



Article

Positive Ground State Solutions for Fractional (p, q) -Laplacian Choquard Equation with Singularity and Upper Critical Exponent

Zhenyu Bai ¹ and Chuanzhi Bai ^{2,*} 

¹ School of Mathematics and Statistics, Guizhou University, Guiyang 550025, China; gs.zybo24@gzu.edu.cn

² School of Mathematics and Statistics, Huaiyin Normal University, Huaian 223300, China

* Correspondence: czbai@hytc.edu.cn

Abstract

We prove the existence of a positive ground state solution for a fractional (p, q) -Laplacian Choquard equation that features both a singularity and an upper critical exponent. The proof relies on a combination of the Nehari manifold technique and Ekeland's variational principle.

Keywords: fractional (p, q) Laplacian; Choquard equation; singularity; variational method; Nehari manifold

1. Introduction

In the past ten years, many scholars have focused on the following Choquard equation

$$-\Delta u + V(x)u = \lambda(I_\alpha * |u|^p)|u|^{p-2}u + h(x, u), \quad x \in \mathbb{R}^N, \quad (1)$$

where $N \in \mathbb{N}$ and $\alpha \in (0, N)$. If $N = 3$, $V(x) = \lambda = 1$, $p = \alpha = 2$ and $h \equiv 0$, then (1) becomes the famous Choquard–Pekar equation, which was proposed in 1954 by Pekar [1] to describe the quantum theory of a polaron at rest, and later, it appeared in 1976 proposed by Lieb [2] in the study of an electron trapped in its own hole. It was also proposed in 1996 by Penrose [3] as a model for the self-gravitational collapse of a quantum mechanical wave function. For more details about the mathematical and physical background of Choquard–Pekar equation, we refer to [4,5], and references therein.

Many scholars have studied the special cases of problem (1). When $N \geq 3$, $V(x) = \lambda = 1$ and $h(x, u) \equiv 0$, Moroz and Van Schaftingen [6] showed the existence of ground state solutions of problem (1) with $p \in (2_\alpha^*, 2_\alpha^*)$, where $2_\alpha^* = \frac{N+\alpha}{N}$ and $2_\alpha^* = \frac{N+\alpha}{N-2}$ are the lower and upper critical exponents of the Choquard equation, respectively. Ghimenti and Van Schaftingen [7] considered the existence of odd symmetric solutions and least energy sign-changing solutions to (1) in subcritical cases $p \in (2_\alpha^*, 2_\alpha^*)$ and $p \in (2, 2_\alpha^*)$ respectively. When $N \geq 3$, $V(x) = 1$ and $p > 1$, Seok [8] investigated problem (1) with a critical local term and obtained the existence of radially symmetric nontrivial solution and concentration results as $\alpha \rightarrow 0$. When $N \geq 3$, $p \in (2_\alpha^*, 2_\alpha^*)$ and $h(x, u) = |u|^{q-2}u$, $q \in (2, \frac{2N}{N-2})$, Li et al. [9] considered the existence and qualitative properties of solutions for problem (1). If $V(x) = 1 + \mu g(x)$ satisfies some conditions and $\mu < 0$, Zhong and Tang [10] studied the existence of ground state sign-changing solutions for problem (1) with a critical pure power nonlinearity. Recently, Yu and Chen [11] investigated the following Choquard equation with singularity



Academic Editors: Giuseppe Di Fazio and Giovanni Molica Bisci

Received: 22 January 2026

Revised: 23 March 2026

Accepted: 10 April 2026

Published: 16 April 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and

conditions of the [Creative Commons](https://creativecommons.org/licenses/by/4.0/)

Attribution (CC BY) license.

$$\begin{cases} -\Delta u + V(x)u = \lambda(I_\alpha * |u|^p)|u|^{p-2}u + f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases} \quad (2)$$

where I_α is the Riesz potential of order $\alpha \in (0, 3)$, $\lambda > 0$, $1 + \frac{\alpha}{3} \leq p < 3 + \alpha$ and $0 < \gamma < 1$. Under some mild assumptions on V and f , the existence and multiplicity of positive solutions of problem (2) are obtained via variational method and the idea of Nehari manifold technique. For other related results about the Choquard equation on $\mathbb{R}^N$, we refer the readers to [12–15] and the references within it. As far as we know, until now, few results are known to Choquard equations with singularity on unbounded domains.

In recent years, numerous research efforts have involved extending semilinear Choquard equations to quasi-linear Choquard equations and fractional Choquard equations. In [16], by using the penalization method and Lyusternik–Schnirelmann theory, Alves and Yang studied the multiplicity and concentration behaviour of positive solutions for a quasi-linear Choquard equation. d’Avenia et al. in [17] investigated a class of fractional Choquard equations, the regularity, existence, nonexistence, symmetry as well as decays properties are obtained. Very recently, Ambrosio et al. [18] proved the existence of a least energy solution for a fractional Choquard Kirchhoff type equation by applying suitable variational methods.

Over the past two decades, significant theoretical achievements and application breakthroughs have been made in the study of positive ground state solutions for fractional Laplacian equations; see [19–22] and the references therein. It is known that the (p, q) -Laplace operators with unbalanced growth can be viewed as a special case of double phase problems, which traces back to the work of Ball [23], where he studied nonlinear elastic models and their qualitative properties. This kind of operator is widely used in chemistry, physics, biology and many other fields. Ambrosio-Repovš [24] studied the multiplicity and concentration results for a class of (p, q) -Laplace operator problems. Xie et al. [25] investigated the existence of solutions for the (p, q) -Laplacian equation with nonlocal Choquard reaction by means of variational methods. Ambrosio and Isernia [26] studied the existence of ground state solutions for a class of (p, q) -Choquard equations and analyzed the regularity, symmetry, and decay properties of these solutions. Zhang and Deng [27] considered a fractional elliptic system involving (p, q) -Laplacian and critical exponent, and obtained a nontrivial solution with the help of the Nehari manifold and fibering maps.

Recently, Fan [28] investigated the existence results of the following fractional (p, q) -Laplacian equations with subcritical or critical Hardy exponents:

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = f(x)|u|^{m-2}u + \frac{|u|^{r-2}u}{|x|^\alpha}, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \quad (3)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, and $0 < s_2 < s_1 < 1$, $1 < q < p < m < r \leq p_\alpha^* = \frac{(N-\alpha)p}{N-ps_1}$, $0 \leq \alpha < ps_1 < N$. For other related results about (p, q) -Laplacian equations, we refer the readers to [29–32] and the references therein.

As far as we know, no study has yet addressed the combination of fractional (p, q) -Laplace operators, Choquard equations, the upper critical exponent, and singularities on unbounded domains. Motivated by above-mentioned results, we consider the following fractional (p, q) -Choquard equations with singularity and upper critical exponent

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u + V(x)(|u|^{p-2}u + |u|^{q-2}u) \\ \quad = \lambda(I_\alpha * |u|^{q_{s_2}^*(\alpha)})|u|^{q_{s_2}^*(\alpha)-2}u + f(x)u^{-\gamma}, & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N, \end{cases} \quad (4)$$

where $N \geq 1$, $0 < s_1 \leq s_2 < 1$, $1 < p < q < \frac{N}{s_2}$, $0 < \gamma < 1$, $\lambda > 0$, $q_{s_2}^*(\alpha) = \frac{q(N+\alpha)}{2(N-q s_2)} > 1$, and I_α is the Riesz potential with order $\alpha \in (0, N)$. $(-\Delta)_p^s$ is the fractional p -Laplacian operator defined for any $u \in C_c^\infty(\mathbb{R}^N)$ by

$$\begin{aligned} (-\Delta)_p^s u(x) &= 2 \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy, \quad x \in \mathbb{R}^N, \\ &= 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy, \quad x \in \mathbb{R}^N, \end{aligned} \quad (5)$$

where P.V. denotes the Cauchy principal value, and $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x - y| < \varepsilon\}$.

Throughout the paper, in order to state our main result, we assume V and f satisfy the following assumptions:

(V₁) $V \in C(\mathbb{R}^N)$ satisfies $\inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0$, where V_0 is a constant;

(V₂) $\text{meas}\{x \in \mathbb{R}^N : -\infty < V(x) \leq b\} < +\infty$ for all $b \in \mathbb{R}$.

(f) $f \in L^{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}}(\mathbb{R}^N)$ is a nonnegative function.

For convenience, let

$$\begin{aligned} \Phi(\lambda) &:= \min \left\{ \frac{2q_{s_2}^*(\alpha)-p}{2q_{s_2}^*(\alpha)-1+\gamma} \left[\frac{p-1+\gamma}{\lambda(2q_{s_2}^*(\alpha)-1+\gamma)C(N,\alpha)S_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)}} \right]^{\frac{p-1+\gamma}{2q_{s_2}^*(\alpha)-p}}, \right. \\ &\quad \left. \frac{2q_{s_2}^*(\alpha)-q}{2q_{s_2}^*(\alpha)-1+\gamma} \left[\frac{q-1+\gamma}{\lambda(2q_{s_2}^*(\alpha)-1+\gamma)C(N,\alpha)S_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)}} \right]^{\frac{q-1+\gamma}{2q_{s_2}^*(\alpha)-q}} \right\}, \\ M_\lambda &:= 2^{-\frac{q(2q_{s_2}^*(\alpha)-1+\gamma)}{2q_{s_2}^*(\alpha)-q}} S_{q_{s_2}^*(\alpha)}^{\gamma-1} \Phi(\lambda), \end{aligned} \quad (6)$$

where S_ν ($\nu = q_{s_2}^*, q_{s_2}^*(\alpha)$) and $C(N, \alpha)$ will be given in (9) and (10), respectively. And

$$\lambda_* = S_H^{\frac{N+\alpha}{N-q s_2}} \left(\frac{\alpha + q s_2}{q(N+\alpha)K_* + (2-2^{2-q})(\alpha + q s_2)} \right)^{\frac{\alpha + q s_2}{N-q s_2}}, \quad (7)$$

where

$$\begin{aligned} K_* &= \frac{p-1+\gamma}{p} \left(\frac{2q_{s_2}^*(\alpha)-1+\gamma}{2(1-\gamma)q_{s_2}^*(\alpha)} \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} S_{q_{s_2}^*(\alpha)}^{1-\gamma} \right)^{\frac{p}{p-1+\gamma}} \\ &\quad \times \left(\frac{2(1-\gamma)q q_{s_2}^*(\alpha)}{(2q_{s_2}^*(\alpha)-q)p2^{2-p-q}} \right)^{\frac{1-\gamma}{p-1+\gamma}}, \end{aligned} \quad (8)$$

and S_H will be given in (11).

Thanks to variational method, Nehari manifold technique, and Ekeland variational principle, we will prove the following theorem, which extends the main result of Yu and Chen [11] (Theorem 1.1) in two aspects: the fractional (p, q) -Laplacian and the upper critical exponent.

Theorem 1. Assume that $0 < \lambda < \lambda_*$, $0 < \gamma < 1$, $0 < \alpha < N$, $0 < s_1 \leq s_2 < 1$, $p > 1 + s_1$, $q > 1 + s_2$, $\max\left\{p, \frac{2N}{N+\alpha+2s_2}\right\} < q < \frac{N}{s_2}$ and (V₁), (V₂), (f) hold. Moreover, assume that $0 < \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} < M_\lambda$; then problem (4) admits at least one positive ground state solution.

This article is organized as follows. In Section 2, we give some notations and preliminaries. Section 3 is devoted to proving the main result of this paper, Theorem 1.

2. Preliminaries

In this section, we briefly recall the relevant definitions and provide some useful technical lemmas.

For any $s \in (0, 1)$, $1 < p < \infty$, we define the Gagliardo seminorm as follows

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{1/p}.$$

The fractional Sobolev space is defined by

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < +\infty\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left([u]_{s,p}^p + \int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}.$$

We now introduce the following working Sobolev space

$$X := \left\{ u \in W^{s_1,p}(\mathbb{R}^N) \cap W^{s_2,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)(|u|^p + |u|^q) d\mu < +\infty \right\}$$

equipped with the norm $\|u\| = \|u\|_{s_1,p,V} + \|u\|_{s_2,q,V}$, where

$$\|u\|_{s_1,p,V} = \left([u]_{s_1,p}^p + \int_{\mathbb{R}^N} V(x)|u|^p dx \right)^{\frac{1}{p}}, \quad \|u\|_{s_2,q,V} = \left([u]_{s_2,q}^q + \int_{\mathbb{R}^N} V(x)|u|^q dx \right)^{\frac{1}{q}}.$$

By Theorem 6.5 in [33] and Lemma 3.4 in [34], and from the assumptions (V_1) and (V_2) , we have the following embedding property.

Lemma 1 ([33,34]). *The space X continuously embedded into $W^{s_1,p}(\mathbb{R}^N) \cap W^{s_2,q}(\mathbb{R}^N)$. Therefore X continuously embedded in $L^v(\mathbb{R}^N)$ for any $v \in [p, q_{s_2}^*]$ and compactly embedded in $L^v(\mathbb{R}^N)$ for $v \in [p, q_{s_2}^*)$. Moreover, there exists constant $S_v > 0$ such that*

$$\|u\|_{L^v(\mathbb{R}^N)} \leq S_v \|u\|, \quad \text{for all } v \in [p, q_{s_2}^*], \quad (9)$$

where $q_{s_2}^* = \frac{Nq}{N-q_{s_2}}$.

To solve problem (4), we need the following Hardy–Littlewood–Sobolev inequality.

Lemma 2 ([35]). *Suppose that $0 < \alpha < N$, $1 < r, \iota < +\infty$ and $1/r + 1/\iota + (N - \alpha)/N = 2$. If $f \in L^r(\mathbb{R}^N)$, $g \in L^\iota(\mathbb{R}^N)$, then there exists a positive constant $C = C(N, \alpha, r, \iota)$ such that*

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{f(x)g(y)}{|x - y|^{N-\alpha}} dx dy \leq C \|f\|_r \|g\|_\iota.$$

Here, $\|\cdot\|_p$ denotes the $L^p(\mathbb{R}^N)$ -norm. If $r = \iota = \frac{2N}{N+\alpha}$, then

$$C = C(N, \alpha) = \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left\{ \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(N)} \right\}^{-\frac{\alpha}{N}}. \quad (10)$$

Remark 1. Let $f(x) = g(x) = |u(x)|^\tau$ for $\tau > 0$. By Hardy–Littlewood–Sobolev inequality, $\int_{\mathbb{R}^N} (I_\alpha * |u|^\tau) |u|^\tau dx$ is well defined if $u \in L^{\tau}(\mathbb{R}^N)$ for $r > 1$ with $\frac{2}{r} + \frac{N-\alpha}{N} = 2$. Thus, by Lemma 1, if $u \in X$, then it is provided that $r\tau \in [p, q_{s_2}^*]$. Hence, τ has to satisfy

$$p_{s_1}^\sharp(\alpha) = \frac{p(N+\alpha)}{2N} \leq \tau \leq \frac{q(N+\alpha)}{2(N-s_2q)} = q_{s_2}^*(\alpha).$$

Here, $p_{s_1}^\sharp(\alpha)$ is said to be the lower critical exponent and $q_{s_2}^*(\alpha)$ is called the upper critical exponent.

We define

$$S_H := \inf_{u \in X \setminus \{0\}} \frac{[u]_{s_2, q}^q}{\left(\int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \right)^{\frac{q}{2q_{s_2}^*(\alpha)}}}. \quad (11)$$

From Hardy–Littlewood–Sobolev inequality, Hölder's inequality and Lemma 1, we obtain

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \leq C(N, \alpha) \|u\|_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)} \leq C(N, \alpha) S_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)} \|u\|_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)}. \quad (12)$$

$$\begin{aligned} \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx &\leq \left(\int_{\mathbb{R}^N} |f(x)|^{\frac{q_{s_2}^*(\alpha)}{q_{s_2}^*(\alpha)-1+\gamma}} dx \right)^{\frac{q_{s_2}^*(\alpha)-1+\gamma}{q_{s_2}^*(\alpha)}} \left(\int_{\mathbb{R}^N} |u|^{q_{s_2}^*(\alpha)} dx \right)^{\frac{1-\gamma}{q_{s_2}^*(\alpha)}} \\ &= \|f\|_{\frac{q_{s_2}^*(\alpha)}{q_{s_2}^*(\alpha)-1+\gamma}} \|u\|_{q_{s_2}^*(\alpha)}^{1-\gamma} \leq \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} S_{q_{s_2}^*(\alpha)}^{1-\gamma} \|u\|_{q_{s_2}^*(\alpha)}^{1-\gamma}. \end{aligned} \quad (13)$$

Furthermore, for any $u, v \in X$, owing to (13), one obtains

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx - \int_{\mathbb{R}^N} f(x) |v|^{1-\gamma} dx \right| \\ &\leq \|f\|_{\frac{q_{s_2}^*(\alpha)}{q_{s_2}^*(\alpha)-1+\gamma}} \|u - v\|_{q_{s_2}^*(\alpha)}^{1-\gamma} \leq \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} S_{q_{s_2}^*(\alpha)}^{1-\gamma} \|u - v\|_{q_{s_2}^*(\alpha)}^{1-\gamma}. \end{aligned} \quad (14)$$

The following lemma regarding the properties of the convolution term $I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}$.

Lemma 3 ([36]). Suppose that $u_n \rightharpoonup u$ in X . Then for any $v \in X$, we have

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n v dx \rightarrow \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)-2} u v dx.$$

as $n \rightarrow \infty$.

Similar to the proof of Lemma 2.2 in [37], we have the following result.

Lemma 4. If $\{u_n\}$ is a bounded sequence in $L^{q_{s_2}^*}(\mathbb{R}^N)$ such that $u_n \rightarrow u$ almost everywhere in $\mathbb{R}^N$ as $n \rightarrow \infty$, then the following holds:

$$\begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx - \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^{q_{s_2}^*(\alpha)}) |u_n - u|^{q_{s_2}^*(\alpha)} dx \\ &\rightarrow \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \end{aligned}$$

as $n \rightarrow \infty$.

To prove the main result of this paper, we need to rewrite the singular integral in (5) into the form of (15) in the following lemma. The proof is similar to that of Lemma 3.2 in [38]. For the convenience of the readers, we provide the proof here.

Lemma 5. Let $s \in (0, 1)$, $p > 1 + s$ and let $(-\Delta)_p^s$ be the fractional p -Laplacian operator defined by (5). Then, for any $u \in C_c^\infty(\mathbb{R}^N)$, $x \in \mathbb{R}^N$,

$$(-\Delta)_p^s u(x) = - \int_{\mathbb{R}^N} \frac{|u(x+y) - u(x)|^{p-2} (u(x+y) - u(x))}{|y|^{N+sp}} dy - \int_{\mathbb{R}^N} \frac{|u(x-y) - u(x)|^{p-2} (u(x-y) - u(x))}{|y|^{N+sp}} dy. \quad (15)$$

Proof. By choosing $z = y - x$ in (5), one has

$$\begin{aligned} (-\Delta)_p^s u(x) &= -2 \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2} (u(y) - u(x))}{|x-y|^{N+sp}} dy \\ &= -2 \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x+z) - u(x)|^{p-2} (u(x+z) - u(x))}{|z|^{N+sp}} dz. \end{aligned} \quad (16)$$

Furthermore, by substituting $\bar{z} = -z$ in last term of (16), we obtain

$$\begin{aligned} \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x+z) - u(x)|^{p-2} (u(x+z) - u(x))}{|z|^{N+sp}} dz \\ = \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x-\bar{z}) - u(x)|^{p-2} (u(x-\bar{z}) - u(x))}{|\bar{z}|^{N+sp}} d\bar{z} \end{aligned}$$

and so after rename $\bar{z}$ as z

$$\begin{aligned} 2 \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x+z) - u(x)|^{p-2} (u(x+z) - u(x))}{|z|^{N+sp}} dz \\ = \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x+z) - u(x)|^{p-2} (u(x+z) - u(x))}{|z|^{N+sp}} dz \\ + \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x-z) - u(x)|^{p-2} (u(x-z) - u(x))}{|z|^{N+sp}} dz. \end{aligned} \quad (17)$$

Hence, if we relabeling z as y in (16) and (17), we can write the fractional p -Laplacian operator in (5) as

$$\begin{aligned} (-\Delta)_p^s u(x) &= - \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x+y) - u(x)|^{p-2} (u(x+y) - u(x))}{|y|^{N+sp}} dy \\ &\quad - \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x-y) - u(x)|^{p-2} (u(x-y) - u(x))}{|y|^{N+sp}} dy. \end{aligned}$$

The singularity of the integral at the origin in the above representation can be removed. Indeed, for any $u \in C_c^\infty(\mathbb{R}^N)$, we use the second-order Taylor expansion to obtain the following inequality

$$\begin{aligned} \frac{|u(x+y) - u(x)|^{p-2} (u(x+y) - u(x)) + |u(x-y) - u(x)|^{p-2} (u(x-y) - u(x))}{|y|^{N+sp}} \\ \leq \frac{\left(\|Du\|_{L^\infty} + \frac{1}{2} \|D^2u\|_{L^\infty} |y| \right)^{p-2} \|D^2u\|_{L^\infty}}{|y|^{N+2s-p}}, \end{aligned}$$

which is integrable near 0 (since $p > 1 + s$). Thus, we can get rid of the P.V. and write (15). $\square$

The energy functional corresponding to problem (4) is defined for any $u \in X$ by

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p} \|u\|_{s_1, p, V}^p + \frac{1}{q} \|u\|_{s_2, q, V}^q - \frac{\lambda}{2q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \\ &\quad - \frac{1}{1-\gamma} \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx. \end{aligned} \quad (18)$$

Obviously, the functional J_λ is well defined and is of class $C^1(X, \mathbb{R})$. We see that $u \in X$ is a weak solution of problem (4) if $u > 0$ in $\mathbb{R}^N$ and for any $v \in X$,

$$\begin{aligned} \langle J'_\lambda(u), v \rangle &= \langle u, v \rangle_{s_1, p} + \int_{\mathbb{R}^N} V(x) |u|^{p-2} u v dx + \langle u, v \rangle_{s_2, q} + \int_{\mathbb{R}^N} V(x) |u|^{q-2} u v dx \\ &\quad - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)-2} u v dx - \int_{\mathbb{R}^N} f(x) u^{-\gamma} v dx \\ &= 0, \end{aligned} \quad (19)$$

where

$$\langle u, v \rangle_{s_1, p} = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+ps_1}} dx dy,$$

and

$$\langle u, v \rangle_{s_2, q} = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{q-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{N+qs_2}} dx dy.$$

The Nehari manifold $\mathcal{N}_\lambda$ is closed linked to the behavior of functions of the form $H_\lambda : t \rightarrow \mathcal{J}_\lambda(tu)$ for $t > 0$ that named fibering maps [39]. If $u \in X$, we have

$$\begin{aligned} H_\lambda(t) &= \frac{t^p}{p} \|u\|_{s_1, p, V}^p + \frac{t^q}{q} \|u\|_{s_2, q, V}^q - \frac{\lambda t^{2q_{s_2}^*(\alpha)}}{2q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \\ &\quad - \frac{t^{1-\gamma}}{1-\gamma} \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx, \end{aligned}$$

$$\begin{aligned} H'_\lambda(t) &= t^{p-1} \|u\|_{s_1, p, V}^p + t^{q-1} \|u\|_{s_2, q, V}^q \\ &\quad - \lambda t^{2q_{s_2}^*(\alpha)-1} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx - t^{-\gamma} \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx, \end{aligned}$$

and

$$\begin{aligned} H''_\lambda(t) &= (p-1)t^{p-2} \|u\|_{s_1, p, V}^p + (q-1)t^{q-2} \|u\|_{s_2, q, V}^q \\ &\quad - \lambda(2q_{s_2}^*(\alpha)-1)t^{2q_{s_2}^*(\alpha)-2} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \\ &\quad + \gamma t^{-\gamma-1} \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx. \end{aligned}$$

Obviously,

$$\begin{aligned} tH'_\lambda(t) &= t^p \|u\|_{s_1, p, V}^p + t^q \|u\|_{s_2, q, V}^q \\ &\quad - \lambda t^{2q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx - t^{1-\gamma} \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx \\ &= \langle \mathcal{J}'_\lambda(tu), tu \rangle, \end{aligned}$$

which implies that for $u \in X \setminus \{0\}$ and $t > 0$, $H'_\lambda(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda$. In particular, $H'_\lambda(1) = 0$ if and only if $u \in \mathcal{N}_\lambda$, where

$$\mathcal{N}_\lambda = \left\{ u \in X : \|u\|_{s_1, p, V}^p + \|u\|_{s_2, q, V}^q - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx - \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx = 0 \right\}. \quad (20)$$

Moreover, we may split $\mathcal{N}_\lambda$ as follows

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : H''_\lambda(1) > 0\} \\ &= \left\{ u \in \mathcal{N}_\lambda : p\|u\|_{s_1, p, V}^p + q\|u\|_{s_2, q, V}^q - (1-\gamma) \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx \right. \\ &\quad \left. - 2\lambda q_{s_2}^*(\alpha) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx > 0 \right\}, \end{aligned}$$

$$\begin{aligned}\mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : H_\lambda''(1) = 0\} \\ &= \left\{ u \in \mathcal{N}_\lambda : p\|u\|_{s_1,p,V}^p + q\|u\|_{s_2,q,V}^q - (1-\gamma) \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \right. \\ &\quad \left. - 2\lambda q_{s_2}^*(\alpha) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx = 0 \right\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : H_\lambda''(1) < 0\} \\ &= \left\{ u \in \mathcal{N}_\lambda : p\|u\|_{s_1,p,V}^p + q\|u\|_{s_2,q,V}^q - (1-\gamma) \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \right. \\ &\quad \left. - 2\lambda q_{s_2}^*(\alpha) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx < 0 \right\},\end{aligned}$$

for any $\lambda > 0$.

Obviously, for each $u \in \mathcal{N}_\lambda$, we can obtain

$$\begin{aligned}& p\|u\|_{s_1,p,V}^p + q\|u\|_{s_2,q,V}^q - (1-\gamma) \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \\ & \quad - 2\lambda q_{s_2}^*(\alpha) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \\ &= (p - 2q_{s_2}^*(\alpha))\|u\|_{s_1,p,V}^p + (q - 2q_{s_2}^*(\alpha))\|u\|_{s_2,q,V}^q \\ & \quad + (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \\ &= (p - 1 + \gamma)\|u\|_{s_1,p,V}^p + (q - 1 + \gamma)\|u\|_{s_2,q,V}^q \\ & \quad - \lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx.\end{aligned}\tag{21}$$

3. Main Results

In this section, we prove the existence and multiplicity of positive solutions for problem (4).

Lemma 6. Assume that $0 < \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} < M_\lambda$, where M_λ is defined in Theorem 1; then $\mathcal{N}_\lambda^\pm \neq \emptyset$ and $\mathcal{N}_\lambda^0 = \{0\}$.

Proof. For each $u \in X \setminus \{0\}$ with $\|u\| \leq 1$ and $t \geq 0$, we have

$$\begin{aligned}t \frac{dJ_\lambda(tu)}{dt} &= t^p \|u\|_{s_1,p,V}^p + t^q \|u\|_{s_2,q,V}^q - t^{1-\gamma} \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \\ &\quad - \lambda t^{2q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \\ &= t^{1-\gamma} \left[t^{p-1+\gamma} \|u\|_{s_1,p,V}^p + t^{q-1+\gamma} \|u\|_{s_2,q,V}^q - \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \right. \\ &\quad \left. - \lambda t^{2q_{s_2}^*(\alpha)-1+\gamma} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \right] \\ &\equiv t^{1-\gamma} [h(t) - \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx],\end{aligned}\tag{22}$$

where

$$h(t) = t^{p-1+\gamma} \|u\|_{s_1,p,V}^p + t^{q-1+\gamma} \|u\|_{s_2,q,V}^q - \lambda t^{2q_{s_2}^*(\alpha)-1+\gamma} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx.$$

Thanks to $2q_{s_2}^*(\alpha) = \frac{q(N+\alpha)}{N-s_2q} > q > p$, one has $h(0) = 0$ and $\lim_{t \rightarrow +\infty} h(t) = -\infty$. Since

$$\begin{aligned}h'(t) &= (p-1+\gamma)t^{p-2+\gamma} \|u\|_{s_1,p,V}^p + (q-1+\gamma)t^{q-2+\gamma} \|u\|_{s_2,q,V}^q \\ &\quad - \lambda(2q_{s_2}^*(\alpha) - 1 + \gamma)t^{2q_{s_2}^*(\alpha)-2+\gamma} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \\ &= t^{p-2+\gamma} [A + Bt^{q-p} - C_\lambda t^{2q_{s_2}^*(\alpha)-p}],\end{aligned}$$

where

$$A = (p - 1 + \gamma) \|u\|_{s_1, p, V}^p, \quad B = (q - 1 + \gamma) \|u\|_{s_2, q, V}^q,$$

and

$$C_\lambda = \lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx.$$

□

In the following, we will prove that there exist two distinct positive numbers t_λ^+, t_λ^- such that $h'(t_\lambda^+) > 0 > h'(t_\lambda^-)$. Now, let $g(t) = A + Bt^{q-p} - C_\lambda t^{2q_{s_2}^*(\alpha)-p}$, then

$$g'(t) = (q - p)Bt^{q-p-1} - (2q_{s_2}^*(\alpha) - p)C_\lambda t^{2q_{s_2}^*(\alpha)-p-1},$$

which implies that $g'(t) > 0$, for $0 < t < t^*$, and $g'(t) < 0$, for $t > t^*$, where

$$t^* = \left(\frac{(q - p)B}{(2q_{s_2}^*(\alpha) - p)C_\lambda} \right)^{\frac{1}{2q_{s_2}^*(\alpha) - q}}.$$

Since $g(0) = A > 0$, $\max_{t \in [0, +\infty)} g(t) = g(t^*) > 0$, and $\lim_{t \rightarrow +\infty} g(t) = -\infty$, we deduce that there exists $t_* > t^*$ such that $g(t_*) = 0$, $g(t) > 0$ for $0 < t < t_*$, and $g(t) < 0$ for $t > t_*$. That is, $h'(t_*) = 0$, $h'(t) > 0$ for $0 < t < t_*$, and $h'(t) < 0$ for $t > t_*$. Thus, $\max_{0 \leq t < +\infty} h(t) = h(t_*)$.

We set

$$h_1(t) = t^{p-1+\gamma} \|u\|_{s_1, p, V}^p - \lambda t^{2q_{s_2}^*(\alpha)-1+\gamma} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx,$$

and

$$h_2(t) = t^{q-1+\gamma} \|u\|_{s_2, q, V}^q - \lambda t^{2q_{s_2}^*(\alpha)-1+\gamma} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx.$$

Obviously, $h_1(0) = h_2(0) = 0$, $\lim_{t \rightarrow +\infty} h_1(t) = -\infty$, $\lim_{t \rightarrow +\infty} h_2(t) = -\infty$, $h_1(t)$ achieves its maximum at

$$t_1 = \left(\frac{(p - 1 + \gamma) \|u\|_{s_1, p, V}^p}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx} \right)^{\frac{1}{2q_{s_2}^*(\alpha) - p}}$$

with

$$\begin{aligned} \max_{t \in [0, +\infty)} h_1(t) &= h_1(t_1) \\ &= \frac{2q_{s_2}^*(\alpha) - p}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u\|_{s_1, p, V}^p \left[\frac{(p - 1 + \gamma) \|u\|_{s_1, p, V}^p}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx} \right]^{\frac{p-1+\gamma}{2q_{s_2}^*(\alpha) - p}}. \end{aligned} \quad (23)$$

And $h_2(t)$ achieves its maximum at

$$t_2 = \left(\frac{(q - 1 + \gamma) \|u\|_{s_2, q, V}^q}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx} \right)^{\frac{1}{2q_{s_2}^*(\alpha) - q}}$$

with

$$\begin{aligned} \max_{t \in [0, +\infty)} h_2(t) &= h_2(t_2) \\ &= \frac{2q_{s_2}^*(\alpha) - q}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u\|_{s_2, q, V}^q \left[\frac{(q - 1 + \gamma) \|u\|_{s_2, q, V}^q}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx} \right]^{\frac{q-1+\gamma}{2q_{s_2}^*(\alpha) - q}}. \end{aligned} \quad (24)$$

From (12), (13), (23), (24), $\frac{q(2q_{s_2}^*(\alpha)-1+\gamma)}{2q_{s_2}^*(\alpha)-q} > \frac{p(2q_{s_2}^*(\alpha)-1+\gamma)}{2q_{s_2}^*(\alpha)-p}$, and the following elementary inequality

$$a^s + b^s \geq 2^{1-s}(a+b)^s, \quad a, b \geq 0, s > 1, \quad (25)$$

we obtain

$$\begin{aligned} & h(t_*) - \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \\ & \geq \frac{1}{2} \left(\max_{t \in [0, +\infty)} h_1(t) + \max_{t \in [0, +\infty)} h_2(t) \right) - \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \\ & = \frac{1}{2} \left(\frac{2q_{s_2}^*(\alpha) - p}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u\|_{s_1, p, V}^p \right. \\ & \quad \times \left[\frac{(p-1+\gamma)\|u\|_{s_1, p, V}^p}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx} \right]^{\frac{p-1+\gamma}{2q_{s_2}^*(\alpha)-p}} \\ & \quad + \frac{2q_{s_2}^*(\alpha) - q}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u\|_{s_2, q, V}^q \\ & \quad \times \left[\frac{(q-1+\gamma)\|u\|_{s_2, q, V}^q}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx} \right]^{\frac{q-1+\gamma}{2q_{s_2}^*(\alpha)-q}} \Bigg) \\ & \quad - \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \\ & \geq \frac{1}{2} \left(\frac{2q_{s_2}^*(\alpha) - p}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u\|_{s_1, p, V}^p \right. \\ & \quad \times \left[\frac{(p-1+\gamma)\|u\|_{s_1, p, V}^p}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) C(N, \alpha) S_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)} \|u\|^{2q_{s_2}^*(\alpha)}} \right]^{\frac{p-1+\gamma}{2q_{s_2}^*(\alpha)-p}} \\ & \quad + \frac{2q_{s_2}^*(\alpha) - q}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u\|_{s_2, q, V}^q \\ & \quad \times \left[\frac{(q-1+\gamma)\|u\|_{s_2, q, V}^q}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) C(N, \alpha) S_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)} \|u\|^{2q_{s_2}^*(\alpha)}} \right]^{\frac{q-1+\gamma}{2q_{s_2}^*(\alpha)-q}} \Bigg) - \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \\ & = \frac{1}{2} \left(\frac{2q_{s_2}^*(\alpha) - p}{2q_{s_2}^*(\alpha) - 1 + \gamma} \left[\frac{p-1+\gamma}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) C(N, \alpha) S_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)}} \right]^{\frac{p-1+\gamma}{2q_{s_2}^*(\alpha)-p}} \right. \\ & \quad \times \left(\frac{\|u\|_{s_1, p, V}}{\|u\|} \right)^{\frac{p(2q_{s_2}^*(\alpha)-1+\gamma)}{2q_{s_2}^*(\alpha)-p}} \|u\|^{1-\gamma} \\ & \quad + \frac{2q_{s_2}^*(\alpha) - q}{2q_{s_2}^*(\alpha) - 1 + \gamma} \left[\frac{q-1+\gamma}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) C(N, \alpha) S_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)}} \right]^{\frac{q-1+\gamma}{2q_{s_2}^*(\alpha)-q}} \\ & \quad \times \left(\frac{\|u\|_{s_2, q, V}}{\|u\|} \right)^{\frac{q(2q_{s_2}^*(\alpha)-1+\gamma)}{2q_{s_2}^*(\alpha)-q}} \|u\|^{1-\gamma} \Bigg) - \int_{\mathbb{R}^N} f(x)|u|^{1-\gamma} dx \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2} \Phi(\lambda) \left(\left(\frac{\|u\|_{s_1,p,V}}{\|u\|} \right)^{\frac{q(2q_{s_2}^*(\alpha)-1+\gamma)}{2q_{s_2}^*(\alpha)-q}} + \left(\frac{\|u\|_{s_2,q,V}}{\|u\|} \right)^{\frac{q(2q_{s_2}^*(\alpha)-1+\gamma)}{2q_{s_2}^*(\alpha)-q}} \right) \|u\|^{1-\gamma} \\
&\quad - \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx \\
&\geq \frac{1}{2} \Phi(\lambda) 2^{-\frac{2q_{s_2}^*(\alpha)(q-1)+q\gamma}{2q_{s_2}^*(\alpha)-q}} \left(\frac{\|u\|_{s_1,p,V}}{\|u\|} + \frac{\|u\|_{s_2,q,V}}{\|u\|} \right)^{\frac{q(2q_{s_2}^*(\alpha)-1+\gamma)}{2q_{s_2}^*(\alpha)-q}} \|u\|^{1-\gamma} \\
&\quad - \|f\| \frac{q_{s_2}^*(\alpha)}{q_{s_2}^*(\alpha)-1+\gamma} S_{q_{s_2}^*(\alpha)}^{1-\gamma} \|u\|^{1-\gamma} \\
&= \left[\Phi(\lambda) 2^{-\frac{q(2q_{s_2}^*(\alpha)-1+\gamma)}{2q_{s_2}^*(\alpha)-q}} - \|f\| \frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)} S_{q_{s_2}^*(\alpha)}^{1-\gamma} \right] \|u\|^{1-\gamma} \\
&> 0,
\end{aligned} \tag{26}$$

since $0 < \|f\| \frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)} < M_\lambda$. Consequently, there exist two points $0 < t_\lambda^+ < t_* < t_\lambda^-$ such that

$$h(t_\lambda^+) = h(t_\lambda^-) = \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx$$

and

$$h'(t_\lambda^+) > 0 > h'(t_\lambda^-),$$

which implies that $t_\lambda^+ u \in N_\lambda^+$ and $t_\lambda^- u \in N_\lambda^-$. Thus, $N_\lambda^\pm \neq \emptyset$.

On the other hand, we will prove that $N_\lambda^0 = \{0\}$. In fact, if there exists $u_0 \in N_\lambda^0$ and $u_0 \neq 0$. Then by (21) and (26), we have

$$\begin{aligned}
0 &< \frac{1}{2} \left(\frac{2q_{s_2}^*(\alpha) - p}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u_0\|_{s_1,p,V}^p \right. \\
&\quad \times \left[\frac{(p-1+\gamma) \|u_0\|_{s_1,p,V}^p}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)} dx} \right]^{\frac{p-1+\gamma}{2q_{s_2}^*(\alpha)-p}} \\
&\quad + \frac{2q_{s_2}^*(\alpha) - q}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u_0\|_{s_2,q,V}^q \\
&\quad \times \left[\frac{(q-1+\gamma) \|u_0\|_{s_2,q,V}^q}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)} dx} \right]^{\frac{q-1+\gamma}{2q_{s_2}^*(\alpha)-q}} \Big) \\
&\quad - \int_{\mathbb{R}^N} f(x) |u_0|^{1-\gamma} dx \\
&< \frac{1}{2} \left(\frac{2q_{s_2}^*(\alpha) - p}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u_0\|_{s_1,p,V}^p + \frac{2q_{s_2}^*(\alpha) - q}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u_0\|_{s_2,q,V}^q \right) \\
&\quad - \int_{\mathbb{R}^N} f(x) |u_0|^{1-\gamma} dx \\
&= \frac{1}{2(2q_{s_2}^*(\alpha) - 1 + \gamma)} \left[(2q_{s_2}^*(\alpha) - p) \|u_0\|_{s_1,p,V}^p + (2q_{s_2}^*(\alpha) - q) \|u_0\|_{s_2,q,V}^q \right. \\
&\quad \left. - (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x) |u_0|^{1-\gamma} dx \right] \\
&= 0
\end{aligned}$$

which is a contradiction. Thus, we conclude that $N_\lambda^0 = \{0\}$. The proof is completed.

Lemma 7. Assume that $0 < \|f\| \frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)} < M_\lambda$; then N_λ^- is a closed set in X .

Proof. From Lemma 6, we have $N_\lambda^- \neq \emptyset$. Suppose $\{u_n\} \subset \mathcal{N}_\lambda^-$ be a sequence satisfying $u_n \rightarrow u_*$ in X . That is,

$$\|u_n - u_*\| = \|u_n - u_*\|_{s_1,p,V} + \|u_n - u_\lambda\|_{s_2,q,V} = o_n(1),$$

which implies that

$$\|u_n - u_*\|_{s_1,p,V}^p = o_n(1), \quad \|u_n - u_\lambda\|_{s_2,q,V}^q = o_n(1). \quad (27)$$

Applying the Brezis–Lieb lemma [13], we can easily get

$$[u_n - u_*]_{s_1,p}^p = [u_n]_{s_1,p}^p - [u_*]_{s_1,p}^p + o_n(1), \quad [u_n - u_*]_{s_2,q}^q = [u_n]_{s_2,q}^q - [u_*]_{s_2,q}^q + o_n(1). \quad (28)$$

$$\int_{\mathbb{R}^N} V(x) |u_n - u_*|^p dx = \int_{\mathbb{R}^N} V(x) |u_n|^p dx - \int_{\mathbb{R}^N} V(x) |u_*|^p dx + o_n(1), \quad (29)$$

and

$$\int_{\mathbb{R}^N} V(x) |u_n - u_*|^q dx = \int_{\mathbb{R}^N} V(x) |u_n|^q dx - \int_{\mathbb{R}^N} V(x) |u_*|^q dx + o_n(1). \quad (30)$$

From (28)–(30), we deduce that

$$\|u_n - u_*\|_{s_1,p,V}^p = \|u_n\|_{s_1,p,V}^p - \|u_*\|_{s_1,p,V}^p + o_n(1), \quad (31)$$

and

$$\|u_n - u_*\|_{s_2,q,V}^q = \|u_n\|_{s_2,q,V}^q - \|u_*\|_{s_2,q,V}^q + o_n(1). \quad (32)$$

Thanks to Lemma 4, (12), (14), (20), (27), (31) and (32), we have

$$\begin{aligned} \|u_*\|_{s_1,p,V}^p + \|u_*\|_{s_2,q,V}^q &= \lim_{n \rightarrow \infty} (\|u_n\|_{s_1,p,V}^p + \|u_n\|_{s_2,q,V}^q) \\ &= \lim_{n \rightarrow \infty} \left[\lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx + \int_{\mathbb{R}^N} f(x) |u_n|^{1-\gamma} dx \right] \\ &= \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_*|^{q_{s_2}^*(\alpha)}) |u_*|^{q_{s_2}^*(\alpha)} dx + \int_{\mathbb{R}^N} f(x) |u_*|^{1-\gamma} dx \end{aligned}$$

and

$$\begin{aligned} &(p - 2q_{s_2}^*(\alpha)) \|u_*\|_{s_1,p,V}^p + (q - 2q_{s_2}^*(\alpha)) \|u_*\|_{s_2,q,V}^q \\ &\quad + (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\Omega} f(x) |u_*|^{1-\gamma} dx \\ &= \lim_{n \rightarrow \infty} \left[(p - 2q_{s_2}^*(\alpha)) \|u_n\|_{s_1,p,V}^p + (q - 2q_{s_2}^*(\alpha)) \|u_n\|_{s_2,q,V}^q \right. \\ &\quad \left. + (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\Omega} f(x) |u_n|^{1-\gamma} dx \right] \leq 0, \end{aligned}$$

which means that $u_* \in \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$.

Since $u_n \in \mathcal{N}_\lambda^-$, it follows from (21) that

$$\begin{aligned} &(p - 1 + \gamma) \|u_n\|_{s_1,p,V}^p + (q - 1 + \gamma) \|u_n\|_{s_2,q,V}^q \\ &\quad < \lambda (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx \\ &\quad \leq \lambda (2q_{s_2}^*(\alpha) - 1 + \gamma) C(N, \alpha) S_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)} \|u_n\|_{s_2,q,V}^{2q_{s_2}^*(\alpha)}. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we yield that

$$\begin{aligned} &(p - 1 + \gamma) \|u_*\|_{s_1,p,V}^p + (q - 1 + \gamma) \|u_*\|_{s_2,q,V}^q \\ &\quad \leq \lambda (2q_{s_2}^*(\alpha) - 1 + \gamma) C(N, \alpha) S_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)} \|u_*\|_{s_2,q,V}^{2q_{s_2}^*(\alpha)}. \end{aligned} \quad (33)$$

On the other hand, if $\|u_*\| \leq 1$, then by (25), we deduce that

$$\begin{aligned} & (p-1+\gamma)\|u_*\|_{s_1,p,V}^p + (q-1+\gamma)\|u_*\|_{s_2,q,V}^q \\ & > (p-1+\gamma)\|u_*\|_{s_1,p,V}^q + (p-1+\gamma)\|u_*\|_{s_2,q,V}^q \\ & \geq (p-1+\gamma)2^{1-q}(\|u_*\|_{s_1,p,V} + \|u_*\|_{s_2,q,V})^q \\ & = (p-1+\gamma)2^{1-q}\|u_*\|^q. \end{aligned} \quad (34)$$

Combining (33) and (34) yields

$$\|u_*\| > \left(\frac{(p+\gamma-1)2^{1-q}}{\lambda(2q_{s_2}^*(\alpha) + \gamma - 1)C(N, \alpha)S_{q_{s_2}^*(\alpha)}^{2q_{s_2}^*(\alpha)}} \right)^{\frac{1}{2q_{s_2}^*(\alpha)-q}} > 0.$$

This and Lemma 6 lead to $u_* \notin \mathcal{N}_\lambda^0$. Thus, $u_* \in \mathcal{N}_\lambda^-$ and so $\mathcal{N}_\lambda^-$ is a closed set in X . The proof is complete. $\square$

Lemma 8. For any $u \in \mathcal{N}_\lambda^\pm$, then there exists $\varepsilon > 0$ and a continuous function $K = K(\omega) > 0$, $\forall \omega \in X$ with $\|\omega\| < \varepsilon$ satisfying that $K(0) = 1$, $K(\omega)(u + \omega) \in \mathcal{N}_\lambda^\pm$.

Proof. For any $u \in \mathcal{N}_\lambda^+$ (we can similarly consider case $u \in \mathcal{N}_\lambda^-$), we define $G : X \times \mathbb{R} \rightarrow \mathbb{R}$ as follows

$$\begin{aligned} G(\omega, t) &= t^p\|u + \omega\|_{s_1,p,V}^p + t^q\|u + \omega\|_{s_2,q,V}^q \\ &\quad - \lambda t^{2q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u + \omega|^{q_{s_2}^*(\alpha)}) |u + \omega|^{q_{s_2}^*(\alpha)} dx - t^{1-\gamma} \int_{\mathbb{R}^N} f(x) |u + \omega|^{1-\gamma} dx. \end{aligned}$$

Owing to $u \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we get $G(0, 1) = 0$ and

$$\begin{aligned} G_t(0, 1) &= p\|u\|_{s_1,p,V}^p + q\|u\|_{s_2,q,V}^q \\ &\quad - 2\lambda q_{s_2}^*(\alpha) \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx - (1-\gamma) \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx > 0. \end{aligned}$$

By means of implicit function theorem for G at the point $(0, 1)$, we obtain that there exists $\varepsilon_0 > 0$ such that for $\omega \in X$, $\|\omega\| < \varepsilon_0$, the equation $G(\omega, t) = 0$ has a unique continuous solution $t = K(\omega) > 0$ satisfying that $K(0) = 1$ and $G(\omega, K(\omega)) = 0$, i.e., $K(\omega)(u + \omega) \in \mathcal{N}_\lambda$. Furthermore, since $G_t(0, 1) > 0$ and

$$\begin{aligned} G_t(\omega, K(\omega)) &= pK^{p-1}(\omega)\|u + \omega\|_{s_1,p,V}^p + qK^{q-1}(\omega)\|u + \omega\|_{s_2,q,V}^q \\ &\quad - 2\lambda q_{s_2}^*(\alpha)K^{2q_{s_2}^*(\alpha)-1}(\omega) \int_{\mathbb{R}^N} (I_\alpha * |u + \omega|^{q_{s_2}^*(\alpha)}) |u + \omega|^{q_{s_2}^*(\alpha)} dx \\ &\quad - (1-\gamma)K^{-\gamma}(\omega) \int_{\mathbb{R}^N} f(x) |u + \omega|^{1-\gamma} dx \\ &= K^{-1}(\omega) \left[pK^p(\omega)\|u + \omega\|_{s_1,p,V}^p + qK^q(\omega)\|u + \omega\|_{s_2,q,V}^q \right. \\ &\quad \left. - 2\lambda q_{s_2}^*(\alpha)K^{2q_{s_2}^*(\alpha)}(\omega) \int_{\mathbb{R}^N} (I_\alpha * |u + \omega|^{q_{s_2}^*(\alpha)}) |u + \omega|^{q_{s_2}^*(\alpha)} dx \right. \\ &\quad \left. - (1-\gamma)K^{1-\gamma}(\omega) \int_{\mathbb{R}^N} f(x) |u + \omega|^{1-\gamma} dx \right], \end{aligned}$$

we can choose $0 < \varepsilon < \varepsilon_0$ potentially small such that for $\omega \in X$ with $\|\omega\| < \varepsilon$,

$$\begin{aligned} & pK^p(\omega)\|u + \omega\|_{s_1,p,V}^p + qK^q(\omega)\|u + \omega\|_{s_2,q,V}^q \\ & \quad - 2\lambda q_{s_2}^*(\alpha)K^{2q_{s_2}^*(\alpha)}(\omega) \int_{\mathbb{R}^N} (I_\alpha * |u + \omega|^{q_{s_2}^*(\alpha)}) |u + \omega|^{q_{s_2}^*(\alpha)} dx \\ & \quad - (1-\gamma)K^{1-\gamma}(\omega) \int_{\mathbb{R}^N} f(x) |u + \omega|^{1-\gamma} dx > 0, \end{aligned}$$

that is

$$K(\omega)(u + \omega) \in \mathcal{N}_\lambda^+, \quad \text{for each } \omega \in X, \|\omega\| < \varepsilon.$$

The proof is complete. $\square$

Lemma 9. *The functional J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. Moreover, if $0 < \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} < M_\lambda$, then the minimal value $m_\lambda = \inf_{u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda(u) < 0$.*

Proof. For each $u \in \mathcal{N}_\lambda$, from (13), (18) and (20), we obtain

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p} \|u\|_{s_1,p,V}^p + \frac{1}{q} \|u\|_{s_2,q,V}^q - \frac{\lambda}{2q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u|^{q_{s_2}^*(\alpha)}) |u|^{q_{s_2}^*(\alpha)} dx \\ &\quad - \frac{1}{1-\gamma} \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx \\ &= \left(\frac{1}{p} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|u\|_{s_1,p,V}^p + \left(\frac{1}{q} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|u\|_{s_2,q,V}^q \\ &\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx \\ &\geq \left(\frac{1}{p} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|u\|_{s_1,p,V}^p + \left(\frac{1}{q} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|u\|_{s_2,q,V}^q \\ &\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} S_{q_{s_2}^*(\alpha)}^{1-\gamma} \|u\|^{1-\gamma}. \end{aligned}$$

On account of $0 < \gamma < 1 < p < q < 2q_{s_2}^*(\alpha)$, J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. Since $0 < \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} < M_\lambda$, we have $\mathcal{N}_\lambda^\pm \neq \emptyset$ by Lemma 6, also $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$ are two closed sets in X from Lemma 7. Hence $\inf_{\mathcal{N}_\lambda^-} J_\lambda$ and $\inf_{\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda$ are well defined. For each $u \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we can deduce from (21) that

$$\begin{aligned} J_\lambda(u) &= \frac{2q_{s_2}^*(\alpha) - p}{2pq_{s_2}^*(\alpha)} \|u\|_{s_1,p,V}^p + \frac{2q_{s_2}^*(\alpha) - q}{2qq_{s_2}^*(\alpha)} \|u\|_{s_2,q,V}^q \\ &\quad - \frac{2q_{s_2}^*(\alpha) - 1 + \gamma}{2(1-\gamma)q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} f(x) |u|^{1-\gamma} dx \\ &< \frac{2q_{s_2}^*(\alpha) - p}{2pq_{s_2}^*(\alpha)} \|u\|_{s_1,p,V}^p + \frac{2q_{s_2}^*(\alpha) - q}{2qq_{s_2}^*(\alpha)} \|u\|_{s_2,q,V}^q \\ &\quad - \frac{2q_{s_2}^*(\alpha) - p}{2(1-\gamma)q_{s_2}^*(\alpha)} \|u\|_{s_1,p,V}^p - \frac{2q_{s_2}^*(\alpha) - q}{2(1-\gamma)q_{s_2}^*(\alpha)} \|u\|_{s_2,q,V}^q \\ &= - \frac{(2q_{s_2}^*(\alpha) - p)(p - 1 + \gamma)}{2(1-\gamma)pq_{s_2}^*(\alpha)} \|u\|_{s_1,p,V}^p - \frac{(2q_{s_2}^*(\alpha) - q)(q - 1 + \gamma)}{2(1-\gamma)qq_{s_2}^*(\alpha)} \|u\|_{s_2,q,V}^q \\ &< 0 \end{aligned}$$

which implies that $\inf_{\mathcal{N}_\lambda^+} J_\lambda < 0$. Moreover, if $0 < \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} < M_\lambda$, then $\mathcal{N}_\lambda^0 = \{0\}$ from Lemma 6. Thus, $\inf_{\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda = \inf_{\mathcal{N}_\lambda^+} J_\lambda < 0$. The proof is complete. $\square$

Throughout Lemmas 10–12, $\{u_n\}$ denotes the sequence from the proof of Theorem 1.

Lemma 10. *There exists a constant $C_2 > 0$ such that the following alternative holds true:*

(i) if $\{u_n\} \subset \mathcal{N}_\lambda^+$, for any $n \in \mathbb{N}$, one has

$$\begin{aligned} &(p - 2q_{s_2}^*(\alpha)) \|u_n\|_{s_1,p,V}^p + (q - 2q_{s_2}^*(\alpha)) \|u_n\|_{s_2,q,V}^q \\ &\quad + (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x) |u_n|^{1-\gamma} dx \geq C_2; \end{aligned}$$

(ii) if $\{u_n\} \subset \mathcal{N}_\lambda^-$, for any $n \in \mathbb{N}$, one has

$$(p - 2q_{s_2}^*(\alpha))\|u_n\|_{s_1,p,V}^p + (q - 2q_{s_2}^*(\alpha))\|u_n\|_{s_2,q,V}^q + (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x)|u_n|^{1-\gamma} dx \leq -C_2.$$

Proof. We here only prove the case (i). Since $u_n \in \mathcal{N}_\lambda^+$, from (14) and (21), it is sufficient to prove

$$(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x)|u_0|^{1-\gamma} dx > \liminf_{n \rightarrow \infty} [(2q_{s_2}^*(\alpha) - p)\|u_n\|_{s_1,p,V}^p + (2q_{s_2}^*(\alpha) - q)\|u_n\|_{s_2,q,V}^q]. \quad (35)$$

To argue by contradiction, let us assume that

$$(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x)|u_0|^{1-\gamma} dx = \liminf_{n \rightarrow \infty} [(2q_{s_2}^*(\alpha) - p)\|u_n\|_{s_1,p,V}^p + (2q_{s_2}^*(\alpha) - q)\|u_n\|_{s_2,q,V}^q]. \quad (36)$$

On the other hand, it can be concluded from $u_n \in \mathcal{N}_\lambda^+$ and (21) that

$$(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x)|u_n|^{1-\gamma} dx > (2q_{s_2}^*(\alpha) - p)\|u_n\|_{s_1,p,V}^p + (2q_{s_2}^*(\alpha) - q)\|u_n\|_{s_2,q,V}^q.$$

Letting $n \rightarrow \infty$ and together with (14), we have

$$\begin{aligned} (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x)|u_0|^{1-\gamma} dx &\geq \limsup_{n \rightarrow \infty} [(2q_{s_2}^*(\alpha) - p)\|u_n\|_{s_1,p,V}^p + (2q_{s_2}^*(\alpha) - q)\|u_n\|_{s_2,q,V}^q] \\ &\geq \liminf_{n \rightarrow \infty} [(2q_{s_2}^*(\alpha) - p)\|u_n\|_{s_1,p,V}^p + (2q_{s_2}^*(\alpha) - q)\|u_n\|_{s_2,q,V}^q]. \end{aligned}$$

This is combined with (36) leads to

$$(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x)|u_0|^{1-\gamma} dx = \lim_{n \rightarrow \infty} [(2q_{s_2}^*(\alpha) - p)\|u_n\|_{s_1,p,V}^p + (2q_{s_2}^*(\alpha) - q)\|u_n\|_{s_2,q,V}^q]. \quad (37)$$

Since $u_n \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, one has

$$\|u_n\|_{s_1,p,V}^p + \|u_n\|_{s_2,q,V}^q = \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx + \int_{\mathbb{R}^N} f(x)|u_n|^{1-\gamma} dx.$$

Passing to the limit as $n \rightarrow \infty$ and together with (14), (37), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[\lambda (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx \right] \\ = \lim_{n \rightarrow \infty} [(p - 1 + \gamma)\|u_n\|_{s_1,p,V}^p + (q - 1 + \gamma)\|u_n\|_{s_2,q,V}^q]. \end{aligned} \quad (38)$$

From (26), we deduce that

$$\begin{aligned}
 0 &< \left[\Phi(\lambda) 2^{-\frac{q(2q_{s_2}^*(\alpha)-1+\gamma)}{2q_{s_2}^*(\alpha)-q}} - \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-2q)}} S_{q_{s_2}^*(\alpha)}^{1-\gamma} \right] \|u_n\|^{1-\gamma} \\
 &\leq \frac{1}{2} \left(\frac{2q_{s_2}^*(\alpha) - p}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u_n\|_{s_{1,p,V}}^p \right. \\
 &\quad \times \left[\frac{(p-1+\gamma) \|u_n\|_{s_{1,p,V}}^p}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx} \right]^{\frac{p-1+\gamma}{2q_{s_2}^*(\alpha)-p}} \\
 &\quad + \frac{2q_{s_2}^*(\alpha) - q}{2q_{s_2}^*(\alpha) - 1 + \gamma} \|u_n\|_{s_{2,q,V}}^q \\
 &\quad \times \left[\frac{(q-1+\gamma) \|u_n\|_{s_{2,q,V}}^q}{\lambda(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx} \right]^{\frac{q-1+\gamma}{2q_{s_2}^*(\alpha)-q}} \\
 &\quad \left. - \int_{\mathbb{R}^N} f(x) |u_n|^{1-\gamma} dx \right) \quad (39)
 \end{aligned}$$

Letting $n \rightarrow \infty$ in (39) and combining with (37) and (38), we obtain

$$\begin{aligned}
 0 &\leq \frac{1}{2} \left(\frac{2q_{s_2}^*(\alpha) - p}{2q_{s_2}^*(\alpha) - 1 + \gamma} \lim_{n \rightarrow \infty} \|u_n\|_{s_{1,p,V}}^p + \frac{2q_{s_2}^*(\alpha) - q}{2q_{s_2}^*(\alpha) - 1 + \gamma} \lim_{n \rightarrow \infty} \|u_n\|_{s_{2,q,V}}^q \right) \\
 &\quad - \int_{\mathbb{R}^N} f(x) |u_0|^{1-\gamma} dx \\
 &= \frac{1}{2(2q_{s_2}^*(\alpha) - 1 + \gamma)} \left[\lim_{n \rightarrow \infty} \left((2q_{s_2}^*(\alpha) - p) \|u_n\|_{s_{1,p,V}}^p + (2q_{s_2}^*(\alpha) - q) \|u_n\|_{s_{2,q,V}}^q \right) \right. \\
 &\quad \left. - 2(2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x) |u_0|^{1-\gamma} dx \right] \\
 &= -\frac{1}{2} \int_{\mathbb{R}^N} f(x) |u_0|^{1-\gamma} dx < 0,
 \end{aligned}$$

which is a contradiction. Thus (35) and hence (i) holds. The proof is complete. $\square$

For each $0 \leq \psi \in X$, we utilize Lemma 8 with $u = u_n \in \mathcal{N}_\lambda^+$ and $w = \eta\psi$, $\eta > 0$ small enough, we can find $k_{n,\psi}(\eta) = K(\eta\psi)$ such that $k_{n,\psi}(0) = 1$ and $k_{n,\psi}(\eta)(u_n + \eta\psi) \in \mathcal{N}_\lambda^+$. But, we don't know if $k_{n,\psi}(\eta)$ is differentiable. For the purpose of proof, we set

$$k'_{n,\psi}(0) = \lim_{\eta \rightarrow 0^+} \frac{k_{n,\psi}(\eta) - 1}{\eta} \in [-\infty, +\infty].$$

If the above limit does not exist, we may choose $\eta_n \rightarrow 0$ (instead of $\eta \rightarrow 0$) with $\eta_n > 0$ such that $k'_{n,\psi}(0) = \lim_{n \rightarrow \infty} \frac{k_{n,\psi}(\eta_n) - 1}{\eta_n} \in [-\infty, +\infty]$. Since $u_n, k_{n,\psi}(\eta)(u_n + \eta\psi) \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we have

$$\|u_n\|_{s_{1,p,V}}^p + \|u_n\|_{s_{2,q,V}}^q - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx - \int_{\mathbb{R}^N} f(x) u_n^{1-\gamma} dx = 0, \quad (40)$$

and

$$\begin{aligned}
 &k_{n,\psi}^p(\eta) \|u_n + \eta\psi\|_{s_{1,p,V}}^p + k_{n,\psi}^q(\eta) \|u_n + \eta\psi\|_{s_{2,q,V}}^q \\
 &\quad - \lambda k_{n,\psi}^{2q_{s_2}^*(\alpha)}(\eta) \int_{\mathbb{R}^N} (I_\alpha * |u_n + \eta\psi|^{q_{s_2}^*(\alpha)}) |u_n + \eta\psi|^{q_{s_2}^*(\alpha)} dx \\
 &\quad - k_{n,\psi}^{1-\gamma}(\eta) \int_{\mathbb{R}^N} f(x) (u_n + \eta\psi)^{1-\gamma} dx = 0.
 \end{aligned} \quad (41)$$

Lemma 11. Let $\{u_n\} \subset \mathcal{N}_\lambda^\pm$. Then $k'_{n,\psi}(0)$ is uniformly bounded for each $\psi \in X$ with $\psi \geq 0$.

Proof. We here only give the proof of the case $\{u_n\} \subset \mathcal{N}_\lambda^+$. From (40) and (41), we deduce that

$$\begin{aligned}
0 &= [k_{n,\psi}^p(\eta) - 1] \|u_n + \eta\psi\|_{s_1,p,V}^p + [k_{n,\psi}^q(\eta) - 1] \|u_n + \eta\psi\|_{s_2,q,V}^q \\
&\quad - \lambda [k_{n,\psi}^{2q_{s_2}^*(\alpha)}(\eta) - 1] \int_{\mathbb{R}^N} (I_\alpha * |u_n + \eta\psi|^{q_{s_2}^*(\alpha)}) |u_n + \eta\psi|^{q_{s_2}^*(\alpha)} dx \\
&\quad - [k_{n,\psi}^{1-\gamma}(\eta) - 1] \int_{\mathbb{R}^N} f(x) (u_n + \eta\psi)^{1-\gamma} dx + [\|u_n + \eta\psi\|_{s_1,p,V}^p - \|u_n\|_{s_1,p,V}^p] \\
&\quad + [\|u_n + \eta\psi\|_{s_2,q,V}^q - \|u_n\|_{s_2,q,V}^q] - \int_{\mathbb{R}^N} f(x) [(u_n + \eta\psi)^{1-\gamma} - u_n^{1-\gamma}] dx \\
&\quad - \lambda \int_{\mathbb{R}^N} [(I_\alpha * |u_n + \eta\psi|^{q_{s_2}^*(\alpha)}) |u_n + \eta\psi|^{q_{s_2}^*(\alpha)} - (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)}] dx \\
&\leq [k_{n,\psi}(\eta) - 1] \left\{ \frac{k_{n,\psi}^p(\eta) - 1}{k_{n,\psi}(\eta) - 1} \|u_n + \eta\psi\|_{s_1,p,V}^p + \frac{k_{n,\psi}^q(\eta) - 1}{k_{n,\psi}(\eta) - 1} \|u_n + \eta\psi\|_{s_2,q,V}^q \right. \\
&\quad - \lambda \frac{k_{n,\psi}^{2q_{s_2}^*(\alpha)}(\eta) - 1}{k_{n,\psi}(\eta) - 1} \int_{\mathbb{R}^N} (I_\alpha * |u_n + \eta\psi|^{q_{s_2}^*(\alpha)}) |u_n + \eta\psi|^{q_{s_2}^*(\alpha)} dx \\
&\quad \left. - \frac{k_{n,\psi}^{1-\gamma}(\eta) - 1}{k_{n,\psi}(\eta) - 1} \int_{\mathbb{R}^N} f(x) (u_n + \eta\psi)^{1-\gamma} dx \right\} \\
&\quad + [\|u_n + \eta\psi\|_{s_1,p,V}^p - \|u_n\|_{s_1,p,V}^p] + [\|u_n + \eta\psi\|_{s_2,q,V}^q - \|u_n\|_{s_2,q,V}^q] \\
&\quad - \lambda \int_{\mathbb{R}^N} [(I_\alpha * |u_n + \eta\psi|^{q_{s_2}^*(\alpha)}) |u_n + \eta\psi|^{q_{s_2}^*(\alpha)} - (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)}] dx.
\end{aligned}$$

Dividing above inequalities by $\eta > 0$ and passing to the limit as $\eta \rightarrow 0^+$, we obtain by using (21) that

$$\begin{aligned}
0 &\leq k'_{n,\psi}(0) \left\{ p \|u_n\|_{s_1,p,V}^p + q \|u_n\|_{s_2,q,V}^q - (1 - \gamma) \int_{\mathbb{R}^N} f(x) u_n^{1-\gamma} dx \right. \\
&\quad \left. - 2q_{s_2}^*(\alpha) \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx \right\} \\
&\quad + p \langle u_n, \psi \rangle_{s_1,p} + p \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx \\
&\quad + q \langle u_n, \psi \rangle_{s_2,q} + q \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \\
&\quad - 2q_{s_2}^*(\alpha) \lambda \int_{\mathbb{R}^N} (I_\alpha * u_n^{q_{s_2}^*(\alpha)}) u_n^{q_{s_2}^*(\alpha)-2} u_n \psi dx \\
&= k'_{n,\psi}(0) \left\{ (p - 2q_{s_2}^*(\alpha)) \|u_n\|_{s_1,p,V}^p + (q - 2q_{s_2}^*(\alpha)) \|u_n\|_{s_2,q,V}^q \right. \\
&\quad \left. + (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} f(x) u_n^{1-\gamma} dx \right\} \\
&\quad + p \langle u_n, \psi \rangle_{s_1,p} + p \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx \\
&\quad + q \langle u_n, \psi \rangle_{s_2,q} + q \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \\
&\quad - 2q_{s_2}^*(\alpha) \lambda \int_{\mathbb{R}^N} (I_\alpha * u_n^{q_{s_2}^*(\alpha)}) u_n^{q_{s_2}^*(\alpha)-2} u_n \psi dx,
\end{aligned} \tag{42}$$

from which it follows that $k'_{n,\psi}(0) \neq -\infty$ based on Lemma 10 (i). Now we prove that $k'_{n,\psi}(0) \neq +\infty$. If not, we assume that $k'_{n,\psi}(0) = +\infty$, then $k_{n,\psi}(\eta) > 1$ for n sufficiently large and $\eta > 0$ small. Making use of condition (ii) (which appears in the proof of Theorem 1) with $z = k_{n,\psi}(\eta)(u_n + \eta\psi)$ result in

$$\begin{aligned}
&\frac{1}{n} [k_{n,\psi}(\eta) - 1] \|u_n\| + \frac{\eta}{n} k_{n,\psi}(\eta) \|\psi\| \\
&\geq \frac{1}{n} \|u_n - k_{n,\psi}(\eta)(u_n + \eta\psi)\| \geq J_\lambda(u_n) - J_\lambda[k_{n,\psi}(\eta)(u_n + \eta\psi)].
\end{aligned} \tag{43}$$

Thus, it follows from $u_n \in \mathcal{N}_\lambda^+ \subset \mathcal{N}$, (18) and (43) that

$$\begin{aligned} \frac{\|\psi\|}{n} k_{n,\psi}(\eta) &\geq \frac{k_{n,\psi}(\eta) - 1}{\eta} \left\{ -\frac{\|u_n\|}{n} - \left(\frac{1}{p} - \frac{1}{1-\gamma} \right) \frac{k_{n,\psi}^p(\eta) - 1}{k_{n,\psi}(\eta) - 1} \|u_n + \eta\psi\|_{s_1,p,V}^p \right. \\ &\quad - \left(\frac{1}{q} - \frac{1}{1-\gamma} \right) \frac{k_{n,\psi}^q(\eta) - 1}{k_{n,\psi}(\eta) - 1} \|u_n + \eta\psi\|_{s_2,q,V}^q \\ &\quad + \lambda \left(\frac{1}{2q_{s_2}^*(\alpha)} - \frac{1}{1-\gamma} \right) \frac{k_{n,\psi}^{2q_{s_2}^*(\alpha)}(\eta) - 1}{k_{n,\psi}(\eta) - 1} \int_{\mathbb{R}^N} (I_\alpha * |u_n + \eta\psi|^{q_{s_2}^*(\alpha)}) |u_n + \eta\psi|^{q_{s_2}^*(\alpha)} dx \Big\} \\ &\quad - \left(\frac{1}{p} - \frac{1}{1-\gamma} \right) \frac{\|u_n + \eta\psi\|_{s_1,p,V}^p - \|u_n\|_{s_1,p,V}^p}{\eta} \\ &\quad - \left(\frac{1}{q} - \frac{1}{1-\gamma} \right) \frac{\|u_n + \eta\psi\|_{s_2,q,V}^q - \|u_n\|_{s_2,q,V}^q}{\eta} \\ &\quad + \lambda \left(\frac{1}{2q_{s_2}^*(\alpha)} - \frac{1}{1-\gamma} \right) \\ &\quad \times \int_{\mathbb{R}^N} \frac{(I_\alpha * |u_n + \eta\psi|^{q_{s_2}^*(\alpha)}) |u_n + \eta\psi|^{q_{s_2}^*(\alpha)} - (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)}}{\eta} dx. \end{aligned}$$

Letting $\eta \rightarrow 0^+$, we use the continuity of $k_{n,\psi}(\eta)$, (21), Lemmm 10 (i) and $\|u_n\| \leq C_1$ to obtain the following inequality

$$\begin{aligned} \frac{\|\psi\|}{n} &\geq k'_{n,\psi}(0) \left\{ -\frac{\|u_n\|}{n} - \left(1 - \frac{p}{1-\gamma} \right) \|u_n\|_{s_1,p,V}^p - \left(1 - \frac{q}{1-\gamma} \right) \|u_n\|_{s_2,q,V}^q \right. \\ &\quad + \lambda \left(1 - \frac{2q_{s_2}^*(\alpha)}{1-\gamma} \right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx \Big\} \\ &\quad - \left(1 - \frac{p}{1-\gamma} \right) \left(\langle u_n, \psi \rangle_{s_1,p} + \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx \right) \\ &\quad - \left(1 - \frac{q}{1-\gamma} \right) \left(\langle u_n, \psi \rangle_{s_2,q} + \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \right) \\ &\quad + \lambda \left(1 - \frac{2q_{s_2}^*(\alpha)}{1-\gamma} \right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n \psi dx \\ &= k'_{n,\psi}(0) \left\{ -\frac{\|u_n\|}{n} + \frac{1}{1-\gamma} \left((p-1+\gamma) \|u_n\|_{s_1,p,V}^p + (q-1+\gamma) \|u_n\|_{s_2,q,V}^q \right. \right. \\ &\quad \left. \left. - \lambda (2q_{s_2}^*(\alpha) - 1 + \gamma) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx \right) \right\} \\ &\quad - \left(1 - \frac{p}{1-\gamma} \right) \left(\langle u_n, \psi \rangle_{s_1,p} + \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx \right) \\ &\quad - \left(1 - \frac{q}{1-\gamma} \right) \left(\langle u_n, \psi \rangle_{s_2,q} + \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \right) \\ &\quad + \lambda \left(1 - \frac{2q_{s_2}^*(\alpha)}{1-\gamma} \right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n \psi dx \\ &\geq k'_{n,\psi}(0) \left\{ -\frac{C_1}{n} + \frac{C_2}{1-\gamma} \right\} \\ &\quad - \left(1 - \frac{p}{1-\gamma} \right) \left(\langle u_n, \psi \rangle_{s_1,p} + \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx \right) \\ &\quad - \left(1 - \frac{q}{1-\gamma} \right) \left(\langle u_n, \psi \rangle_{s_2,q} + \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \right) \\ &\quad + \lambda \left(1 - \frac{2q_{s_2}^*(\alpha)}{1-\gamma} \right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n \psi dx, \end{aligned} \tag{44}$$

that cannot occur due to the fact that $k'_{n,\psi}(0) = +\infty$ and $-\frac{C_1}{n} + \frac{C_2}{1-\gamma} > 0$ for n large enough. Thus, $k'_{n,\psi}(0) \neq +\infty$. To sum up, $|k'_{n,\psi}(0)| < +\infty$. Furthermore, Lemma 10 (i) combining with $\|u\| \leq C_1$, (42) and (44) also implies that

$$|k'_{n,\psi}(0)| \leq C_3, \tag{45}$$

for n sufficiently large and a suitable positive constant C_3 . This completes the proof. $\square$

Lemma 12. Let $\lambda > 0$ and $\{u_n\} \subset \mathcal{N}_\lambda^\pm$. Then, for each $\psi \in X$, it holds

$$\begin{aligned} & \langle u_n, \psi \rangle_{s_1, p} + \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx + \langle u_n, \psi \rangle_{s_2, q} + \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \\ & - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n \psi dx - \int_{\mathbb{R}^N} f(x) |u_n|^{-\gamma} \psi dx \\ & := H(u_n, \psi) = o_n(1), \end{aligned} \quad (46)$$

as $n \rightarrow \infty$.

Proof. Let $t > 0$ small enough, $\psi \in X$. From (43), it follows that

$$\begin{aligned} & \frac{1}{n} [k_{n, \psi}(t) - 1] \|u_n\| + \frac{t}{n} k_{n, \psi}(t) \|\psi\| \\ & \geq \frac{1}{n} \|u_n - k_{n, \psi}(t)(u_n + t\psi)\| \geq J_\lambda(u_n) - J_\lambda[k_{n, \psi}(t)(u_n + t\psi)], \end{aligned} \quad (47)$$

and

$$\begin{aligned} & J_\lambda(u_n) - J_\lambda[k_{n, \psi}(t)(u_n + t\psi)] \\ & = \frac{1 - k_{n, \psi}^p(t)}{p} \|u_n\|_{s_1, p, V}^p + \frac{1 - k_{n, \psi}^q(t)}{q} \|u_n\|_{s_2, q, V}^q \\ & + \lambda \frac{k_{n, \psi}^{2q_{s_2}^*(\alpha)}(t) - 1}{2q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_n + t\psi|^{q_{s_2}^*(\alpha)}) |u_n + t\psi|^{q_{s_2}^*(\alpha)} dx \\ & + \frac{k_{n, \psi}^{1-\gamma}(t) - 1}{1 - \gamma} \int_{\mathbb{R}^N} f(x) |u_n + t\psi|^{1-\gamma} dx \\ & + \frac{k_{n, \psi}^p(t)}{p} (\|u_n\|_{s_1, p, V}^p - \|u_n + t\psi\|_{s_1, p, V}^p) + \frac{k_{n, \psi}^q(t)}{q} (\|u_n\|_{s_2, q, V}^q - \|u_n + t\psi\|_{s_2, q, V}^q) \\ & + \frac{\lambda}{2q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} [I_\alpha * |u_n + t\psi|^{q_{s_2}^*(\alpha)}) |u_n + t\psi|^{q_{s_2}^*(\alpha)} - I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)}] dx \\ & + \frac{1}{1 - \gamma} \int_{\mathbb{R}^N} f(x) [|u_n + t\psi|^{1-\gamma} - |u_n|^{1-\gamma}] dx. \end{aligned}$$

Combining this with (40) and (47), dividing by $t > 0$ and passing to the $\liminf$ as $t \rightarrow 0^+$, based on the continuity of $k_{n, \psi}(t)$ and Fotou's Lemma, we obtain

$$\begin{aligned} & \frac{|k'_{n, \psi}(0)| \|u_n\| + \|\psi\|}{n} \\ & \geq -k'_{n, \psi}(0) \left[\|u_n\|_{s_1, p, V}^p + \|u_n\|_{s_2, q, V}^q \right. \\ & \quad \left. - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx - \int_{\mathbb{R}^N} f(x) |u_n|^{1-\gamma} dx \right] \\ & - \langle u_n, \psi \rangle_{s_1, p} - \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx - \langle u_n, \psi \rangle_{s_2, q} - \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \\ & + \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n \psi dx \\ & + \liminf_{t \rightarrow 0^+} \frac{1}{1 - \gamma} \int_{\mathbb{R}^N} f(x) \frac{(u_n + t\psi)^{1-\gamma} - u_n^{1-\gamma}}{t} dx \\ & \geq -\langle u_n, \psi \rangle_{s_1, p} - \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx - \langle u_n, \psi \rangle_{s_2, q} - \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \\ & + \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n \psi dx \\ & + \int_{\mathbb{R}^N} f(x) \liminf_{t \rightarrow 0^+} \frac{(u_n + t\psi)^{1-\gamma} - u_n^{1-\gamma}}{(1 - \gamma)t} dx \end{aligned}$$

$$= -\langle u_n, \psi \rangle_{s_1, p} - \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx - \langle u_n, \psi \rangle_{s_2, q} - \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \\ + \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n \psi dx + \int_{\mathbb{R}^N} f(x) u_n^{-\gamma} \psi dx.$$

Thus, for $\psi \in X$ with $\psi \geq 0$, we derive that

$$\begin{aligned} & \langle u_n, \psi \rangle_{s_1, p} + \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx + \langle u_n, \psi \rangle_{s_2, q} + \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \\ & - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n \psi dx - \int_{\mathbb{R}^N} f(x) u_n^{-\gamma} \psi dx \\ & \geq -\frac{|k'_{n, \psi}(0)| \|u_n\| + \|\psi\|}{n}. \end{aligned} \quad (48)$$

Since the above inequality holds for $-\psi$, it follows that

$$\begin{aligned} & \langle u_n, \psi \rangle_{s_1, p} + \int_{\mathbb{R}^N} V(x) |u_n|^{p-2} u_n \psi dx + \langle u_n, \psi \rangle_{s_2, q} + \int_{\mathbb{R}^N} V(x) |u_n|^{q-2} u_n \psi dx \\ & - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n \psi dx - \int_{\mathbb{R}^N} f(x) u_n^{-\gamma} \psi dx \\ & \leq \frac{|k'_{n, \psi}(0)| \|u_n\| + \|\psi\|}{n}. \end{aligned} \quad (49)$$

From $\|u_n\| \leq C_1$ and $|k'_{n, \psi}(0)| \leq C_3$ (Lemma 11) uniformly in n large, we know by (48) and (49) that (46) holds. The proof is complete. $\square$

For convenience, let

$$c_\lambda = \frac{\alpha + qs_2}{q(N + \alpha)} \lambda^{-\frac{N - qs_2}{\alpha + qs_2}} S_H^{\frac{N + \alpha}{\alpha + qs_2}} - K_* - (2 - 2^{2-q}) \frac{\alpha + qs_2}{q(N + \alpha)},$$

where K_* as shown in (8).

Lemma 13. If $\{u_n\} \subset \mathcal{N}_\lambda^\pm$ is a $(PS)_c$ sequence of J_λ with $c < c_\lambda$, then the sequence $\{u_n\}$ has a convergence subsequence.

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence of J_λ , the coerciveness of J_λ on $\mathcal{N}_\lambda$ shows that $\{u_n\}$ is bounded. Up to a subsequence, there exists $u_0 \in X$ such that

$$\begin{aligned} u_n & \rightharpoonup u_0 \quad \text{in } X, \\ u_n & \rightarrow u_0 \quad \text{in } L^r(\mathbb{R}^N) \quad \text{for any } 1 \leq r < q_{s_2}^*, \\ u_n & \rightarrow u_0 \quad \text{a.e. in } \mathbb{R}^N, \quad u_n \leq g \text{ a.e. in } \mathbb{R}^N, \end{aligned} \quad (50)$$

as $n \rightarrow \infty$, with $g \in L^r(\mathbb{R}^N)$ for a fixed $r \in [1, q_{s_2}^*)$. From (31), (32), (14) and Lemma 4, we obtain

$$\|u_n\|_{s_1, p, V}^p = \|u_n - u_0\|_{s_1, p, V}^p + \|u_0\|_{s_1, p, V}^p + o_n(1), \quad (51)$$

$$\|u_n\|_{s_2, q, V}^q = \|u_n - u_0\|_{s_2, q, V}^q + \|u_0\|_{s_2, q, V}^q + o_n(1), \quad (52)$$

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)-2} u_n dx &= \int_{\mathbb{R}^N} (I_\alpha * |u_n - u_0|^{q_{s_2}^*(\alpha)}) |u_n - u_0|^{q_{s_2}^*(\alpha)-2} u_n dx \\ &+ \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 dx + o_n(1), \end{aligned} \quad (53)$$

and

$$\int_{\mathbb{R}^N} f(x) |u_n|^{1-\gamma} dx = \int_{\mathbb{R}^N} f(x) |u_n - u_0|^{1-\gamma} dx + \int_{\mathbb{R}^N} f(x) |u_0|^{1-\gamma} dx + o_n(1). \quad (54)$$

Taking the test function $\psi = u_n - u_0$ in (46), we deduce from (50) to (53), and Lemma 3 that, as $n \rightarrow \infty$,

$$o_n(1) = \|u_n - u_0\|_{s_1,p,V}^p + \|u_n - u_0\|_{s_2,q,V}^q - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_n - u_0|^{q_{s_2}^*(\alpha)}) |u_n - u_0|^{q_{s_2}^*(\alpha)} dx - \int_{\mathbb{R}^N} f(x) u_n^{-\gamma} (u_n - u_0) dx + o_n(1).$$

Hence

$$\lim_{n \rightarrow \infty} \|u_n - u_0\|_{s_1,p,V}^p + \lim_{n \rightarrow \infty} \|u_n - u_0\|_{s_2,q,V}^q = \lambda \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n - u_0|^{q_{s_2}^*(\alpha)}) |u_n - u_0|^{q_{s_2}^*(\alpha)} dx + \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x) u_n^{-\gamma} (u_n - u_0) dx. \quad (55)$$

Let

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n - u_0|^{q_{s_2}^*(\alpha)}) |u_n - u_0|^{q_{s_2}^*(\alpha)} dx := I^{2q_{s_2}^*(\alpha)}. \quad (56)$$

Since by (50) one has $u_n^{1-\gamma} \leq g^{1-\gamma}$ a.e. in $\mathbb{R}^N$, by Lebesgue dominated convergence theorem

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x) u_n^{1-\gamma} dx = \int_{\mathbb{R}^N} f(x) u_0^{1-\gamma} dx. \quad (57)$$

Moreover, we obtain by Fatou's lemma that

$$\int_{\mathbb{R}^N} f(x) u_0^{1-\gamma} dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x) u_n^{-\gamma} u_0 dx. \quad (58)$$

It follows from (55) to (58) that

$$\lim_{n \rightarrow \infty} \|u_n - u_0\|_{s_1,p,V}^p + \lim_{n \rightarrow \infty} \|u_n - u_0\|_{s_2,q,V}^q \leq \lambda I^{2q_{s_2}^*(\alpha)}. \quad (59)$$

It is clear that $I \geq 0$. If $I = 0$, then by (59) one yields that $u_n \rightarrow u_0$ in X as $n \rightarrow \infty$, concluding the proof. Hence, let us assume by contradiction that $I > 0$. By (11), we have

$$\|u_n - u_0\|_{s_1,p,V}^p + \|u_n - u_0\|_{s_2,q,V}^q \geq [u_n - u_0]_{s_2,q}^q \geq S_H I^q. \quad (60)$$

Combining (59) and (60), we get

$$\lambda I^{2q_{s_2}^*(\alpha)-q} \geq S_H. \quad (61)$$

Substituting (61) into (60), we have

$$\lim_{n \rightarrow \infty} \|u_n - u_0\|_{s_1,p,V}^p + \lim_{n \rightarrow \infty} \|u_n - u_0\|_{s_2,q,V}^q \geq \left(\frac{S_H^{2q_{s_2}^*(\alpha)}}{\lambda^q} \right)^{\frac{1}{2q_{s_2}^*(\alpha)-q}}. \quad (62)$$

From (46), (51), (52) and (54), we derive

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{2q_{s_2}^*(\alpha)} H(u_n, u_n) &= \left(\frac{1}{p} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|u_n\|_{s_1,p,V}^p + \left(\frac{1}{q} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|u_n\|_{s_2,q,V}^q \\ &\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \int_{\mathbb{R}^N} f(x) |u_n|^{1-\gamma} dx + o_n(1) \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{p} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|u_n - u_0\|_{s_1,p,V}^p + \left(\frac{1}{q} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|u_n - u_0\|_{s_2,q,V}^q \\
&\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \int_{\mathbb{R}^N} f(x) |u_n - u_0|^{1-\gamma} dx \\
&\quad + \left(\frac{1}{p} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|u_0\|_{s_1,p,V}^p + \left(\frac{1}{q} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|u_0\|_{s_2,q,V}^q \\
&\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \int_{\mathbb{R}^N} f(x) |u_0|^{1-\gamma} dx + o_n(1) \\
&\geq \left(\frac{1}{q} - \frac{1}{2q_{s_2}^*(\alpha)} \right) (\|u_n - u_0\|_{s_1,p,V}^p + \|u_n - u_0\|_{s_2,q,V}^q) \\
&\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \int_{\mathbb{R}^N} f(x) |u_n - u_0|^{1-\gamma} dx \\
&\quad + \left(\frac{1}{q} - \frac{1}{2q_{s_2}^*(\alpha)} \right) (\|u_0\|_{s_1,p,V}^p + \|u_0\|_{s_2,q,V}^q) \\
&\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} S_{q_{s_2}^*(\alpha)}^{1-\gamma} \|u_0\|^{1-\gamma} + o_n(1).
\end{aligned} \tag{63}$$

Note that from (25), we have

$$\begin{aligned}
&\|u_0\|_{s_1,p,V}^p + \|u_0\|_{s_2,q,V}^q \\
&\geq 2^{1-p} (1 + \|u_0\|_{s_1,p,V})^p + 2^{1-q} (1 + \|u_0\|_{s_2,q,V})^q - 2 \\
&\geq 2^{1-q} [(1 + \|u_0\|_{s_1,p,V})^p + (1 + \|u_0\|_{s_2,q,V})^q] - 2 \\
&\geq 2^{2-p-q} (2 + \|u_0\|_{s_1,p,V} + \|u_0\|_{s_2,q,V})^p - 2 \\
&= 2^{2-p-q} (2 + \|u_0\|)^p - 2 \geq 2^{2-p-q} \|u_0\|^p + 2^{2-q} - 2.
\end{aligned} \tag{64}$$

We define

$$\begin{aligned}
Q(t) &= \left(\frac{1}{q} - \frac{1}{2q_{s_2}^*(\alpha)} \right) 2^{2-p-q} t^p \\
&\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} S_{q_{s_2}^*(\alpha)}^{1-\gamma} t^{1-\gamma}.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
Q(t) &\geq -\frac{p-1+\gamma}{p} \left(\frac{2q_{s_2}^*(\alpha)-1+\gamma}{2(1-\gamma)q_{s_2}^*(\alpha)} \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2q)}} S_{q_{s_2}^*(\alpha)}^{1-\gamma} \right)^{\frac{p}{p-1+\gamma}} \\
&\quad \times \left(\frac{2(1-\gamma)q_{s_2}^*(\alpha)}{(2q_{s_2}^*(\alpha)-q)p2^{2-p-q}} \right)^{\frac{1-\gamma}{p-1+\gamma}} \\
&= -K_*.
\end{aligned} \tag{65}$$

Substituting (64) into (63) and using the estimate (65), we obtain

$$\begin{aligned}
&J_\lambda(u_n) - \frac{1}{2q_{s_2}^*(\alpha)} H(u_n, u_n) \\
&\geq \left(\frac{1}{q} - \frac{1}{2q_{s_2}^*(\alpha)} \right) (\|u_n - u_0\|_{s_1,p,V}^p + \|u_n - u_0\|_{s_2,q,V}^q) \\
&\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2q_{s_2}^*(\alpha)} \right) \int_{\mathbb{R}^N} f(x) |u_n - u_0|^{1-\gamma} dx \\
&\quad - K_* - (2 - 2^{2-q}) \left(\frac{1}{q} - \frac{1}{2q_{s_2}^*(\alpha)} \right) + o_n(1).
\end{aligned}$$

Letting $n \rightarrow \infty$, it follows from (50) and (62) that

$$c \geq \frac{\alpha + qs_2}{q(N+\alpha)} \lambda^{-\frac{N-qs_2}{\alpha+qs_2}} S_H^{\frac{N+\alpha}{\alpha+qs_2}} - K_* - (2 - 2^{2-q}) \frac{\alpha + qs_2}{q(N+\alpha)},$$

which contradicts the assumption $c < c_\lambda$. This concludes the proof. $\square$

Remark 2. If $0 < \lambda < \lambda_*$ (λ_* as in (7)), then $c_\lambda > 0$.

The proof of the main result of this paper is given below.

Proof of Theorem 1. Employing Ekeland variational principle to construct a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda^+$ satisfying

$$(i) J_\lambda(u_n) < \inf_{\mathcal{N}_\lambda^+} J_\lambda + \frac{1}{n},$$

$$(ii) J_\lambda(z) \geq J_\lambda(u_n) - \frac{1}{n} \|u_n - z\|, \quad \forall z \in \mathcal{N}_\lambda^+.$$

From $J_\lambda(|u_n|) = J_\lambda(u_n)$ and Lemma 9, we may assume that $u_n \geq 0$. The coerciveness of J_λ on $\mathcal{N}_\lambda$ shows that $\|u_n\| \leq C_1$ for some positive constant C_1 . Thus, there exists a subsequence of $\{u_n\}$ (still represented by $\{u_n\}$) and $u_0 \in X$ such that $u_n \rightharpoonup u_0$ in X and $u_n \rightarrow u_0$ a.e. in $\mathbb{R}^N$. As a consequence, applying Lemma 9 with $c = m_\lambda < 0 < c_\lambda$ (by Remark 2), we yields that $u_n \rightarrow u_0$ in X , up to a subsequence. Moreover, by the weakly lower semi-continuity of the norm, (14) and (53), we obtain

$$\begin{aligned} J_\lambda(u_0) &= \frac{1}{p} \|u_0\|_{s_1,p,V}^p + \frac{1}{q} \|u_0\|_{s_2,q,V}^q - \frac{1}{1-\gamma} \int_{\mathbb{R}^N} f(x) |u_0|^{1-\gamma} dx \\ &\quad - \frac{\lambda}{2q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)} dx \\ &\leq \liminf_{n \rightarrow \infty} \left[\frac{1}{p} \|u_n\|_{s_1,p,V}^p + \frac{1}{q} \|u_n\|_{s_2,q,V}^q - \frac{1}{1-\gamma} \int_{\mathbb{R}^N} f(x) |u_n|^{1-\gamma} dx \right. \\ &\quad \left. - \frac{\lambda}{2q_{s_2}^*(\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{q_{s_2}^*(\alpha)}) |u_n|^{q_{s_2}^*(\alpha)} dx \right] \\ &= \liminf_{n \rightarrow \infty} J_\lambda(u_n) = \inf_{\mathcal{N}_\lambda^+} J_\lambda < 0, \end{aligned} \quad (66)$$

which implies that $u_0 \geq 0$ and $u_0 \neq 0$. Moreover, Lemma 10 show that $u_0 \in \mathcal{N}_\lambda^+$. Now, we will prove that u_0 is a solution of problem (4). For any $v \in X$ and $\varepsilon > 0$, set $\psi_\varepsilon = u_0 + \varepsilon v$ and $\Omega_\varepsilon = \{x \in \mathbb{R}^N : \psi_\varepsilon \leq 0\}$. Applying inequality (48) with $\psi = \psi_\varepsilon^+$, we deduce that

$$\begin{aligned} 0 &\leq \frac{1}{\varepsilon} \left\{ \langle u_0, \psi_\varepsilon^+ \rangle_{s_1,p} + \int_{\mathbb{R}^N} V(x) |u_0|^{p-2} u_0 \psi_\varepsilon^+ dx + \langle u_0, \psi_\varepsilon^+ \rangle_{s_2,q} \right. \\ &\quad \left. + \int_{\mathbb{R}^N} V(x) |u_0|^{q-2} u_0 \psi_\varepsilon^+ dx \right. \\ &\quad \left. - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 \psi_\varepsilon^+ dx - \int_{\mathbb{R}^N} f(x) u_0^{-\gamma} \psi_\varepsilon^+ dx \right\}. \end{aligned} \quad (67)$$

By direct calculation, we obtain

$$\begin{aligned} \langle u_0, \psi_\varepsilon^+ \rangle_{s_1,p} &= \iint_{(\mathbb{R}^N \setminus \Omega_\varepsilon) \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (\psi_\varepsilon(x) - \psi_\varepsilon(y))}{|x - y|^{N+ps_1}} dx dy \\ &\quad + \iint_{(\mathbb{R}^N \setminus \Omega_\varepsilon) \times \Omega_\varepsilon} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (\psi_\varepsilon^+(x) - \psi_\varepsilon^+(y))}{|x - y|^{N+ps_1}} dx dy \\ &\leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^p}{|x - y|^{N+ps_1}} dx dy + \varepsilon \langle u_0, v \rangle_{s_1,p} \\ &\quad - \varepsilon \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+ps_1}} dx dy \\ &\quad + 2 \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x - y|^{N+ps_1}} dx dy. \end{aligned} \quad (68)$$

Similarly, we have

$$\begin{aligned} \langle u_0, \psi_\varepsilon^+ \rangle_{s_2, q} &\leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^q}{|x - y|^{N+qs_2}} dx dy + \varepsilon \langle u_0, v \rangle_{s_2, q} \\ &\quad - \varepsilon \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{|u_0(x) - u_0(y)|^{q-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+qs_2}} dx dy \\ &\quad + 2 \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{q-2} (u_0(x) - u_0(y)) (\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x - y|^{N+qs_2}} dx dy. \end{aligned} \quad (69)$$

Moreover, we deduce that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} V(x) |u_0|^{p-2} u_0 \psi_\varepsilon dx &= \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} V(x) |u_0|^p dx + \varepsilon \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} V(x) |u_0|^{p-2} u_0 v dx \\ &\leq \int_{\mathbb{R}^N} V(x) |u_0|^p dx + \varepsilon \int_{\mathbb{R}^N} V(x) |u_0|^{p-2} u_0 v dx - \varepsilon \int_{\Omega_\varepsilon} V(x) |u_0|^{p-2} u_0 v dx, \end{aligned} \quad (70)$$

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} V(x) |u_0|^{q-2} u_0 \psi_\varepsilon dx \\ \leq \int_{\mathbb{R}^N} V(x) |u_0|^q dx + \varepsilon \int_{\mathbb{R}^N} V(x) |u_0|^{q-2} u_0 v dx - \varepsilon \int_{\Omega_\varepsilon} V(x) |u_0|^{q-2} u_0 v dx, \end{aligned} \quad (71)$$

$$\begin{aligned} -\lambda \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 \psi_\varepsilon dx \\ = -\lambda \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 \psi_\varepsilon dx + \lambda \int_{\Omega_\varepsilon} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 \psi_\varepsilon dx \\ = -\lambda \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)} dx - \lambda \varepsilon \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 v dx \\ + \lambda \int_{\Omega_\varepsilon} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 \psi_\varepsilon dx, \end{aligned} \quad (72)$$

$$\begin{aligned} - \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} f(x) u_0^{-\gamma} \psi_\varepsilon dx &= - \int_{\mathbb{R}^N} f(x) u_0^{-\gamma} \psi_\varepsilon dx + \int_{\Omega_\varepsilon} f(x) u_0^{-\gamma} \psi_\varepsilon dx \\ &= - \int_{\mathbb{R}^N} f(x) u_0^{1-\gamma} dx - \varepsilon \int_{\mathbb{R}^N} f(x) u_0^{-\gamma} v dx + \int_{\Omega_\varepsilon} f(x) u_0^{-\gamma} \psi_\varepsilon dx, \end{aligned} \quad (73)$$

$$\begin{aligned} \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x - y|^{N+ps_1}} dx dy \\ = \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (-\psi_\varepsilon(x))}{|x - y|^{N+ps_1}} dx dy \\ \leq \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (-\psi_\varepsilon(x) + \psi_\varepsilon(y))}{|x - y|^{N+ps_1}} dx dy \\ = \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) [-(u_0(x) - u_0(y)) - \varepsilon(v(x) - v(y))]}{|x - y|^{N+ps_1}} dx dy \\ = - \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^p}{|x - y|^{N+ps_1}} dx dy \\ - \varepsilon \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+ps_1}} dx dy \\ \leq - \varepsilon \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+ps_1}} dx dy \\ \leq \varepsilon \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \left| \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+ps_1}} \right| dx dy, \end{aligned} \quad (74)$$

and

$$\begin{aligned} \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{q-2} (u_0(x) - u_0(y)) (\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x - y|^{N+qs_2}} dx dy \\ \leq \varepsilon \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \left| \frac{|u_0(x) - u_0(y)|^{q-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+qs_2}} \right| dx dy. \end{aligned} \quad (75)$$

Substitute equations (68)–(75) into (67), we obtain

$$\begin{aligned}
0 &\leq \frac{1}{\varepsilon} \left\{ \|u_0\|_{s_1,p,V}^p + \|u_0\|_{s_2,q,V}^q - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)} dx - \int_{\mathbb{R}^N} f(x) u_0^{1-\gamma} dx \right\} \\
&\quad + \left\{ \langle u_0, v \rangle_{s_1,p} + \int_{\mathbb{R}^N} V(x) |u_0|^{p-2} u_0 v dx + \langle u_0, v \rangle_{s_2,q} + \int_{\mathbb{R}^N} V(x) |u_0|^{q-2} u_0 v dx \right. \\
&\quad \left. - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 v dx - \int_{\mathbb{R}^N} f(x) u_0^{-\gamma} v dx \right\} \\
&\quad - \left\{ \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+ps_1}} dx dy \right. \\
&\quad \left. + \int_{\Omega_\varepsilon} V(x) |u_0|^{p-2} u_0 v dx \right. \\
&\quad \left. + \iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{|u_0(x) - u_0(y)|^{q-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+qs_2}} dx dy \right. \\
&\quad \left. + \int_{\Omega_\varepsilon} V(x) |u_0|^{q-2} u_0 v dx \right\} \\
&\quad + \frac{1}{\varepsilon} \left\{ \lambda \int_{\Omega_\varepsilon} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 \psi_\varepsilon dx + \int_{\Omega_\varepsilon} f(x) u_0^{-\gamma} \psi_\varepsilon dx \right\} \\
&\quad + \frac{2}{\varepsilon} \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x - y|^{N+ps_1}} dx dy \\
&\quad + \frac{2}{\varepsilon} \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \frac{|u_0(x) - u_0(y)|^{q-2} (u_0(x) - u_0(y)) (\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x - y|^{N+qs_2}} dx dy \\
&\leq \left\{ \langle u_0, v \rangle_{s_1,p} + \int_{\mathbb{R}^N} V(x) |u_0|^{p-2} u_0 v dx + \langle u_0, v \rangle_{s_2,q} + \int_{\mathbb{R}^N} V(x) |u_0|^{q-2} u_0 v dx \right. \\
&\quad \left. - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 v dx - \int_{\mathbb{R}^N} f(x) u_0^{-\gamma} v dx \right\} \\
&\quad - \left[\iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+ps_1}} dx dy \right. \\
&\quad \left. + \int_{\Omega_\varepsilon} V(x) |u_0|^{p-2} u_0 v dx \right] \\
&\quad - \left[\iint_{\Omega_\varepsilon \times \Omega_\varepsilon} \frac{|u_0(x) - u_0(y)|^{q-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+qs_2}} dx dy \right. \\
&\quad \left. + \int_{\Omega_\varepsilon} V(x) |u_0|^{q-2} u_0 v dx \right] \\
&\quad + 2 \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \left| \frac{|u_0(x) - u_0(y)|^{p-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+ps_1}} \right| dx dy \\
&\quad + 2 \iint_{\Omega_\varepsilon \times (\mathbb{R}^N \setminus \Omega_\varepsilon)} \left| \frac{|u_0(x) - u_0(y)|^{q-2} (u_0(x) - u_0(y)) (v(x) - v(y))}{|x - y|^{N+qs_2}} \right| dx dy.
\end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$ to the above inequality and employing the fact $|\Omega_\varepsilon| \rightarrow 0$ as $\varepsilon \rightarrow 0^+$, we can deduce that

$$\begin{aligned}
&\langle u_0, v \rangle_{s_1,p} + \int_{\mathbb{R}^N} V(x) |u_0|^{p-2} u_0 v dx + \langle u_0, v \rangle_{s_2,q} + \int_{\mathbb{R}^N} V(x) |u_0|^{q-2} u_0 v dx \\
&\quad - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 v dx - \int_{\mathbb{R}^N} f(x) u_0^{-\gamma} v dx \geq 0, \quad \forall v \in X.
\end{aligned}$$

This inequality remains true for $-v$. Therefore, we conclude

$$\begin{aligned}
&\langle u_0, v \rangle_{s_1,p} + \int_{\mathbb{R}^N} V(x) |u_0|^{p-2} u_0 v dx + \langle u_0, v \rangle_{s_2,q} + \int_{\mathbb{R}^N} V(x) |u_0|^{q-2} u_0 v dx \\
&\quad - \lambda \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{q_{s_2}^*(\alpha)}) |u_0|^{q_{s_2}^*(\alpha)-2} u_0 v dx - \int_{\mathbb{R}^N} f(x) u_0^{-\gamma} v dx = 0, \quad \forall v \in X.
\end{aligned} \tag{76}$$

Owing to (76), it is easy to know that

$$(-\Delta)_p^{s_1} u_0 + (-\Delta)_q^{s_2} u_0 + V(x) (|u_0|^{p-2} u_0 + |u_0|^{q-2} u_0) \geq 0. \tag{77}$$

Suppose that there exists $x_0 \in \mathbb{R}^N$ such that $u_0(x_0) = 0$, then from (77), we have that $(-\Delta)_p^{s_1} u_0(x_0) + (-\Delta)_q^{s_2} u_0(x_0) \geq 0$. On the other hand, since $u_0 \geq 0$, $u_0 \not\equiv 0$, $p > 1 + s_1$ and $q > 1 + s_2$, we obtain from Lemma 5 that

$$\begin{aligned} & (-\Delta)_p^{s_1} u_0(x_0) + (-\Delta)_q^{s_2} u_0(x_0) \\ &= - \int_{\mathbb{R}^N} \frac{|u_0(x_0 + y) - u_0(x_0)|^{p-2} (u_0(x_0 + y) - u_0(x_0))}{|y|^{N+s_1 p}} dy \\ &\quad - \int_{\mathbb{R}^N} \frac{|u_0(x_0 - y) - u_0(x_0)|^{p-2} (u_0(x_0 - y) - u_0(x_0))}{|y|^{N+s_1 p}} dy \\ &\quad - \int_{\mathbb{R}^N} \frac{|u_0(x_0 + y) - u_0(x_0)|^{q-2} (u_0(x_0 + y) - u_0(x_0))}{|y|^{N+s_2 q}} dy \\ &\quad - \int_{\mathbb{R}^N} \frac{|u_0(x_0 - y) - u_0(x_0)|^{q-2} (u_0(x_0 - y) - u_0(x_0))}{|y|^{N+s_2 q}} dy \\ &= - \int_{\mathbb{R}^N} \frac{|u_0(x_0 + y)|^{p-2} u_0(x_0 + y) + |u_0(x_0 - y)|^{p-2} u_0(x_0 - y)}{|y|^{N+s_1 p}} dy \\ &\quad - \int_{\mathbb{R}^N} \frac{|u_0(x_0 + y)|^{q-2} u_0(x_0 + y) + |u_0(x_0 - y)|^{q-2} u_0(x_0 - y)}{|y|^{N+s_2 q}} dy \\ &< 0, \end{aligned}$$

which is a contradiction. Thus, $u_0 > 0$ in $\mathbb{R}^N$ and $u_0 \in \mathcal{N}_\lambda^+$ is a solution of problem (4) with $J_\lambda(u_0) = \inf_{\mathcal{N}_\lambda^+} J_\lambda$ when $0 < \|f\|_{\frac{q(N+\alpha)}{q(N+\alpha)-2(1-\gamma)(N-s_2 q)}} < M_\lambda$. The proof of Theorem 1 is completed. $\square$

Remark 3. From [40], we know that the Critical exponent problem involving the p -Laplacian and Choquard terms is an important question, but until now, it is unknown whether the optimal constant S_H in (11) can be achieved, and if it can be achieved, the form of u is also unknown (see open problem 3 in [40]). Therefore, according to the existing methods [41,42], we are currently unable to obtain a second solution in $\mathcal{N}_\lambda^-$.

Author Contributions: Conceptualization, Z.B.; Methodology, Z.B.; formal analysis, Z.B.; investigation, Z.B. and C.B.; writing—original draft preparation, Z.B.; writing—review and editing, Z.B. and C.B. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Natural Science Foundation of China (11571136).

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: The authors express their gratitude to the editor and reviewers for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflicts of interest in this work.

References

1. Pekar, S. *Untersuchungen über die Elektronentheorie der Kristalle*; Akademie Verlag: Berlin, Germany, 1954.
2. Lieb, E.H. Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Stud. Appl. Math.* **1976**, *57*, 93–105. [\[CrossRef\]](#)
3. Penrose, R. On gravity's role in quantum state reduction. *Gen. Relativ. Gravit.* **1996**, *28*, 581–600. [\[CrossRef\]](#)
4. Moroz, M.; Penrose, R.; Tod, P. Spherically-symmetric solutions of the Schrödinger-Newton equations. *Class. Quantum Gravity* **1998**, *15*, 2733–2742. [\[CrossRef\]](#)
5. Moroz, V.; Van Schaftingen, J. A guide to the Choquard equation. *J. Fixed Point Theory Appl.* **2017**, *19*, 773–813. [\[CrossRef\]](#)
6. Moroz, V.; Van Schaftingen, J. Groundstates of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics. *J. Funct. Anal.* **2013**, *265*, 153–184. [\[CrossRef\]](#)
7. Ghimenti, M.; Van Schaftingen, J. Nodal solutions for the Choquard equation. *J. Funct. Anal.* **2016**, *271*, 107–135. [\[CrossRef\]](#)
8. Seok, J. Nonlinear Choquard equations involving a critical local term. *Appl. Math. Lett.* **2017**, *63*, 77–87. [\[CrossRef\]](#)
9. Li, X.; Ma, S.; Zhang, G. Existence and qualitative properties of solutions for Choquard equations with a local term. *Nonlinear Anal. Real World Appl.* **2019**, *45*, 1–25. [\[CrossRef\]](#)

10. Zhong, X.; Tang, C. Ground state sign-changing solutions for a class of subcritical Choquard equations with a critical pure power nonlinearity in $\mathbb{R}^N$. *Comput. Math. Appl.* **2018**, *76*, 23–34. [\[CrossRef\]](#)
11. Yu, S.; Chen, J. Multiple and asymptotical behavior of solutions to a Choquard equation with singularity. *J. Math. Anal. Appl.* **2022**, *511*, 126047. [\[CrossRef\]](#)
12. Mukherjee, T.; Sreenadh, K. Positive solutions for nonlinear Choquard equation with singular nonlinearity. *Complex Var. Elliptic Equ.* **2017**, *62*, 1044–1071. [\[CrossRef\]](#)
13. Sang, Y.; Hsu, T. Fractional Kirchhoff-Choquard system with upper critical exponent and singular nonlinearity. *J. Pseudo-Differ. Oper. Appl.* **2022**, *13*, 10. [\[CrossRef\]](#)
14. Chen, W.; Rădulescu, V.D.; Zhang, B. Fractional Choquard-Kirchhoff problems with critical nonlinearity and Hardy potential. *Anal. Math. Phys.* **2021**, *11*, 132. [\[CrossRef\]](#)
15. Ambrosio, V.; Autuori, G.; Isernia, T. Existence of least energy solutions for a quasilinear Choquard equation. *Commun. Pure Appl. Anal.* **2024**, *23*, 1661–1678. [\[CrossRef\]](#)
16. Alves, C.O.; Yang, M. Investigating the multiplicity and concentration behaviour of solutions for a quasilinear Choquard equation via the penalization method. *Proc. R. Soc. Edinb. Sect. A* **2016**, *146*, 23–58. [\[CrossRef\]](#)
17. d’Avenia, P.; Siciliano, G.; Squassina, M. On fractional Choquard equations. *Math. Model. Methods Appl. Sci.* **2015**, *25*, 1447–1476. [\[CrossRef\]](#)
18. Ambrosio, V.; Isernia, T.; Temperini, L. Least energy solutions for nonlinear fractional Choquard–Kirchhoff equations in $\mathbb{R}^N$. *Discrete Contin. Dyn. Syst.* **2025**, *45*, 4817–4851.
19. Bisci, G.M.; Rădulescu, V.D.; Servadei, R. *Variational Methods for Nonlocal Fractional Problems*; Cambridge University Press: Cambridge, UK, 2016.
20. Lamred, H.; Nasri, Y. Fractional elliptic equations involving Hardy potential and critical growth. *J. Elliptic Parabol. Equ.* **2025**, *11*, 1537–1568. [\[CrossRef\]](#)
21. Al-Raei, M. Applying fractional quantum mechanics to systems with electrical screening effects. *Chaos Solitons Fractals* **2021**, *150*, 111209. [\[CrossRef\]](#)
22. Gao, Y.; Liu, L.; Wei, N.; Gu, H.; Wu, Y. New existence of multiple solutions for fractional Kirchhoff equations with logarithmic nonlinearity. *Fractal Fract.* **2025**, *9*, 646. [\[CrossRef\]](#)
23. Ball, J. Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. *Philos. Trans. R. Soc. A Math. Phys. Eng. Sci.* **1982**, *306*, 557–611. [\[CrossRef\]](#)
24. [\[CrossRef\]](#) Ambrosio, V.; Repovš, D. Multiplicity and concentration results for a (p, q) -Laplacian problem in $\mathbb{R}^N$. *Z. Angew. Math. Phys.* **2021**, *72*, 33. [\[CrossRef\]](#)
25. Xie, X.; Wang, T.; Zhang, W. Existence of solutions for the (p, q) -Laplacian equation with nonlocal Choquard reaction. *Appl. Math. Lett.* **2023**, *135*, 108418. [\[CrossRef\]](#)
26. Ambrosio, V.; Isernia, T. Ground state solutions for a (p, q) -Choquard equation with a general nonlinearity. *J. Differ. Equ.* **2024**, *401*, 428–468. [\[CrossRef\]](#)
27. Zhang, A.; Deng, Z. An existence result for a fractional elliptic system involving (p, q) -Laplacian and critical exponent. *Complex Var. Elliptic Equ.* **2023**, *68*, 746–762. [\[CrossRef\]](#)
28. Fan, Z. On fractional (p, q) -Laplacian equations involving subcritical or critical Hardy exponents. *J. Pseudo-Differ. Oper. Appl.* **2022**, *13*, 63. [\[CrossRef\]](#)
29. Alves, C.O.; Ambrosio, V.; Isernia, T. Existence, multiplicity and concentration for a class of fractional p & q -Laplacian problems in $\mathbb{R}^N$. *Commun. Pure Appl. Anal.* **2019**, *18*, 2009–2045.
30. Ambrosio, V.; Isernia, T. A multiplicity result for a (p, q) -Schrödinger-Kirchhoff type equation. *Ann. Mat. Pura Appl.* **2022**, *201*, 943–984. [\[CrossRef\]](#)
31. Baldelli, L.; Brizi, Y.; Filippucci, R. Multiplicity results for (p, q) -Laplacian equations with critical exponent in $\mathbb{R}^N$ and negative energy. *Calc. Var.* **2021**, *60*, 8. [\[CrossRef\]](#)
32. [\[CrossRef\]](#) Chen, W.; Feng, D. Critical fractional (p, q) -Kirchhoff type problem with a generalized Choquard nonlinearity and magnetic field. *Bull. Malays. Math. Sci. Soc.* **2024**, *47*, 33. [\[CrossRef\]](#)
33. De Araujo, A.L.A.; Medeiros, V. Eigenvalue for a problem involving the fractional (p, q) -Laplacian operator and nonlinearity with a singular and a supercritical Sobolev growth. *Anal. Math. Phys.* **2024**, *14*, 16. [\[CrossRef\]](#)
34. Lin, X.; Zheng, S. Multiplicity and asymptotic behavior of solutions to fractional (p, q) -Kirchhoff type problems with critical Sobolev-Hardy exponent. *Electron. J. Differ. Equ.* **2021**, *66*, 1–20. [\[CrossRef\]](#)
35. Zuo, J.; Zhang, W.; Rădulescu, V.D. Multiplicity and concentration properties for (p, q) -Kirchhoff non-autonomous problems with Choquard nonlinearity. *Bull. Sci. math.* **2024**, *191*, 103398. [\[CrossRef\]](#)
36. Ao, Y. Existence of solutions for a class of nonlinear Choquard equations with critical growth. *Appl. Anal.* **2021**, *100*, 465–481. [\[CrossRef\]](#)

37. Gao, F.; Yang, M. The Brezis-Nirenberg type critical problem for the nonlinear Choquard equation. *Sci. China Math.* **2018**, *61*, 1219–1242. [[CrossRef](#)]
38. Di Nezza, E.; Palatucci, G.; Valdinoci, E. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **2012**, *136*, 521–573. [[CrossRef](#)]
39. Ambrosio, V.; Isernia, T. On a fractional p -Laplacian problem with critical Sobolev-Hardy exponents. *Mediterr. J. Math.* **2018**, *15*, 219. [[CrossRef](#)]
40. Mukherjee, T.; Sreenadh, K. Critical growth elliptic problems with Choquard type nonlinearity: A survey. *arXiv* **2018**, arXiv:1811.04353. [[CrossRef](#)]
41. Fiscella, A.; Mishra, P.K. The Nehari manifold for fractional Kirchhoff problems involving singular and critical terms. *Nonlinear Anal.* **2019**, *186*, 6–32. [[CrossRef](#)]
42. Lei, C.; Suo, H.; Chu, C. Multiple positive solutions for a Schrodinger-Newton system with singularity and critical growth. *Electron. J. Differ. Equ.* **2018**, *86*, 1–15.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.