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Multifractal Analysis of Unstable Polynomial Entropies in Partially Hyperbolic Systems

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Abstract

We investigate the multifractal structure of unstable local polynomial entropies for arbitrary Borel probability measures in partially hyperbolic dynamical systems. We introduce the notions of Bowen unstable polynomial entropy, unstable local polynomial entropy, and a measure-theoretic polynomial entropy in the unstable direction and study their fundamental properties. Our main result establishes a multifractal relation for the level sets determined by unstable local polynomial entropies, showing that the Bowen unstable polynomial entropy of these sets is described by a Legendre-type formula involving the corresponding measure-theoretic polynomial entropy.

Keywords: multifractal analysis; Bowen unstable polynomial entropy; unstable local polynomial entropy

1. Introduction

Let (X, d, f) be a topological dynamical system, where (X, d) is a compact metric space and $f : X \to X$ is continuous. We denote by $\mathcal{M}(X)$ the collection of all Borel probability measures on X. A central theme in multifractal analysis is to describe the local geometric and dynamical behavior of measures by decomposing the phase space into subsets determined by the asymptotic values of suitable local invariants.

One of the most basic quantities in this setting is the pointwise dimension of a measure $\mu \in \mathcal{M}(X)$ at a point $x \in X$, provided the limit exists, defined as

$$d_\mu(x) = \lim_{\epsilon \to 0} \frac{\log \mu(B(x, \epsilon))}{\log \epsilon},$$

where $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. For a given $\alpha \geq 0$, the associated level set

$$X_\alpha = \{x \in X : d_\mu(x) = \alpha\}$$

captures the points at which the measure exhibits the same local scaling behavior. A fundamental problem is to evaluate the size and complexity of such sets using geometric or dynamical invariants, including Hausdorff dimension, topological entropy, and topological pressure.

A systematic approach to this problem was developed by Olsen [1,2], who introduced generalized Hausdorff dimensions $\dim_\mu^q(\cdot)$ for $q \in \mathbb{R}$ and established a multifractal formalism relating $\dim_H(X_\alpha)$ to the Legendre transform of these dimensions. This frame-

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work provides a bridge between local measure-theoretic quantities and global geometric characteristics.

In dynamical systems, the evolution under iteration of the map f plays a crucial role, and metric balls are naturally replaced by *Bowen balls*. Given $x \in X$, $\epsilon > 0$, and $n \in \mathbb{N}$, the Bowen ball of order n is defined by

$$B_n(x,\epsilon) = \{y \in X : d(f^i(x), f^i(y)) < \epsilon \text{ for all } 0 \le i \le n-1\}.$$

Using this notion, Brin and Katok [3] introduced the lower and upper local entropies of a measure μ at a point x, defined respectively by

$$\underline{h}_\mu(f,x) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x,\epsilon)), \quad \overline{h}_\mu(f,x) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x,\epsilon)).$$

When these two quantities coincide, their common value is denoted by $h_\mu(f,x)$ and referred to as the local entropy at x. For $\alpha \ge 0$, one may then consider the level set

$$K_\alpha(\mu) = \{x \in X : h_\mu(f,x) = \alpha\}.$$

Building on ideas from multifractal geometry and dimension theory, Takens and Verbitski [4] developed a general multifractal framework for local entropies by introducing a suitable entropy function associated with a measure. Within this approach, they derived a relation between the Bowen topological entropy [5] of the level sets $K_\alpha(\mu)$ and the corresponding multifractal spectrum. Related multifractal phenomena for dynamical quantities have since been investigated from different perspectives; see, for instance, the work of Yan and Chen [6] on recurrence times.

Polynomial entropy provides an alternative measure of dynamical complexity by capturing subexponential growth rates. This concept was introduced by Marco [7] and further developed in various contexts [8–21]. For partially hyperbolic systems, the dynamics exhibits distinct behaviors along stable, center, and unstable directions. To capture this anisotropic structure, Hu, Hua and Wu [22] introduced unstable metric entropy and unstable topological entropy and established a corresponding variational principle. Subsequently, Tian and Wu [23,24] developed Bowen unstable topological entropy and unstable topological pressure for general subsets using a Carathéodory–Pesin construction. Further extensions include unstable measure-theoretic pressure for sub-additive potentials [25] and nonlinear unstable pressure on subsets [26].

Motivated by these developments, the present paper investigates multifractal properties associated with polynomial entropy in the unstable direction. We introduce the notions of Bowen unstable polynomial entropy, unstable local polynomial entropy, and unstable polynomial entropy associated with a measure. Our main results establish a multifractal description of the level sets determined by unstable local polynomial entropies and provide a precise relation between their Bowen unstable polynomial entropy and the corresponding multifractal spectrum. These results extend classical multifractal theory to a polynomial growth framework in partially hyperbolic dynamics.

To illustrate the potential applicability of these concepts, we note that evaluating polynomial entropy metrics can inform real-world systems, including the following:

- Control input bounds for stabilization of parabolic PDE–ODE systems [27];
- Modeling of vortex-induced vibrations in bridge decks [28];
- Stochastic control in mean-field jump-diffusion systems [29];
- Predictive modeling of complex aerodynamic trajectories via time–frequency analysis [30].

These examples highlight that the proposed multifractal and polynomial entropy framework is not only of theoretical interest but also has potential applications in modern applied mathematics, engineering, and physics.

2. Preliminaries

Throughout this paper, $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all integers and positive integers, respectively, and we write $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\mathcal{C}_X$ denote the family of all finite open covers of X. For two covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, we say that $\mathcal{V}$ refines $\mathcal{U}$, written $\mathcal{U} \preceq \mathcal{V}$, if each element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$. The join of $\mathcal{U}$ and $\mathcal{V}$ is defined by

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U},\, V \in \mathcal{V}\},$$

which is a common refinement of both $\mathcal{U}$ and $\mathcal{V}$.

Given $n \in \mathbb{N}$ and a cover $\mathcal{U} \in \mathcal{C}_X$, we define the iterated join

$$\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{U}) = \{U_{i_0} \cap f^{-1}U_{i_1} \cap \cdots \cap f^{-(n-1)}U_{i_{n-1}} : U_{i_j} \in \mathcal{U}\}.$$

For $n \in \mathbb{N}$ and $\epsilon > 0$, the dynamical metric d_n on X is given by

$$d_n(x,y) = \max_{0 \le i < n} d(f^i(x), f^i(y)),$$

and the associated Bowen ball is

$$B_n(x,\epsilon) = \{y \in X : d_n(x,y) < \epsilon\}.$$

If $\mathcal{U}$ is a finite open cover of X, we write $\operatorname{diam}(\mathcal{U}) = \max\{\operatorname{diam}(U) : U \in \mathcal{U}\}$.

For $n \geq 1$, let $\mathcal{W}_n(\mathcal{U})$ denote the collection of all finite strings $\mathbf{U} = U_{i_0} \cdots U_{i_{n-1}}$ with symbols taken from $\mathcal{U}$. The length of $\mathbf{U}$ is denoted by $m(\mathbf{U}) = n$, and we set $\mathcal{W}(\mathcal{U}) = \bigcup_{n \geq 1} \mathcal{W}_n(\mathcal{U})$. To each string $\mathbf{U} \in \mathcal{W}(\mathcal{U})$, we associate the set

$$X(\mathbf{U}) = \{x \in X : f^j(x) \in U_{i_j},\ j = 0, 1, \ldots, m(\mathbf{U}) - 1\}.$$

Let $Z \subset X$ be nonempty and let $s \in \mathbb{R}$. For $N \in \mathbb{N}$, define

$$\mathcal{M}(Z, s, \mathcal{U}, N) = \inf_{\mathcal{G}} \sum_{\mathbf{U} \in \mathcal{G}} e^{-sm(\mathbf{U})},$$

where the infimum is taken over all collections $\mathcal{G} \subset \bigcup_{j \geq N} \mathcal{W}_j(\mathcal{U})$ satisfying $\bigcup_{\mathbf{U} \in \mathcal{G}} X(\mathbf{U}) \supset Z$. Since $\mathcal{M}(Z, s, \mathcal{U}, N)$ is nondecreasing in N, we define

$$\mathcal{M}(Z, s, \mathcal{U}) = \lim_{N \to \infty} \mathcal{M}(Z, s, \mathcal{U}, N),$$

and set

$$h_{\mathrm{B}}(f, Z, \mathcal{U}) = \inf\{s : \mathcal{M}(Z, s, \mathcal{U}) = 0\} = \sup\{s : \mathcal{M}(Z, s, \mathcal{U}) = \infty\}.$$

Definition 1. *The Bowen topological entropy of a set $Z \subset X$ is defined by*

$$h_{\mathrm{B}}(f, Z) = \sup_{\mathcal{U} \in \mathcal{C}_X} h_{\mathrm{B}}(f, Z, \mathcal{U}).$$

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2.1. Unstable Topological Entropy of Subsets

In the remainder of this paper, we focus on partially hyperbolic dynamics. Let M be a compact, connected, boundaryless Riemannian manifold and $f : M \to M$ a C^1 diffeomorphism. The map f is said to be *partially hyperbolic* if the tangent bundle admits a Df-invariant splitting

$$TM = E^s \oplus E^c \oplus E^u,$$

such that vectors in these subbundles exhibit uniform contraction, intermediate behavior, and uniform expansion, respectively. More precisely, with respect to a suitable Riemannian metric, vectors $v^\sigma \in E^\sigma(x)$ ($\sigma = s, c, u$) satisfy

$$\|D_x f v^s\| < \|D_x f v^c\| < \|D_x f v^u\|,$$

together with uniform contraction on E^s and uniform expansion on E^u. The bundles E^s and E^u are integrable, giving rise to the stable and unstable foliations W^s and W^u. We refer to [31] for further background.

Let $\mathcal{M}(M)$ denote the space of Borel probability measures on M, and $\mathcal{M}_f(M)$ the subset of f-invariant measures. The Riemannian structure induces a metric d^u on unstable manifolds. For $\delta > 0$, denote by $W^u(x,\delta)$ the open d^u-ball in $W^u(x)$ centered at x with radius δ. For sufficiently small δ, the metrics d^u and d are equivalent on $\overline{W^u(x,\delta)}$.

Tian and Wu [23] introduced unstable analogues of Bowen topological entropy for arbitrary subsets.

Definition 2 ([23])**.** *The Bowen unstable topological entropy of $Z \subset M$ is defined by*

$$h^u_{\mathrm{B}}(f,Z) = \lim_{\delta\to 0} \sup_{x\in M} h_{\mathrm{B}}(f, \overline{W^u(x,\delta)} \cap Z).$$

An equivalent formulation can be given using coverings by unstable Bowen balls. For $y \in W^u(x)$, define

$$d^u_n(x,y) = \max_{0\le j\le n-1} d^u(f^j(x), f^j(y)),$$

and let

$$B^u_n(x,\epsilon) = \{y \in W^u(x) : d^u_n(x,y) \le \epsilon\}.$$

For $Z \subset \overline{W^u(x,\delta)}$, $s \ge 0$, $N \in \mathbb{N}$ and $\epsilon > 0$, set

$$\mathcal{M}(Z,s,\epsilon,N) = \inf \sum_i e^{-sn_i},$$

where the infimum is taken over all families $\{B^u_{n_i}(x_i,\epsilon)\}$ with $x_i \in \overline{W^u(x,\delta)}$, $n_i \ge N$, and whose union covers Z. Taking limits as $N \to \infty$ and $\epsilon \to 0$ yields a critical value $\bar{h}_{\mathrm{B}}(f,Z)$.

Proposition 1 ([23])**.** *For any $Y \subset \overline{W^u(x,\delta)}$, one has $\bar{h}_{\mathrm{B}}(f,Y) = h_{\mathrm{B}}(f,Y)$. Consequently,*

$$h^u_{\mathrm{B}}(f,Z) = \lim_{\delta\to 0} \sup_{x\in M} \bar{h}_{\mathrm{B}}(f, \overline{W^u(x,\delta)} \cap Z).$$

The unstable entropy can also be characterized using separated and spanning sets. For $\epsilon > 0$ and $n \in \mathbb{N}$, a subset $F \subset \overline{W^u(x,\delta)} \cap Z$ is called (n,ϵ) u-separated if $d^u_n(x,y) > \epsilon$ for all distinct $x, y \in F$. Similarly, a set $E \subset W^u(x)$ is called (n,ϵ) u-spanning if

$$\overline{W^u(x,\delta)} \cap Z \subset \bigcup_{y\in E} B^u_n(y,\epsilon).$$

Let $r^u(Z,\epsilon,n,x,\delta)$ and $s^u(Z,\epsilon,n,x,\delta)$ denote the maximal and minimal cardinalities of such sets, respectively. These quantities satisfy

$$r^u(Z,2\epsilon,n,x,\delta) \le s^u(Z,\epsilon,n,x,\delta) \le r^u(Z,\epsilon,n,x,\delta).$$

Theorem 1 ([23])**.** *For any $Z \subset M$ and all sufficiently small $\delta > 0$,*

$$h^u_{\mathrm{B}}(f,Z) = \sup_{x\in M} h_{\mathrm{B}}(f,\overline{W^u(x,\delta)} \cap Z).$$

2.2. Bowen Unstable Polynomial Entropy

In this subsection, we introduce a polynomial version of unstable entropy for subsets of partially hyperbolic systems.

Let $Z \subset M$ be a nonempty subset and let $s \in \mathbb{R}$. For a finite open cover $\mathcal{U}$ of M and $N \in \mathbb{N}$, define

$$\mathcal{M}^{\mathrm{pol}}(Z,s,\mathcal{U},N) = \inf_{\mathcal{G}} \sum_{\mathbf{U}\in\mathcal{G}} \left(\frac{1}{m(\mathbf{U})}\right)^s,$$

where the infimum is taken over all collections $\mathcal{G} \subset \bigcup_{j\ge N} \mathcal{W}_j(\mathcal{U})$ such that $\bigcup_{\mathbf{U}\in\mathcal{G}} X(\mathbf{U}) \supset Z$. It is straightforward to verify that $\mathcal{M}^{\mathrm{pol}}(\cdot,s,\mathcal{U},N)$ defines a finite outer measure on X. Moreover, for fixed Z, s and $\mathcal{U}$, the quantity $\mathcal{M}^{\mathrm{pol}}(Z,s,\mathcal{U},N)$ is monotone nondecreasing in N. Consequently, the limit

$$\mathcal{M}^{\mathrm{pol}}(Z,s,\mathcal{U}) = \lim_{N\to\infty} \mathcal{M}^{\mathrm{pol}}(Z,s,\mathcal{U},N)$$

exists.

We associate with this construction the critical value

$$h^{\mathrm{pol}}_{\mathrm{B}}(f,Z,\mathcal{U}) = \inf\{s : \mathcal{M}^{\mathrm{pol}}(Z,s,\mathcal{U}) = 0\} = \sup\{s : \mathcal{M}^{\mathrm{pol}}(Z,s,\mathcal{U}) = \infty\}.$$

Definition 3. *The Bowen polynomial entropy of Z is defined by*

$$h^{\mathrm{pol}}_{\mathrm{B}}(f,Z) = \sup_{\mathcal{U}} h^{\mathrm{pol}}_{\mathrm{B}}(f,Z,\mathcal{U}),$$

where the supremum is taken over all finite open covers of M.

Using an argument analogous to the one developed in [32], one can show that $h^{\mathrm{pol}}_{\mathrm{B}}(f,Z,\mathcal{U})$ stabilizes as the mesh of the cover tends to zero. This leads to the equivalent formulation below.

Definition 4. *Let $f : M \to M$ be a C^1 partially hyperbolic diffeomorphism. Then*

$$h^{\mathrm{pol}}_{\mathrm{B}}(f,Z) = \lim_{\mathrm{diam}(\mathcal{U})\to 0} h^{\mathrm{pol}}_{\mathrm{B}}(f,Z,\mathcal{U}).$$

Motivated by the definition of unstable topological entropy in [23], we now introduce its polynomial counterpart.

Definition 5. *The Bowen unstable polynomial entropy of $Z \subset M$ is defined by*

$$h^{\mathrm{pol},u}_{\mathrm{B}}(f,Z) = \lim_{\delta\to 0} \sup_{x\in M} h^{\mathrm{pol}}_{\mathrm{B}}(f,\overline{W^u(x,\delta)} \cap Z).$$

An equivalent description can be given in terms of coverings by unstable Bowen balls. Let $Z \subset \overline{W^u(x,\delta)}$, $s \geq 0$, $N \in \mathbb{N}$ and $\epsilon > 0$. Define

$$\mathcal{M}^{\mathrm{pol}}(Z,s,\epsilon,N) = \inf \sum_i \left(\frac{1}{n_i}\right)^s,$$

where the infimum is taken over all finite or countable families $\{B^u_{n_i}(x_i,\epsilon)\}$ satisfying $x_i \in \overline{W^u(x,\delta)}$, $n_i \geq N$, and $\bigcup_i B^u_{n_i}(x_i,\epsilon) \supset Z$. As before, monotonicity in N allows us to define

$$\mathcal{M}^{\mathrm{pol}}(Z,s,\epsilon) = \lim_{N\to\infty} \mathcal{M}^{\mathrm{pol}}(Z,s,\epsilon,N), \qquad \mathcal{M}^{\mathrm{pol}}(Z,s) = \lim_{\epsilon\to 0} \mathcal{M}^{\mathrm{pol}}(Z,s,\epsilon).$$

The associated critical exponent is denoted by

$$\bar{h}^{\mathrm{pol}}_{\mathrm{B}}(f,Z) = \inf\{s : \mathcal{M}^{\mathrm{pol}}(Z,s) = 0\} = \sup\{s : \mathcal{M}^{\mathrm{pol}}(Z,s) = \infty\}.$$

Proposition 2. *For any* $Y \subset \overline{W^u(x,\delta)}$,

$$\bar{h}^{\mathrm{pol}}_{\mathrm{B}}(f,Y) = h^{\mathrm{pol}}_{\mathrm{B}}(f,Y).$$

Consequently, for every $Z \subset M$,

$$h^{\mathrm{pol},u}_{\mathrm{B}}(f,Z) = \lim_{\delta\to 0} \sup_{x\in M} \bar{h}^{\mathrm{pol}}_{\mathrm{B}}(f,\overline{W^u(x,\delta)} \cap Z).$$

Proof. Since M is compact and the unstable foliation depends continuously on the base point, there exists $C > 1$ such that, for all sufficiently small δ and all $y,z \in \overline{W^u(x,\delta)}$,

$$d(y,z) \leq d^u(y,z) \leq Cd(y,z).$$

Let $\mathcal{U}$ be a finite open cover of M and let $\delta(\mathcal{U})$ denote its Lebesgue number. If $x \in X(\mathbf{U})$ for some $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$, then

$$B^u_n\Big(x, \tfrac{1}{2}\delta(\mathcal{U})\Big) \subset X(\mathbf{U}) \subset B^u_n(x, 2C\,\mathrm{diam}(\mathcal{U})).$$

From this inclusion one easily derives, for any $s \geq 0$,

$$\mathcal{M}^{\mathrm{pol}}(Y,s,2C\,\mathrm{diam}(\mathcal{U}),N) \leq \mathcal{M}^{\mathrm{pol}}(Y,s,\mathcal{U},N) \leq \mathcal{M}^{\mathrm{pol}}(Y,s,\tfrac{1}{2}\delta(\mathcal{U}),N).$$

Passing to the limits $N \to \infty$ and $\mathrm{diam}(\mathcal{U}) \to 0$ yields

$$\bar{h}^{\mathrm{pol}}_{\mathrm{B}}(f,Y) \leq h^{\mathrm{pol}}_{\mathrm{B}}(f,Y) \leq \bar{h}^{\mathrm{pol}}_{\mathrm{B}}(f,Y),$$

which proves the claim. □

The next lemma shows that the limit $\delta \to 0$ in Definition 5 is in fact unnecessary.

Lemma 1. *Let* $f : M \to M$ *be a* C^1 *partially hyperbolic diffeomorphism and* $Z \subset M$*. Then, for all sufficiently small* $\delta > 0$,

$$h^{\mathrm{pol},u}_{\mathrm{B}}(f,Z) = \sup_{x\in M} h^{\mathrm{pol}}_{\mathrm{B}}(f,\overline{W^u(x,\delta)} \cap Z).$$

The quantity $h_{\mathrm{B}}^{\mathrm{pol},u}(f,\cdot)$ shares many structural properties with Bowen entropy and with Hausdorff-type dimensions. Using standard arguments as in [32], one obtains the following basic properties.

Proposition 3. *Let $f : M \to M$ be a C^1 partially hyperbolic diffeomorphism. Then*

(1) $h_{\mathrm{B}}^{\mathrm{pol},u}(f,\emptyset) \le 0$*;*

(2) *if $Z_1 \subset Z_2 \subset M$, then $h_{\mathrm{B}}^{\mathrm{pol},u}(f,Z_1) \le h_{\mathrm{B}}^{\mathrm{pol},u}(f,Z_2)$;*

(3) *for any sequence $\{Z_i\}_{i\ge1}$ of subsets of M,*

$$h_{\mathrm{B}}^{\mathrm{pol},u}\left(f,\bigcup_{i=1}^{\infty} Z_i\right) = \sup_i h_{\mathrm{B}}^{\mathrm{pol},u}(f,Z_i).$$

2.3. Lower Local Polynomial Entropy

Let $\mu \in \mathcal{M}(M)$. Inspired by the work of Brin and Katok [3], we define the lower local polynomial entropy by

$$\underline{h}_{\mu}^{\mathrm{pol}}(f,x) := \lim_{\epsilon\to0}\liminf_{n\to\infty} -\frac{1}{\log n}\log\mu(B_n(x,\epsilon)),$$

and the upper local polynomial entropy by

$$\overline{h}_{\mu}^{\mathrm{pol}}(f,x) := \lim_{\epsilon\to0}\limsup_{n\to\infty} -\frac{1}{\log n}\log\mu(B_n(x,\epsilon)).$$

We say that the local polynomial entropy exists at x if

$$\underline{h}_{\mu}^{\mathrm{pol}}(f,x) = \overline{h}_{\mu}^{\mathrm{pol}}(f,x).$$

In this case, the common value is denoted by $h_{\mu}^{\mathrm{pol}}(f,x)$.

Moreover, for $\alpha \ge 0$, we define the level set of local polynomial entropy by

$$K_{\alpha}^{\mathrm{pol}}(\mu) = \{x \in M : h_{\mu}^{\mathrm{pol}}(f,x) = \alpha\}.$$

In this paper, we focus on local polynomial entropies and the associated multifractal spectra defined via Bowen balls. More precisely, we study the size of the level sets $K_{\alpha}^{\mathrm{pol}}(\mu)$.

2.4. (q,μ)-Entropy

Following Takens and Verbitski [4], which is based on ideas of Olsen [1] and the formalism developed in [32], we recall the definition of (q,μ)-entropy.

Let μ be a non-atomic Borel probability measure on M. Without loss of generality, we assume that μ assigns positive measure to every nonempty open set. For any at the most countable collection $\mathcal{G} = \{B_{n_i}(x_i,\epsilon)\}$ and any $q,s \in \mathbb{R}$, define the (q,s)-free energy of $\mathcal{G}$ by

$$F_{\mu}(\mathcal{G},q,s) := \sum_i \mu(B_{n_i}(x_i,\epsilon))^q \exp(-sn_i).$$

For a nonempty set $Z \subseteq M$, $q,s \in \mathbb{R}$, $\epsilon > 0$, and $N \in \mathbb{N}$, define

$$\mathcal{M}_{\mu}^{c}(Z,q,s,\epsilon,N) := \inf_{\mathcal{G}} F_{\mu}(\mathcal{G},q,s),$$

where the infimum is taken over all finite or countable collections $\mathcal{G} = \{B_{n_i}(x_i, \epsilon)\}$ with $x_i \in Z$ and $n_i \geq N$ such that $Z \subseteq \bigcup_i B_{n_i}(x_i, \epsilon)$. We also set

$$\mathcal{M}_{\mu}^{c}(\emptyset, q, s, \epsilon, N) = 0$$

for all q, s, ϵ, and N.

Since $\mathcal{M}_{\mu}^{c}(Z, q, s, \epsilon, N)$ is nondecreasing in N, the following limit exists:

$$\mathcal{M}_{\mu}^{c}(Z, q, s, \epsilon) = \lim_{N \to \infty} \mathcal{M}_{\mu}^{c}(Z, q, s, \epsilon, N).$$

As $\mathcal{M}_{\mu}^{c}(Z, q, s, \epsilon)$ is not necessarily monotone with respect to Z, we define

$$\mathcal{M}_{\mu}(Z, q, s, \epsilon) := \sup_{Z' \subset Z} \mathcal{M}_{\mu}^{c}(Z', q, s, \epsilon).$$

There exists a critical value $h_{\mu}(f, q, Z, \epsilon) \in [-\infty, \infty]$ such that

$$\mathcal{M}_{\mu}(Z, q, s, \epsilon) = \begin{cases} 0, & s > h_{\mu}(f, q, Z, \epsilon), \\ \infty, & s < h_{\mu}(f, q, Z, \epsilon). \end{cases}$$

Definition 6 ([4]). *The (q, μ)-entropy of Z is defined by*

$$h_{\mu}(f, q, Z) = \limsup_{\epsilon \to 0} h_{\mu}(f, q, Z, \epsilon).$$

3. Unstable (q, μ)-Polynomial Entropy of Noncompact Sets

In this section, we extend the notion of (q, μ)-entropy introduced in [4] to the polynomial growth setting, define the unstable (q, μ)-polynomial entropy, and investigate its basic properties.

3.1. Definition of Unstable (q, μ)-Polynomial Entropy

We first introduce the unstable (q, s)-polynomial free energy. Let μ be a non-atomic Borel probability measure on M. Without loss of generality, we assume that μ assigns a positive measure to every nonempty open set. For any finite or countable collection $\mathcal{G} = \{B_{n_i}^{u}(x_i, \epsilon)\}_i$ and any $q, s \in \mathbb{R}$, define the unstable (q, s)-polynomial free energy of $\mathcal{G}$ by

$$F_{\mu}^{\mathrm{pol},u}(\mathcal{G}, q, s) := \sum_i \mu(B_{n_i}^{u}(x_i, \epsilon))^q \left(\frac{1}{n_i}\right)^s.$$

Given a nonempty set $Z \subseteq M$, $q, s \in \mathbb{R}$, $\epsilon > 0$, and $N \in \mathbb{N}$, define

$$\mathcal{M}_{\mu,c}^{\mathrm{pol},u}(Z, q, s, \epsilon, N) := \inf_{\mathcal{G}} F_{\mu}^{\mathrm{pol},u}(\mathcal{G}, q, s),$$

where the infimum is taken over all finite or countable collections $\mathcal{G} = \{B_{n_i}^{u}(x_i, \epsilon)\}_i$ with $x_i \in Z$ and $n_i \geq N$ such that $Z \subseteq \bigcup_i B_{n_i}^{u}(x_i, \epsilon)$. We further set

$$\mathcal{M}_{\mu,c}^{\mathrm{pol},u}(\emptyset, q, s, \epsilon, N) = 0$$

for all q, s, ϵ and N.

Since $\mathcal{M}_{\mu,c}^{\mathrm{pol},u}(Z, q, s, \epsilon, N)$ is nondecreasing in N, the following limit exists:

$$\mathcal{M}_{\mu,c}^{\mathrm{pol},u}(Z, q, s, \epsilon) := \lim_{N \to \infty} \mathcal{M}_{\mu,c}^{\mathrm{pol},u}(Z, q, s, \epsilon, N).$$

As we restrict to covers whose centers belong to the set Z, the quantity $\mathcal{M}_{\mu,c}^{\mathrm{pol},u}(Z,q,s,\epsilon)$ is not necessarily monotone with respect to Z. To restore monotonicity, we define

$$\mathcal{M}_{\mu}^{\mathrm{pol},u}(Z,q,s,\epsilon) := \sup_{Z'\subset Z} \mathcal{M}_{\mu,c}^{\mathrm{pol},u}(Z',q,s,\epsilon).$$

We now list some basic properties of the set function $\mathcal{M}_{\mu}^{\mathrm{pol},u}(Z,q,s,\epsilon)$.

Proposition 4. *For any $q,s \in \mathbb{R}$, the following properties hold:*

(1) $\mathcal{M}_{\mu}^{\mathrm{pol},u}(\emptyset,q,s,\epsilon) = 0$;

(2) *if $Z_1 \subset Z_2$, then* $\mathcal{M}_{\mu}^{\mathrm{pol},u}(Z_1,q,s,\epsilon) \le \mathcal{M}_{\mu}^{\mathrm{pol},u}(Z_2,q,s,\epsilon)$;

(3) *for any sequence $\{Z_i\}_{i=1}^{\infty} \subset M$,*

$$\mathcal{M}_{\mu}^{\mathrm{pol},u}\left(\bigcup_{i=1}^{\infty} Z_i,q,s,\epsilon\right) \le \sum_{i=1}^{\infty} \mathcal{M}_{\mu}^{\mathrm{pol},u}(Z_i,q,s,\epsilon).$$

The following facts, which will be used later, are straightforward to verify, and we omit their proofs.

Fact 1. *The set function $\mathcal{M}_{\mu}^{\mathrm{pol},u}(\cdot,q,s,\epsilon)$ defines an outer measure on M.*

Fact 2.

- *If $\mathcal{M}_{\mu}^{\mathrm{pol},u}(Z,q,t,\epsilon) < \infty$ and $s > t$, then $\mathcal{M}_{\mu}^{\mathrm{pol},u}(Z,q,s,\epsilon) = 0$.*
- *If $\mathcal{M}_{\mu}^{\mathrm{pol},u}(Z,q,t,\epsilon) > 0$ and $s < t$, then $\mathcal{M}_{\mu}^{\mathrm{pol},u}(Z,q,s,\epsilon) = \infty$.*

As a consequence, there exists a critical value $h_{\mu}^{\mathrm{pol},u}(f,q,Z,\epsilon) \in [-\infty,\infty]$ such that

$$\mathcal{M}_{\mu}^{\mathrm{pol},u}(Z,q,s,\epsilon) = \begin{cases} 0, & s > h_{\mu}^{\mathrm{pol},u}(f,q,Z,\epsilon), \\ \infty, & s < h_{\mu}^{\mathrm{pol},u}(f,q,Z,\epsilon). \end{cases}$$

Definition 7. *The unstable (q,μ)-polynomial entropy of Z is defined by*

$$h_{\mu}^{\mathrm{pol},u}(f,q,Z) := \lim_{\delta\to 0} \sup_{x\in M} h_{\mu}^{\mathrm{pol},u}(f,q,\overline{W^u(x,\delta)} \cap Z),$$

where

$$h_{\mu}^{\mathrm{pol},u}(f,q,\overline{W^u(x,\delta)} \cap Z) = \limsup_{\epsilon\to 0} h_{\mu}^{\mathrm{pol},u}(f,q,\overline{W^u(x,\delta)} \cap Z,\epsilon).$$

3.2. Properties of the Unstable (q,μ)-Polynomial Entropy

The unstable (q,μ)-polynomial entropy $h_{\mu}^{\mathrm{pol},u}(f,q,Z,\epsilon)$ and $h_{\mu}^{\mathrm{pol},u}(f,q,Z)$ share many structural properties with the classical (q,μ)-entropy introduced by Takens and Verbitski [4]. Since the arguments are analogous, we state these properties below and omit the detailed proofs.

Proposition 5. *Let $f: M \to M$ be a C^1 partially hyperbolic diffeomorphism. Then for any $q \in \mathbb{R}$ and $\epsilon > 0$ the following hold:*

(1) $h_{\mu}^{\mathrm{pol},u}(f,q,\emptyset,\epsilon) = -\infty$;

(2) *if $Z_1 \subseteq Z_2$, then*

$$h_{\mu}^{\mathrm{pol},u}(f,q,Z_1,\epsilon) \le h_{\mu}^{\mathrm{pol},u}(f,q,Z_2,\epsilon);$$

(3) *for any sequence* $\{Z_i\}_{i=1}^{\infty} \subset M$*,*

$$h_{\mu}^{\mathrm{pol},u}\left(f, q, \bigcup_{i=1}^{\infty} Z_i, \epsilon\right) = \sup_{i \geq 1} h_{\mu}^{\mathrm{pol},u}(f, q, Z_i, \epsilon).$$

Combining Definition 7 with Proposition 5, we immediately obtain the following properties of $h_{\mu}^{\mathrm{pol},u}(f, q, Z)$.

Proposition 6. *Let* $f : M \to M$ *be a* C^1 *partially hyperbolic diffeomorphism. Then for any* $q \in \mathbb{R}$ *the following statements hold:*

(1) $h_{\mu}^{\mathrm{pol},u}(f, q, \emptyset) = -\infty$*;*

(2) *if* $Z_1 \subseteq Z_2$*, then*

$$h_{\mu}^{\mathrm{pol},u}(f, q, Z_1) \leq h_{\mu}^{\mathrm{pol},u}(f, q, Z_2);$$

(3) *for any sequence* $\{Z_i\}_{i=1}^{\infty} \subset M$*,*

$$h_{\mu}^{\mathrm{pol},u}\left(f, q, \bigcup_{i=1}^{\infty} Z_i\right) = \sup_{i \geq 1} h_{\mu}^{\mathrm{pol},u}(f, q, Z_i).$$

The following lemma shows that the limit $\delta \to 0$ in Definition 7 is not essential, which is analogous to Lemma 1.

Lemma 2. *Let* $f : M \to M$ *be a* C^1 *partially hyperbolic diffeomorphism and let* $Z \subset M$*. Then for any* $\delta > 0$*,*

$$h_{\mu}^{\mathrm{pol},u}(f, q, Z) = \sup_{x \in M} h_{\mu}^{\mathrm{pol},u}\left(f, q, \overline{W^u(x, \delta)} \cap Z\right).$$

Proof. The inequality

$$h_{\mu}^{\mathrm{pol},u}(f, q, Z) \leq \sup_{x \in M} h_{\mu}^{\mathrm{pol},u}\left(f, q, \overline{W^u(x, \delta)} \cap Z\right)$$

follows directly from Definition 7. We now prove the reverse inequality.

Fix $\delta > 0$ and let $\varepsilon > 0$. Choose $y \in M$ such that

$$\sup_{x \in M} h_{\mu}^{\mathrm{pol},u}\left(f, q, \overline{W^u(x, \delta)} \cap Z\right) \leq h_{\mu}^{\mathrm{pol},u}\left(f, q, \overline{W^u(y, \delta)} \cap Z\right) + \frac{\varepsilon}{2}.$$

Next, choose $\delta_1 \in (0, \delta)$ such that

$$h_{\mu}^{\mathrm{pol},u}(f, q, Z) \geq \sup_{x \in M} h_{\mu}^{\mathrm{pol},u}\left(f, q, \overline{W^u(x, \delta_1)} \cap Z\right) - \frac{\varepsilon}{2}.$$

Since $\overline{W^u(y, \delta)}$ is compact, there exist finitely many points $y_1, \ldots, y_k \in \overline{W^u(y, \delta)}$ (depending only on δ, δ_1, and the Riemannian metric) such that

$$\overline{W^u(y, \delta)} \subseteq \bigcup_{i=1}^{k} \overline{W^u(y_i, \delta_1)}.$$

Consequently,

$$\begin{aligned}\sup_{x\in M} h_{\mu}^{\mathrm{pol},u}\Big(f,q,\overline{W^u(x,\delta)}\cap Z\Big) &\le \max_{1\le i\le k} h_{\mu}^{\mathrm{pol},u}\Big(f,q,\overline{W^u(y_i,\delta_1)}\cap Z\Big)+\frac{\varepsilon}{2}\\ &\le \sup_{x\in M} h_{\mu}^{\mathrm{pol},u}\Big(f,q,\overline{W^u(x,\delta_1)}\cap Z\Big)+\frac{\varepsilon}{2}\\ &\le h_{\mu}^{\mathrm{pol},u}(f,q,Z)+\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$ completes the proof. □

4. Main Results

Let $\mathcal{U}$ be a finite open cover of a compact metric space (M,d), and denote by $\delta(\mathcal{U})$ a Lebesgue number of $\mathcal{U}$. We begin with two elementary facts.

Fact 3. *If $\epsilon_1<\dfrac{\delta(\mathcal{U})}{2}$, then for every $x\in M$ and each $n\in\mathbb{N}$ there exists a string $\mathbf{U}_x=U_{i_1}U_{i_2}\cdots U_{i_n}$ of length n such that*

$$B_n^u(x,\epsilon_1)\subseteq X(\mathbf{U}_x).$$

Fact 4. *For any string $\mathbf{U}=U_{i_1}U_{i_2}\cdots U_{i_n}$ and any $x\in X(\mathbf{U})$, we have*

$$X(\mathbf{U})\subseteq B^u_{m(\mathbf{U})}(x,\epsilon_2)$$

for every $\epsilon_2>2\,\mathrm{diam}(\mathcal{U})$.

4.1. Coincidence of the Bowen Unstable Polynomial Entropy and the Unstable $(0,\mu)$-Polynomial Entropy

When $q=0$, the measure μ does not enter the definition of the unstable (q,μ)-polynomial entropy. This observation leads to the following result.

Theorem 2. *Let $f:M\to M$ be a C^1 partially hyperbolic diffeomorphism, and let $\mu\in\mathcal{M}(M)$ be non-atomic and positive on every nonempty open set. Then for any subset $Z\subset M$,*

$$h_{\mu}^{\mathrm{pol},u}(f,0,Z)=h_{\mathrm{B}}^{\mathrm{pol},u}(f,Z).$$

Proof. We may assume $Z\neq\emptyset$. We first show that

$$h_{\mathrm{B}}^{\mathrm{pol},u}(f,Z)\le h_{\mu}^{\mathrm{pol},u}(f,0,Z).$$

Fix $\delta>0$, $x\in M$, and a finite open cover $\mathcal{U}$ of M. Let $0<\epsilon<\delta(\mathcal{U})/2$. Consider an arbitrary centered cover

$$\mathcal{G}=\{B^u_{n_i}(x_i,\epsilon)\}_i$$

of $\overline{W^u(x,\delta)}\cap Z$ with $x_i\in\overline{W^u(x,\delta)}\cap Z$ and $n_i>N$. By Fact 3, for each i there exists a string $\mathbf{U}_i$ with $m(\mathbf{U}_i)=n_i$ such that

$$B^u_{n_i}(x_i,\epsilon)\subseteq X(\mathbf{U}_i).$$

Let $\Gamma_{\mathcal{G}}=\{\mathbf{U}_i\}$. Then $\Gamma_{\mathcal{G}}$ covers $\overline{W^u(x,\delta)}\cap Z$ and

$$F_{\mu}^{\mathrm{pol},u}(\mathcal{G},0,s)=\sum_{\mathbf{U}_i\in\Gamma_{\mathcal{G}}}\left(\frac{1}{m(\mathbf{U}_i)}\right)^{s}.$$

Since $\mathcal{G}$ is arbitrary,

$$\mathcal{M}_{\mu,c}^{\mathrm{pol},u}(\overline{W^u(x,\delta)}\cap Z,0,s,\epsilon,N)\geq\mathcal{M}^{\mathrm{pol}}(\overline{W^u(x,\delta)}\cap Z,\mathcal{U},s,N).$$

Letting $N\to\infty$ yields

$$\mathcal{M}^{\mathrm{pol}}(\overline{W^u(x,\delta)}\cap Z,\mathcal{U},s)\leq\mathcal{M}_{\mu}^{\mathrm{pol},u}(\overline{W^u(x,\delta)}\cap Z,0,s,\epsilon).$$

Consequently,

$$h_{\mathrm{B}}^{\mathrm{pol}}(f,\overline{W^u(x,\delta)}\cap Z,\mathcal{U})\leq h_{\mu}^{\mathrm{pol},u}(f,0,\overline{W^u(x,\delta)}\cap Z,\epsilon).$$

Letting $\epsilon\to 0$ and then taking the supremum over $x\in M$ and the limit $\delta\to 0$, we obtain

$$h_{\mathrm{B}}^{\mathrm{pol},u}(f,Z)\leq h_{\mu}^{\mathrm{pol},u}(f,0,Z).$$

We now prove the reverse inequality. Assume that there exists $\gamma>0$ such that

$$h_{\mu}^{\mathrm{pol},u}(f,0,Z)-h_{\mathrm{B}}^{\mathrm{pol},u}(f,Z)>3\gamma.$$

Then there exist $\delta>0$ and $x\in M$ satisfying

$$h_{\mu}^{\mathrm{pol},u}(f,0,\overline{W^u(x,\delta)}\cap Z)-h_{\mathrm{B}}^{\mathrm{pol},u}(f,\overline{W^u(x,\delta)}\cap Z)>\tfrac{5}{2}\gamma.$$

Choose $\epsilon_0>0$ such that

$$h_{\mu}^{\mathrm{pol},u}(f,0,\overline{W^u(x,\delta)}\cap Z,\epsilon_0)-h_{\mathrm{B}}^{\mathrm{pol},u}(f,\overline{W^u(x,\delta)}\cap Z)>2\gamma.$$

Let $\mathcal{U}$ be a finite open cover with $\mathrm{diam}(\mathcal{U})<\epsilon_0/2$ and

$$h_{\mu}^{\mathrm{pol},u}(f,0,\overline{W^u(x,\delta)}\cap Z,\epsilon_0)-h_{\mathrm{B}}^{\mathrm{pol}}(f,\overline{W^u(x,\delta)}\cap Z,\mathcal{U})>\gamma. \tag{1}$$

Let $Z'\subset\overline{W^u(x,\delta)}\cap Z$ and let Γ be any collection of strings covering Z'. For each $\mathbf{U}\in\Gamma$, choose $x_{\mathbf{U}}\in X(\mathbf{U})\cap Z'$. By Fact 4,

$$X(\mathbf{U})\subset B^u_{m(\mathbf{U})}(x_{\mathbf{U}},\epsilon_0).$$

Thus,

$$\mathcal{M}_{\mu,c}^{\mathrm{pol},u}(Z',0,s,\epsilon_0)\leq\mathcal{M}^{\mathrm{pol}}(Z',\mathcal{U},s)\leq\mathcal{M}^{\mathrm{pol}}(\overline{W^u(x,\delta)}\cap Z,\mathcal{U},s).$$

Taking the supremum over all Z' yields

$$\mathcal{M}_{\mu}^{\mathrm{pol},u}(\overline{W^u(x,\delta)}\cap Z,0,s,\epsilon_0)\leq\mathcal{M}_{\mathrm{B}}^{\mathrm{pol}}(\overline{W^u(x,\delta)}\cap Z,\mathcal{U},s).$$

Hence,

$$h_{\mu}^{\mathrm{pol},u}(f,0,\overline{W^u(x,\delta)}\cap Z,\epsilon_0)\leq h_{\mathrm{B}}^{\mathrm{pol}}(f,\overline{W^u(x,\delta)}\cap Z,\mathcal{U}),$$

which contradicts (1). The proof is complete. □

4.2. Relation Between the Bowen Polynomial Entropy and the (q,μ)-Entropies of the Level Sets K_α

Theorem 3. *Let $f:M\to M$ be a C^1 partially hyperbolic diffeomorphism and let $\mu\in\mathcal{M}(M)$ be non-atomic and positive on every nonempty open set. Then for any $\alpha\geq 0$ and any $q\in\mathbb{R}$,*

$$h_{\mathrm{B}}^{\mathrm{pol},u}(f,K_{\alpha}^{\mathrm{pol},u}(\mu))=q\alpha+h_{\mu}^{\mathrm{pol},u}(f,q,K_{\alpha}^{\mathrm{pol},u}(\mu)).$$

Proof. Fix $\alpha \geq 0$ and consider the corresponding level set

$$\begin{aligned} K_{\alpha}^{\mathrm{pol},u}(\mu) &= \{x \in M : h_{\mu}^{\mathrm{pol},u}(f,x) = \alpha\} \\ &= \left\{x \in M : \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{-\log \mu(B_n^u(x,\epsilon))}{\log n} = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{-\log \mu(B_n^u(x,\epsilon))}{\log n} = \alpha \right\}, \end{aligned}$$

where $h_{\mu}^{\mathrm{pol},u}(f,x)$ denotes the unstable local polynomial entropy at x.

Fix a decreasing sequence $\{\epsilon_L\}_{L \geq 1}$ with $\epsilon_L \to 0$ as $L \to \infty$. This sequence will be fixed throughout the proof. Let $r > 0$ and define

$$K_{\alpha,L}^{\mathrm{pol},u} = \left\{ x \in K_{\alpha}^{\mathrm{pol},u}(\mu) : \alpha - r < \liminf_{n \to \infty} \frac{-\log \mu(B_n^u(x,\epsilon_L))}{\log n} \right\}.$$

Clearly,

$$K_{\alpha,L}^{\mathrm{pol},u} \subseteq K_{\alpha,L+1}^{\mathrm{pol},u}, \qquad K_{\alpha}^{\mathrm{pol},u}(\mu) = \bigcup_{L=1}^{\infty} K_{\alpha,L}^{\mathrm{pol},u}.$$

Moreover, since the function $\frac{-\log \mu(B_n^u(x,\epsilon))}{\log n}$ is monotone in ϵ, for every $x \in K_{\alpha}^{\mathrm{pol},u}(\mu)$ and every $\epsilon > 0$,

$$\limsup_{n \to \infty} \frac{-\log \mu(B_n^u(x,\epsilon))}{\log n} \leq \alpha.$$

Fix $x \in K_{\alpha,L}^{\mathrm{pol},u}$. Then there exists $N_0 = N_0(x,r,\epsilon_L) \in \mathbb{N}$ such that

$$\alpha - r < \frac{-\log \mu(B_n^u(x,\epsilon_L))}{\log n} < \alpha + r, \qquad \text{for all } n \geq N_0.$$

Define

$$K_{\alpha,L,N}^{\mathrm{pol},u} = \{x \in K_{\alpha,L}^{\mathrm{pol},u} : N_0(x,r,\epsilon_L) < N\}. \tag{2}$$

Then

$$K_{\alpha,L,N}^{\mathrm{pol},u} \subseteq K_{\alpha,L,N+1}^{\mathrm{pol},u}, \qquad K_{\alpha,L}^{\mathrm{pol},u} = \bigcup_{N=1}^{\infty} K_{\alpha,L,N}^{\mathrm{pol},u}.$$

Let $\mathcal{U}$ be a finite open cover of M. By the basic properties of Bowen polynomial entropy, we have

$$h_{\mathrm{B}}^{\mathrm{pol}}(f, \overline{W^u(x,\delta)} \cap K_{\alpha}^{\mathrm{pol},u}(\mu), \mathcal{U}) = \lim_{L \to \infty} \lim_{N \to \infty} h_{\mathrm{B}}^{\mathrm{pol}}(f, \overline{W^u(x,\delta)} \cap K_{\alpha,L,N}^{\mathrm{pol},u}, \mathcal{U}).$$

We may assume that $K_{\alpha}^{\mathrm{pol},u}(\mu) \neq \emptyset$, since otherwise there is nothing to prove. The proof is divided into two parts.

Upper bound.

We show that

$$h_{\mathrm{B}}^{\mathrm{pol},u}(f, K_{\alpha}^{\mathrm{pol},u}(\mu)) \leq q\alpha + h_{\mu}^{\mathrm{pol},u}(f, q, K_{\alpha}^{\mathrm{pol},u}(\mu)).$$

Lemma 3. *Let $\mathcal{U}$ be an arbitrary finite open cover of M and let $K_{\alpha,L,N}^{\mathrm{pol},u}$ be defined as in (2). Assume that $\epsilon_L < \delta(\mathcal{U})/2$, where $\delta(\mathcal{U})$ denotes the Lebesgue number of $\mathcal{U}$. Then for any $x \in M$, $\delta > 0$, and any real numbers s, t satisfying*

$$s \geq q\alpha + |q|r + t,$$

we have

$$\mathcal{M}^{\mathrm{pol}}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, \mathcal{U}, s\big) \le \mathcal{M}^{\mathrm{pol},u}_{\mu,c}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, q, t, \epsilon_L\big).$$

Proof. Fix $n > N$ and let $\mathcal{G} = \{B^u_{n_i}(x_i,\epsilon_L)\}_i$ be an arbitrary cover of $K^{\mathrm{pol},u}_{\alpha,L,N}$, where $x_i \in \overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}$ and $n_i \ge n$ for all i. By Fact 3, for each i there exists a word $\mathbf{U}_i$ with length $m(\mathbf{U}_i) = n_i$ such that $B^u_{n_i}(x_i,\epsilon_L) \subset X(\mathbf{U}_i)$. Consequently,

$$\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N} \subset \bigcup_i B^u_{n_i}(x_i,\epsilon_L) \subset \bigcup_i X(\mathbf{U}_i),$$

and the collection $\Gamma_{\mathcal{G}} = \{\mathbf{U}_i\}$ forms a cover of $K^{\mathrm{pol},u}_{\alpha,L,N}$.

Since $x_i \in K^{\mathrm{pol},u}_{\alpha,L,N}$ and $n_i \ge n > N$, we have

$$\left(\frac{1}{n_i}\right)^{\alpha+r} \le \mu(B^u_{n_i}(x_i,\epsilon_L)) \le \left(\frac{1}{n_i}\right)^{\alpha-r}.$$

If $q \ge 0$, then $\mu(B^u_{n_i}(x_i,\epsilon_L))^q \ge (1/n_i)^{q(\alpha+r)}$, and hence

$$\begin{aligned}\sum_i \mu(B^u_{n_i}(x_i,\epsilon_L))^q \left(\frac{1}{n_i}\right)^t &\ge \sum_i \left(\frac{1}{n_i}\right)^{q\alpha+qr+t} \ge \sum_{\mathbf{U}_i\in\Gamma_{\mathcal{G}}} \left(\frac{1}{n_i}\right)^s \\ &\ge \mathcal{M}^{\mathrm{pol}}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, \mathcal{U}, s, n\big),\end{aligned} \tag{3}$$

provided that $s \ge q\alpha + qr + t$.

If $q \le 0$, then $\mu(B^u_{n_i}(x_i,\epsilon_L))^q \ge (1/n_i)^{q(\alpha-r)}$, and similarly

$$\begin{aligned}\sum_i \mu(B^u_{n_i}(x_i,\epsilon_L))^q \left(\frac{1}{n_i}\right)^t &\ge \sum_i \left(\frac{1}{n_i}\right)^{q\alpha-qr+t} \ge \sum_{\mathbf{U}_i\in\Gamma_{\mathcal{G}}} \left(\frac{1}{n_i}\right)^s \\ &\ge \mathcal{M}^{\mathrm{pol}}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, \mathcal{U}, s, n\big),\end{aligned} \tag{4}$$

whenever $s \ge q\alpha - qr + t$.

Combining (3) and (4), we conclude that

$$\mathcal{M}^{\mathrm{pol}}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, \mathcal{U}, s, n\big) \le \mathcal{M}^{\mathrm{pol},u}_{\mu,c}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, q, t, \epsilon_L, n\big).$$

Letting $n \to \infty$ yields the desired inequality. □

We now prove the upper bound by contradiction. Assume that there exists $q_0 \in \mathbb{R}$ such that

$$\gamma = \frac{1}{4}\Big(h^{\mathrm{pol},u}_{\mathrm{B}}(f, K^{\mathrm{pol},u}_{\alpha}(\mu)) - q_0\alpha - h^{\mathrm{pol},u}_{\mu}(f, q_0, K^{\mathrm{pol},u}_{\alpha}(\mu))\Big) > 0.$$

Since Bowen polynomial entropy can be computed via arbitrarily fine open covers, there exists a finite open cover $\mathcal{U}$ of M such that

$$h^{\mathrm{pol}}_{\mathrm{B}}\big(f, \overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha}(\mu), \mathcal{U}\big) > q_0\alpha + h^{\mathrm{pol},u}_{\mu}\big(f, q_0, \overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha}(\mu)\big) + 3\gamma.$$

Let $r > 0$ be arbitrary if $q_0 = 0$, and let $r = \gamma/(2|q_0|)$ if $q_0 \neq 0$. Using the decompositions of $K_\alpha^{\mathrm{pol},u}(\mu)$ and the definition of $h_\mu^{\mathrm{pol},u}$, we may choose for L and N to be sufficiently large that $\epsilon_L < \delta(\mathcal{U})/2$ and the following inequalities are held:

$$\begin{aligned} &h_{\mathrm{B}}^{\mathrm{pol}}(f, \overline{W^u(x,\delta)} \cap K_{\alpha,L,N}^{\mathrm{pol},u}, \mathcal{U}) > q_0\alpha + h_\mu^{\mathrm{pol},u}(f, q_0, \overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu)) + 2\gamma, \\ &h_\mu^{\mathrm{pol},u}(f, q_0, \overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu)) + \frac{\gamma}{2} > h_\mu^{\mathrm{pol},u}(f, q_0, \overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu), \epsilon_L). \end{aligned} \tag{5}$$

From the definition of Bowen polynomial entropy, this implies that

$$\mathcal{M}^{\mathrm{pol}}\Big(\overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu), \mathcal{U}, q_0\alpha + h_\mu^{\mathrm{pol},u}(f, q_0, \overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu)) + 2\gamma\Big) = +\infty.$$

Applying Lemma 3 with

$$s = q_0\alpha + h_\mu^{\mathrm{pol},u}(f, q_0, \overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu)) + 2\gamma$$

and

$$t = h_\mu^{\mathrm{pol},u}(f, q_0, \overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu)) + \gamma - |q_0| r,$$

we obtain

$$\mathcal{M}_\mu^{\mathrm{pol},u}\big(\overline{W^u(x,\delta)} \cap K_{\alpha,L,N}^{\mathrm{pol},u}, q_0, t, \epsilon_L\big) = +\infty, \tag{6}$$

which contradicts the definition of $h_\mu^{\mathrm{pol},u}(f, q_0, \overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu))$. This contradiction completes the proof of the upper bound.

Lower bound.

We prove that

$$h_{\mathrm{B}}^{\mathrm{pol},u}(f, K_\alpha^{\mathrm{pol},u}(\mu)) \geq q\alpha + h_\mu^{\mathrm{pol},u}(f, q, K_\alpha^{\mathrm{pol},u}(\mu)).$$

Lemma 4. *Let $r > 0$ and let $K_{\alpha,L,N}^{\mathrm{pol},u}$ be defined as above for some $L, N \in \mathbb{N}$. Assume that $\mathcal{U}$ is a finite open cover of M satisfying* $\mathrm{diam}(\mathcal{U}) < \epsilon_L/2$. *Then for any $x \in M$, $\delta > 0$, and any real numbers s, t such that*

$$s \leq q\alpha - |q| r + t,$$

we have

$$\mathcal{M}_\mu^{\mathrm{pol},u}\big(\overline{W^u(x,\delta)} \cap K_{\alpha,L,N}^{\mathrm{pol},u}, q, t, \epsilon_L\big) \leq \mathcal{M}^{\mathrm{pol}}\big(\overline{W^u(x,\delta)} \cap K_{\alpha,L,N}^{\mathrm{pol},u}, \mathcal{U}, s\big).$$

Proof. Fix $L, N \in \mathbb{N}$ and let $Z \subset \overline{W^u(x,\delta)} \cap K_{\alpha,L,N}^{\mathrm{pol},u}$ be a nonempty subset. For any $n > N$, let Γ be an arbitrary collection of words covering Z such that $m(\mathbf{U}) \geq n$ for all $\mathbf{U} \in \Gamma$. Without loss of generality, we may assume that $X(\mathbf{U}) \cap Z \neq \emptyset$ and choose a point $x_{\mathbf{U}} \in X(\mathbf{U}) \cap Z$ for each $\mathbf{U} \in \Gamma$.

Since $\mathrm{diam}(\mathcal{U}) < \epsilon_L/2$, it follows that

$$x_{\mathbf{U}} \in X(\mathbf{U}) \cap Z \subset B_{m(\mathbf{U})}^u(x_{\mathbf{U}}, \epsilon_L).$$

Thus the family $\{B_{m(\mathbf{U})}^u(x_{\mathbf{U}}, \epsilon_L) : \mathbf{U} \in \Gamma\}$ forms a centered cover of Z.

Because $x_{\mathbf{U}} \in K_{\alpha,L,N}^{\mathrm{pol},u}$ and $m(\mathbf{U}) > N$, we have

$$\left(\frac{1}{m(\mathbf{U})}\right)^{\alpha+r} \leq \mu(B_{m(\mathbf{U})}^u(x_{\mathbf{U}}, \epsilon_L)) \leq \left(\frac{1}{m(\mathbf{U})}\right)^{\alpha-r}.$$

If $q \geq 0$, then

$$\begin{aligned}\mathcal{M}^{\mathrm{pol},u}_{\mu,c}(Z,q,t,\epsilon_L,n) &\leq \sum_{\mathbf{U}\in\Gamma} \mu\big(B^u_{m(\mathbf{U})}(x_{\mathbf{U}},\epsilon_L)\big)^q \left(\frac{1}{m(\mathbf{U})}\right)^t \\ &\leq \sum_{\mathbf{U}\in\Gamma} \left(\frac{1}{m(\mathbf{U})}\right)^{q\alpha-qr+t} \leq \sum_{\mathbf{U}\in\Gamma} \left(\frac{1}{m(\mathbf{U})}\right)^s,\end{aligned}$$

provided that $s \leq q\alpha - qr + t$. Since Γ is arbitrary, we obtain

$$\mathcal{M}^{\mathrm{pol},u}_{\mu,c}(Z,q,t,\epsilon_L,n) \leq \mathcal{M}^{\mathrm{pol}}(Z,\mathcal{U},s,n).$$

Letting $n \to \infty$ yields

$$\mathcal{M}^{\mathrm{pol},u}_{\mu}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, q, t, \epsilon_L\big) \leq \mathcal{M}^{\mathrm{pol}}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, \mathcal{U}, s\big). \tag{7}$$

If $q \leq 0$, then

$$\mu\big(B^u_{m(\mathbf{U})}(x_{\mathbf{U}},\epsilon_L)\big)^q \leq \left(\frac{1}{m(\mathbf{U})}\right)^{q(\alpha+r)}.$$

Consequently,

$$\mathcal{M}^{\mathrm{pol},u}_{\mu,c}(Z,q,t,\epsilon_L,n) \leq \sum_{\mathbf{U}\in\Gamma} \left(\frac{1}{m(\mathbf{U})}\right)^{q\alpha+qr+t} \leq \sum_{\mathbf{U}\in\Gamma} \left(\frac{1}{m(\mathbf{U})}\right)^s,$$

whenever $s \leq q\alpha + qr + t$. Proceeding as above, we obtain

$$\mathcal{M}^{\mathrm{pol},u}_{\mu}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, q, t, \epsilon_L\big) \leq \mathcal{M}^{\mathrm{pol}}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, \mathcal{U}, s\big). \tag{8}$$

Combining (7) and (8) completes the proof. □

We now prove the lower bound by contradiction. Assume that there exists $q_0 \in \mathbb{R}$ such that

$$\gamma = \frac{1}{4}\Big(q_0\alpha + h^{\mathrm{pol},u}_{\mu}(f,q_0,K^{\mathrm{pol},u}_{\alpha}(\mu)) - h^{\mathrm{pol},u}_{\mathrm{B}}(f,K^{\mathrm{pol},u}_{\alpha}(\mu))\Big) > 0.$$

By the definition of unstable (q,μ)-polynomial entropy, we have

$$q_0\alpha + h^{\mathrm{pol},u}_{\mu}(f,q_0,\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha}(\mu)) - h^{\mathrm{pol},u}_{\mathrm{B}}(f,\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha}(\mu)) > 3\gamma.$$

Let $\lambda > 0$ be arbitrary if $q_0 = 0$, and let $\lambda = \gamma/(2|q_0|)$ otherwise. Using the approximation by the sets $K^{\mathrm{pol},u}_{\alpha,L,N}$ and Definition 7 together with Proposition 4, we may choose for L and N to be sufficiently large that

$$h^{\mathrm{pol},u}_{\mu}(f,q_0,\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N},\epsilon_L) > h^{\mathrm{pol},u}_{\mathrm{B}}(f,\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha}(\mu)) + 2\gamma - q_0\alpha.$$

By the definition of unstable (q,μ)-polynomial entropy, this implies

$$\mathcal{M}^{\mathrm{pol},u}_{\mu}\big(\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha,L,N}, q_0, h^{\mathrm{pol},u}_{\mathrm{B}}(f,\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha}(\mu)) + 2\gamma - q_0\alpha, \epsilon_L\big) = +\infty.$$

On the other hand, by the definition of Bowen polynomial entropy, there exists a finite open cover $\mathcal{U}$ of M such that $\mathrm{diam}(\mathcal{U}) < \epsilon_L/2$ and

$$h^{\mathrm{pol}}_{\mathrm{B}}(f,\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha}(\mu),\mathcal{U}) < h^{\mathrm{pol}}_{\mathrm{B}}(f,\overline{W^u(x,\delta)} \cap K^{\mathrm{pol},u}_{\alpha}(\mu)) + \frac{\gamma}{2}.$$

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Let

$$t = h_{\mathrm{B}}^{\mathrm{pol}}\big(f, \overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu)\big) + 2\gamma - q_0\alpha, \qquad s = h_{\mathrm{B}}^{\mathrm{pol}}\big(f, \overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu)\big) + \gamma - |q_0|\lambda.$$

Applying Lemma 4, we obtain

$$\mathcal{M}^{\mathrm{pol}}\big(\overline{W^u(x,\delta)} \cap K_{\alpha,L,N}^{\mathrm{pol},u}, \mathcal{U}, s\big) = +\infty. \tag{9}$$

However,

$$s \geq h_{\mathrm{B}}^{\mathrm{pol}}\big(f, \overline{W^u(x,\delta)} \cap K_\alpha^{\mathrm{pol},u}(\mu)\big) + \frac{\gamma}{2} > h_{\mathrm{B}}^{\mathrm{pol}}\big(f, \overline{W^u(x,\delta)} \cap K_{\alpha,L,N}^{\mathrm{pol},u}, \mathcal{U}\big),$$

which implies

$$\mathcal{M}^{\mathrm{pol}}\big(\overline{W^u(x,\delta)} \cap K_{\alpha,L,N}^{\mathrm{pol},u}, \mathcal{U}, s\big) = 0,$$

contradicting (9). This contradiction completes the proof of the lower bound. □

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