = \left\{ t \in \partial\mathsf{T} : \lim_{n \to \infty} \frac{S_n \mathsf{X}(t)}{S_n \widetilde{\mathsf{X}}(t)} = \alpha, \quad \lim_{n \to \infty} \frac{S_n \mathsf{Y}(t)}{S_n \widetilde{\mathsf{Y}}(t)} = \beta \right\}, \qquad \alpha, \beta \in \mathbb{R}. \tag{1}$$

The multifractal properties of these sets, including the determination of their Hausdorff and packing dimensions, were first investigated in [5] (see also [3,6]). Level sets and branching measures on Galton–Watson trees have also been studied through local dimensions and multifractal spectra; see, for instance, [7–9]. Level sets of this type have since been widely studied in a variety of frameworks [10–15], often via the multifractal formalism, which connects their dimension to the Legendre transform of an appropriate pressure function. Related fine properties of branching measures, including thin and thick points and uniform limit laws, were considered in [16,17]. A standard approach consists of constructing a suitable Mandelbrot-type random measure supported on the sets under consideration [3,18]. Related questions concerning branching measures, harmonic measures and geometric properties of the boundary of Galton–Watson trees were considered in [19–21]. For further developments on the construction of auxiliary measures in related contexts, we refer the reader to [22–26].

We introduce, for $q, (\alpha, \beta) \in \mathbb{R}^2$, the function $\widetilde{\mathcal{P}}_{\alpha,\beta}(q)$ by

$$\widetilde{\mathcal{P}}_{\alpha,\beta}(q) = \log \mathbb{E}\Big( \sum_{i=1}^{N} \exp\left( \langle q | (\mathsf{X}_i - \alpha \widetilde{\mathsf{X}}_i,\ \mathsf{Y}_i - \beta \widetilde{\mathsf{Y}}_i) \rangle \right) \Big) := \log \mathbb{E}\Big( S_{\alpha,\beta}(q) \Big). \tag{2}$$

We assume that

$$\forall\, q, (\alpha, \beta) \in \mathbb{R}^2, \text{such that } \widetilde{\mathcal{P}}^*_{\alpha,\beta}(\nabla \widetilde{\mathcal{P}}_{\alpha}(q)) > 0, \quad \exists\, \gamma > 1,\ \mathbb{E}(S_{\alpha,\beta}(q)^{\gamma}) < \infty \tag{3}$$

and

$$\forall \gamma \in \mathbb{R}, \quad \mathbb{E}\Big(\sum_{i=1}^{N} e^{-\gamma Z}\Big) < \infty, \tag{4}$$

for all $Z \in \mathcal{R}$. The Legendre transform of a mapping $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is defined as the concave and upper semi-continuous function:

$$f^*(\alpha) := \inf_{q \in \mathbb{R}^d} \Big( f(q) - \langle q, \alpha \rangle \Big).$$

Now, we define the set of our study; that is, for which the set $E(\alpha, \beta) \neq \emptyset$,

$$\mathcal{K} = \Big\{ (\alpha, \beta) \in \mathbb{R}^2 : \ \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0) > 0 \Big\}$$

and we exclude degenerate situations by assuming that, for every $(\alpha, \beta) \in \mathcal{K}$, the random vectors

$$\Gamma(i, \alpha, \beta) = (\mathsf{X}_i - \alpha \widetilde{\mathsf{X}}_i, \ \mathsf{Y}_i - \beta \widetilde{\mathsf{Y}}_i), \qquad 1 \leq i \leq N, \tag{5}$$

are not almost surely (a.s.) constant.

In Section 3, we show that, a.s., for every $(\alpha, \beta) \in \mathcal{K}$,

$$\dim E(\alpha, \beta) = Dim E(\alpha, \beta) = \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0),$$

where dim and *Dim* denote the Hausdorff and packing dimensions, respectively (see Theorem 2). This result can be obtained by suitably adapting the arguments developed in [6] [Theorem 1.1] (see also [3,5]). Nevertheless, for the sake of completeness and to facilitate comparison with the results established in Section 4, we provide a full proof. This approach also allows for a clearer parallel between the applications to homogeneous and inhomogeneous Mandelbrot measures presented in Sections 3 and 4, respectively.

The convergence of the ratios of random walks does not capture their full asymptotic behavior; a finer description requires precise control of the associated rate of convergence. This point of view is also connected with previous works on thin and thick points and fine geometric properties of branching measures on Galton–Watson trees; see, for example, [16,17,27,28]. More precisely, let $(r_n)_{n \geq 1}$ be a sequence of positive real numbers such that $r_n = o(n)$. In the dyadic setting $\partial \mathsf{T} = \{0, 1\}^{\mathbb{N}}$, Kahane and Fan [29] determined, for every $\alpha$, a.s. the Hausdorff dimension of the refined level sets $\widetilde{E}_{\alpha,r}$ defined by

$$\widetilde{E}_{\alpha,r} = \Big\{ t \in \partial \mathsf{T} : \ S_n \mathsf{X}(t) - n\alpha \sim r_n \text{ as } n \to \infty \Big\},$$

where the asymptotic equivalence means that $S_n \mathsf{X}(t) - n\alpha - r_n = o(1)$. Their results hold under the assumptions

$$\sqrt{n \log \log n} = o(r_n) \qquad \text{and} \qquad r_n - r_{n-1} = o(1).$$

These conditions are satisfied, for instance, when $r_n = n^\beta$ with $\beta \in (1/2, 1)$. Subsequently, in [30], these results were extended in several directions, either by computing a.s. the Hausdorff dimension of $\widetilde{E}_{\alpha,r}$ for all $\alpha$ in a given set, or by considering more general refinements of the level sets $\widetilde{E}_{\alpha,r}$. In the present work, we investigate a more general framework by introducing a refined analysis of the two-dimensional level sets $E(\alpha, \beta)$. To this end, let

$$\boldsymbol{s} = (\boldsymbol{s}_n)_{n \geq 0}, \qquad \boldsymbol{s}_n = (r_n, \widetilde{r}_n) \in \mathbb{R}^2,$$

be a deterministic vector-valued sequence satisfying the asymptotic conditions

$$\boldsymbol{s}_n = o(n) \quad \text{and} \quad \boldsymbol{\eta}_n := \boldsymbol{s}_n - \boldsymbol{s}_{n-1} = o(1), \tag{6}$$

where, for a sequence $(\boldsymbol{v}_n) \subset \mathbb{R}^2$, the notation $\boldsymbol{v}_n = o(n)$ means $\lim_{n\to\infty} \frac{\|\boldsymbol{v}_n\|}{n} = 0$. In Section 4, we provide sufficient conditions on the sequence $(\boldsymbol{s}_n)$ ensuring that the refined level set

$$E_{\boldsymbol{s}}(\alpha,\beta) = \left\{ t \in \partial \mathsf{T} : \; S_n \mathsf{X}(t) - \alpha S_n \widetilde{\mathsf{X}}(t) \sim r_n, \; S_n \mathsf{Y}(t) - \beta S_n \widetilde{\mathsf{Y}}(t) \sim \widetilde{r}_n \text{ as } n \to \infty \right\}$$

has maximal Hausdorff and packing dimensions. While the estimation of the dimension of the set $E(\alpha,\beta)$ relies on the construction of a homogeneous Mandelbrot measure simultaneously adapted to $\alpha$ and $\beta$, this approach is no longer applicable in the refined setting of Section 4. Instead, we construct an inhomogeneous Mandelbrot measure supported on $E_{\boldsymbol{s}}(\alpha,\beta)$, which allows us to obtain a sharp lower bound for its Hausdorff dimension.

The classical multifractal analysis developed in [6,31] mainly concerns standard level sets and homogeneous Mandelbrot measures, while the refined one-dimensional framework investigated in [26,27] deals with scalar deviation sequences. In contrast, the present paper studies refined vector-valued level sets associated with ratios of branching random walks on Galton–Watson trees together with prescribed convergence speeds. More precisely, instead of considering classical level sets defined by the convergence of a single normalized branching random walk, or one-dimensional refined deviation sets of the form $\{S_n X(t) - n\alpha \sim r_n\}$, the paper investigates multidimensional refined sets of the form $E_s(\alpha,\beta)$, where the deviation sequence

$$s_n = (r_n, \widetilde{r}_n) \in \mathbb{R}^2$$

is vector-valued. This framework requires the simultaneous control of several correlated branching random walks together with nontrivial asymptotic deviation constraints. In particular, the classical homogeneous Mandelbrot measure techniques are no longer sufficient to capture the geometry of these refined multidimensional sets. To overcome this difficulty, the paper introduces suitable inhomogeneous Mandelbrot measures depending on the prescribed vector deviation sequence, allowing for the derivation of sharp Hausdorff and packing dimension formulas for the sets $E_{\mathsf{s}}(\alpha,\beta)$.

This article is organized as follows. In the next section, we recall the definitions and notations used throughout the paper and present the necessary preliminary results. Section 3 is devoted to the statement and proof of our main theorem concerning the Hausdorff and packing dimensions of the set $E(\alpha,\beta)$. Finally, in Section 4, we establish and prove results related to the dimensional properties of the refined sets $E_{\boldsymbol{s}}(\alpha,\beta)$.

## 2. Notation and Preliminaries Results

### *2.1. Hausdorff and Packing Dimensions*

Let $K \subseteq \mathbb{N}_+^{\mathbb{N}_+}$ be a $\sigma$–compact metric space endowed with a metric $d$. For any $x \in K$ and $r > 0$, we denote by $B(x,r)$ the closed ball centered at $x$ with radius $r$. We briefly recall the definitions of Hausdorff and packing measures. Let $s > 0$ and $E \subseteq K$. The $s$-dimensional Hausdorff measure of $E$ is defined by

$$\mathcal{H}^s(E) = \lim_{\delta\to 0^+} \inf \left\{ \sum_{i\in\mathbb{N}} \operatorname{diam}(\mathsf{U}_i)^s : E \subseteq \bigcup_i \mathsf{U}_i, \; \operatorname{diam}(\mathsf{U}_i) \le \delta \right\},$$

where the infimum is taken over all countable coverings $\{U_i\}_{i\in\mathbb{N}}$ of $E$. Similarly, the $s$-dimensional packing pre-measure of $E$ is given by

$$\overline{P}^s(E) = \lim_{\delta\to 0^+} \sup\left\{ \sum_{i\in\mathbb{N}} \operatorname{diam}(B_i)^s : \{B_i = B(x_i, r_i)\}_{i\in\mathbb{N}} \text{ is a packing of } E,\ \operatorname{diam}(B_i) \le \delta \right\},$$

where the supremum runs over all countable families of disjoint balls centered in $E$. The packing measure $P^s$ is then defined by

$$P^s(E) = \inf\left\{ \sum_{i\in\mathbb{N}} \overline{P}^s(E_i) : E \subseteq \bigcup_i E_i \right\}.$$

The Hausdorff and packing dimensions of $E$ are respectively defined by

$$\dim E = \inf\{s > 0 : \mathcal{H}^s(E) = 0\}, \qquad DimE = \inf\{s > 0 : P^s(E) = 0\},$$

with the convention that $\inf \varnothing = \infty$ (see [32,33]).

Let $\mu$ be a finite positive Borel measure supported on $K$. The lower and upper Hausdorff dimensions of $\mu$ are defined by

$$\underline{\dim}(\mu) = \operatorname*{ess\,inf}_{\mu} \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}, \qquad \overline{\dim}(\mu) = \operatorname*{ess\,sup}_{\mu} \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}.$$

Similarly, the lower and upper packing dimensions of $\mu$ are given by

$$\underline{Dim}(\mu) = \operatorname*{ess\,inf}_{\mu} \limsup_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}, \qquad \overline{Dim}(\mu) = \operatorname*{ess\,sup}_{\mu} \limsup_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}.$$

If $\underline{\dim}(\mu) = \overline{\dim}(\mu)$ (resp. $\underline{Dim}(\mu) = \overline{Dim}(\mu)$), the common value is denoted by $\dim \mu$ (resp. $Dim\mu$). In this case, the measure $\mu$ is said to be exact dimensional [31].

### 2.2. Galton–Watson Tree

Let

$$\mathcal{S} := \left(N,\ (X_1, Y_1), (\widetilde{X}_1, \widetilde{Y}_1), (X_2, Y_2), (\widetilde{X}_2, \widetilde{Y}_2), \dots\right)$$

be a random vector taking values in $\mathbb{N}_+ \times (\mathbb{R}^2 \times \mathbb{R}_+^2)^{\mathbb{N}_+}$. We consider a family

$$\left\{ \left(N_u, (X_{u1}, Y_{u1}), (\widetilde{X}_{u1}, \widetilde{Y}_{u1}), \dots\right) \right\}_{u\in\bigcup_{n\ge 0}\mathbb{N}_+^n}$$

of independent copies of $\mathcal{S}$, indexed by the set of all finite words over the alphabet $\mathbb{N}_+$, where $\mathbb{N}_+^0 = \{\varnothing\}$ denotes the empty word. The associated Galton–Watson tree $\mathsf{T}$ is constructed from the offspring variables $\{N_u\}$ as follows: the root $\varnothing$ belongs to $\mathsf{T}$, and if $u \in \mathsf{T}$ and $i \in \mathbb{N}_+$, then the concatenation $ui$ is a vertex of $\mathsf{T}$ if and only if $1 \le i \le N_u$. For any finite word $u$, we denote by $\mathsf{T}(u)$ the Galton–Watson subtree rooted at $u$, generated by the offspring variables $\{N_{uv}\}_v$. For $n \ge 1$, the set of vertices of generation $n$ in $\mathsf{T}(u)$ is denoted by

$$\mathsf{T}_n(u) = \mathsf{T}(u) \cap \mathbb{N}_+^n.$$

Throughout this work, we assume that $\mathbb{E}(N) > 1$ so that the Galton–Watson tree is supercritical. Moreover, without loss of generality, we suppose that extinction does not occur, that is, $\mathbb{P}(N \ge 1) = 1$. For an infinite word $t = t_1 t_2 \cdots \in \mathbb{N}_+^{\mathbb{N}_+}$ and an integer $n \ge 0$, we

set $t_{|n} = t_1 \cdots t_n \in \mathbb{N}_+^n$. If $u \in \mathbb{N}_+^n$, its length is denoted by $|u| = n$, with the convention $t_{|0} = \emptyset$. The cylinder associated with $u$ is defined by

$$[u] = \{t \in \mathbb{N}_+^{\mathbb{N}_+} : t_{||u|} = u\}.$$

The space $\mathbb{N}_+^{\mathbb{N}_+}$ is endowed with the standard ultrametric distance

$$d_1(u,v) = \exp\Big(-\sup\{|w| : u \in [w],\ v \in [w]\}\Big), \tag{7}$$

with the convention $\exp(-\infty) = 0$. The boundary of the Galton–Watson tree $\mathsf{T}$ is then defined as the compact set

$$\partial\mathsf{T} = \bigcap_{n\geq 1} \bigcup_{u\in\mathsf{T}_n} [u],$$

which consists of all infinite words $t = t_1 t_2 \cdots$ over $\mathbb{N}_+$ such that $t_{|n} \in \mathsf{T}$ for every $n \geq 0$.

### *2.3. Mandelbrot Measures*

Let $\big(N, (U_1, \Lambda_1), (U_2, \Lambda_2), \ldots\big)$ be a random vector taking values in $\mathbb{N}_+ \times (\mathbb{R}^d \times \mathbb{R})^{\mathbb{N}_+}$. Assume that the offspring variable $N$ satisfies the standing assumptions introduced earlier and that the following conditions hold:

$$\begin{cases} \mathbb{E}\Big(\displaystyle\sum_{i=1}^{N} \mathrm{e}^{\Lambda_i}\Big) = 1, \\ \mathbb{E}\Big(\displaystyle\sum_{i=1}^{N} \Lambda_i\, \mathrm{e}^{\Lambda_i}\Big) < 0, \\ \mathbb{E}\Big[\Big(\displaystyle\sum_{i=1}^{N} \mathrm{e}^{\Lambda_i}\Big) \log^+\Big(\displaystyle\sum_{i=1}^{N} \mathrm{e}^{\Lambda_i}\Big)\Big] < \infty. \end{cases} \tag{8}$$

Moreover, we assume the integrability condition

$$\mathbb{E}\Big(\sum_{i=1}^{N} \|U_i\|\, \mathrm{e}^{\Lambda_i}\Big) < \infty. \tag{9}$$

Let

$$\big\{(N_u, (U_{u1}, \Lambda_{u1}, (U_{u2}, \Lambda_{u2}, \ldots)))\big\}_u$$

be a family of independent copies of the above random vector, indexed by finite words $u = u_1 \cdots u_n$ with $n \geq 0$ and $u_i \in \mathbb{N}_+$ and defined on a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$. Under assumption (8), classical results on branching random walks (Kahane [24] in the deterministic offspring case and Biggins [34] in general) ensure that, a.s., for every $n \geq 1$ and every vertex $u \in \mathbb{N}_+^n$, the sequence

$$Z_p(u) = \sum_{v\in\mathsf{T}_p(u)} \exp\big(S_{n+p}\Lambda(uv) - S_n\Lambda(u)\big), \qquad p \geq 1,$$

converges to a strictly positive limit denoted by $Z(u)$. If condition (8) is violated, the limit exists but vanishes a.s. Using the family $\{(N_u, (U_{ui}, \Lambda_{ui}))\}$ we define a random measure $\nu$ on the $\sigma$-algebra $\mathcal{C}$ generated by the cylinder sets of $\mathbb{N}_+^{\mathbb{N}_+}$ by

$$\nu([u]) = \begin{cases} \exp\big(S_n\Lambda(u)\big)\, Z(u), & \text{if } u \in \mathsf{T}_n, \\ 0, & \text{otherwise.} \end{cases}$$

The measure $\nu$ is supported on the boundary $\partial\mathsf{T}$ and is referred to as a *Mandelbrot measure*. The classical branching measure corresponds to the particular choice $\Lambda_i = -\log \mathbb{E}(N)$ for $1 \le i \le N$. Condition (9) allows for one to apply the strong law of large numbers to the branching random walk $S_n U(t)$ under the Peyrière measure $\mathcal{Q}$, defined on $\mathcal{A}' \otimes \mathcal{C}$ by

$$\mathcal{Q}(E) = \mathbb{E}_{\mathbb{P}'}\left( \int \mathbf{1}_E(\omega', t)\, \nu(\mathrm{d}t) \right).$$

As a consequence (see, e.g., [5,24]), we obtain the following result.

**Proposition 1.** *a.s., for $\nu$-almost every $t \in \partial\mathsf{T}$,*

$$\lim_{n\to\infty} \frac{S_n U(t)}{n} = \mathbb{E}\left( \sum_{i=1}^{N} U_i\, \mathrm{e}^{\Lambda_i} \right),$$

Moreover, since $\mathbb{E}(Z) < \infty$ (see [5,24]), one has, a.s., for $\nu$-a.e. $t \in \partial\mathsf{T}$, $\limsup_{n\to\infty} \frac{\log Z(t_{|n})}{-n} \le 0$. It follows that a.s., for $\nu$-almost every $t \in \partial\mathsf{T}$,

$$\limsup_{n\to\infty} \frac{\log \nu([t_{|n}])}{-n} \le -\mathbb{E}\left( \sum_{i=1}^{N} \Lambda_i\, \mathrm{e}^{\Lambda_i} \right).$$

Using Kahane's approach for canonical cascades on homogeneous trees [24], we get the following result.

**Theorem 1.** *Assume in addition that $\mathbb{E}(Z \log^+(Z)) < \infty$. Then, a.s., for $\nu$-almost every $t \in \partial\mathsf{T}$,*

$$\liminf_{n\to\infty} \frac{\log \nu([t_{|n}])}{-n} = -\mathbb{E}\left( \sum_{i=1}^{N} \Lambda_i\, \mathrm{e}^{\Lambda_i} \right).$$

### *2.4. Inhomogeneous Mandelbrot Measures*

Let $N$ be an $\mathbb{N}_+$-valued random variable and let

$$\left(N, \Lambda_{k,1}, \Lambda_{k,2}, \ldots\right), \qquad k \ge 1,$$

be a sequence of random vectors taking values in $\mathbb{N}_+ \times \mathbb{R}^{\mathbb{N}_+}$. We assume that the normalization condition

$$\mathbb{E}\left( \sum_{i=1}^{N} \mathrm{e}^{\Lambda_{k,i}} \right) = 1 \quad \text{holds for all } k \ge 1.$$

For each $k \ge 1$, we consider a family

$$\left\{ (N_u, \Lambda_{k,u1}, \Lambda_{k,u2}, \ldots) \right\}_{u \in \mathbb{N}_+^{k-1}}$$

of independent copies of $\left(N, \Lambda_{k,1}, \Lambda_{k,2}, \ldots\right)$. We assume that all these random vectors, indexed by $u \in \bigcup_{k\ge 0} \mathbb{N}_+^{k-1}$, are mutually independent. A.s., for every $n \ge 1$ and every vertex $u \in \mathbb{N}_+^n$, the sequence

$$Z_p(u) = \sum_{v \in \mathsf{T}_p(u)} \exp\left( \Lambda_{n+1,uv_1} + \Lambda_{n+2,uv_1v_2} + \cdots + \Lambda_{n+p,uv_1\cdots v_p} \right), \qquad p \ge 1,$$

converges to a nonnegative limit denoted by $Z(u)$. Using the collection

$$\left\{ (N_u, \Lambda_{k,u1}, \Lambda_{k,u2}, \ldots) \right\}_{k\ge 1,\, u \in \mathbb{N}_+^{k-1}},$$

we define an inhomogeneous Mandelbrot measure $\nu$ on the $\sigma$-algebra $\mathcal{C}$ generated by the cylinder sets of $\mathbb{N}_+^{\mathbb{N}_+}$ by

$$\nu([u]) = \begin{cases} \exp\left(\Lambda_{1,u_1} + \Lambda_{2,u_1u_2} + \cdots + \Lambda_{n,u_1\cdots u_n}\right) Z(u), & \text{if } u \in \mathsf{T}_n, \\ 0, & \text{otherwise.} \end{cases}$$

The measure $\nu$ is supported on the boundary $\partial\mathsf{T}$. It follows that if there exists $\gamma > 1$ such that

$$\sup_{k\geq 1} \mathbb{E}\left(\sum_{i=1}^{N} \mathrm{e}^{\gamma\Lambda_{k,i}}\right) + \sup_{k\geq 1} \mathbb{E}\left(\left(\sum_{i=1}^{N} \mathrm{e}^{\Lambda_{k,i}}\right)^{\gamma}\right) < \infty, \tag{10}$$

then the measure $\nu$ is a.s. nontrivial. Assume furthermore that

$$\sum_{k\geq 1} k^{-2}\mathbb{E}\left(\sum_{i=1}^{N} \Lambda_{k,i}^2\, \mathrm{e}^{\Lambda_{k,i}}\right) < \infty. \tag{11}$$

**Proposition 2.** *Under the above assumptions, a.s., for $\nu$-almost every $t \in \partial\mathsf{T}$, one has*

$$\begin{aligned} \liminf_{n\to\infty} \frac{-1}{n} \sum_{k=1}^{n} \mathbb{E}\left(\sum_{i=1}^{N} \Lambda_{k,i}\, \mathrm{e}^{\Lambda_{k,i}}\right) &= \liminf_{n\to\infty} \frac{\log \nu([t_{|n}])}{-n} \\ &\leq \limsup_{n\to\infty} \frac{\log \nu([t_{|n}])}{-n} = \limsup_{n\to\infty} \frac{-1}{n} \sum_{k=1}^{n} \mathbb{E}\left(\sum_{i=1}^{N} \Lambda_{k,i}\, \mathrm{e}^{\Lambda_{k,i}}\right). \end{aligned}$$

These bounds determine both the Hausdorff and packing dimensions of $\nu$. Finally, following the martingale approach (see, for example, [5]), one obtains the asymptotic behavior of the associated branching random walk.

**Proposition 3.** *Assume in addition that*

$$\sum_{k\geq 1} k^{-2}\mathbb{E}\left(\sum_{i=1}^{N} \|U_{k,i}\|^2\, \mathrm{e}^{\Lambda_{k,i}}\right) < \infty.$$

*Then, a.s., for $\nu$-almost every $t \in \partial\mathsf{T}$,*

$$\lim_{n\to\infty} \frac{1}{n}\left(S_nU(t) - \sum_{k=1}^{n} \mathbb{E}\left(\sum_{i=1}^{N} U_i\, \mathrm{e}^{\Lambda_{k,i}}\right)\right) = 0.$$

The above results highlight the effectiveness of inhomogeneous Mandelbrot measures in the study of level sets of infinite branches along which the branching random walk $S_nU(t)$ exhibits prescribed asymptotic behaviors.

## 3. Hausdorff and Packing Dimension of the Set $E(\alpha, \beta)$

In this section we will compute the Hausdorff and packing dimension of the set $E(\alpha, \beta)$. More precisely, we will prove the following result.

**Theorem 2.** *Assume (3) and (4). Suppose that $\partial\mathsf{T}$ is endowed with the distance $d_1$ defined by (7) a.s. Then, for every $(\alpha, \beta) \in \mathcal{K}$, we have a.s.*

$$\dim E(\alpha, \beta) = Dim E(\alpha, \beta) = \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0).$$

*In particular, we have $E(\alpha, \beta) \neq \emptyset$.*

We state the following result which generalizes [6] [Lemma 2.1].

**Lemma 1.** *Assume* (4). *Let* $(a_n)_{n\geq 1}$ *be a sequence of positive numbers such that* $a_n \to \infty$ *and* $\liminf_{n\to\infty} \frac{a_n}{n} > 0$. *Then, there exist constants* $0 < c_1 < c_2 < \infty$ *such that, a.s., there exists an integer* $n_0$ *satisfying*

$$c_1 a_n \leq S_n Z(u) \leq c_2 a_n \qquad \text{for all } n \geq n_0 \text{ and all } u \in \mathsf{T}_n, \text{ and all } Z \in \{\widetilde{\mathsf{X}}, \widetilde{\mathsf{Y}}\}.$$

*In particular,* $\left\{t \in \partial\mathsf{T} : S_n \mathsf{X}(t)/S_n\widetilde{\mathsf{X}}(t) \to \alpha\right\} = \left\{t \in \partial\mathsf{T} : S_n(\mathsf{X} - \alpha\widetilde{\mathsf{X}})(t)/n \to 0\right\}$.

**Proof.** Set $m_n = \inf_{u\in\mathsf{T}_n} S_n Z(u)$ and $M_n = \sup_{u\in\mathsf{T}_n} S_n Z(u)$. Fix $\gamma > 0$ such that $\mathbb{E}\big(\sum_{i=1}^N e^{\gamma Z_i}\big) < \infty$, which exists by (4). For any $c > 0$, by Markov's inequality,

$$\begin{aligned}\mathbb{P}\big(M_n \geq c\, a_n\big) &= \mathbb{P}\big(\exists\, u \in \mathsf{T}_n :\ S_n Z(u) \geq c\, a_n\big)\\ &\leq \mathbb{E}\left(\sum_{u\in\mathsf{T}_n} e^{\gamma S_n Z(u)}\right) e^{-\gamma c a_n} = \left(\mathbb{E}\left(\sum_{i=1}^N e^{\gamma Z_i}\right)\right)^n e^{-\gamma c a_n},\end{aligned}$$

by independence across generations. Since $\liminf_{n\to\infty} a_n/n > 0$, choose $c > 0$ large enough so that

$$\gamma c \liminf_{n\to\infty} \frac{a_n}{n} > \log \mathbb{E}\left(\sum_{i=1}^N e^{\gamma Z_i}\right).$$

Therefore, there exist constants $A > 0$ and $\delta > 0$ such that

$$\mathbb{P}(M_n \geq c\, a_n) \leq A\, e^{-\delta n} \quad \text{for all } n \geq 1.$$

for all $n$ large enough. Hence, $\sum_n \mathbb{P}(M_n \geq c\, a_n) < \infty$, and by the Borel–Cantelli lemma, a.s. $M_n \leq c\, a_n$ for all $n$ large enough. This yields the upper bound with $c_2 = c$. Now, we will prove the lower bound, for this, fix $\gamma > 0$. For any $c > 0$,

$$\begin{aligned}\mathbb{P}(\exists\, u \in \mathsf{T}_n :\ S_n Z(u) \leq c\, a_n) &= \mathbb{P}\Big(\exists\, u \in \mathsf{T}_n :\ e^{-S_n Z(u)} \geq e^{-c a_n}\Big)\\ &\leq e^{\gamma c a_n}\, \mathbb{E}\left(\sum_{u\in\mathsf{T}_n} e^{-\gamma S_n Z(u)}\right)\\ &= e^{\gamma c a_n}\left(\mathbb{E}\left(\sum_{i=1}^N e^{-\gamma Z_i}\right)\right)^n.\end{aligned}$$

by Markov's inequality. Using (4), there exists $\gamma > 0$ such that

$$m_\gamma := \mathbb{E}\left(\sum_{i=1}^N e^{-\gamma Z_i}\right) < 1.$$

Hence,

$$\mathbb{P}(\exists\, u \in \mathsf{T}_n :\ S_n Z(u) \leq c\, a_n) \leq \exp\big(\gamma c a_n + n \log m_\gamma\big).$$

Since $\liminf_{n\to\infty} a_n/n > 0$, one can choose $c > 0$ small enough so that

$$\gamma c \liminf_{n\to\infty} \frac{a_n}{n} < -\log m_\gamma.$$

It follows that there exist constants $A > 0$ and $\delta > 0$ such that

$$\mathbb{P}(\exists\, u \in \mathsf{T}_n :\ S_n Z(u) \leq c\, a_n) \leq A\, e^{-\delta n} \quad \text{for all } n \geq 1.$$

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The series $\sum_{n\geq 1} Ae^{-\delta n}$ converges; by the Borel–Cantelli lemma, a.s., for all $n$ large enough, $\forall u \in \mathsf{T}_n$, $S_n Z(u) \geq c\, a_n$. Setting $c_1 = c$ completes the proof. □

### 3.1. Upper Bound of Packing Dimension

It is well known that, for any set $E \subset \mathbb{R}^d$, we have $\dim E \leq Dim E$. Then, we only have to prove that, with probability 1, $Dim\, E(\alpha,\beta) \leq \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0)$ and $\dim\, E(\alpha,\beta) \geq \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0)$.

**Proposition 4.** *For any $(\alpha,\beta) \in \mathbb{R}^2$, a.s.,*

$$Dim\, E(\alpha,\beta) \leq \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0).$$

**Proof.** Fix $(\alpha,\beta) \in \mathbb{R}^2$ and $\varepsilon > 0$. Observe first that

$$E(\alpha,\beta) \subset \bigcup_{N\geq 1} F_N,$$

where

$$F_N = \bigcap_{n\geq N} \Big\{ t \in \partial\mathsf{T} : \big\| S_n(\mathsf{X}(t),\mathsf{Y}(t)) - S_n(\alpha\widetilde{\mathsf{X}}(t),\beta\widetilde{\mathsf{Y}}(t)) \big\| \leq \varepsilon n \Big\}.$$

For $v \in \mathsf{T}_n$, we introduce the notation

$$\Gamma_n(v,\alpha,\beta) = S_n(\mathsf{X}(t),\mathsf{Y}(t)) - S_n(\alpha\widetilde{\mathsf{X}}(t),\beta\widetilde{\mathsf{Y}}(t)), \qquad v = t_{|n}.$$

Then,

$$\begin{aligned} E(\alpha,\beta) &\subset \bigcap_{\varepsilon>0} \bigcap_{N\geq 1} \bigcup_{n\geq N} \{ t \in \partial\mathsf{T} : \|\Gamma_n(v,\alpha,\beta)\| \leq \varepsilon n \} \\ &\subset \bigcap_{q\in\mathbb{R}^2} \bigcap_{\varepsilon>0} \bigcap_{N\geq 1} \bigcup_{n\geq N} \{ t \in \partial\mathsf{T} : \big|\langle q, \Gamma_n(v,\alpha,\beta)\rangle\big| \leq \varepsilon n \|q\| \}. \end{aligned}$$

Fix $q \in \mathbb{R}^2$ and $N \geq 1$. Let $u \in \mathsf{T}$ with $|u| \geq N$. For any $N' \geq |u|$, every $e^{-N'}$-packing of $[u] \cap F_N$ is contained in

$$\bigcup_{n\geq N'} \bigcup_{\substack{v\in\mathsf{T}_n \\ |\langle q,\Gamma_n(v,\alpha,\beta)\rangle| \leq \varepsilon n\|q\|}} [v].$$

Let $s = \widetilde{\mathcal{P}}_{\alpha,\beta}(q) + \varepsilon\|q\| + \eta$, with $\eta > 0$. Then,

$$\begin{aligned} \sup_{\mathcal{I}} \sum_{j\in\mathcal{I}} |[v_j]|^s &\leq \sum_{n\geq N'} \sum_{\substack{v\in\mathsf{T}_n \\ |\langle q,\Gamma_n(v,\alpha,\beta)\rangle| \leq \varepsilon n\|q\|}} |[v]|^s \\ &\leq \sum_{n\geq N'} \sum_{v\in\mathsf{T}_n} \exp\Big( \langle q, \Gamma_n(v,\alpha,\beta)\rangle - \big(s - \widetilde{\mathcal{P}}_{\alpha,\beta}(q) - \varepsilon\|q\|\big)n - \widetilde{\mathcal{P}}_{\alpha,\beta}(q)n \Big) \\ &\leq \sum_{n\geq N'} \sum_{v\in\mathsf{T}_n} \exp\Big( \langle q, \Gamma_n(v,\alpha,\beta)\rangle - \big(\widetilde{\mathcal{P}}_{\alpha,\beta}(q) + \eta\big)n \Big). \end{aligned}$$

By the definition of $\widetilde{\mathcal{P}}_{\alpha,\beta}(q)$, we have

$$\begin{aligned} &\mathbb{E}\left[ \sum_{n\geq 1} \sum_{v\in\mathsf{T}_n} \exp\Big( \langle q, \Gamma_n(v,\alpha,\beta)\rangle - \big(\widetilde{\mathcal{P}}_{\alpha,\beta}(q) + \eta\big)n \Big) \right] \\ &= \sum_{n\geq 1} \mathbb{E}\left( \sum_{i=1}^{N} \exp\Big( \langle q, (\mathsf{X}_i - \alpha\widetilde{\mathsf{X}}_i, \mathsf{Y}_i - \beta\widetilde{\mathsf{Y}}_i)\rangle - \big(\widetilde{\mathcal{P}}_{\alpha,\beta}(q) + \eta\big) \Big) \right)^n < \infty. \end{aligned}$$

Hence, a.s., $\overline{P}^s([u] \cap F_N) = 0$. Since this holds for every $u$ with $|u| \geq N$, we obtain $P^s(F_N) = 0$ a.s. As this is true for all $N \geq 1$, it follows that

$$P^s(E(\alpha, \beta)) = 0 \quad \text{a.s.}$$

Therefore,

$$Dim\, E(\alpha, \beta) \leq \widetilde{\mathcal{P}}_{\alpha,\beta}(q) + \varepsilon\|q\| + \eta \quad \text{for all } \varepsilon > 0,\ \eta > 0.$$

Letting $\varepsilon, \eta \to 0$ yields

$$Dim\, E(\alpha, \beta) \leq \inf_{q \in \mathbb{R}^2} \widetilde{\mathcal{P}}_{\alpha,\beta}(q) \quad \text{a.s.}$$

□

### 3.2. Lower Bound of Hausdorff Dimension

For each $(q, (\alpha, \beta)) \in \mathbb{R}^2 \times \mathbb{R}^2$, a Mandelbrot measure $\mu_{q,\alpha,\beta}$ on $\partial\mathsf{T}$ is associated with the vectors $((N_u, \exp(\langle q|\Gamma(u1, \alpha, \beta)\rangle) - \widetilde{\mathcal{P}}_{\alpha,\beta}(q)), \exp(\langle q|\Gamma(u2, \alpha, \beta)\rangle) - \widetilde{\mathcal{P}}_{\alpha,\beta}(q), \ldots)$, $u \in \bigcup_{n\geq 0} \mathbb{N}_+^n$, and this measure is nondegenerate if and only if

$$\widetilde{\mathcal{P}}^*_{\alpha,\beta}(\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q)) = \widetilde{\mathcal{P}}_{\alpha,\beta}(q) - \langle q|\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q)\rangle > 0.$$

Since $\widetilde{\mathcal{P}}_{\alpha,\beta}$ is strictly convex, its gradient can vanish at most at one point. Moreover, whenever the centered vectors $(\mathsf{X}_i - \alpha\widetilde{\mathsf{X}}_i, \mathsf{Y}_i - \beta\widetilde{\mathsf{Y}}_i)$ take both positive and negative values, continuity of the gradient ensures the existence of a unique $q \in \mathbb{R}^2$ such that $\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q) = 0$. In addition, $(\alpha, \beta) \mapsto q_{\alpha,\beta}$ is analytic. Indeed, we consider the map $(\alpha, \beta, q) \mapsto \nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q)$. Its differential with respect to $q$ at $(\alpha, \beta, q_{\alpha,\beta})$ and the Hessian matrix, which coincides with the covariance matrix of $(\mathsf{X}_i - \alpha\widetilde{\mathsf{X}}_i, \mathsf{Y}_i - \beta\widetilde{\mathsf{Y}}_i)$. This matrix is invertible unless $(\mathsf{X}_i - \alpha\widetilde{\mathsf{X}}_i, \mathsf{Y}_i - \beta\widetilde{\mathsf{Y}}_i)$ is a.s. constant, which is ruled out by the nondegeneracy assumption (5). Therefore, the implicit function theorem applies and yields that $(\alpha, \beta) \mapsto q_{\alpha,\beta}$ is analytic on $\mathcal{K}$.

Let $(\alpha, \beta) \in \mathcal{K}$ and construct the Mandelbrot measure $\mu_{q,\alpha,\beta}$ on $\partial\mathsf{T}$ as in Section 3.2, such that $\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q_{\alpha,\beta}) = 0$ and $\widetilde{\mathcal{P}}^*_{\alpha,\beta}\big(\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q_{\alpha,\beta})\big) > 0$. This implies, in particular, that this measure is a.s. positive. Moreover (see Proposition 1), with probability 1, we have

$$\lim_{n\to\infty} \frac{S_n(\mathsf{X}(t) - \alpha\widetilde{\mathsf{X}}(t))}{n} = \alpha_q(\alpha, \beta) \quad \lim_{n\to\infty} \frac{S_n(\mathsf{Y}(t) - \beta\widetilde{\mathsf{Y}}(t))}{n} = \widetilde{\alpha}_q(\alpha, \beta)$$

$\mu_{q,\alpha,\beta}$-a.e., where

$$\alpha_q(\alpha, \beta) = \mathbb{E}\Big(\sum_{i=1}^{N} (\mathsf{X}_i - \alpha\widetilde{\mathsf{X}}_i) \exp(\langle q|\Gamma(i, \alpha, \beta)\rangle) - \widetilde{\mathcal{P}}_{\alpha,\beta}(q)\Big).$$

$$\widetilde{\alpha}_q(\alpha, \beta) = \mathbb{E}\Big(\sum_{i=1}^{N} (\mathsf{Y}_i - \beta\widetilde{\mathsf{Y}}_i) \exp(\langle q|\Gamma(i, \alpha, \beta)\rangle) - \widetilde{\mathcal{P}}_{\alpha,\beta}(q)\Big).$$

Note that $\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q) = (\alpha_q(\alpha, \beta), \widetilde{\alpha}_q(\alpha, \beta))$. It follows, using Lemma 1, we have $\mu_{q,\alpha,\beta}(E(\alpha, \beta)) = 1$. In addition, under the assumption (3), we have that the Hausdorff dimension of the measure $\mu_{q,\alpha,\beta}$ is $\widetilde{\mathcal{P}}^*_{\alpha,\beta}(0)$ (Theorem 1) so that, from the mass distribution principle [32] [Theorem 4.2]), we have a.s.

$$\dim E(\alpha, \beta) \geq \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0).$$

Now, consider a special case of our level sets and consider

$$\widetilde{E}(\alpha, \beta) = \Big\{t \in \partial\mathsf{T} : \lim_{n\to\infty} \frac{S_n\mathsf{X}(t)}{n} = \alpha, \quad \lim_{n\to\infty} \frac{S_n\mathsf{Y}(t)}{n} = \beta\Big\}.$$

A direct consequence of Theorem 2 is the following result. First, we define the associated pressure function by

$$\widetilde{\mathcal{P}}(q) = \log \mathbb{E}\left( \sum_{i=1}^{N} \exp\left( \langle q, (\mathsf{X}_i, \mathsf{Y}_i) \rangle \right) \right), \qquad q \in \mathbb{R}^2. \tag{12}$$

**Corollary 1.** *Under the same assumptions as in Theorem 2, a.s., for every $(\alpha, \beta) \in \mathcal{K}$, the Hausdorff and packing dimensions of the level set $\widetilde{E}(\alpha, \beta)$ coincide and are given by*

$$\dim \widetilde{E}(\alpha, \beta) = Dim \widetilde{E}(\alpha, \beta) = \widetilde{\mathcal{P}}^*(\alpha, \beta).$$

**Example 1.** *In this example, we consider the classical multifractal analysis of binomial measures. Let $\mathsf{T}$ be a supercritical Galton–Watson tree with deterministic offspring number $N = 2$. Let $0 < p_1, p_2 < 1$, and let $\eta_1$ and $\eta_2$ be two binomial measures with parameters $p_1$ and $p_2$, respectively. They are defined recursively by*

$$\eta_i([u1]) = p_i\, \eta_i([u]), \qquad \eta_i([u0]) = (1 - p_i)\, \eta_i([u]),$$

*for all words $u \in \mathsf{T}_{n-1}$ and $i = 1, 2$ (see, e.g., [35–37]). We define*

$$\mathsf{X}_u = -\log \eta_1([u]), \qquad \mathsf{Y}_u = -\log \eta_2([u]), \qquad \widetilde{\mathsf{X}}_u = \widetilde{\mathsf{Y}}_u = 1.$$

*With this choice, the level set $E(\alpha, \beta)$ introduced in (1) reduces to*

$$E(\alpha, \beta) = \left\{ t \in \partial \mathsf{T} : \lim_{n \to \infty} -\frac{\log \eta_1([t_{|n}])}{n} = \alpha, \quad \lim_{n \to \infty} -\frac{\log \eta_2([t_{|n}])}{n} = \beta \right\}, \qquad (\alpha, \beta) \in \mathbb{R}^2.$$

*This set consists of boundary points where the two binomial measures $\eta_1$ and $\eta_2$ have prescribed local dimensions $\alpha$ and $\beta$, respectively. The pressure function $\widetilde{\mathcal{P}}$ defined in (12) is given by*

$$\widetilde{\mathcal{P}}(q_1, q_2) = \log 2 + \log\left( \tfrac{1}{2} \left( p_1^{q_1} p_2^{q_2} + (1 - p_1)^{q_1} (1 - p_2)^{q_2} \right) \right), \qquad (q_1, q_2) \in \mathbb{R}^2.$$

*By Corollary 1, we obtain*

$$\dim E(\alpha, \beta) = Dim E(\alpha, \beta) = \widetilde{\mathcal{P}}^*(\alpha, \beta) = \inf_{(q_1, q_2) \in \mathbb{R}^2} \left( \widetilde{\mathcal{P}}(q_1, q_2) - q_1 \alpha - q_2 \beta \right).$$

**Example 2.** *Let $\mathsf{T}$ be a supercritical Galton–Watson tree with deterministic offspring number $N \geq 2$. Let $(\widetilde{W}_u)_{u \in \mathsf{T}}$ and $(C_u)_{u \in \mathsf{T}}$ be two families of positive independent and identical distribution random variables, independent of each other, satisfying the assumptions stated in Section 2.3. Under these conditions, one can define two exact-dimensional multiplicative cascade measures on $\partial \mathsf{T}$ by*

$$\widetilde{\mu}([t_1 \cdots t_n]) = \prod_{k=1}^{n} \widetilde{W}_{t_1 \cdots t_k}, \qquad \nu([t_1 \cdots t_n]) = \prod_{k=1}^{n} C_{t_1 \cdots t_k}.$$

*Let $\gamma \in \mathbb{R}$ be chosen so that the function $\mu$ defined by*

$$\mu([t_1 \cdots t_n]) = \prod_{k=1}^{n} (\widetilde{W}_{t_1 \cdots t_k})^{\gamma}\, C_{t_1 \cdots t_k}$$

*is a well-defined (nondegenerate) multiplicative cascade measure on $\partial \mathsf{T}$. In this example, we consider*

$$\mathsf{X}_u = -\log C_u, \qquad \widetilde{\mathsf{X}}_u = -\log \widetilde{W}_u, \qquad \mathsf{Y}_u = \widetilde{\mathsf{Y}}_u = 1.$$

*Then,*

$$S_n\mathsf{X}(t) = -\log \nu([t_{|n}]), \qquad S_n\widetilde{\mathsf{X}}(t) = -\log \widetilde{\mu}([t_{|n}])$$

*and consider the ratio level set*

$$E(\alpha) = \Big\{ t \in \partial\mathsf{T} : \lim_{n\to\infty} \frac{S_n\mathsf{X}(t)}{S_n\widetilde{\mathsf{X}}(t)} = \alpha \Big\}, \qquad \alpha = -\gamma.$$

*Since* $S_n\mathsf{X}(t) - \alpha S_n\widetilde{\mathsf{X}}(t) = -\log \mu([t_{|n}])$, *we obtain*

$$E(\alpha) = \Big\{ t \in \partial\mathsf{T} : \lim_{n\to\infty} \frac{-\log \mu([t_{|n}])}{S_n\widetilde{\mathsf{X}}(t)} = 0 \Big\}.$$

*Under the standing assumptions on* $(\widetilde{W}_u)$, *one typically has*

$$\lim_{n\to\infty} \frac{S_n\widetilde{\mathsf{X}}(t)}{n} = \dim \widetilde{\mu}(t) \quad \textit{for } \widetilde{\mu}\textit{-a.e. } t.$$

*Therefore, the study of* $E(\alpha)$ *reduces to the analysis of the set*

$$\Big\{ t \in \partial\mathsf{T} : \lim_{n\to\infty} \frac{-\log \mu([t_{|n}])}{n} = 0 \Big\},$$

*which is a classical object in the multifractal analysis of multiplicative cascades and appears, in particular, in the work of Cole [38] and its extensions (see also [39]).*

## 4. Hausdorff and Packing Dimension of $E_{\mathsf{s}}(\alpha, \beta)$

In this section, we present our main result concerning the size of the refined level set $E_s(\alpha, \beta)$ (see Theorem 3 below). We emphasize that the approach developed in [30] to determine the Hausdorff and packing dimensions of the one-dimensional sets $\widetilde{E}_{\alpha,r}$ does not extend to the present vector-valued setting and therefore does not yield information on the dimension of $E_s(\alpha, \beta)$.

In addition, the proof of Theorem 3 follows a different strategy from that used in Section 3. For the classical level sets $E(\alpha, \beta)$, the dimension estimates are obtained through the construction of a homogeneous Mandelbrot measure associated with fixed parameters $(\alpha, \beta)$. This approach relies on the fact that the asymptotic constraints defining $E(\alpha, \beta)$ are stationary and therefore can be encoded by a single family of weights. In contrast, the refined sets $E_{\mathsf{s}}(\alpha, \beta)$ involve prescribed deviation sequences $s_n = (r_n, \widetilde{r}_n)$, which introduce a non-stationary behavior along the generations of the tree. Consequently, the homogeneous construction of Section 3 is no longer sufficient. To overcome this difficulty, we construct a suitable inhomogeneous Mandelbrot measure whose weights depend on the prescribed deviation sequence. The proof then proceeds in three main steps. First, we define the auxiliary inhomogeneous Mandelbrot measure and verify the required moment conditions. Next, we establish the asymptotic behavior of the associated branching random walks and compute the Hausdorff and packing dimensions of the measure. Finally, we show that the measure is supported on $E_{\mathsf{s}}(\alpha, \beta)$, which yields the desired lower bound for the dimension of $E_{\mathsf{s}}(\alpha, \beta)$.

Let $(\boldsymbol{s}_n)_{n\geq 1}$ be the sequence introduced in Section 1, and assume that condition (6) holds. In addition, we impose the following summability condition:

$$\sum_{n\geq 1} \exp\left( -\epsilon \sum_{k=1}^{n} k^{-2} \|\boldsymbol{\eta}_k\|^2 \right) < +\infty, \qquad \forall\, \epsilon > 0. \tag{13}$$

**Theorem 3.** *Let* $\mathsf{s} = (\mathsf{s}_n)_{n\geq 1}$ *be a positive sequence such that* (6) *and* (13) *are satisfied. Then, a.s., for all* $(\alpha, \beta) \in \mathcal{K}$,

$$\dim E_{\mathsf{s}}(\alpha,\beta) = Dim\, E_{\mathsf{s}}(\alpha,\beta) = \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0) = \inf_{q\in\mathbb{R}^2} \widetilde{\mathcal{P}}_{\alpha,\beta}(q).$$

Recall that

$$\dim E_{\boldsymbol{s}}(\alpha,\beta) \leq Dim E(\alpha,\beta) \leq \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0).$$

Therefore, in order to prove Theorem 3, it is sufficient to construct, for each $\omega$ belonging to a suitable family of Mandelbrot measures, a probability measure $\mu^{\boldsymbol{s}}_{\omega}$ supported on $E_{\boldsymbol{s}}(\alpha,\beta)$ and to compute its Hausdorff and packing dimensions. More precisely, we shall show that

$$\mu^{\boldsymbol{s}}_{\omega}\big(E_{\boldsymbol{s}}(\alpha,\beta)\big) = 1,$$

and that, almost surely, for $\mu^{\boldsymbol{s}}_{\omega}$–almost every $t \in E_{\boldsymbol{s}}(\alpha,\beta)$,

$$\dim \mu^{\boldsymbol{s}}_{\omega} = \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0).$$

Then, by applying [32] [Theorem 4.2], we obtain the lower bound

$$\dim E_{\boldsymbol{s}}(\alpha,\beta) \geq \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0) = \inf_{q\in\mathbb{R}^2} \widetilde{\mathcal{P}}_{\alpha,\beta}(q),$$

which, combined with the upper bound, completes the proof of Theorem 3.

### *4.1. Construction of Inhomogeneous Mandelbrot Measures*

We consider the set

$$\mathcal{J} = \Big\{(q,(\alpha,\beta) \in \mathbb{R}^2 \times \mathcal{K}) :\ \widetilde{\mathcal{P}}^*_{\alpha,\beta}(\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q)) > 0\Big\}.$$

Recall that, for each $(\alpha,\beta) \in \mathcal{K}$, the existence of unique $q := q_{\alpha,\beta}$ such that $\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q_{\alpha,\beta}) = 0$. Moreover, $(\alpha,\beta) \in int(\mathcal{K}) \longmapsto q_{\alpha,\beta}$ is analytic. For each $\omega = (q,(\alpha,\beta)) \in \mathcal{J}$, we will construct an auxiliary measure $\mu_{\omega}$. We define, for $k \geq 1$, $q_k$ as the unique real $t$, such that

$$\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(t) = \boldsymbol{\eta}_k.$$

We consider the Mandelbrot measure $\mu_{\omega}$ on $\partial\mathsf{T}$ is associated with the vectors

$$(N_u, \exp(\langle q_k|\Gamma(u1,\alpha,\beta)\rangle) - \widetilde{\mathcal{P}}_{\alpha,\beta}(q_k)),\ldots),$$

$u \in \bigcup_{n\geq 0}\mathbb{N}^n_+$. This measure is nondegenerate if and only if

$$\widetilde{\mathcal{P}}^*_{\alpha,\beta}(\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q)) = \widetilde{\mathcal{P}}_{\alpha,\beta}(q) - \langle q|\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q)\rangle > 0.$$

Let

$$\Lambda_{k,i} = \langle q_k, \Gamma(i,\alpha,\beta)\rangle - \widetilde{\mathcal{P}}_{\alpha,\beta}(q_k), \qquad k \geq 1,\ 1 \leq i \leq N, \tag{14}$$

By Assumptions (3) and (4), the variables $\Lambda_{k,i}$ admit uniform exponential moments and depend linearly on $Z_i \in \mathcal{R}$ with coefficients $(q_k)_k$ contained in a compact set. Since, for any $\delta > 0$, there exists $C_\delta > 0$ such that $\|Z_i\|^2 \leq C_\delta\, e^{\delta\|Z_i\|}$, an application of Hölder's inequality yields the existence of $\gamma > 1$ for which

$$\sup_{k\geq 1} \mathbb{E}\left(\sum_{i=1}^{N} \|Z_i\|^2\, e^{\gamma\Lambda_{k,i}}\right) < \infty. \tag{15}$$

Clearly, for $U_i = \Gamma(i, \alpha, \beta)$ we have, for all $i$,

$$\sum_{i=1}^{N} \|U_i\|^2 \, \mathrm{e}^{\Lambda_{k,i}} \quad \leq \quad C \sum_{Z \in \mathcal{R}} \sum_{i=1}^{N} \|Z_i\|^2 \, \mathrm{e}^{\Lambda_{k,i}} \leq^{(15)} C_1$$

Therefore, (9) is satisfied and then, by Proposition 3, we obtain a.s., for $\mu_\omega$-almost every $t \in \partial \mathsf{T}$,

$$\lim_{n \to \infty} \frac{1}{n} \left( S_n U(t) - \sum_{k=1}^{n} \mathbb{E}\left( \sum_{i=1}^{N} U_i \, \mathrm{e}^{\Lambda_{k,i}} \right) \right) = 0.$$

*4.2. Hausdorff and Packing Dimension of the Measure $\mu_\omega^{\mathsf{s}}$*

Recall the weights defining the Mandelbrot measure $\Lambda_{k,i}$ given by (14). Using assumption (3) and the definition (14), one can get

$$\sup_{k \geq 1} \mathbb{E}\left( \sum_{i=1}^{N} \mathrm{e}^{\gamma \Lambda_{k,i}} \right) < \infty.$$

Moreover, by the inequality $(\sum_{i=1}^{N} a_i)^\gamma \leq N^{\gamma-1} \sum_{i=1}^{N} a_i^\gamma$ for $\gamma \geq 1$, we obtain

$$\mathbb{E}\left( \left( \sum_{i=1}^{N} \mathrm{e}^{\Lambda_{k,i}} \right)^\gamma \right) \leq \mathbb{E}\left( N^{\gamma-1} \sum_{i=1}^{N} \mathrm{e}^{\gamma \Lambda_{k,i}} \right).$$

so that

$$\sup_{k \geq 1} \mathbb{E}\left( \left( \sum_{i=1}^{N} \mathrm{e}^{\Lambda_{k,i}} \right)^\gamma \right) < \infty.$$

which gives the condition (10) is satisfied. Now, observe that

$$\boldsymbol{\eta}_k = \nabla \widetilde{\mathcal{P}}_{\alpha,\beta}(q_k) = \mathbb{E}\left( \sum_{i=1}^{N} \Gamma(i, \alpha, \beta) \, \mathrm{e}^{\Lambda_{k,i}} \right),$$

and then we have the following lemma.

**Lemma 2.** *There exists a constant $C > 0$ such that, for all $k \geq 1$ and all $1 \leq i \leq N$,*

$$\Lambda_{k,i}^2 \leq C\Big( \langle q_k, \Gamma(i, \alpha, \beta) - \boldsymbol{\eta}_k \rangle^2 + \|\boldsymbol{\eta}_k\|^2 + 1 \Big),$$

**Proof.** For each $k \geq 1$ and $1 \leq i \leq N$, we decompose $\Lambda_{k,i}$ as

$$\Lambda_{k,i} = \underbrace{\langle q_k, \Gamma(i, \alpha, \beta) - \boldsymbol{\eta}_k \rangle}_{:= A_{k,i}} + \underbrace{\big( \langle q_k, \boldsymbol{\eta}_k \rangle - \widetilde{\mathcal{P}}_{\alpha,\beta}(q_k) \big)}_{:= B_k}.$$

Using the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$ valid for all $a, b \in \mathbb{R}$, we obtain

$$\Lambda_{k,i}^2 \leq 2A_{k,i}^2 + 2B_k^2.$$

Since the sequence $(q_k)_{k \geq 1}$ remains in a compact subset of $\mathbb{R}^2$, there exists $M > 0$ such that $\|q_k\| \leq M$ for all $k$. Moreover, the function $\widetilde{\mathcal{P}}_{\alpha,\beta}$ is continuous on compact sets. Therefore,

$$|B_k| \leq |\langle q_k, \boldsymbol{\eta}_k \rangle| + |\widetilde{\mathcal{P}}_{\alpha,\beta}(q_k)| \leq M \|\boldsymbol{\eta}_k\| + C_0,$$

for some constant $C_0 > 0$. Consequently,

$$B_k^2 \le 2M^2\|\boldsymbol{\eta}_k\|^2 + 2C_0^2 \le C_1\big(\|\boldsymbol{\eta}_k\|^2 + 1\big),$$

for a suitable constant $C_1 > 0$. Combining the above inequalities yields

$$\Lambda_{k,i}^2 \le 2\langle q_k, \Gamma(i,\alpha,\beta) - \boldsymbol{\eta}_k\rangle^2 + 2C_1\big(\|\boldsymbol{\eta}_k\|^2 + 1\big).$$

Absorbing all constants into a single constant $C > 0$ completes the proof. □

**Corollary 2.** *There exists a constant $C' > 0$ such that, for all $k \ge 1$,*

$$\mathbb{E}\left(\sum_{i=1}^{N} \Lambda_{k,i}^2\, \mathrm{e}^{\Lambda_{k,i}}\right) \le C'\big(1 + \|\boldsymbol{\eta}_k\|^2\big).$$

**Proof.** By Lemma 2, there exists a constant $C > 0$ such that, for all $k \ge 1$ and all $1 \le i \le N$,

$$\Lambda_{k,i}^2 \le C\Big(\langle q_k, \Gamma(i,\alpha,\beta) - \boldsymbol{\eta}_k\rangle^2 + \|\boldsymbol{\eta}_k\|^2 + 1\Big)$$

and then

$$\sum_{i=1}^{N} \Lambda_{k,i}^2\, \mathrm{e}^{\Lambda_{k,i}} \le C\sum_{i=1}^{N}\langle q_k, \Gamma(i,\alpha,\beta) - \boldsymbol{\eta}_k\rangle^2\, \mathrm{e}^{\Lambda_{k,i}} + C\big(\|\boldsymbol{\eta}_k\|^2 + 1\big)\sum_{i=1}^{N} \mathrm{e}^{\Lambda_{k,i}}.$$

It follows that

$$\mathbb{E}\left(\sum_{i=1}^{N} \Lambda_{k,i}^2\, \mathrm{e}^{\Lambda_{k,i}}\right) \le C\mathbb{E}\left(\sum_{i=1}^{N}\langle q_k, \Gamma(i,\alpha,\beta) - \boldsymbol{\eta}_k\rangle^2\, \mathrm{e}^{\Lambda_{k,i}}\right) + C\big(\|\boldsymbol{\eta}_k\|^2 + 1\big)\underbrace{\mathbb{E}\left(\sum_{i=1}^{N} \mathrm{e}^{\Lambda_{k,i}}\right)}_{=1}.$$

Although $\mathbb{E}\Big(\sum_{i=1}^{N}(\Gamma(i,\alpha,\beta) - \boldsymbol{\eta}_k)\, \mathrm{e}^{\Lambda_{k,i}}\Big) = 0$, the quadratic term does not vanish in general. However, since the sequence $(q_k)_k$ remains in a compact set and the increments $\Gamma(i,\alpha,\beta)$ admit uniform exponential moments by Assumptions (3) and (4), this variance is uniformly integrable. Consequently, there exists a constant $C_0 > 0$ such that

$$\mathbb{E}\left(\sum_{i=1}^{N}\langle q_k, \Gamma(i,\alpha,\beta) - \boldsymbol{\eta}_k\rangle^2\, \mathrm{e}^{\Lambda_{k,i}}\right) \le C_0 \quad \text{for all } k \ge 1.$$

Combining the above estimates yields

$$\mathbb{E}\left(\sum_{i=1}^{N} \Lambda_{k,i}^2\, \mathrm{e}^{\Lambda_{k,i}}\right) \le C'\big(1 + \|\boldsymbol{\eta}_k\|^2\big),$$

for a suitable constant $C' > 0$, which completes the proof. □

In the following we give the Hausdorff and packing dimension of the measure $\mu_\omega^{\mathbf{s}}$. To this end we need to verify the condition (11) and then we need the following result.

**Lemma 3.** *Let $(b_k)_{k\ge1}$ be a sequence of nonnegative real numbers such that*

$$\sum_{k=1}^{n} b_k = O(\log n).$$

*Assume that*

$$\sum_{n\geq 1} \exp\left(-\epsilon \sum_{k=1}^{n} b_k\right) < \infty \qquad \textit{for all } \epsilon > 0. \tag{16}$$

*Then,* $\sum_{k\geq 1} b_k < \infty$.

**Proof.** Set $B_n = \sum_{k=1}^{n} b_k$ and assume by contradiction that

$$\sum_{k\geq 1} b_k = +\infty,$$

so that $B_n \to +\infty$ as $n \to \infty$. By assumption, there exists a constant $C > 0$ such that

$$B_n \leq C \log n \quad \text{for all sufficiently large } n.$$

Fix $\epsilon > 0$. Then, for $n$ large enough,

$$\exp(-\epsilon B_n) \geq \exp(-\epsilon C \log n) = n^{-\epsilon C}.$$

Choose $\epsilon > 0$ such that $\epsilon C \leq 1$. This is always possible since $\epsilon$ is arbitrary. For such a choice, we obtain

$$\exp(-\epsilon B_n) \geq \frac{1}{n} \quad \text{for infinitely many } n.$$

It follows that $\sum_{n\geq 1} \exp(-\epsilon B_n) = \infty$, which contradicts assumption (16). Therefore,

$$\sum_{k\geq 1} b_k < \infty.$$

□

**Corollary 3.** *Assume* (13)*, and then*

$$\dim \mu_\omega^{\mathsf{s}} := \lim_{n\to\infty} \frac{\log(\mu_\omega^s[t_{|n}])}{\log(\operatorname{diam}([t_{|n}]))} = \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0).$$

**Proof.** Applying Lemma 3 for $b_k = k^{-2}\|\boldsymbol{\eta}_k\|^2$ to get (11) is satisfied under (13). It follows, by Proposition 2, that a.s., for $\mu_\omega^{\mathsf{s}}$–almost every $t \in \partial\mathsf{T}$,

$$\begin{aligned}
\dim \mu_\omega^{\mathsf{s}} &= \lim_{n\to\infty} \frac{-1}{n} \sum_{k=1}^{n} \mathbb{E}\left(\sum_{i=1}^{N} \Lambda_{k,i}\, \mathrm{e}^{\Lambda_{k,i}}\right) \\
&= \lim_{n\to\infty} \frac{-1}{n} \sum_{k=1}^{n} \mathbb{E}\left(\sum_{i=1}^{N} \langle q_k | \Gamma(i,\alpha,\beta)\rangle\, \mathrm{e}^{\Lambda_{k,i}}\right) - \widetilde{\mathcal{P}}_{\alpha,\beta}(q_k) \\
&= \lim_{n\to\infty} \frac{-1}{n} \sum_{k=1}^{n} \langle q_k | \nabla \widetilde{\mathcal{P}}_{\alpha,\beta}(q_k)\rangle - \widetilde{\mathcal{P}}_{\alpha,\beta}(q_k) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \widetilde{\mathcal{P}}^*_{\alpha,\beta}(\nabla \widetilde{\mathcal{P}}_{\alpha,\beta}(q_k)) = \widetilde{\mathcal{P}}^*_{\alpha,\beta}(0).
\end{aligned}$$

□

**Remark 1.** *Theorem 3 provides a refined version of Theorem 2. Indeed, the sets $E_{\mathbf{s}}(\alpha,\beta)$ are obtained from the classical level sets $E(\alpha,\beta)$ by imposing additional asymptotic constraints on the convergence rates:*

$$S_n\mathsf{X}(t)-\alpha S_n\widetilde{\mathsf{X}}(t)\sim r_n,\qquad S_n\mathsf{Y}(t)-\beta S_n\widetilde{\mathsf{Y}}(t)\sim \widetilde{r}_n.$$

*It is clear that every point of $E_s(\alpha,\beta)$ belongs to $E(\alpha,\beta)$, and therefore,*

$$E_{\mathbf{s}}(\alpha,\beta)\subset E(\alpha,\beta).$$

*However, Theorem 3 shows that, under assumptions (6) and (13), the Hausdorff and packing dimensions remain unchanged:*

$$\dim E_{\mathbf{s}}(\alpha,\beta)=DimE_{\mathbf{s}}(\alpha,\beta)=\dim E(\alpha,\beta)=DimE(\alpha,\beta)=\widetilde{\mathcal{P}}^{*}_{\alpha,\beta}(0).$$

*Hence, the refined level sets have the same multifractal spectrum as the classical level sets, despite the presence of prescribed convergence-rate constraints.*

## 5. Examples

1. Let $\beta\in(1/2,1)$ and define the deterministic sequence

$$\boldsymbol{s}_n=(r_n,\widetilde{r}_n)\quad\text{with}\quad r_n=n^{\beta},\qquad \widetilde{r}_n=n^{\beta}.$$

Then, $\boldsymbol{s}_n=o(n)$ and that $\|\boldsymbol{\eta}_n\|=O(n^{\beta-1})=o(1)$. Hence, condition (6) is satisfied. Moreover,

$$\sum_{k=1}^{n}k^{-2}\|\boldsymbol{\eta}_k\|^2\le c\sum_{k=1}^{n}k^{-2}\,k^{2(\beta-1)}$$

for some positive constant $c$, which converges as $n\to\infty$. Consequently, condition (13) holds. For this choice of $\boldsymbol{s}$, Theorem 3 applies and yields, almost surely, for all $(\alpha,\beta)\in\mathcal{K}$,

$$\dim E_{\boldsymbol{s}}(\alpha,\beta)=DimE_{\boldsymbol{s}}(\alpha,\beta)=\widetilde{\mathcal{P}}^{*}_{\alpha,\beta}(0).$$

2. Assume that the offspring number $N\ge 2$ is deterministic. The random variables satisfy

$$\widetilde{\mathsf{X}}_u=\widetilde{\mathsf{Y}}_u=1\quad\text{a.s.},$$

and

$$\mathbb{P}(\mathsf{X}_u=x)=\mathbb{P}(\mathsf{X}_u=-x)=\tfrac{1}{2},\qquad \mathbb{P}(\mathsf{Y}_u=y)=\mathbb{P}(\mathsf{Y}_u=-y)=\tfrac{1}{2},$$

where $x>0$ and $y>0$. For $(\alpha,\beta)\in\mathbb{R}^2$ and $q=(q_1,q_2)\in\mathbb{R}^2$, the associated pressure function is given by

$$\widetilde{\mathcal{P}}_{\alpha,\beta}(q)=\log N+\log\left(\frac{e^{q_1(x-\alpha)}+e^{-q_1(x+\alpha)}}{2}\right)+\log\left(\frac{e^{q_2(y-\beta)}+e^{-q_2(y+\beta)}}{2}\right).$$

The gradient of $\widetilde{\mathcal{P}}_{\alpha,\beta}$ is given by

$$\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q)=\begin{pmatrix}\dfrac{(x-\alpha)e^{q_1(x-\alpha)}-(x+\alpha)e^{-q_1(x+\alpha)}}{e^{q_1(x-\alpha)}+e^{-q_1(x+\alpha)}}\\[2ex]\dfrac{(y-\beta)e^{q_2(y-\beta)}-(y+\beta)e^{-q_2(y+\beta)}}{e^{q_2(y-\beta)}+e^{-q_2(y+\beta)}}\end{pmatrix}.$$

Since the minimum of $\widetilde{\mathcal{P}}_{\alpha,\beta}$ is attained at the unique point $q^\star = (q_1^\star, q_2^\star)$ satisfying $\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q^\star) = 0$, we get

$$q_1^\star = \frac{1}{2x}\log\left(\frac{x+\alpha}{x-\alpha}\right), \qquad q_2^\star = \frac{1}{2y}\log\left(\frac{y+\beta}{y-\beta}\right),$$

which is well defined if and only if $\alpha \in (-x, x)$ and $\beta \in (-y, y)$. Therefore,

$$\widetilde{\mathcal{P}}^*_{\alpha,\beta}(0) \quad = \quad \inf_{q\in\mathbb{R}^2} \widetilde{\mathcal{P}}_{\alpha,\beta}(q) = \log N + h_\alpha(x) + h_\beta(y),$$

where

$$h_\beta(y) = y\, h\left(\frac{\beta}{y}\right) \text{ and } h(u) = \frac{1}{2}\Big[(1+u)\log(1+u) + (1-u)\log(1-u)\Big].$$

In particular, when $\alpha = \beta = 0$, we have $\widetilde{\mathcal{P}}^*_{0,0}(0) = \log N = \dim \partial\mathsf{T}$.

3. Assume that, for each vertex $u \in \mathsf{T}$, the random vectors $(\mathsf{X}_u, \mathsf{Y}_u, \widetilde{\mathsf{X}}_u, \widetilde{\mathsf{Y}}_u)$ are i.i.d. and given by

$$\mathsf{X}_u = a_1 + \sigma_1 G_u, \qquad \widetilde{\mathsf{X}}_u = b_1 > 0,$$
$$\mathsf{Y}_u = a_2 + \sigma_2 H_u, \qquad \widetilde{\mathsf{Y}}_u = b_2 > 0,$$

where $(G_u, H_u)$ are independent standard Gaussian random variables. Let $N \geq 2$ be deterministic. For $(\alpha, \beta) \in \mathbb{R}^2$ and $q = (q_1, q_2) \in \mathbb{R}^2$, the associated pressure function is given explicitly by

$$\widetilde{\mathcal{P}}_{\alpha,\beta}(q) = \log N + q_1(a_1 - \alpha b_1) + q_2(a_2 - \beta b_2) + \frac{1}{2}(\sigma_1^2 q_1^2 + \sigma_2^2 q_2^2).$$

A direct computation yields

$$\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q) = \begin{pmatrix} (a_1 - \alpha b_1) + \sigma_1^2 q_1 \\ (a_2 - \beta b_2) + \sigma_2^2 q_2 \end{pmatrix}.$$

Therefore, $\nabla\widetilde{\mathcal{P}}_{\alpha,\beta}(q^\star) = 0$ for

$$q_1^\star = -\frac{a_1 - \alpha b_1}{\sigma_1^2}, \qquad q_2^\star = -\frac{a_2 - \beta b_2}{\sigma_2^2}$$

and then

$$\widetilde{\mathcal{P}}^*_{\alpha,\beta}(0) = \inf_{q\in\mathbb{R}^2} \widetilde{\mathcal{P}}_{\alpha,\beta}(q) = \log N - \frac{(a_1 - \alpha b_1)^2}{2\sigma_1^2} - \frac{(a_2 - \beta b_2)^2}{2\sigma_2^2}.$$

Consequently, the admissible parameter set $\mathcal{K}$ can be written explicitly as

$$\mathcal{K} = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \frac{(a_1 - \alpha b_1)^2}{2\sigma_1^2} + \frac{(a_2 - \beta b_2)^2}{2\sigma_2^2} \leq \log N\right\},$$

which is an ellipse in $\mathbb{R}^2$. For any sequence $\boldsymbol{s}$ satisfying assumptions (6)–(13), Theorem 3 applies and yields, almost surely, for every $(\alpha, \beta) \in \mathcal{K}$,

$$\dim E_{\boldsymbol{s}}(\alpha, \beta) = Dim E_{\boldsymbol{s}}(\alpha, \beta) = \log N - \frac{(a_1 - \alpha b_1)^2}{2\sigma_1^2} - \frac{(a_2 - \beta b_2)^2}{2\sigma_2^2}.$$

## 6. Conclusions and Concluding Remarks

In this paper, we developed a multifractal framework for the analysis of ratios of branching random walks on the boundary of a Galton–Watson tree. Our approach goes beyond the classical study of level sets defined by simple convergence, by incorporating refined asymptotic behaviors and prescribed convergence speeds. This allows us to capture a richer geometric structure, which cannot be accessed through standard additive normalizations. By constructing suitable homogeneous and inhomogeneous Mandelbrot measures, we obtained almost sure formulas for the Hausdorff and packing dimensions of the associated level sets. These results extend and unify several known situations, including classical multifractal analysis of multiplicative cascades, ratios of measures, and Birkhoff averages arising in dynamical systems.

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