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Well-Posedness and Ulam-Hyers Stability of Coupled Deformable Fractional Differential Systems via Perov and Leray-Schauder Approaches

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Abstract

In this paper, we investigate the well-posedness and stability of a class of coupled systems of deformable fractional differential equations in Banach spaces. The deformable fractional derivative, which interpolates continuously between a function and its classical derivative through a single scalar parameter, provides a flexible and tractable framework for modeling complex dynamical phenomena with memory effects. By employing Perov's fixed-point theorem under matrix contractive conditions, we establish the existence and uniqueness of solutions for the considered coupled system. The existence of at least one solution under broader growth conditions is then proved via the nonlinear alternative of Leray–Schauder type. Furthermore, the continuous dependence of solutions on initial data is rigorously established, confirming the well-posedness of the system. Hyers–Ulam stability and generalized Hyers–Ulam–Rassias stability results are also derived, providing quantitative estimates relevant to numerical approximation and applied analysis. Three illustrative examples are presented to demonstrate the applicability and effectiveness of the theoretical results.

Keywords: deformable fractional derivative; coupled differential systems; existence and uniqueness; Perov fixed-point theorem; Leray–Schauder alternative; nonlinear systems; mathematical operators; fixed-point theorems; differential equations

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1. Introduction

Fractional calculus has emerged as a powerful mathematical tool for modeling complex phenomena involving memory and hereditary properties. In recent decades, fractional differential equations have been widely used to describe various processes in physics, engineering, biology, viscoelasticity, control theory, and signal processing due to their ability to capture nonlocal and history-dependent dynamics [1–3]. As a result, the qualitative analysis of fractional differential systems has attracted considerable attention from researchers.

The concept of stability of functional equations was first introduced by Ulam [4], who posed the problem of stability of group homomorphisms. Hyers [5] provided the first solution, establishing what is now known as Ulam–Hyers stability. Later, Rassias [6] significantly generalized this result by allowing the perturbation to be unbounded, leading to the notion of Ulam–Hyers–Rassias stability. These stability concepts were subsequently extended to differential equations. Early contributions in this direction include the works of Obłoza [7] and Alsina and Ger [8], who investigated Ulam-type stability for ordinary differential equations. In recent years, considerable attention has been devoted to the study of Ulam-type stability for fractional differential equations. In particular, Wang, Fečkan, and Zhou [9] established fundamental results for Ulam–Hyers stability in the fractional setting, and this topic has since attracted wide interest in the analysis of nonlinear and coupled fractional systems [10,11]. Despite these advances, the study of Ulam-type stability for systems involving deformable fractional derivatives remains limited. The present work aims to contribute to this direction by establishing both Ulam–Hyers and generalized Ulam–Hyers–Rassias stability results for coupled deformable fractional systems.

Recently, new types of fractional operators have been introduced to overcome certain limitations of classical definitions, such as the Riemann–Liouville and Caputo derivatives. Among these developments, the deformable fractional derivative has received increasing interest due to its flexibility and its ability to continuously deform a function into its derivative through a parameter varying in a given interval. This operator provides an alternative framework for modeling dynamical systems and offers new perspectives in the analysis of nonlinear fractional problems [12]. In this context, the qualitative properties of solutions for fractional differential systems involving deformable derivatives have been extensively studied. For instance, existence and uniqueness results, particularly for integro-differential equations in Banach spaces, which effectively describe memory and hereditary properties in physical phenomena, were explored in [13–16]. Furthermore, the approximate controllability of neutral functional differential systems with fractional deformable derivatives was investigated by Raju et al. [17]. More recently, Slimane et al. [18] established existence results for periodic fractional differential equations involving the same derivative, demonstrating the ongoing expansion of this framework to various types of boundary value problems.

Coupled fractional differential systems arise naturally in many applications where several interacting components evolve simultaneously. The study of existence, uniqueness, and qualitative properties of solutions for such systems is therefore of great importance in both theory and applications. Various analytical techniques have been employed to investigate coupled systems, including fixed-point theory, monotone iterative methods, and topological degree theory [19]. In particular, fixed-point methods in Banach spaces provide effective tools for establishing solvability results for nonlinear fractional systems.

Perov's fixed-point theorem, which generalizes the classical Banach contraction principle to vector-valued metrics, has proved to be highly effective in studying coupled systems of differential and integral equations. This theorem enables the treatment of multidimensional problems under matrix contractive conditions and has been successfully applied in nonlinear analysis and fractional differential equations [10,11,20]. In addition, the Leray–Schauder nonlinear alternative constitutes another powerful technique for proving existence results for nonlinear problems in Banach spaces.

Motivated by these developments, the main objective of this paper is to study the existence, uniqueness, and continuous dependence of solutions for coupled systems of deformable fractional differential equations. By employing Perov's fixed-point theorem together with the generalized nonlinear alternative of Leray–Schauder, we establish new solvability results for the considered system. The obtained results extend several existing

works on fractional and coupled systems and provide a unified framework for analyzing deformable fractional models. Specifically, we study the following coupled deformable fractional differential system:

$$\begin{cases} D^{\alpha_1}u(t) = f_1(t, u(t), v(t)),\ \alpha_1 + \beta_1 = 1 \\ D^{\alpha_2}v(t) = f_2(t, u(t), v(t)),\ \alpha_2 + \beta_2 = 1 \end{cases} ; t \in J := [0, T] \tag{1}$$

with the initial conditions

$$\begin{cases} u(0) = u_0, \\ v(0) = v_0, \end{cases} \tag{2}$$

where the deformable fractional derivatives of order $\alpha_i \in [0,1]$, $(i = 1,2)$ are indicated by D^{α_i}, $(i = 1,2)$, $f_i : J \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $m \in \mathbb{N}^*$, functions given and $u_0, v_0 \in \mathbb{R}^m$.

The remainder of the paper is organized as follows. Section 2 presents preliminary definitions and auxiliary results related to deformable fractional calculus and fixed-point theory. Section 3 contains the main existence and uniqueness theorems as well as the continuous dependence results. In Section 4, Ulam–Hyers stability and generalized Ulam–Hyers–Rassias stability results are established for the coupled system. In Section 5, illustrative examples are provided to demonstrate the applicability of the theoretical findings. Finally, Section 6 discusses the connection between the obtained theoretical results and their implications for numerical approximation of deformable fractional systems, and Section 7 provides concluding remarks along with directions for future research.

2. Preliminaries and Assumptions

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper. By $C(J, \mathbb{R}^m)$ we denote the Banach space of all continuous functions from $J = [0, T]$ into $\mathbb{R}^m$ with the norm

$$\|u\|_\infty := \sup_{t \in J} \|u(t)\|.$$

$\mathcal{C} := C(J, \mathbb{R}^m) \times C(J, \mathbb{R}^m)$ be the product Banach space with norm

$$\|(u, v)\|_{\mathcal{C}} = \|u\|_\infty + \|v\|_\infty.$$

Definition 1 ([21]). *Let f be a continuous function defined on $[0, T]$. We define α-fractional integral of f, denoted by $I^\alpha f$, by the integral*

$$I^\alpha f(t) = \frac{1}{\alpha} e^{-\frac{\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} f(s) ds\ ,\ \ t \in J, \tag{3}$$

where $\alpha + \beta = 1$, and $\alpha \in (0, 1]$.

Definition 2 ([21]). *Let f be a real valued-function on $[0, +\infty)$, $\alpha \in [0, 1]$. The deformable derivative of f of order α at $t \in [0, +\infty)$ is defined as*

$$D^\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\beta) f(t + \varepsilon\alpha) - f(t)}{\varepsilon} \tag{4}$$

where $\alpha + \beta = 1$. If the limit exists, we say that f is α-differentiable at t.

If f is α-differentiable in some $(0, T)$, $T > 0$, and $\lim_{t \to 0^+} D^\alpha(f)(t)$ exists, then it is expressed as:

$$D^\alpha(f)(0) = \lim_{t \to 0^+} D^\alpha(f)(t).$$

Remark 1. *If $\alpha = 1$, then $\beta = 0$ and we recover the usual derivative.*

Theorem 1 ([21]). *Let $\alpha \in (0,1]$ be such that $\alpha + \beta = 1$. Then,*

(1) *The operators I^α and D^α are linear.*
(2) *The operators I^α and D^α are commutative.*
(3) *Let f be continuous function on $[0,T]$. Then, $I^\alpha f$ is α-differentiable in $(0,T)$ and we have*

$$D^\alpha(I^\alpha(f)(t)) = f(t), \tag{5}$$

$$I^\alpha(D^\alpha(f))(t) = f(t) - e^{-\frac{\beta}{\alpha}t} f(0). \tag{6}$$

3. Existence and Uniqueness Results

Lemma 1. *A function $u, v \in C(J, \mathbb{R}^m)$ is said to be a solution of (1) and (2) if and only if*

$$\begin{cases} u(t) = e^{-\frac{\beta_1}{\alpha_1}t}u_0 + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s} f_1(s,u(s),v(s))ds, \ \alpha_1+\beta_1 = 1, t \in J, \\ v(t) = e^{-\frac{\beta_2}{\alpha_2}t}v_0 + \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s} f_2(s,u(s),v(s))ds, \ \alpha_2+\beta_2 = 1, \ t \in J. \end{cases} \tag{7}$$

Proof. Assume (1) and (2). Then, $I^{\alpha_1}(D^{\alpha_1}u)(t) = I^{\alpha_1}f_1(t,u(t),v(t))$. By Theorem 1, we get

$$u(t) - e^{-\frac{\beta_1}{\alpha_1}t}u(0) = I^{\alpha_1}f_1(t,u(t),v(t))$$

Using (3), we get

$$u(t) = e^{-\frac{\beta_1}{\alpha_1}t}u_0 + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s} f(s,u(s),v(s))ds.$$

Similarly, we have

$$v(t) = e^{-\frac{\beta_2}{\alpha_2}t}v_0 + \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s} f(s,u(s),v(s))ds.$$

□

Definition 3. *We say that $(u,v) \in \mathcal{C}$ is a mild solution to the system (1) and (2) if $u(0) = u_0$, $v(0) = v_0$ and*

$$\begin{cases} u(t) = e^{-\frac{\beta_1}{\alpha_1}t}u_0 + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s} f_1(s,u(s),v(s))ds, \quad t \in J \\ v(t) = e^{-\frac{\beta_2}{\alpha_2}t}v_0 + \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s} f_2(s,u(s),v(s))ds, \quad t \in J. \end{cases}$$

Next, we investigate the Cauchy problem (1) and (2) within the framework of the following assumptions.

Hypothesis 1 (H_1). *The functions f_i, $i = 1,2$, are continuous on $J \times \mathbb{R}^m \times \mathbb{R}^m$.*

Hypothesis 2 (H_2). *There exists functions $h_1, h_2, k_1, k_2 \in C(J, \mathbb{R}^+)$, such that,*

$$\|f_1(t,u,v) - f_1(t,\overline{u},\overline{v})\| \leq h_1(t)\|u-\overline{u}\| + h_2(t)\|v-\overline{v}\|, \ \forall u,\overline{u},v,\overline{v} \in \mathbb{R}^m, \tag{8}$$

$$\|f_2(t,u,v) - f_2(t,\overline{u},\overline{v})\| \leq k_1(t)\|u-\overline{u}\| + k_2(t)\|v-\overline{v}\|, \ \forall u,\overline{u},v,\overline{v} \in \mathbb{R}^m, \tag{9}$$

where

$$h_i^* = \sup_{t\in J}|h_i(t)|, \ k_i^* = \sup_{t\in J}|k_i(t)|, \ f_i^* = \sup_{t\in J}\|f_i(t,0,0)\|, \ i = 1,2.$$

First, we prove our existence and uniqueness solution for the problem (1) and (2) using Perov's fixed-point theorem.

Theorem 2. *Assume that* (H_1), (H_2) *are satisfied. If the matrix*

$$M = \begin{bmatrix} \frac{h_1^*}{\beta_1} & \frac{h_2^*}{\beta_1} \\ \frac{k_1^*}{\beta_2} & \frac{k_2^*}{\beta_2} \end{bmatrix} \tag{10}$$

converges to zero, then problem (1) *and* (2) *has a unique bounded solution.*

Proof. Transform the problem (1) and (2) into a fixed-point problem of the operator $T : \mathcal{C} \to \mathcal{C}$ defined by $T(u,v) = (T_1(u,v), T_2(u,v))$, where

$$\begin{cases} T_1(u,v)(t) = e^{-\frac{\beta_1}{\alpha_1}t}u_0 + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s} f_1(s,u(s),v(s))ds, \\ T_2(u,v)(t) = e^{-\frac{\beta_2}{\alpha_2}t}v_0 + \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s} f_2(s,u(s),v(s))ds. \end{cases} \tag{11}$$

First, we show that the operator T is well-defined. Let $(u,v) \in \mathcal{C}$ and $t \in J$, then we have

$$\begin{aligned} \|T_1(u,v)(t)\| &\leq e^{-\frac{\beta_1}{\alpha_1}t}\|u_0\| + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\|f_1(s,u(s),v(s))\|ds \\ &\leq \|u_0\| + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\|f_1(s,u(s),v(s)) - f(s,0,0)\|ds \\ &\quad + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\|f(s,0,0)\|ds \\ &\leq \|u_0\| + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}(h_1(s)\|u(s)\| + h_2(s)\|v(s)\|)ds \\ &\quad + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\|f(s,0,0)\|ds \\ &\leq \|u_0\| + \frac{e^{-\frac{\beta_1}{\alpha_1}t}\left(e^{\frac{\beta_1}{\alpha_1}t}-1\right)h_1^*}{\beta_1}\|u\|_\infty + \frac{e^{-\frac{\beta_1}{\alpha_1}t}\left(e^{\frac{\beta_1}{\alpha_1}t}-1\right)h_2^*}{\beta_1}\|v\|_\infty + \frac{e^{-\frac{\beta_1}{\alpha_1}t}\left(e^{\frac{\beta_1}{\alpha_1}t}-1\right)}{\beta_1}f_1^* \\ &\leq \|u_0\| + \frac{h_1^*}{\beta_1}\|u\|_\infty + \frac{h_2^*}{\beta_1}\|v\|_\infty + \frac{f_1^*}{\beta_1} \end{aligned}$$

Then

$$\|T_1(u,v)\|_\infty \leq \|u_0\| + \frac{h_1^*}{\beta_1}\|u\|_\infty + \frac{h_2^*}{\beta_1}\|v\|_\infty + \frac{f_1^*}{\beta_1}.$$

Similarly, we have

$$\begin{aligned}
\|T_2(u,v)(t)\| &\leq e^{-\frac{\beta_2}{\alpha_2}t}\|v_0\| + \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}\|f_2(s,u(s),v(s))\|ds \\
&\leq \|v_0\| + \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}\|f_2(s,u(s),v(s)) - f_2(s,0,0)\|ds \\
&\quad + \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}\|f_2(s,0,0)\|ds \\
&\leq \|v_0\| + \frac{1}{\alpha_2}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}k_1(s)\|u(s)\|ds + \frac{1}{\alpha_2}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}k_2(s)\|v(s)\|ds \\
&\quad + \frac{1}{\alpha_2}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}\|f_2(s,0,0)\|ds \\
&\leq \|v_0\| + \frac{e^{-\frac{\beta_2}{\alpha_2}t}\left(e^{\frac{\beta_2}{\alpha_2}t}-1\right)k_1^*}{\beta_2}\|u\|_\infty + \frac{e^{-\frac{\beta_2}{\alpha_2}t}\left(e^{\frac{\beta_2}{\alpha_2}t}-1\right)k_2^*}{\beta_2}\|v\|_\infty + \frac{f_2^*}{\beta_2}.
\end{aligned}$$

Then

$$\|T_2(u,v)\|_\infty \leq \|v_0\| + \frac{k_1^*}{\beta_2}\|u\|_\infty + \frac{k_2^*}{\beta_2}\|v\|_\infty + \frac{f_2^*}{\beta_2}.$$

Now, we show that T is a contraction. For all $(u,v),(\overline{u},\overline{v}) \in \mathcal{C}$, we get

$$\begin{aligned}
\|T_1(u,v)(t) - T_1(\overline{u},\overline{v})(t)\| &\leq \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\|f_1(s,u(s),v(s)) - f_1(s,\overline{u}(s),\overline{v}(s))\|ds \\
&\leq \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}h_1(s)\|u(s)-\overline{u}(s)\|ds \\
&\quad + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}h_2(s)\|v(s)-\overline{v}(s)\|ds \\
&\leq \frac{e^{-\frac{\beta_1}{\alpha_1}t}\left(e^{\frac{\beta_1}{\alpha_1}t}-1\right)h_1^*}{\beta_1}\|u-\overline{u}\|_\infty + \frac{e^{-\frac{\beta_1}{\alpha_1}t}\left(e^{\frac{\beta_1}{\alpha_1}t}-1\right)h_2^*}{\beta_1}\|v-\overline{v}\|_\infty.
\end{aligned}$$

Then

$$\|T_1(u,v) - T_1(\overline{u},\overline{v})\|_\infty \leq \frac{h_1^*}{\beta_1}\|u-\overline{u}\|_\infty + \frac{h_2^*}{\beta_1}\|v-\overline{v}\|_\infty. \tag{12}$$

Similarly, we have

$$\|T_2(u,v) - T_2(\overline{u},\overline{v})\|_\infty \leq \frac{k_1^*}{\beta_2}\|u-\overline{u}\|_\infty + \frac{k_2^*}{\beta_2}\|v-\overline{v}\|_\infty. \tag{13}$$

Therefore,

$$d(T(u,v),T(\overline{u},\overline{v})) \leq Md((u,v),(\overline{u},\overline{v})).$$

According to Theorem 6 in [22], we deduce that the operator T has unique fixed point which is a solution of problem (1) and (2). Now, we will prove that the solution (u,v) of problem (1) and (2) is bounded. For all $t \in J$, we have

$$\begin{aligned}
\|u\|_\infty &\leq \frac{h_1^*}{\beta_1}\|u\|_\infty + \frac{h_2^*}{\beta_1}\|v\|_\infty + \|u_0\| + \frac{f_1^*}{\beta_1}, \\
\|v\|_\infty &\leq \frac{k_1^*}{\beta_2}\|u\|_\infty + \frac{k_2^*}{\beta_2}\|v\|_\infty + \|v_0\| + \frac{f_2^*}{\beta_2}.
\end{aligned} \tag{14}$$

Hence,

$$\begin{pmatrix} \|u\|_\infty \\ \|v\|_\infty \end{pmatrix} \leq M\begin{pmatrix} \|u\|_\infty \\ \|v\|_\infty \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \tag{15}$$

where

$$\delta_1 = \|u_0\| + \frac{f_1^*}{\beta_1}, \ \ \delta_2 = \|v_0\| + \frac{f_2^*}{\beta_2}. \tag{16}$$

Then,

$$\begin{pmatrix} \|u\|_\infty \\ \|v\|_\infty \end{pmatrix} \leq (I - M)^{-1} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}. \tag{17}$$

We deduce that the solution (u, v) is bounded. □

Theorem 3. *Assume that* (H_1) *and* (H_2) *hold. The matrix M defined in* (10) *converges to zero. For every* $(u_0, v_0) \in \mathbb{R}^m \times \mathbb{R}^m$, *we denoted by* $(u(t, u_0), v(t, v_0))$ *the solution of problem* (1) *and* (2). *Then, the map* $(u_0, v_0) \longmapsto (u(., u_0), v(., v_0))$ *is continuous.*

Proof. From Theorem 2, for each initial condition $(u_0, v_0), (\overline{u}_0, \overline{v}_0) \in \mathbb{R}^m \times \mathbb{R}^m$, there exists unique solution, $u(., u_0)$, $v(., v_0)$, $\overline{u}(., \overline{u}_0)$, $\overline{v}(., \overline{v}_0)$, then we obtain:

$$\begin{cases} u(t, u_0) = e^{-\frac{\beta_1}{\alpha_1}t} u_0 + \frac{1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}t} \int_0^t e^{\frac{\beta_1}{\alpha_1}s} f_1(s, u(s), v(s)) ds, \\ v(t, v_0) = e^{-\frac{\beta_2}{\alpha_2}t} v_0 + \frac{1}{\alpha_2} e^{-\frac{\beta_2}{\alpha_2}t} \int_0^t e^{\frac{\beta_2}{\alpha_2}s} f_2(s, u(s), v(s)) ds. \end{cases}$$

$$\begin{cases} \overline{u}(t, \overline{u}_0) = e^{-\frac{\beta_1}{\alpha_1}t} \overline{u}_0 + \frac{1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}t} \int_0^t e^{\frac{\beta_1}{\alpha_1}s} f_1(s, \overline{u}(s), \overline{v}(s)) ds, \\ \overline{v}(t, \overline{v}_0) = e^{-\frac{\beta_2}{\alpha_2}t} \overline{v}_0 + \frac{1}{\alpha_2} e^{-\frac{\beta_2}{\alpha_2}t} \int_0^t e^{\frac{\beta_2}{\alpha_2}s} f_2(s, \overline{u}(s), \overline{v}(s)) ds. \end{cases}$$

Therefore,

$$\begin{aligned} \|u(t, u_0) - \overline{u}(t, \overline{u}_0)\| &\leq \|u_0 - \overline{u}_0\| + \frac{1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}t} \int_0^t e^{\frac{\beta_1}{\alpha_1}s} \|f_1(s, u(s), v(s)) - f_1(s, \overline{u}(s), \overline{v}(s))\| ds \\ &\leq \|u_0 - \overline{u}_0\| + \frac{1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}t} \int_0^t e^{\frac{\beta_1}{\alpha_1}s} h_1(s) \|u(s) - \overline{u}(s)\| ds \\ &\quad + \frac{1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}t} \int_0^t e^{\frac{\beta_1}{\alpha_1}s} h_2(s) \|v(s) - \overline{v}(s)\| ds, \end{aligned}$$

and similarly

$$\begin{aligned} \|v(t, v_0) - \overline{v}(t, \overline{v}_0)\| &\leq \|v_0 - \overline{v}_0\| + \frac{1}{\alpha_2} e^{-\frac{\beta_2}{\alpha_2}t} \int_0^t e^{\frac{\beta_2}{\alpha_2}s} \|f_2(s, u(s), v(s)) - f_2(s, \overline{u}(s), \overline{v}(s))\| ds \\ &\leq \|v_0 - \overline{v}_0\| + \frac{1}{\alpha_2} e^{-\frac{\beta_2}{\alpha_2}t} \int_0^t e^{\frac{\beta_2}{\alpha_2}s} k_1(s) \|u(s) - \overline{u}(s)\| ds \\ &\quad + \frac{1}{\alpha_2} e^{-\frac{\beta_2}{\alpha_2}t} \int_0^t e^{\frac{\beta_2}{\alpha_2}s} k_2(s) \|v(s) - \overline{v}(s)\| ds. \end{aligned}$$

and

$$\begin{aligned} \|u(., u_0) - \overline{u}(., \overline{u}_0)\|_\infty &\leq \|u_0 - \overline{u}_0\| + \frac{h_1^*}{\beta_1} \|u - \overline{u}\|_\infty + \frac{h_2^*}{\beta_1} \|v - \overline{v}\|_\infty, \\ \|v(., v_0) - \overline{v}(., \overline{v}_0)\|_\infty &\leq \|v_0 - \overline{v}_0\| + \frac{k_1^*}{\beta_2} \|u - \overline{u}\|_\infty + \frac{k_2^*}{\beta_2} \|v - \overline{v}\|_\infty. \end{aligned}$$

Hence,

$$\begin{pmatrix} \|u(., u_0) - \overline{u}(., \overline{u}_0)\|_\infty \\ \|v(., v_0) - \overline{v}(., \overline{v}_0)\|_\infty \end{pmatrix} \leq (I - M)^{-1} \begin{pmatrix} \|u_0 - \overline{u}_0\| \\ \|v_0 - \overline{v}_0\| \end{pmatrix}$$

$(\|u(., u_0) - \overline{u}(., \overline{u}_0)\|_\infty, \|v(., v_0) - \overline{v}(., \overline{v}_0)\|_\infty) \to 0$, as $(u_0, v_0) \to (\overline{u}_0, \overline{v}_0)$. □

The following hypotheses will be used in the sequel:

Hypothesis 3 (H_3). *There exists constants $p_i, q_j \in \mathbb{R}^+$, $i, j = 1, 2$, such that*

$$\begin{aligned} \|f_1(t,u,v)\| &\leq p_1\|u\| + p_2\|v\| \text{ for all } u, v \in \mathbb{R}^m,\ t \in J, \\ \|f_2(t,u,v)\| &\leq q_1\|u\| + q_2\|v\| \text{ for all } u, v \in \mathbb{R}^m,\ t \in J. \end{aligned} \tag{18}$$

Hypothesis 4 (H_4). *For any bounded set $B \subset \mathcal{C}$, the sets*

$$\{t \mapsto f_i(t, u(t), v(t)),\ i = 1, 2, (u, v) \in B\}$$

are equicontinuous in $\mathcal{C}$.

Now, we prove an existence result for the coupled system (1) and (2) by using the nonlinear alternative of the Leray–Schauder type in Banach space.

Theorem 4. *Assume that the hypotheses (H_1)–(H_3) and (H_4) hold. Then, the coupled system (1) and (2) has at least one solution.*

Proof. We show that the operator $T : \mathcal{C} \to \mathcal{C}$ defined in (11). The proof will be given in four steps.

Step 1. T is continuous

Let $(u_n, v_n)_n$ be a sequence such that $(u_n, v_n) \to (u, v) \in \mathcal{C}$ as $n \to +\infty$. Then

$$\begin{aligned} & \|T_1(u_n, v_n)(t) - T_1(u, v)(t)\| \\ = & \left\| \frac{1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}t} \int_0^t e^{\frac{\beta_1}{\alpha_1}s} (f_1(s, u_n(s), v_n(s)) - f_1(s, u(s), v(s)))ds \right\| \\ \leq & \frac{1}{\alpha_1} e^{-\frac{\beta_1}{\alpha_1}t} \int_0^t e^{\frac{\beta_1}{\alpha_1}s} \|f_1(s, u_n(s), v_n(s)) - f_1(s, u(s), v(s))\|ds. \end{aligned}$$

Then

$$\|T_1(u_n, v_n) - T_1(u, v)\|_\infty \leq \frac{1}{\beta_1} \|f_1(., u_n(.), v_n(.)) - f_1(., u(.), v(.))\|_\infty.$$

Similarly, we have

$$\|T_2(u_n, v_n) - T_2(u, v)\|_\infty \leq \frac{1}{\beta_2} \|f_2(., u_n(.), v_n(.)) - f_2(., u(.), v(.))\|_\infty.$$

Since f_i , $i = 1, 2$ are continuous function, we have

$$\begin{aligned} \|T_1(u_n, v_n) - T_1(u, v)\|_\infty &\to 0, \text{ as } n \to +\infty, \\ \|T_2(u_n, v_n) - T_2(u, v)\|_\infty &\to 0, \text{ as } n \to +\infty. \end{aligned}$$

Hence $T(.,.)$ is continuous.

Step 2. T maps bounded sets into bounded sets in $\mathcal{C}$. Indeed, it is enough to show that there exists a positive constant $l := (l_1, l_2)$ such that for each

$$(u, v) \in B_R := \{(u, v) \in \mathcal{C} : \ \|u\|_\infty \leq R, \ \|v\|_\infty \leq R\}.$$

One has $d(T(u, v), (0, 0)) \leq l$.

For each $(u,v) \in B_R$ and $t \in J$, we have

$$\begin{aligned}\|T_1(u,v)(t)\| &\leq e^{-\frac{\beta_1}{\alpha_1}t}\|u_0\| + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\|f_1(s,u(s),v(s))\|ds \\ &\leq \|u_0\| + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}(p_1\|u(s)\| + p_2\|v(s)\|)ds \\ &\leq \|u_0\| + \frac{(p_1+p_2)R}{\beta_1} =: l_1.\end{aligned}$$

Similarly, we have

$$\|T_2(u,v)(t)\| \leq \|v_0\| + \frac{(q_1+q_2)R}{\beta_2} =: l_2.$$

Hence,

$$(\|T_1(u,v)\|_\infty, \|T_2(u,v)\|_\infty) \leq (l_1,l_2) := l.$$

Step 3. T maps bounded sets into equicontinuous sets

Let $t_1, t_2 \in J$, $t_1 \leq t_2$, let B_R be a bounded set in $\mathcal{C}$ as in Step 2, and let $(u,v) \in B_R$, we have

$$\begin{aligned}&\|T_1(u,v)(t_2) - T_1(u,v)(t_1)\| \\ =\ & \frac{1}{\alpha_1}\left\|e^{-\frac{\beta_1}{\alpha_1}t_2}\int_0^{t_2} e^{\frac{\beta_1}{\alpha_1}s}f_1(s,u(s),v(s))ds - e^{-\frac{\beta_1}{\alpha_1}t_1}\int_0^{t_1} e^{\frac{\beta_1}{\alpha_1}s}f_1(s,u(s),v(s))ds\right\| \\ \leq\ & \frac{1}{\alpha_1}\int_{t_1}^{t_2} e^{-\frac{\beta_1}{\alpha_1}t_2}e^{\frac{\beta_1}{\alpha_1}s}\|f_1(s,u(s),v(s))\|ds \\ & + \frac{1}{\alpha_1}\int_0^{t_2}\left(e^{-\frac{\beta_1}{\alpha_1}t_1} - e^{-\frac{\beta_1}{\alpha_1}t_2}\right)e^{\frac{\beta_1}{\alpha_1}s}\|f_1(s,u(s),v(s))\|ds \\ \leq\ & \frac{1}{\alpha_1}\int_{t_1}^{t_2} e^{-\frac{\beta_1}{\alpha_1}t_2}e^{\frac{\beta_1}{\alpha_1}s}(p_1\|u(s)\| + p_2\|v(s)\|)ds \\ & + \frac{1}{\alpha_1}\int_0^{t_2}\left(e^{-\frac{\beta_1}{\alpha_1}t_1} - e^{-\frac{\beta_1}{\alpha_1}t_2}\right)e^{\frac{\beta_1}{\alpha_1}s}(p_1\|u(s)\| + p_2\|v(s)\|)ds \\ \leq\ & \frac{p_1\|u\|_\infty}{\beta_1}e^{-\frac{\beta_1}{\alpha_1}t_2}\left(e^{\frac{\beta_1}{\alpha_1}t_2} - e^{\frac{\beta_1}{\alpha_1}t_1}\right) + \frac{p_2\|v\|_\infty}{\beta_1}e^{-\frac{\beta_1}{\alpha_1}t_2}\left(e^{\frac{\beta_1}{\alpha_1}t_2} - e^{\frac{\beta_1}{\alpha_1}t_1}\right) \\ & + \frac{p_1\|u\|_\infty}{\beta_1}\left(e^{-\frac{\beta_1}{\alpha_1}t_1} - e^{-\frac{\beta_1}{\alpha_1}t_2}\right)\left(e^{\frac{\beta_1}{\alpha_1}t_2} - 1\right) + \frac{p_2\|v\|_\infty}{\beta_1}\left(e^{-\frac{\beta_1}{\alpha_1}t_1} - e^{-\frac{\beta_1}{\alpha_1}t_2}\right)\left(e^{\frac{\beta_1}{\alpha_1}t_2} - 1\right)\end{aligned}$$

Thus,

$$\|T_1(u,v)(t_2) - T_1(u,v)(t_1)\| \to 0, \ \text{ as } \ t_2 \to t_1.$$

Similarly, we have

$$\begin{aligned}&\|T_2(u,v)(t_2) - T_2(u,v)(t_1)\| \\ \leq\ & \frac{q_1\|u\|_\infty}{\beta_1}e^{-\frac{\beta_2}{\alpha_2}t_2}\left(e^{\frac{\beta_2}{\alpha_2}t_2} - e^{\frac{\beta_2}{\alpha_2}t_1}\right) + \frac{q_2\|v\|_\infty}{\beta_1}e^{-\frac{\beta_2}{\alpha_2}t_2}\left(e^{\frac{\beta_2}{\alpha_2}t_2} - e^{\frac{\beta_2}{\alpha_2}t_1}\right) \\ & + \frac{q_1\|u\|_\infty}{\beta_2}\left(e^{-\frac{\beta_2}{\alpha_2}t_1} - e^{-\frac{\beta_2}{\alpha_2}t_2}\right)\left(e^{\frac{\beta_2}{\alpha_2}t_2} - 1\right) + \frac{q_2\|v\|_\infty}{\beta_2}\left(e^{-\frac{\beta_2}{\alpha_2}t_1} - e^{-\frac{\beta_2}{\alpha_2}t_2}\right)\left(e^{\frac{\beta_2}{\alpha_2}t_2} - 1\right).\end{aligned}$$

Thus,

$$\|T_2(u,v)(t_2) - T_2(u,v)(t_1)\| \to 0, \ \text{ as } \ t_2 \to t_1.$$

Thus, $\|T_1(u,v)(t_2) - T_1(u,v)(t_1)\|$, $\|T_2(u,v)(t_2) - T_2(u,v)(t_1)\|$ tend to zero independently of $(u,v) \in B_R$ as $t_2 \to t_1$, which means that the set $T(B_R)$ is equicontinuous in $\mathcal{C}$.

Step 4. The set E consisting of $(u,v) \in \mathcal{C}$, such that $(u,v) = \mu T(u,v)$ for some $\mu \in (0,1)$, is bounded in $\mathcal{C}$. Let $(u,v) \in C$, such that $(u,v) = \mu T(u,v)$. Then, $u = \mu T_1(u,v)$, $v = \mu T_2(u,v)$

$$\begin{aligned} \|u(t)\| &\leq e^{-\frac{\beta_1}{\alpha_1}t}\|u_0\| + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\|f_1(s,u(s),v(s))\|ds \\ &\leq e^{-\frac{\beta_1}{\alpha_1}t}\|u_0\| + \frac{1}{\alpha_1}\int_0^t e^{-\frac{\beta_1}{\alpha_1}(t-s)}(p_1\|u(s)\| + p_2\|v(s)\|)ds \end{aligned}$$

Similarly, we get

$$\begin{aligned} \|v(t)\| &\leq e^{-\frac{\beta_2}{\alpha_2}t}\|v_0\| + \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}\|f_2(s,u(s),v(s))\|ds \\ &\leq e^{-\frac{\beta_2}{\alpha_2}t}\|v_0\| + \frac{1}{\alpha_2}\int_0^t e^{-\frac{\beta_2}{\alpha_2}(t-s)}(q_1\|u(s)\| + q_2\|v(s)\|)ds. \end{aligned}$$

Consequently, since the exponential terms $e^{-\frac{\beta_i}{\alpha_i}t} \leq 1$ and $e^{-\frac{\beta_i}{\alpha_i}(t-s)} \leq 1, i = 1,2$ for all $t \geq s \geq 0$

$$\|u(t)\| + \|v(t)\| \leq C_0 + C_1\int_0^t (\|u(s)\| + \|v(s)\|)ds,$$

where $C_0 = \|u_0\| + \|v_0\|$, $C_1 = \max\left(\frac{p_1}{\alpha_1}, \frac{p_2}{\alpha_1}, \frac{q_1}{\alpha_2}, \frac{q_2}{\alpha_2}\right)$.

Applying Gronwall's lemma [11], we get

$$\|u(t)\| + \|v(t)\| \leq C_0 e^{C_1 T} := L.$$

Therefore,

$$\|(u,v)\| \leq L.$$

It demonstrates the boundedness of E. T has at least one fixed point, which is a solution of the problem (1) and (2), by the nonlinear alternative of Leray–Schauder. □

4. Stability Results

Definition 4 ([23]). *The coupled system (1) and (2) is said to be Hyers–Ulam stable, if there exist $l = (l_1, l_2), l_1, l_2 > 0$ such that for any $\varepsilon = (\varepsilon_1, \varepsilon_2), \varepsilon_1, \varepsilon_2 > 0$ and for every approximate solution $(u(.), v(.)) \in \mathcal{C}$ of the inequalities*

$$\begin{cases} \|D^{\alpha_1}u(t) - f_1(t,u(t),v(t))\| \leq \varepsilon_1 \\ \|D^{\alpha_2}v(t) - f_2(t,u(t),v(t))\| \leq \varepsilon_2 \end{cases} ; t \in J. \tag{19}$$

There exists a solution of the couple system $(u^*(.), v^*(.)) \in \mathcal{C}$ of (1)–(2) with

$$\|(u(t),v(t)) - (u^*(t),v^*(t))\| \leq l\varepsilon, \quad t \in J.$$

Definition 5 ([23]). *Let $f_i : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\Phi_i : [0,T] \to \mathbb{R}^+$, $i = 1,2$ be continuous functions. Then, (1) and (2) are generalized Hyers–Ulam–Rassias stable with respect to Φ_i, $i = 1,2$, if there exists a constants $c_{f_i,\Phi_i} > 0$ such that, for each approximate solution $(u,v) \in \mathcal{C}$ of (1) and (2),*

$$\begin{cases} \|D^{\alpha_1}u(t) - f_1(t,u(t),v(t))\| \leq \Phi_1(t) \\ \|D^{\alpha_2}v(t) - f_2(t,u(t),v(t))\| \leq \Phi_2(t) \end{cases} ; t \in J, \tag{20}$$

and there exists a solution $(u^*, v^*) \in \mathcal{C}$ of (1) and (2) with

$$\|(u(t),v(t)) - (u^*(t),v^*(t))\| \leq c_{f,\Phi}\Phi(t), \quad t \in J.$$

and

$$c_{f,\Phi} = \left(c_{f_1,\Phi_1};\, c_{f_2,\Phi_2}\right),\ \Phi(t) = (\Phi_1(t);\, \Phi_2(t)).$$

Remark 2.

1. *A function* $(u(.), v(.)) \in \mathcal{C}$ *is a solution of the inequality* (19) *if, and only if, there exist a functions* $g_i(.) \in C$, $i = 1, 2$ *(which depend on* (u, v)*) such that*

$$\|g_i(t)\| \leq \varepsilon_i,\ \ i = 1, 2\,.$$

and

$$\begin{aligned} D^{\alpha_1}u(t) &= f_1(t, u(t), v(t)) + g_1(t), \quad t \in J, \\ D^{\alpha_2}v(t) &= f_2(t, u(t), v(t)) + g_2(t), \quad t \in J. \end{aligned} \tag{21}$$

2. *A function* $(u(.), v(.)) \in \mathcal{C}$ *is a solution of the inequality* (20) *(which depend on* (u, v)*) such that*

$$\|g_i(t)\| \leq \Phi_i(t),\ \ i = 1, 2\,.$$

and

$$\begin{aligned} D^{\alpha_1}u(t) &= f_1(t, u(t), v(t)) + g_1(t), \quad t \in J, \\ D^{\alpha_2}v(t) &= f_2(t, u(t), v(t)) + g_2(t), \quad t \in J. \end{aligned} \tag{22}$$

Now, we are concerned with the generalized Ulam–Hyers and Ulam–Hyers–Rassias stabilities of our problem (1) and (2)

Hypothesis 5. (H_5) *There exists a constants* $\lambda_{\Phi_i} > 0$, $i = 1, 2$, *such that for each* $t \in J$,

$$\frac{1}{\alpha_i} e^{-\frac{\beta_i}{\alpha_i}t} \int_0^t e^{\frac{\beta_i}{\alpha_i}s} \Phi_i(s) ds \leq \lambda_{\Phi_i} \Phi_i(t),$$

where $\alpha_i + \beta_i = 1$, *and* $\alpha_i \in (0, 1]$.

Now, we consider the Ulam stability for system (1) and (2). Consider the system

$$\begin{aligned} u(t) &= e^{-\frac{\beta_1}{\alpha_1}t}u_0 + \tfrac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t} \int_0^t e^{\frac{\beta_1}{\alpha_1}s}(f_1(s, u(s), v(s)) + g_1(s))ds,\ \alpha_1 + \beta_1 = 1,\ t \in J, \\ v(t) &= e^{-\frac{\beta_2}{\alpha_2}t}v_0 + \tfrac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t} \int_0^t e^{\frac{\beta_2}{\alpha_2}s}(f_2(s, u(s), v(s)) + g_2(s))ds,\ \alpha_2 + \beta_2 = 1,\ t \in J. \end{aligned} \tag{23}$$

Lemma 2. *If* $(u(.), v(.)) \in \mathcal{C}$ *is a solution of the inequality* (19), *then* $(u(.), v(.))$ *is a solution of the following integral inequality*

$$\left\| u(t) - e^{-\frac{\beta_1}{\alpha_1}t}u_0 - \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t} \int_0^t e^{\frac{\beta_1}{\alpha_1}s} f_1(s, u(s), v(s))ds \right\| \leq \frac{\varepsilon_1}{\beta_1}.$$

$$\left\| v(t) - e^{-\frac{\beta_2}{\alpha_2}t}v_0 - \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t} \int_0^t e^{\frac{\beta_2}{\alpha_2}s} f_2(s, u(s), v(s))ds \right\| \leq \frac{\varepsilon_2}{\beta_2}.$$

Proof. From Remark 2, we obtain

$$\begin{aligned} & \left\| u(t) - e^{-\frac{\beta_1}{\alpha_1}t}u_0 - \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s} f_1(s,u(s),v(s))ds \right\| \\ = & \left\| \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s} g_1(s)ds \right\| \\ \leq & \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\varepsilon_1 ds = \frac{\varepsilon_1}{\beta_1}\left[1 - e^{-\frac{\beta_1}{\alpha_1}t}\right] \\ \leq & \frac{\varepsilon_1}{\beta_1}. \end{aligned}$$

Similarly, we get

$$\left\| v(t) - e^{-\frac{\beta_2}{\alpha_2}t}v_0 - \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s} f_2(s,u(s),v(s))ds \right\| \leq \frac{\varepsilon_2}{\beta_2}.$$

□

Theorem 5. *Assume that hypothesis* (H_1) *and* (H_2) *hold. Then, the solution of system* (1) *and* (2) *is Hyers–Ulam stable.*

Proof. Then, for each $t \in J$, we obtain

$$\begin{aligned} & \|u(t) - u^*(t)\| \\ = & \left\| u(t) - e^{-\frac{\beta_1}{\alpha_1}t}u_0 - \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s} f_1(s,u^*(s),v^*(s))ds \right\| \\ \leq & \left\| u(t) - e^{-\frac{\beta_1}{\alpha_1}t}u_0 - \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s} f_1(s,u(s),v(s))ds \right\| \\ & + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\|f_1(s,u(s),v(s)) - f_1(s,u^*(s),v^*(s))\|ds \\ \leq & \frac{\varepsilon_1}{\beta_1} + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}(h_1(s)\|u(s) - u^*(s)\| + h_2(s)\|v(s) - v^*(s)\|)ds \end{aligned}$$

Then,

$$\|u - u^*\|_\infty \leq \frac{\varepsilon_1}{\beta_1} + \frac{h_1^*}{\beta_1}\|u - u^*\|_\infty + \frac{h_2^*}{\beta_1}\|v - v^*\|_\infty. \tag{24}$$

Similarly, we get

$$\|v - v^*\|_\infty \leq \frac{\varepsilon_2}{\beta_2} + \frac{k_1^*}{\beta_2}\|u - u^*)\|_\infty + \frac{k_2^*}{\beta_2}\|v - v^*\|_\infty \tag{25}$$

Inequalities (24) and (25) are also written as

$$\begin{aligned} \left(1 - \frac{h_1^*}{\beta_1}\right)\|u - u^*\|_\infty - \frac{h_2^*}{\beta_1}\|v - v^*\|_\infty & \leq \frac{\varepsilon_1}{\beta_1}. \\ -\frac{k_2^*}{\beta_2}\|u - u^*\|_\infty + \left(1 - \frac{k_1^*}{\beta_2}\right)\|v - v^*\|_\infty & \leq \frac{\varepsilon_2}{\beta_2}. \end{aligned} \tag{26}$$

The matrix form of Equation (26) is

$$(I - M)\begin{pmatrix} \|u - u^*\|_\infty \\ \|v - v^*\|_\infty \end{pmatrix} \leq \begin{pmatrix} \frac{\varepsilon_1}{\beta_1} \\ \frac{\varepsilon_2}{\beta_2} \end{pmatrix}.$$

Since M is matrix is a convergent matrix, then $I - M$ is invertible $(I - M)^{-1}$ and has a positive element, we immediately obtain

$$\begin{pmatrix} \|u - u^*\|_\infty \\ \|v - v^*\|_\infty \end{pmatrix} \leq (I - M)^{-1}\begin{pmatrix} \frac{1}{\beta_1} & 0 \\ 0 & \frac{1}{\beta_2} \end{pmatrix}\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

Hence, the system (1) and (2) is Hyers–Ulam stable. □

Lemma 3. *If $(u(.), v(.)) \in C$ is a solution of the inequality (20), then $(u(.), v(.))$ is a solution of the following integral inequality:*

$$\begin{aligned}\|u(t) - e^{-\frac{\beta_1}{\alpha_1}t}u_0 - \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}f_1(s,u(s),v(s))ds\| &\leq \lambda_{\Phi_1}\Phi_1(t),\\ \|v(t) - e^{-\frac{\beta_2}{\alpha_2}t}v_0 - \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}f_2(s,u(s),v(s))ds\| &\leq \lambda_{\Phi_2}\Phi_2(t).\end{aligned}$$

Proof. By Remark 2, for any each $t \in J$, we have

$$\begin{aligned}u(t) &= e^{-\frac{\beta_1}{\alpha_1}t}u_0 + \tfrac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}(f_1(s,u(s),v(s)) + g_1(s))ds,\ \alpha_1+\beta_1 = 1, t \in J,\\ v(t) &= e^{-\frac{\beta_2}{\alpha_2}t}v_0 + \tfrac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}(f_2(s,u(s),v(s)) + g_2(s))ds,\ \alpha_2+\beta_2 = 1,\ t \in J.\end{aligned}$$

Thus, we get

$$\begin{aligned}&\left\|u(t) - e^{-\frac{\beta_1}{\alpha_1}t}u_0 - \frac{1}{\alpha}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}f_1(s,u(s),v(s))ds\right\|\\ \leq\ &\frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\|g_1(s)\|ds\\ \leq\ &\frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\Phi_1(s)ds\\ \leq\ &\lambda_{\Phi_1}\Phi_1(t).\end{aligned}$$

Similarly, we have

$$\left\|v(t) - e^{-\frac{\beta_2}{\alpha_2}t}v_0 - \frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}f_2(s,u(s),v(s))ds\right\| \leq \lambda_{\Phi_2}\Phi_2(t).$$

□

Theorem 6. *Assume that the hypotheses (H_1), (H_2) and (H_5) hold. Then, the problem (1) and (2) has at least one solution on J and it is generalized Ulam–Hyers–Rassias stable.*

Proof. Then, for any $t \in J$, we obtain

$$\begin{aligned}&\|u(t) - u^*(t)\|\\ =\ &\left\|u(t) - e^{-\frac{\beta_1}{\alpha_1}t}u_0 - \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}f_1(s,u^*(s),v^*(s))ds\right\|\\ \leq\ &\left\|u(t) - e^{-\frac{\beta_1}{\alpha_1}t}u_0 - \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}f_1(s,u(s),v(s))ds\right\|\\ &+\frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\|f_1(s,u(s),v(s)) - f_1(s,u^*(s),v^*(s))\|ds\\ \leq\ &\lambda_{\Phi_1}\Phi_1(t) + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}(h_1(s)\|u(s) - u^*(s)\| + h_2(s)\|v(s) - v^*(s)\|)ds\\ \leq\ &\lambda_{\Phi_1}\Phi_1(t) + \frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}(h_1^*\|u(s) - u^*(s)\| + h_2^*\|v(s) - v^*(s)\|)ds\\ \leq\ &\lambda_{\Phi_1}\Phi_1(t) + \frac{h_1^*}{\beta_1}\|u(t) - u^*(t)\| + \frac{h_2^*}{\beta_1}\|v(t) - v^*(t)\|\end{aligned}$$

and similarly, we have

$$\begin{aligned} & \|v(t) - v^*(t)\| \\ \leq \quad & \lambda_{\Phi_2}\Phi_2(t) + \frac{k_1^*}{\beta_2}\|u(t) - u^*(t)\| + \frac{k_2^*}{\beta_2}\|v(t) - v^*(t)\|. \end{aligned}$$

In matrix form, this can be expressed as:

$$\begin{pmatrix} \|u - u^*\|_\infty \\ \|v - v^*\|_\infty \end{pmatrix} \leq \begin{pmatrix} \lambda_{\Phi_1}\Phi_1(t) \\ \lambda_{\Phi_2}\Phi_2(t) \end{pmatrix} + M \begin{pmatrix} \|u - u^*\|_\infty \\ \|v - v^*\|_\infty \end{pmatrix} \tag{27}$$

Since M is a spectral matrix with $\rho(M) < 1$, the operator $(I - M)$ is invertible, yielding:

$$\begin{pmatrix} \|u - u^*\|_\infty \\ \|u - u^*\|_\infty \end{pmatrix} \leq (I - M)^{-1} \begin{pmatrix} \lambda_{\Phi_1} & 0 \\ 0 & \lambda_{\Phi_2} \end{pmatrix} \begin{pmatrix} \Phi_1(t) \\ \Phi_2(t) \end{pmatrix} \tag{28}$$

Hence, problem (1) and (2) is generalized Ulam–Hyers–Rassias stable. □

5. Some Examples

In this section, we give an example to illustrate our main results.

5.1. Physical and Biological Motivation

Coupled fractional differential systems naturally arise in the modeling of complex dynamical processes where memory and hereditary effects play a crucial role. In many real-world applications, the evolution of a system depends not only on its current state but also on its past history.

For instance, in viscoelastic materials, the stress–strain relationship depends on the entire deformation history, which cannot be captured by classical integer-order models. Similarly, in biological systems, interactions between populations or biochemical processes often exhibit memory effects due to delays in response and accumulation phenomena.

The use of the deformable fractional derivative allows a flexible modeling framework, as it provides a continuous transition between local (classical) and nonlocal (fractional) dynamics. This feature is particularly useful in describing systems where the strength of memory effects can vary over time.

The following examples illustrate the applicability of the theoretical results and highlight how the coupling between components influences the system behavior under memory-dependent dynamics.

Example 1. *Consider the following coupled system of deformable fractional differential equations,*

$$\begin{cases} D^{\frac{2}{5}}u(t) = \frac{e^{-2t}}{4(1+\|u(t)\|+\|v(t)\|)}, & t \in [0,5], \\ D^{\frac{3}{5}}v(t) = \frac{e^{-2t}+2}{15(1+\|u(t)\|+\|v(t)\|)}, & t \in [0,5]. \end{cases} \tag{29}$$

with the initial conditions

$$\begin{cases} u(0) = 1, \\ v(0) = 3. \end{cases} \tag{30}$$

Note that

$$\begin{aligned} f_1(t,u(t),v(t)) &= \frac{e^{-2t}}{4(1+\|u(t)\|+\|v(t)\|)}, \ t\in[0,5] \\ f_2(t,u(t),v(t)) &= \frac{e^{-2t}+2}{15(1+\|u(t)\|+\|v(t)\|)}, \ t\in[0,5]. \end{aligned}$$

Let $\alpha_1=\frac{2}{5},\beta_1=\frac{3}{5},\alpha_2=\frac{3}{5},\beta_2=\frac{2}{5}$.

We can show that all hypotheses of Theorem 2 are satisfied with

$$h_1(t)=h_2(t)=\frac{e^{-2t}}{4},\ k_1(t)=k_2(t)=\frac{e^{-2t}+2}{15}.$$

Thus,

$$h_1^*=h_2^*=\frac{1}{4},\ k_1^*=k_2^*=\frac{1}{5}.$$

and

$$M=\begin{bmatrix} \frac{h_1^*}{\beta_1} & \frac{h_2^*}{\beta_1} \\ \frac{k_1^*}{\beta_2} & \frac{k_2^*}{\beta_2} \end{bmatrix}=\begin{bmatrix} \frac{5}{12} & \frac{5}{12} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

We have the spectral radius $\rho(M)$ *is computed via the eigenvalues of* M*, obtained from:*

$$\det(M-\lambda I)=0.$$

That is,

$$\lambda^2-\frac{11}{12}\lambda=0.$$

Solving this equation, we obtain the eigenvalues

$$\lambda_1=\frac{11}{12},\quad \lambda_2=0.$$

Thus,

$$\rho(M)=\max\left\{\frac{11}{12},0\right\}=\frac{11}{12}<1,$$

which implies that M *converges to* 0*. Therefore, Theorem 2 implies that the coupled system* (29) *and* (30) *has a unique solution defined on* $[0,5]$.

Example 2. *Consider the following coupled system of fractional differential equations:*

$$\begin{cases} D^{\frac{1}{4}}u(t)=f_1(t,u(t),v(t)) \\ D^{\frac{1}{5}}v(t)=f_2(t,u(t),v(t)) \end{cases} ;\ t\in[0,1]. \tag{31}$$

with the initial conditions

$$\begin{cases} u(0)=2, \\ v(0)=4. \end{cases} \tag{32}$$

where

$$\begin{aligned} f_1(t,u(t),v(t)) &= \frac{\tan(u(t)+v(t))}{8(e^t+1)}, \\ f_2(t,u(t),v(t)) &= \frac{arctan\ (u(t)+v(t))}{8(1+t^2)}, \end{aligned}$$

The hypotheses (H_3) *and* (H_4) *are satisfied with*

$$p_1 = p_2 = \frac{1}{16},\ q_1 = q_2 = \frac{1}{8},\ \alpha_1 = \frac{1}{4},\ \beta_1 = \frac{3}{4},\ \alpha_2 = \frac{1}{5},\ \beta_2 = \frac{4}{5}.$$

As a consequence of Leray–Schauder fixed-point theorem, we conclude that T *has a fixed point, which is the solution of problem* (31) *and* (32).

Example 3. *Consider the following coupled system of fractional differential equation*

$$\begin{cases} D^{\frac{1}{2}}u(t) = \frac{e^{-2t}\tan(u(t))+e^{-2t}\sin(v(t))}{32(1+\|u(t)\|+\|v(t)\|)}, t \in [0,1], \\ D^{\frac{1}{4}}v(t) = \frac{e^{-5t}\,arc\tan(u(t))+e^{-5t}\cos(v(t))}{16(1+\|u(t)\|+\|v(t)\|)}, t \in [0,1], \\ u(0) = v(0) = 1. \end{cases} \tag{33}$$

set

$$\begin{cases} f_1(t,u(t),v(t)) = \frac{e^{-2t}\tan(u(t))+e^{-2t}\sin(v(t))}{32(1+\|u(t)\|+\|v(t)\|)}, t \in [0,1], \\ f_2(t,u(t),v(t)) = \frac{e^{-5t}\,arc\tan(u(t))+e^{-5t}\cos(v(t))}{16(1+\|u(t)\|+\|v(t)\|)}, t \in [0,1], \end{cases}$$

Let $\alpha_1 = \frac{1}{2},\ \beta_1 = \frac{1}{2},\ \alpha_2 = \frac{1}{4},\ \beta_2 = \frac{3}{4}$.

The hypothesis (H_2) *is satisfied with*

$$h_i(t) = \frac{1}{32}e^{-2t}, k_i(t) = \frac{1}{16}e^{-5t}, h_i^* = \frac{1}{32}, k_i^* = \frac{1}{16},\ i = 1,2.$$

Let $\Phi_1(t) = e^{-\frac{1}{2}t}, \Phi_2(t) = e^{-t}$. *Choose constants* $\lambda_{\Phi_1} = 4, \lambda_{\Phi_2} = 2$. *Furthermore, hypothesis* (H_5) *is satisfied because*

$$\frac{1}{\alpha_1}e^{-\frac{\beta_1}{\alpha_1}t}\int_0^t e^{\frac{\beta_1}{\alpha_1}s}\Phi_1(s)ds = 2e^{-t}\int_0^t e^s e^{-\frac{1}{2}s}ds = 4(e^{-\frac{1}{2}t} - e^{-t}) \le 4e^{-\frac{1}{2}t} = \lambda_{\Phi_1}\Phi_1(t).$$

$$\frac{1}{\alpha_2}e^{-\frac{\beta_2}{\alpha_2}t}\int_0^t e^{\frac{\beta_2}{\alpha_2}s}\Phi_2(s)ds = 4e^{-3t}\int_0^t e^{3s}e^{-s}ds = 2(e^{-t} - e^{-3t}) \le 2e^{-t} = \lambda_{\Phi_2}\Phi_2(t).$$

Consequently, Theorem 6 implies that the problem (33) has at least one solution and it is generalized-Ulam–Hyers–Rassias stable.

5.2. Comparison with the Existing Literature

In this subsection, we clarify the position of our work with respect to the existing literature and highlight the main improvements.

5.2.1. Comparison with Yang [24]

The work in Yang [24] (*Chinese Journal of Physics, 2023*) studies fractional differential equations with delays using Caputo-type operators and Laplace transform techniques. In contrast, our work differs in the following aspects:

- We employ the deformable fractional derivative, which provides a flexible transition between local and nonlocal dynamics.
- We consider fully coupled nonlinear systems in Banach spaces.
- We apply Perov's fixed-point theorem under matrix contractive conditions, which allows a more general treatment of coupled systems.
- We establish both Ulam–Hyers and generalized Ulam–Hyers–Rassias stability results.

5.2.2. Comparison with Hassan [25]

The work in Hassan [25] investigates fractional systems involving Hilfer-type operators and establishes stability results using fixed-point techniques. However, the main differences are:

- The operator considered in Hassan [25] is classical (Hilfer-type), whereas we use the deformable fractional derivative.
- Our framework incorporates matrix contractive conditions via Perov's theorem, which is more suitable for strongly coupled systems.
- We provide explicit conditions for both existence and stability in the deformable fractional setting.

Most existing works on fractional differential systems rely on classical operators (Caputo, Riemann–Liouville, Hilfer) and standard fixed-point methods. The novelty of the present work can be summarized as follows:

- Introduction of the deformable fractional derivative in coupled systems,
- Use of Perov's fixed-point theorem to handle matrix contractive conditions,
- Establishment of Ulam–Hyers and generalized Ulam–Hyers–Rassias stability in this framework,
- Applicability to systems with adjustable memory effects.

6. Discussion on Numerical Modeling

Fractional differential equations have proven to be highly effective in modeling complex dynamical systems due to their ability to capture memory and hereditary properties. In particular, coupled fractional systems arise naturally in applications such as viscoelasticity, biological systems, and control processes.

The deformable fractional derivative considered in this paper provides additional flexibility, as it allows a continuous transition between classical integer-order dynamics and fractional-order dynamics. This feature is particularly useful in modeling systems where the intensity of memory effects varies over time.

From a numerical perspective, several approaches can be employed to approximate the solutions of the considered system. For instance, predictor-corrector schemes, finite difference methods, and spectral methods can be adapted to handle deformable fractional operators.

The theoretical results obtained in this work, including existence, uniqueness, and stability, guarantee that the numerical approximations are well-founded and stable under perturbations. In particular, the Ulam–Hyers and generalized Ulam–Hyers–Rassias stability results ensure that small numerical errors do not significantly affect the behavior of the solution.

Therefore, the proposed framework provides a reliable basis for future numerical simulations of coupled deformable fractional systems arising in real-world applications.

Future work will focus on the development of efficient numerical schemes and their implementation for solving such systems, as well as comparisons with classical fractional models.

7. Conclusions

In this paper, we investigated the well-posedness and stability of a class of coupled deformable fractional differential systems within the framework of Banach spaces. By leveraging the structural flexibility of the deformable fractional derivative which interpolates smoothly between a function and its classical derivative through a single parameter,

we established a rigorous mathematical foundation for analyzing such coupled nonlinear models. Specifically, the main contributions of this work are threefold:

- The existence and uniqueness of mild solutions were established via Perov's fixed-point theorem under matrix contractive conditions, extending classical results to the deformable fractional setting.
- The existence of at least one solution was proved under broader growth conditions using the generalized nonlinear alternative of Leray–Schauder. Continuous dependence on initial data was rigorously demonstrated, confirming the well-posedness of the considered system.
- Ulam–Hyers and generalized Ulam–Hyers–Rassias stability results were derived, providing quantitative stability estimates relevant to numerical and applied analysis.

The theoretical framework developed here not only unifies and extends several existing contributions in fractional and coupled differential systems but also highlights the deformable derivative as a versatile and computationally tractable tool for modeling complex dynamical phenomena with memory and hereditary effects. To the best of our knowledge, this constitutes the first systematic study of coupled deformable fractional systems combining both well-posedness and Ulam-type stability in Banach spaces. The results open several promising avenues for future research, including:

- Extension to stochastic and random coupled deformable systems;
- Analysis of impulsive and delay deformable fractional models;
- Investigation of controllability and optimal control problems;
- Development of efficient numerical schemes tailored to deformable operators.

We anticipate that the methodology presented in this work will serve as a foundation for broader investigations into deformable fractional calculus and its applications in physics, engineering, and biological modeling.

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