



Article

Stability and Period-Doubling Bifurcation of Fractional-Order Commensal Symbiosis Model with Allee Effect

Mohammed Bakheet Almatrafi 

Department of Mathematics, College of Science, Taibah University, Al-Madinah Al-Munawarah 42353, Saudi Arabia; mmutrafi@taibahu.edu.sa

Abstract

Differential equations with fractional order play an important role in modeling some natural phenomena. This paper investigates the dynamics of the fractional-order commensal symbiosis model with the Allee effect. This model describes the relationship between prey and predator populations. The piecewise-constant approximation technique is applied to discretize this model. Equilibrium points are established, and local stability conditions are calculated using fractional-order linearization and eigenvalue-based arguments. Moreover, the bifurcation theory is successfully invoked to discuss the period-doubling bifurcation. In particular, sufficient conditions are effectively determined for the emergence of the period-doubling bifurcation. We utilize the hybrid control approach to control the behavior of the considered system. Then, some numerical examples are presented to demonstrate the accuracy and validity of the theoretical results. The findings indicate that fractional order and Allee effects improve system dynamics and substantially improve stability limits and bifurcation structures, providing new insights into how to handle the dynamics of ecological systems.

Keywords: fractional-order commensal symbiosis model; equilibrium points; stability; period-doubling bifurcation; chaos control

1. Introduction

An effective mathematical structure for modeling connections in which one organism benefits from the existence of another without generating a major positive or undesirable effect on the other organism is given by the commensal symbiosis model [1,2]. The presence of the host species usually leads to a greater per-capita growth rate for the benefiting species, whereas the host maintains its own basic population dynamics. The ecologically relevant phenomena that populations are unable to remain below an acceptable density, even under ideal environmental circumstances, can be seen in the model when an Allee effect is included into the growth of the commensal population. These mathematical representations are particularly effective for exploring microbial consortia, interactions between plants and microbes, and cooperative–neutral ecological interactions in which the beneficiary species relies on a microenvironment or resource associated with the host for its long-term existence. Fractional-order derivatives have been increasingly prevalent in environmental simulation recently, since they can offer more precise results than standard integer-order models. Moreover, they incorporate long-behavior dynamics, anomalous diffusion, and sub-exponential growth patterns that can be seen in real biological systems [3].

Various studies, which were used to investigate the dynamic behaviors, stability features, and environmental effects associated with commensal symbiosis, have attracted



Academic Editor: Gani Stamov

Received: 2 March 2026

Revised: 23 March 2026

Accepted: 25 March 2026

Published: 28 March 2026

Copyright: © 2026 by the author.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and

conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

several scholars to describe some biological phenomena such as a fractional-order commensal symbiosis model with the Allee effect. Understanding the long-term behavior of commensal systems is extensively dependent on the investigation of equilibria. Utilizing comparison theorems to deduce the cases in which these equilibria are stable or unstable, Liu et al. [4] analyzed the stability of a system of commensalism and parasitism with harvesting in commensal species. The stability of a commensalism system involving Michaelis–Menten harvesting was examined in [5]. This system also highlighted the criteria for global attractivity and equilibrium stability. More research has concentrated on systems which involve functional responses, like the Holling type. The stability of a commensalism structure incorporating Holling-type responses, for illustration, was successfully explored in [2] to demonstrate how these nonlinear terms effectively impact the equilibrium states of the system. Furthermore, Chen et al. [6] obtained some constraints under which the positive equilibrium of a commensal system with a Holling II functional response is global. The Allee effect may boost the species' ultimate density, pointing to a possible favorable function in some symbiotic situations [7]. In addition, the findings in [8] illustrated that the positive equilibrium is globally stable where the Allee effect was encountered by the first species. This emphasizes the stabilizing impact of the Allee effect on population structure [8]. The period-doubling bifurcations in a discrete-time commensal system with the Allee effect was addressed in [9]. This study emphasized that the possibility of intricate oscillatory dynamics and bifurcations around coexistence points can be beneficially shown [9]. Finally, a commensal scheme with nonlinear dispersal and the Allee effect was successfully investigated in [10]. Zhong et al. [10] discussed the impacts of the Allee effect in complex ecological circumstances and explained how spatial heterogeneity impact stability and bifurcation behavior.

Despite these improvements, some significant gaps persist. The vast majority of studies on ecology utilize straightforward hypotheses about commensal relationships. As a result, the desire to more effectively understand interactions among organisms that do not fully benefit from each other is the motivation that drives the investigation of a fractional-order commensal symbiosis model. Through the utilization of fractional derivatives, scholars can modify the adaptability of fractional-order systems and analyze how various types of input impact stability, bifurcation, chaos, and coexistence conditions. The importance of analyzing a fractional-order commensal symbiosis system stems from its capability to present an improved and adaptable explanation for interactions in ecology in which one species takes advantage while the other stays unaffected. In order to effectively deal with previous challenges, this research discusses the dynamics of a novel fractional-order commensal symbiosis model which demonstrates the complicated nature of long-term ecological connections. This study is successfully carried out to establish equilibrium points, examine local stability, find bifurcation, and investigate how fractional orders affect the system's behavior. Furthermore, some numerical simulations are presented to verify the results obtained.

The fractional-order commensal symbiosis model with the Allee effect is given by

$$\begin{cases} T^\beta u(t) = u(t) \left(A_1 - B_1 u(t) + \frac{Cv(t)^p}{1 + v(t)^p} \right), \\ T^\beta v(t) = v(t) (A_2 - B_2 v(t)) \frac{v(t)}{M + v(t)}, \end{cases} \quad (1)$$

where $0 < \beta < 1$ is the fractional-order parameter and $p \geq 1$. The variables $u(t) > 0$ and $v(t) > 0$ are considered to be the population densities of the prey and predator at time t , respectively. A_1 and A_2 represent the intrinsic prey and predator growth rates, respectively. B_1 describes the competition among individuals of prey and B_2 has a similar meaning to

B_1 . Moreover, C describes the intensity of the cooperative effect of the predator on the prey, and the term $\left(\frac{v(t)}{M+v(t)}\right)$ is the Allee effect, while $M > 0$ represents the constant of the Allee effect. Finally, T^β is a fractional derivative of the conformable type, as defined in [11].

$$T^\beta f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon(t-S)^{1-\beta}) - f(t)}{\epsilon}, \quad 0 < \beta < 1, \quad (2)$$

where $h > 0$ is the discretization parameter. It was shown in [12] that the following fact was evidenced from Equation (2).

$$T^\beta f(t) = (t-S)^{1-\beta} f'(t). \quad (3)$$

2. Discretization Process

The discretization process of system (1) using the piecewise-constant approximation method [13,14] is shown in this section. Applying this technique, we obtain

$$\begin{cases} T^\beta u(t) = u(t) \left(A_1 - B_1 u \left(\left[\frac{t}{S} \right] S \right) + \frac{Cv \left(\left[\frac{t}{S} \right] S \right)^p}{1 + v \left(\left[\frac{t}{S} \right] S \right)^p} \right), \\ T^\beta v(t) = v(t) \left(A_2 - B_2 v \left(\left[\frac{t}{S} \right] S \right) \right) \frac{v \left(\left[\frac{t}{S} \right] S \right)}{M + v \left(\left[\frac{t}{S} \right] S \right)}, \end{cases} \quad (4)$$

where $\left[\frac{t}{S} \right]$ represents the integer part of $t \in [iS, (i+1)S)$, $i = 0, 1, \dots$, and $S > 0$ is a discretization parameter. By applying relation (3), the first equation of system (4) can be rewritten as

$$(t-iS)^{1-\beta} \frac{du(t)}{dt} = u(t) \left(A_1 - B_1 u(iS) + \frac{Cv(iS)^p}{1 + v(iS)^p} \right). \quad (5)$$

Separating variables yields

$$\frac{du(t)}{u(t)} = \left(A_1 - B_1 u(iS) + \frac{Cv(iS)^p}{1 + v(iS)^p} \right) (t-iS)^{\beta-1} dt. \quad (6)$$

Integrating over the interval $[iS, t)$ gives

$$\ln u(t) - \ln u(iS) = \left(A_1 - B_1 u(iS) + \frac{Cv(iS)^p}{1 + v(iS)^p} \right) \frac{(t-iS)^\beta}{\beta}. \quad (7)$$

Taking the limit as $t \rightarrow (i+1)S$ and introducing the notation $u(iS) = u_i$, $v(iS) = v_i$, we obtain

$$u_{i+1} = u_i \exp \left(\left(A_1 - B_1 u_i + \frac{Cv_i^p}{1 + v_i^p} \right) \frac{S^\beta}{\beta} \right). \quad (8)$$

The second equation of system (4) is similarly solved to end up with

$$v_{i+1} = v_i \exp \left(\frac{v_i (A_2 - B_2 v_i) S^\beta}{\beta (M + v_i)} \right). \quad (9)$$

Consequently, the discrete-time counterpart of system (1) can be expressed as the following two-dimensional nonlinear system:

$$\begin{cases} u_{i+1} = u_i \exp\left(\left(A_1 - B_1 u_i + \frac{C v_i^p}{1 + v_i^p}\right) \frac{S^\beta}{\beta}\right), \\ v_{i+1} = v_i \exp\left(\frac{v_i(A_2 - B_2 v_i) S^\beta}{\beta(M + v_i)}\right), \end{cases} \quad (10)$$

where S is the discretization parameter.

3. The Existence and Local Stability of the Fixed Points

In this section, we analyze the stability of fixed points of system (10). First, the equilibrium points of system (10) are determined by solving the following algebraic nonlinear system:

$$\begin{cases} u = u \exp\left(\left(A_1 - B_1 u + \frac{C v^p}{1 + v^p}\right) \frac{S^\beta}{\beta}\right), \\ v = v \exp\left(\frac{v(A_2 - B_2 v) S^\beta}{\beta(M + v)}\right). \end{cases}$$

For all positive parameters A_1, B_1, C, A_2, B_2, M , and $p \geq 1$, model (10) has four fixed points:

$$P_0 = (0, 0), \quad P_1 = \left(\frac{A_1}{B_1}, 0\right), \quad P_2 = \left(0, \frac{A_2}{B_2}\right),$$

and

$$P_3 = \left(\frac{A_1 \left(\frac{A_2}{B_2}\right)^p + C \left(\frac{A_2}{B_2}\right)^p + A_1}{B_1 \left(1 + \left(\frac{A_2}{B_2}\right)^p\right)}, \frac{A_2}{B_2}\right).$$

The Jacobian matrix of system (10) at any equilibrium point $M = (u, v)$ can be written as

$$J_M = \begin{pmatrix} \left(1 - \frac{B_1 u S^\beta}{\beta}\right) \exp\left(\left(A_1 - B_1 u + \frac{C v^p}{1 + v^p}\right) \frac{S^\beta}{\beta}\right) & \frac{p C u S^\beta \exp\left(\left(A_1 - B_1 u + \frac{C v^p}{1 + v^p}\right) \frac{S^\beta}{\beta}\right)}{\beta v^{1-p} (1 + v^p)^2} \\ 0 & \left(1 + \frac{(A_2 M - B_2 v(2M + v)) v S^\beta}{\beta(M + v)^2}\right) \exp\left(\frac{v(A_2 - B_2 v) S^\beta}{\beta(M + v)}\right) \end{pmatrix}. \quad (11)$$

Consequently, the characteristic equation of the previous matrix is

$$\mathcal{W}(\kappa) = \kappa^2 - \mathcal{P}\kappa + \mathcal{Q} = 0,$$

with

$$\begin{aligned} \mathcal{P} &= \left(1 - \frac{B_1 u S^\beta}{\beta}\right) \exp\left(\left(A_1 - B_1 u + \frac{C v^p}{1 + v^p}\right) \frac{S^\beta}{\beta}\right) + \left(1 + \frac{(A_2 M - B_2 v(2M + v)) v S^\beta}{\beta(M + v)^2}\right) \exp\left(\frac{v(A_2 - B_2 v) S^\beta}{\beta(M + v)}\right), \\ \mathcal{Q} &= \left(1 - \frac{B_1 u S^\beta}{\beta}\right) \left(1 + \frac{(A_2 M - B_2 v(2M + v)) v S^\beta}{\beta(M + v)^2}\right) \exp\left(\left(A_1 - B_1 u + \frac{C v^p}{1 + v^p} + \frac{v(A_2 - B_2 v)}{(M + v)}\right) \frac{S^\beta}{\beta}\right). \end{aligned}$$

Next, the stability of the fixed points is examined with the help of Lemma 1 and Lemma 2.

Lemma 1 ([14,15]). Let (u, v) be a fixed point of system (10) with multipliers (eigenvalues of Jacobian matrix) κ_1 and κ_2 . Then,

1. If $|\kappa_1| < 1$ and $|\kappa_2| < 1$, it is a sink point and locally asymptotically stable;
2. If $|\kappa_1| > 1$ and $|\kappa_2| > 1$, it is a source point and locally unstable;

3. If $|\kappa_1| < 1$ and $|\kappa_2| > 1$ or $(|\kappa_1| > 1$ and $|\kappa_2| < 1)$, it is a saddle point;
4. If $|\kappa_1| = 1$ or $|\kappa_2| = 1$, it is non-hyperbolic.

Lemma 2 ([14,16]). Suppose that the polynomial $\mathcal{W}(\kappa) = \kappa^2 - \mathcal{P}\kappa + \mathcal{Q}$, where $\mathcal{W}(1) > 0$, and κ_1 and κ_2 are the two roots of $\mathcal{W}(\kappa) = 0$. Then,

1. $|\kappa_1| < 1$ and $|\kappa_2| < 1$ if and only if $\mathcal{W}(-1) > 0$ and $\mathcal{W}(0) < 1$.
2. $|\kappa_1| > 1$ and $|\kappa_2| > 1$ if and only if $\mathcal{W}(-1) > 0$ and $\mathcal{W}(0) > 1$.
3. $|\kappa_1| < 1$ and $|\kappa_2| > 1$ (or $|\kappa_1| > 1$ and $|\kappa_2| < 1$) if and only if $\mathcal{W}(-1) < 0$.
4. $\kappa_1 = -1$ and $\kappa_2 \neq 1$ if and only if $\mathcal{W}(-1) = 0$ and $\mathcal{P} \neq 0, 2$.
5. κ_1 and κ_2 are complex numbers and $|\kappa_1| = |\kappa_2| = 1$ if and only if $\mathcal{P}^2 - 4\mathcal{Q} < 0$ and $\mathcal{W}(0) = 1$.

Theorem 1. The trivial fixed point $P_0 = (0, 0)$ and predator-free fixed point $P_1 = \left(\frac{A_1}{B_1}, 0\right)$ are non-hyperbolic.

Proof. The matrix (11) at the fixed point $P_0 = (0, 0)$ is written as

$$J_{P_0} = \begin{pmatrix} e^{\frac{A_1 S^\beta}{\beta}} & 0 \\ 0 & 1 \end{pmatrix},$$

whose eigenvalues are

$$\kappa_1 = e^{\frac{A_1 S^\beta}{\beta}} > 1, \quad \kappa_2 = 1.$$

Next, we evaluate matrix (11) at the fixed point $P_1 = \left(\frac{A_1}{B_1}, 0\right)$ to obtain

$$J_{P_1} = \begin{pmatrix} 1 - \frac{A_1 S^\beta}{\beta} & 0 \\ 0 & 1 \end{pmatrix},$$

with

$$\kappa_1 = 1 - \frac{A_1 S^\beta}{\beta}, \quad \kappa_2 = 1.$$

Using Lemma 1, it can be deduced that the points P_0 and P_1 are non-hyperbolic. $\square$

Theorem 2. The prey-free fixed point $P_2 = \left(0, \frac{A_2}{B_2}\right)$ is a saddle point if $0 < S < \left(\frac{2\beta(MB_2 + A_2)}{A_2^2}\right)^{\frac{1}{\beta}}$, a source point if $S > \left(\frac{2\beta(MB_2 + A_2)}{A_2^2}\right)^{\frac{1}{\beta}}$, and non-hyperbolic if $S = \left(\frac{2\beta(MB_2 + A_2)}{A_2^2}\right)^{\frac{1}{\beta}}$.

Proof. The matrix (11) at $P_2 = \left(0, \frac{A_2}{B_2}\right)$ is

$$J_{P_2} = \begin{pmatrix} \text{Exp}\left(A_1 + \left(\frac{CA_2^p}{B_2^p + A_2^p}\right)\frac{S^\beta}{\beta}\right) & 0 \\ 0 & 1 - \left(\frac{A_2^2 S^\beta}{\beta(MB_2 + A_2)}\right) \end{pmatrix},$$

where

$$\kappa_1 = \text{Exp}\left(A_1 + \left(\frac{CA_2^p}{B_2^p + A_2^p}\right)\frac{S^\beta}{\beta}\right) > 1, \quad \kappa_2 = 1 - \left(\frac{A_2^2 S^\beta}{\beta(MB_2 + A_2)}\right).$$

Using Lemma 1, it can be deduced that the point P_2 is a saddle point if $0 < S < \left(\frac{2\beta(MB_2+A_2)}{A_2^2}\right)^{\frac{1}{\beta}}$, a source point if $S > \left(\frac{2\beta(MB_2+A_2)}{A_2^2}\right)^{\frac{1}{\beta}}$, and non-hyperbolic if $S = \left(\frac{2\beta(MB_2+A_2)}{A_2^2}\right)^{\frac{1}{\beta}}$. $\square$

Theorem 3. For the coexistence fixed point $P_3 = \left(\frac{A_1\left(\frac{A_2}{B_2}\right)^p + C\left(\frac{A_2}{B_2}\right)^p + A_1}{B_1\left(1 + \left(\frac{A_2}{B_2}\right)^p\right)}, \frac{A_2}{B_2}\right)$, we have the following results:

1. If $0 < S < \left(\frac{\beta(H_1+H_3-\sqrt{(H_1+H_3)^2-4H_1H_3})}{H_1H_3}\right)^{1/\beta}$, then P_3 is locally asymptotically stable (sink).
2. If $S > \left(\frac{\beta(H_1+H_3+\sqrt{(H_1+H_3)^2-4H_1H_3})}{H_1H_3}\right)^{1/\beta}$, then P_3 is a source point.
3. If $\left(\frac{\beta(H_1+H_3-\sqrt{(H_1+H_3)^2-4H_1H_3})}{H_1H_3}\right)^{1/\beta} < S < \left(\frac{\beta(H_1+H_3+\sqrt{(H_1+H_3)^2-4H_1H_3})}{H_1H_3}\right)^{1/\beta}$, then P_3 is a saddle point.
4. The fixed point P_3 is non-hyperbolic and the roots of the characteristic equation ($\mathcal{W}(\kappa) = 0$) at this point are $\kappa_1 = -1$ and $|\kappa_2| \neq 1$ if

$$S = S_{1,2} = \left(\frac{\beta(H_1 + H_3 \mp \sqrt{(H_1 + H_3)^2 - 4H_1H_3})}{H_1H_3}\right)^{1/\beta} \quad \text{and} \quad S \neq \left(\frac{2\beta}{H_1 + H_3}\right)^{1/\beta}.$$

Proof. The matrix (11) (J_{P_3}) at P_3 can be expressed as

$$J_{P_3} = \begin{pmatrix} \left(1 - \frac{S^\beta}{\beta} H_1\right) & \frac{S^\beta}{\beta} H_2 \\ 0 & \left(1 - \frac{S^\beta}{\beta} H_3\right) \end{pmatrix}. \quad (12)$$

Here, we have

$$H_1 = \left(\frac{A_1\left(\frac{A_2}{B_2}\right)^p + C\left(\frac{A_2}{B_2}\right)^p + A_1}{\left(1 + \left(\frac{A_2}{B_2}\right)^p\right)}\right), \quad H_3 = \left(\frac{A_2^2}{MB_2 + A_2}\right),$$

$$H_2 = \frac{pCB_2}{A_2B_1} \left(\frac{A_1\left(\frac{A_2}{B_2}\right)^{2p} + C\left(\frac{A_2}{B_2}\right)^{2p} + A_1\left(\frac{A_2}{B_2}\right)^p}{\left(1 + \left(\frac{A_2}{B_2}\right)^p\right)^3}\right).$$

Hence, the characteristic polynomial is

$$\mathcal{W}(\kappa) = \kappa^2 - \left(2 - \frac{(H_1 + H_3)S^\beta}{\beta}\right)\kappa + \left(1 - \frac{S^\beta}{\beta} H_1\right)\left(1 - \frac{S^\beta}{\beta} H_3\right).$$

By performing basic calculations, we find

$$\mathcal{W}(1) = \frac{(H_1H_3)S^{2\beta}}{\beta^2} > 0, \quad \mathcal{W}(0) = 1 - \frac{(H_1 + H_3)S^\beta}{\beta} + \frac{(H_1H_3)S^{2\beta}}{\beta^2},$$

$$\text{and} \quad \mathcal{W}(-1) = 4 - \frac{2(H_1 + H_3)S^\beta}{\beta} + \frac{(H_1H_3)S^{2\beta}}{\beta^2}.$$

Applying Lemma 2, we find the requirement. $\square$

4. Period-Doubling Bifurcation Analysis

Here, we use the bifurcation theory presented in [14,17,18] to discuss the conditions under which the considered system encounters a period-doubling bifurcation at the coexistence fixed point

$$P_3 = \left(\frac{A_1 \left(\frac{A_2}{B_2} \right)^p + C \left(\frac{A_2}{B_2} \right)^p + A_1}{B_1 \left(1 + \left(\frac{A_2}{B_2} \right)^p \right)}, \frac{A_2}{B_2} \right).$$

We make the assumption that the bifurcation parameter changes within a small portion of the set

$$\mathcal{B}_1 = \left\{ (A_1, A_2, B_1, B_2, p, C, \beta) \in \mathbb{R}^7 \left| \beta \in (0, 1], S_1 = \left(\frac{\beta(H_1 + H_3 \mp \sqrt{(H_1 + H_3)^2 - 4H_1H_3})}{H_1H_3} \right)^{1/\beta} \right. \right. \\ \left. \left. S \neq \left(\frac{2\beta}{H_1 + H_3} \right)^{1/\beta}, H_1 = \left(\frac{A_1 \left(\frac{A_2}{B_2} \right)^p + C \left(\frac{A_2}{B_2} \right)^p + A_1}{\left(1 + \left(\frac{A_2}{B_2} \right)^p \right)} \right), H_3 = \left(\frac{A_2^2}{MB_2 + A_2} \right) \right\}.$$

Assume that $(A_1, A_2, B_1, B_2, p, C, \beta) \in \mathcal{B}_1$. Then, system (10) is expressed as

$$\begin{cases} u_{i+1} = u_i \exp \left(\left(A_1 - B_1 u_i + \frac{C v_i^p}{1 + v_i^p} \right) \frac{(S_1 + \bar{S})^\beta}{\beta} \right) = K_1(u_i, v_i, \bar{S}), \\ v_{i+1} = v_i \exp \left(\frac{v_i(A_2 - B_2 v_i)(S_1 + \bar{S})^\beta}{\beta(M + v_i)} \right) = K_2(u_i, v_i, \bar{S}). \end{cases} \quad (13)$$

Here, $\bar{S}$ is a small perturbation from S_1 where $\bar{S} \ll 1$. Utilizing the change of variables

$$X_i = u_i - \left(\frac{A_1 \left(\frac{A_2}{B_2} \right)^p + C \left(\frac{A_2}{B_2} \right)^p + A_1}{B_1 \left(1 + \left(\frac{A_2}{B_2} \right)^p \right)} \right) \text{ and } Y_i = v_i - \left(\frac{A_2}{B_2} \right),$$

the fixed point P_3 is simply shifted to the origin. Next, we apply the Taylor series at origin to expand K_1 and K_2 to the third order. Achieving this, system (13) turns into

$$\begin{cases} X_{i+1} = a_{11}X_i + a_{12}Y_i + a_{13}X_i^2 + a_{14}X_iY_i + a_{15}Y_i^2 + b_{11}X_i\bar{S} + b_{12}Y_i\bar{S} + a_{16}X_i^3 + \\ b_{13}\bar{S}^2 + b_{14}X_i^2\bar{S} + b_{15}Y_i^2\bar{S} + a_{17}X_i^2Y_i + a_{18}X_iY_i^2 + b_{16}X_iY_i\bar{S} + b_{17}\bar{S}^3 + \\ b_{18}X_i\bar{S}^2 + b_{19}Y_i\bar{S}^2 + a_{19}Y_i^3 + \mathcal{R}_1(|X_i|, |Y_i|, |\bar{S}|)^4, \\ Y_{i+1} = a_{21}X_i + a_{22}Y_i + a_{23}X_i^2 + a_{24}X_iY_i + a_{25}Y_i^2 + b_{21}X_i\bar{S} + b_{22}Y_i\bar{S} + a_{26}X_i^3 + \\ b_{23}\bar{S}^2 + b_{24}X_i^2\bar{S} + b_{25}Y_i^2\bar{S} + a_{27}X_i^2Y_i + a_{28}X_iY_i^2 + b_{26}X_iY_i\bar{S} + b_{27}\bar{S}^3 + \\ b_{28}X_i\bar{S}^2 + b_{29}Y_i\bar{S}^2 + a_{29}Y_i^3 + \mathcal{R}_2(|X_i|, |Y_i|, |\bar{S}|)^4, \end{cases} \quad (14)$$

where the values of the coefficients are presented in Appendix A. In addition, we obtain the normal form of system (14) at $\bar{S}$ using the transformation

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \kappa_2 - a_{11} \end{pmatrix} \begin{pmatrix} \bar{u}_i \\ \bar{v}_i \end{pmatrix}.$$

Thus, system (14) turns into

$$\begin{cases} \bar{u}_{i+1} = -\bar{x}_n + \tilde{K}_1(\bar{u}_i, \bar{v}_i, \bar{S}), \\ \bar{y}_{n+1} = \kappa_2 \bar{v}_i + \tilde{K}_2(\bar{u}_i, \bar{v}_i, \bar{S}), \end{cases} \quad (15)$$

where

$$\begin{aligned} \tilde{K}_1(\bar{u}_i, \bar{v}_i, \bar{S}) = & \left(\frac{a_{13}(\kappa_2 - a_{11}) - a_{12}a_{23}}{a_{12}(1 + \kappa_2)} \right) X_i^2 + \left(\frac{a_{14}(\kappa_2 - a_{11}) - a_{12}a_{24}}{a_{12}(1 + \kappa_2)} \right) X_i Y_i + \left(\frac{a - 15(\kappa_2 - a_{11}) - a_{12}a_{25}}{a_{12}(1 + \kappa_2)} \right) Y_i^2 \\ & + \left(\frac{b_{11}(\kappa_2 - a_{11}) - a_{12}b_{21}}{a_{12}(1 + \kappa_2)} \right) X_i \bar{S} + \left(\frac{b_{12}(\kappa_2 - a_{11}) - a_{12}b_{22}}{a_{12}(1 + \kappa_2)} \right) Y_i \bar{S} + \left(\frac{a_{16}(\kappa_2 - a_{11}) - a_{12}a_{26}}{a_{12}(1 + \kappa_2)} \right) X_i^3 \\ & + \left(\frac{b_{13}(\kappa_2 - a_{11}) - a_{12}b_{23}}{a_{12}(1 + \kappa_2)} \right) \bar{S}^2 + \left(\frac{b_{14}(\kappa_2 - a_{11}) - a_{12}b_{24}}{a_{12}(1 + \kappa_2)} \right) X_i^2 \bar{S} + \left(\frac{b_{15}(\kappa_2 - a_{11}) - a_{12}b_{25}}{a_{12}(1 + \kappa_2)} \right) w_n^2 \bar{S} \\ & + \left(\frac{a_{17}(\kappa_2 - a_{11}) - a_{12}a_{27}}{a_{12}(1 + \kappa_2)} \right) X_i^2 Y_i + \left(\frac{a_{18}(\kappa_2 - a_{11}) - a_{12}a_{28}}{a_{12}(1 + \kappa_2)} \right) X_i Y_i^2 + \left(\frac{b_{16}(\kappa_2 - a_{11}) - a_{12}b_{26}}{a_{12}(1 + \kappa_2)} \right) X_i Y_i \bar{S} \\ & + \left(\frac{b_{17}(\kappa_2 - a_{11}) - a_{12}b_{27}}{a_{12}(1 + \kappa_2)} \right) \bar{S}^3 + \left(\frac{b_{18}(\kappa_2 - a_{11}) - a_{12}b_{28}}{a_{12}(1 + \kappa_2)} \right) X_i \bar{S}^2 + \left(\frac{b_{19}(\kappa_2 - a_{11}) - a_{12}b_{29}}{a_{12}(1 + \kappa_2)} \right) w_n \bar{S}^2 \\ & + \left(\frac{a_{19}(\kappa_2 - a_{11}) - a_{12}a_{29}}{a_{12}(1 + \kappa_2)} \right) Y_i^3 + \mathcal{R}_1(|X_i|, |Y_i|, |\bar{S}|)^4, \end{aligned}$$

and

$$\begin{aligned} \tilde{K}_2(\bar{u}_i, \bar{v}_i, \bar{S}) = & \left(\frac{a_{13}(1 - a_{11}) + a_{12}a_{23}}{a_{12}(1 + \kappa_2)} \right) X_i^2 + \left(\frac{a_{14}(1 - a_{11}) + a_{12}a_{24}}{a_{12}(1 + \kappa_2)} \right) X_i Y_i + \left(\frac{a - 15(1 - a_{11}) + a_{12}a_{25}}{a_{12}(1 + \kappa_2)} \right) Y_i^2 \\ & + \left(\frac{b_{11}(1 - a_{11}) + a_{12}b_{21}}{a_{12}(1 + \kappa_2)} \right) X_i \bar{S} + \left(\frac{b_{12}(1 - a_{11}) + a_{12}b_{22}}{a_{12}(1 + \kappa_2)} \right) Y_i \bar{S} + \left(\frac{a_{16}(1 - a_{11}) + a_{12}a_{26}}{a_{12}(1 + \kappa_2)} \right) X_i^3 \\ & + \left(\frac{b_{13}(1 - a_{11}) + a_{12}b_{23}}{a_{12}(1 + \kappa_2)} \right) \bar{h}^2 + \left(\frac{b_{14}(1 - a_{11}) + a_{12}b_{24}}{a_{12}(1 + \kappa_2)} \right) X_i^2 \bar{S} + \left(\frac{b_{15}(1 - a_{11}) + a_{12}b_{25}}{a_{12}(1 + \kappa_2)} \right) Y_i^2 \bar{S} \\ & + \left(\frac{a_{17}(1 - a_{11}) + a_{12}a_{27}}{a_{12}(1 + \kappa_2)} \right) X_i^2 Y_i + \left(\frac{a_{18}(1 - a_{11}) + a_{12}a_{28}}{a_{12}(1 + \kappa_2)} \right) X_i Y_i^2 + \left(\frac{b_{16}(1 - a_{11}) + a_{12}b_{26}}{a_{12}(1 + \kappa_2)} \right) X_i Y_i \bar{S} \\ & + \left(\frac{b_{17}(1 - a_{11}) + a_{12}b_{27}}{a_{12}(1 + \kappa_2)} \right) \bar{S}^3 + \left(\frac{b_{18}(1 - a_{11}) + a_{12}b_{28}}{a_{12}(1 + \kappa_2)} \right) X_i \bar{S}^2 + \left(\frac{b_{19}(1 - a_{11}) + a_{12}b_{29}}{a_{12}(1 + \kappa_2)} \right) Y_i \bar{S}^2 \\ & + \left(\frac{a_{19}(1 - a_{11}) + a_{12}a_{29}}{a_{12}(1 + \kappa_2)} \right) Y_i^3 + \mathcal{R}_2(|X_i|, |Y_i|, |\bar{S}|)^4, \end{aligned}$$

with

$$X_i = a_{12}(\bar{u}_i + \bar{v}_i) \quad \text{and} \quad Y_i = -(1 + a_{11})\bar{u}_i + (\kappa_2 - a_{11})\bar{v}_i.$$

Next, we derive the center manifold for system (15), which may be characterized as follows:

$$W^c(0, 0, 0) = \left\{ (\bar{u}, \bar{v}) \in \mathbb{R}^2 : \bar{v} = c_1 \bar{u}^2 + c_2 \bar{u} \bar{S} + c_3 \bar{S}^2 + \mathcal{R}(\bar{u}, \bar{S})^3 \right\},$$

where

$$\begin{aligned} c_1 = & \left(\frac{(a_{13}a_{12} - a_{12}a_{24})(1 + a_{11}) - a_{12}^2a_{23} - (a_{14} - a_{26})(1 + a_{11})^2}{(1 - (\kappa_2)^2)} + \frac{a_{16}(1 + a_{11})^3}{a_{12}(1 - (\kappa_2)^2)} \right), \\ c_2 = & \left(\frac{(b_{11} - b_{22})(1 + a_{11}) - a_{12}b_{12}}{(1 - (\kappa_2)^2)} + \frac{b_{12}(1 + a_{11})^2}{a_{12}(1 - (\kappa_2)^2)} \right), \text{ and } c_3 = \frac{b_{13}(1 - a_{11}) - a_{12}b_{23}}{a_{12}(1 - (\kappa_2)^2)}. \end{aligned}$$

System (15) restricted to the center manifold is

$$\Gamma : \bar{u} \rightarrow -\bar{u} + \epsilon_1 \bar{u}^2 + \epsilon_2 \bar{u} \bar{S} + \epsilon_3 \bar{u}^2 \bar{S} + \epsilon_4 \bar{x} \bar{h}^2 + \epsilon_5 \bar{x}^3 + \mathcal{R}(\bar{u}, \bar{S})^4,$$

where

$$\begin{aligned}\epsilon_1 &= \frac{(\kappa_2 - a_{11})(a_{13}a_{12} - a_{14}(1 + a_{11})) + a_{12}(a_{24}(1 + a_{11}) - a_{23}a_{12})}{(1 + \kappa_2)} \\ &\quad + \frac{(1 + a_{11})^2(a_{15}(\kappa_2 - a_{11}) - a_{24}a_{12})}{a_{12}(1 + \kappa_2)}, \\ \epsilon_2 &= \frac{(\kappa_2 + a_{11})(a_{12}b_{11} - b_{12}(1 + a_{11})) + a_{12}(b_{22}(1 + a_{11}) - b_{21}a_{12})}{a_{12}(1 + \kappa_2)}, \\ \epsilon_3 &= \frac{(\kappa_2 + a_{11})(a_{12}b_{14} - b_{16}(1 + a_{11})) - a_{12}(a_{12}b_{24} + (1 + a_{11})b_{26}) - b_{25}(1 + a_{11})^2}{(1 + \kappa_2)}, \\ \epsilon_4 &= \frac{b_{18}(\mu' + a_{11}) - b_{28}a_{12} + b_{29}(1 + a_{11})}{(1 + \mu')} - \frac{b_{19}(\mu' + a_{11})^2}{a_{12}(1 + \mu')}, \\ \epsilon_5 &= \frac{(\kappa_2 - a_{11})(a_{14}a_{12}^2 + a_{16}(1 + a_{11})^2 - a_{15}a_{12}(1 + a_{11})) - a_{24}a_{12}^3 + a_{25}a_{12}^2(1 + g_{11})}{(1 + \kappa_2)}.\end{aligned}$$

Assume that

$$\mathcal{L}_1 = \left(\frac{\partial^2 \Gamma}{\partial \bar{u} \partial \bar{S}} + \frac{1}{2} \frac{\partial \Gamma}{\partial \bar{S}} \frac{\partial^2 \Gamma}{\partial \bar{u}^2} \right) \Big|_{(0,0)} = \epsilon_2,$$

and

$$\mathcal{L}_2 = \left(\frac{1}{6} \frac{\partial^3 \Gamma}{\partial \bar{u}^3} + \left(\frac{1}{2} \frac{\partial^2 \Gamma}{\partial \bar{u}^2} \right)^2 \right) \Big|_{(0,0)} = \epsilon_5 + \epsilon_1^2.$$

Theorem 4. If $\mathcal{L}_1 \neq 0$ and $\mathcal{L}_2 \neq 0$, then system (10) experiences a P-D bifurcation at the positive fixed point

$$P_3 = \left(\frac{A_1 \left(\frac{A_2}{B_2} \right)^p + C \left(\frac{A_2}{B_2} \right)^p + A_1}{B_1 \left(1 + \left(\frac{A_2}{B_2} \right)^p \right)}, \frac{A_2}{B_2} \right),$$

when the parameter h changes within a small neighborhood of $\mathcal{B}_1$. If $\mathcal{L}_2 > 0$ (respectively $\mathcal{L}_2 < 0$), then system (10) bifurcates from the fixed point P_3 to a two-periodic stable orbit (respectively, unstable).

5. Chaos Control

This section focuses on the implementation of chaos control approaches on system (10). More specifically, two different control mechanisms are implemented to manage bifurcations and reduce chaotic behavior. It is worth noting that studies on the control of chaos in discrete-time dynamical systems are currently quite active. Several methods for controlling bifurcation scenarios and stabilizing chaotic motions in various kinds of discrete-time systems have been developed in recent years [19].

In this paper, we utilize the hybrid control method [20] to control the behavior of system (10). Applying this method leads to the following controlled system:

$$\begin{cases} u_{i+1} = \varrho \left(u_i \exp \left(\left(A_1 - B_1 u_i + \frac{C v_i^p}{1 + v_i^p} \right) \frac{S^\beta}{\beta} \right) \right) + \varrho(u_i - u^*), \\ v_{i+1} = \varrho \left(v_i \exp \left(\frac{v_i (A_2 - B_2 v_i) S^\beta}{\beta (M + v_i)} \right) \right) + \varrho \left(v_i - \frac{A_2}{B_2} \right), \end{cases} \quad (16)$$

where $0 < q < 1$ is a control parameter for the hybrid control method and

$$u^* = \frac{A_1 \left(\frac{A_2}{B_2}\right)^p + C \left(\frac{A_2}{B_2}\right)^p + A_1}{B_1 \left(1 + \left(\frac{A_2}{B_2}\right)^p\right)}.$$

The stability of the system at its positive fixed point determines the controllability of system (16). We calculate the Jacobian matrix $J_M^C(u_i^*, v_i^*)$ of system (16) at its unique positive fixed point P_3 in order to analyze this stability characteristic. Achieving this gives

$$J_{P_3}^C = \begin{pmatrix} \left(1 - \frac{qS^\beta}{\beta} H_1\right) & \frac{qS^\beta}{\beta} H_2 \\ 0 & \left(1 - \frac{qS^\beta}{\beta} H_3\right) \end{pmatrix}. \quad (17)$$

The characteristic polynomial of the Jacobian matrix computed at P_3 can therefore be represented as follows:

$$\mathcal{W}^c(\kappa_c) = \kappa_c^2 - \left(2 - \frac{(H_1 + H_3)qS^\beta}{\beta}\right)\kappa_c + \left(1 - \frac{qS^\beta}{\beta} H_1\right)\left(1 - \frac{qS^\beta}{\beta} H_3\right),$$

where

$$H_1 = \left(\frac{A_1 \left(\frac{A_2}{B_2}\right)^p + C \left(\frac{A_2}{B_2}\right)^p + A_1}{\left(1 + \left(\frac{A_2}{B_2}\right)^p\right)}\right), \quad H_3 = \left(\frac{A_2^2}{MB_2 + A_2}\right).$$

The controllability analysis of system (16) is used as the basis for the following lemma:

Lemma 3. *The point P_3 of system (16) is locally asymptotically stable if the following inequality satisfies*

$$\left|2 - \frac{(H_1 + H_3)qS^\beta}{\beta}\right| < 1 + \left(1 - \frac{qS^\beta}{\beta} H_1\right)\left(1 - \frac{qS^\beta}{\beta} H_3\right) < 2. \quad (18)$$

The state feedback control approach [19] is executed by reformulating system (10) as follows:

$$\begin{cases} u_{i+1} = u_i \exp\left(\left(A_1 - B_1 u_i + \frac{C v_i^p}{1 + v_i^p}\right) \frac{S^\beta}{\beta}\right) - \vee(u_i, v_i), \\ v_{i+1} = v_i \exp\left(\frac{v_i(A_2 - B_2 v_i)S^\beta}{\beta(M + v_i)}\right). \end{cases} \quad (19)$$

Here, the expression

$$\vee_i(u_i, v_i) = \rho_1 \left(u_i - \frac{A_1 \left(\frac{A_2}{B_2}\right)^p + C \left(\frac{A_2}{B_2}\right)^p + A_1}{B_1 \left(1 + \left(\frac{A_2}{B_2}\right)^p\right)}\right) + \rho_2 \left(v_i - \frac{A_2}{B_2}\right),$$

denotes the feedback control force, while ρ_1 and ρ_2 represent the feedback gains that adjust the system dynamics. The Jacobian matrix for the controlled system (19) is given as follows:

$$J_{P_3}^C = \begin{pmatrix} a_{11} - \rho_1 & a_{12} - \rho_2 \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{11} = \left(1 - \frac{S^\beta}{\beta} H_1\right), \quad a_{12} = \frac{S^\beta}{\beta} H_2, \quad a_{21} = 0, \quad a_{22} = \left(1 - \frac{S^\beta}{\beta} H_3\right).$$

In addition, the Jacobian matrix $J_{P_3}^C$'s characteristic polynomial is calculated as follows:

$$\bar{\kappa}^2 - (a_{11} + a_{22} - \rho_1)\bar{\kappa} + a_{22}(a_{11} - \rho_1) - a_{21}(a_{12} - \rho_2) = 0. \quad (20)$$

Let ρ_1 and ρ_2 be the roots of characteristic Equation (20); then we have

$$\bar{\kappa}_1 + \bar{\kappa}_2 = a_{11} + a_{22} - \rho_1, \text{ and } \bar{\kappa}_1 \bar{\kappa}_2 = a_{22}(a_{11} - \rho_1) - a_{21}(a_{12} - \rho_2). \quad (21)$$

The marginal stability boundaries of the relevant controlled system are then obtained by setting $\bar{\kappa}_1 = \pm 1$ and enforcing the condition $\bar{\kappa}_1 \bar{\kappa}_2 = 1$. The eigenvalues $\bar{\kappa}_1$ and $\bar{\kappa}_2$ are guaranteed to be on the boundaries of the open unit disk under the following conditions:

- Assume that $\bar{\kappa}_1 \bar{\kappa}_2 = 1$, then Equation (21) yields the first stability boundary

$$E_1 : \quad a_{22}\rho_1 - a_{21}\rho_2 = a_{22}a_{11} - a_{12}a_{21} - 1.$$

- Setting $\bar{\kappa}_1 = 1$ and using Equation (21), we obtain the second boundary

$$E_2 : \quad (1 - a_{22})\rho_1 + a_{21}\rho_2 = -1 - a_{22}a_{11} + a_{21}a_{12} + a_{11} + a_{22}.$$

- Taking $\bar{\kappa}_1 = -1$ and again applying Equation (21), the third boundary is derived as

$$E_3 : \quad (1 + a_{22})\rho_1 - a_{21}\rho_2 = 1 + a_{22}a_{11} - a_{21}a_{12} + a_{11} + a_{22}.$$

Consequently, for fixed parameter values, the stable eigenvalues of the controlled system are confined to the triangular region in the (ρ_1, ρ_2) -plane bounded by the straight lines E_1 , E_2 , and E_3 .

6. Numerical Simulations

This section provides several numerical examples to show the accuracy and validity of the theoretical results.

Example 1. System (10) is considered in this example with the parameter values $A_1 = 1.2$, $B_1 = 2.5$, $C_1 = 2$, $p = 2$, $A_2 = 2$, $B_2 = 0.21$, $M = 0.3$, $\beta = 0.75$, and $r \in [0, 1.8]$, together with the initial conditions $(u_0, v_0) = (0.1513, 0.4362)$. Suppose that S denotes the bifurcation parameter; then we observe that when $S = 0.3675$, the unique positive fixed point $P_3 = (1.2713, 9.5238)$ loses its stability and system (10) undergoes a period-doubling bifurcation. It is worth noting that the Jacobian matrix of the system computed at this fixed point is given by

$$J_{P_3} = \begin{pmatrix} -1.0000 & 0.0036 \\ 0 & -0.2201 \end{pmatrix}.$$

Therefore, the characteristic polynomial of matrix J_{P_3} is written as

$$\mathcal{W}(\kappa) = \kappa^2 + 1.2201\kappa + 0.2201,$$

whose roots are

$$\kappa_1 = -1.000, \quad \text{and} \quad \kappa_2 = -0.2201.$$

In biological nature, the period-doubling bifurcation takes place when a system begins to alternate between two different dynamic states rather than repeating the same state. It frequently indicates the starting point of complex or chaotic behavior.

Example 2. In this example, we discuss the behavior of system (10) under the parameter values

$$A_1 = 1.2, B_1 = 2.5, C_1 = 2, p = 2, A_2 = 2, B_2 = 0.21, M = 0.3, \beta = 0.75 \text{ and } S = 1.34,$$

and the initial conditions $(u_0, v_0) = (1.3, 8.3)$. When we use these values, system (10) loses the stability of its fixed points through a period-doubling bifurcation. At $S = 1.34$, the system experiences chaotic behaviors as illustrated in Figure 1a,c. It should be noted that the fixed point P_3 becomes stable when the hybrid control technique is used, as can be seen in Figure 1b,d, and the chaotic behavior disappears, as can be seen in Figure 1a. This effectively shows how well the hybrid control approach controls chaotic behaviors and stabilizes the system.

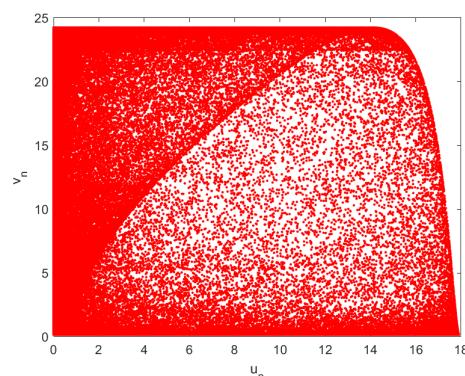
Next, using the same parameter values, we apply the feedback control method to system (10). Under these conditions, the marginal stability of the controlled system (19) is characterized by the following relations:

$$E_1 : -0.962\rho_1 - 0.415\rho_2 = 0.231,$$

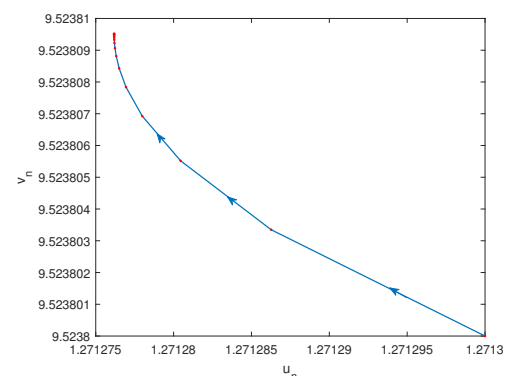
$$E_2 : 0.4235\rho_2 = -0.323,$$

$$E_3 : -1.642\rho_1 - 0.435\rho_2 = 2.4319.$$

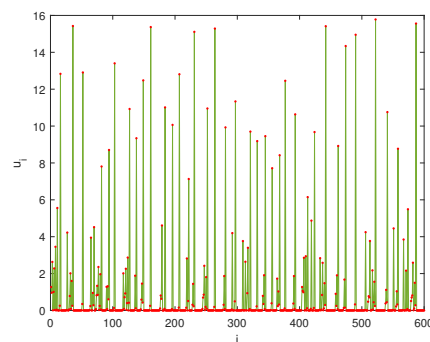
The stability zone of the controlled system is represented as a triangular region in the (ρ_1, ρ_2) -plane that is bounded by the straight lines Γ_1, Γ_2 , and Γ_3 . Figure 2a depicts that region.



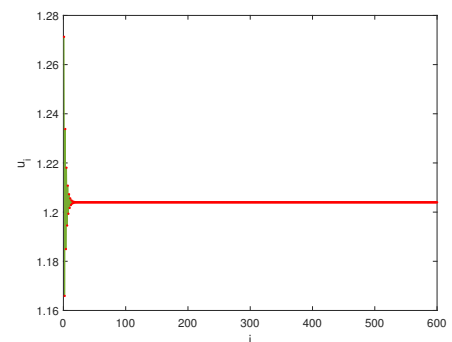
(a) Unstable solution for u_i in system (10) when $S = 1.34$.



(b) Stable solution for u_i in controlling system (16) when $S = 1.34$ and $q = 0.9$.



(c) Phase portrait for $S = 1.34$.



(d) Phase portrait for $S = 1.34$ and $q = 0.39$.

Figure 1. The phase portraits and time series of systems (10) and (16) for different values of S are presented as follows: (a,c) $S = 1.34$; (b,d) $S = 1.34$; and $q = 0.9$, respectively, with parameter values $A_1 = 1.2, B_1 = 2.5, C = 2, p = 2, A_2 = 2, B_2 = 0.21, M = 0.3$, and $\beta = 0.75$.

As observed in Figure 2b, the fixed point P_2 becomes locally asymptotically stable for the inputs $\rho_1 = 1.5$ and $\rho_2 = -0.4$. On the other hand, the point P_3 lies outside of the zone of stability when $\rho_1 = 0$ and $\rho_2 = 0$ are used. This leads to instability, as depicted in Figure 1a. From the perspective of the environment, the feedback control process can be understood as focused management techniques meant to prevent undesired oscillatory or chaotic activity and stabilize the dynamics of populations.

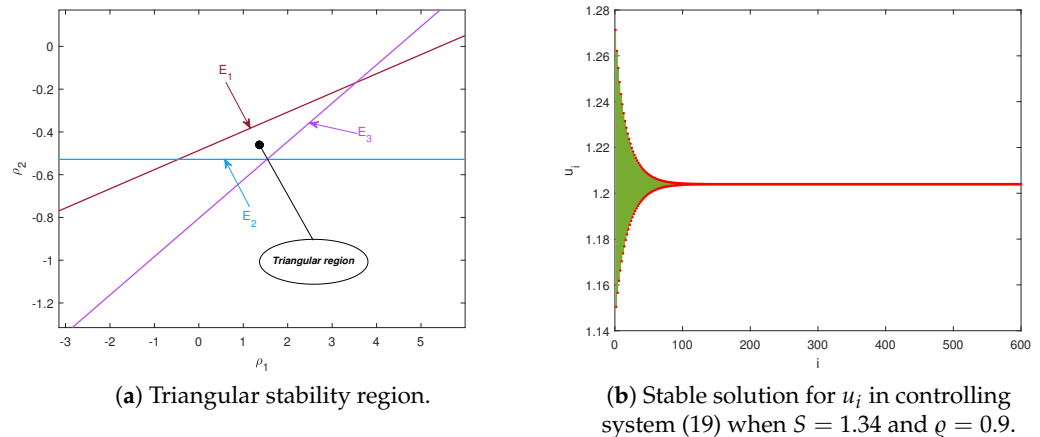


Figure 2. (a) Triangular stability region of the controlled system (19), bounded by E_1 , E_2 , and E_3 . (b) Time evolutions for $(\rho_1, \rho_2) = (1.5, -0.4)$ and $(0, 0)$, respectively. The parameter values are: $A_1 = 1.2$, $B_1 = 2.5$, $C = 2$, $p = 2$, $A_2 = 2$, $B_2 = 0.21$, $M = 0.3$, $S = 1.34$, and $\beta = 0.75$.

7. Results and Discussion

Here, we discuss the main results obtained in this article. The bifurcation graphs and the corresponding maximum Lyapunov exponents (MLEs) are clearly shown in Figures 3–5 for the parameter ranges $S \in [0, 1.6]$, $S \in [0, 0.6]$, and $S \in [0.6, 1.6]$, respectively. We also observe that Figure 3 presents the bifurcation plot together with the associated Lyapunov exponent of system (10) as functions of the bifurcation parameter S . The fixed point P_3 is stable inside the interval $0 < S < 0.3675$ (see Figure 3a,b). This point becomes unstable at $S = S_1 = 0.3675$ as a result of a period-doubling bifurcation. Furthermore, Figure 3b indicates that the Lyapunov exponent assumes both positive and negative values, revealing the alternation between regular and chaotic dynamics of the system. The time series associated with this case are shown in Figures 6 and 7. This reveals a cascade of period-doubling bifurcations. As observed in Figure 6, the system experiences sequential periodic oscillations with periods 2, 4, and 8. Figure 7 clearly illustrates how this trend eventually leads to chaotic behavior. We display the phase portrait of system (10) for $S = 1.5$ and $S = 1.6$ in Figure 8a and 8b, respectively, to additionally verify the presence of chaos. Note that Figure 9 depicts the bifurcation behavior and the associated Lyapunov exponent utilizing the previously specified parameters $A_1 = 1.2$, $B_1 = 2.5$, $C = 2$, $p = 2$, $A_2 = 2$, $B_2 = 0.21$, $M = 0.3$, $S = 0.2$, and $\beta \in [0, 1]$, with $h = 0.2$. System (10) becomes unstable for $\beta \leq 0.6033$. As β increases beyond 0.6033, the system undergoes a subcritical period-doubling bifurcation at $\beta = 0.6033$ and becomes asymptotically stable for $0.6033 < \beta < 1$. Biologically, the period-doubling bifurcation occurs in this system when the system starts to switch between two distinct dynamic states instead of repeating the same state. In general, it signifies the beginning of chaotic or complex behavior. More specifically, a period-doubling bifurcation signifies a change from regular or stable population oscillations to more complicated and erratic dynamics. This pattern of behavior might suggest the establishment of multi-period population cycles in interacting species, which would imply ecological instability or sensitivity to changes in the environment. Such dynamics in commensal symbiosis systems show that, under specific conditions,

the relationship can continue to result in unexpected population fluctuations even when one species benefits without harming the other. In addition, the Allee effect indicates that low-density populations are specifically susceptible, and period-doubling might point to vital limits beyond which populations undergo significant oscillations or even collapse.

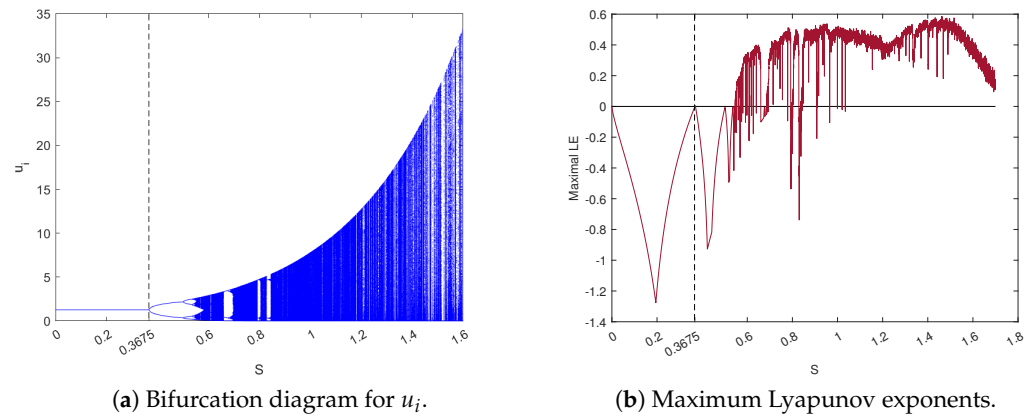


Figure 3. (a) Supercritical period–doubling bifurcation diagram of system (10). (b) Corresponding maximum Lyapunov exponents for the parameter values: $A_1 = 1.2$, $B_1 = 2.5$, $C = 2$, $p = 2$, $A_2 = 2$, $B_2 = 0.21$, $M = 0.3$, $\beta = 0.75$, and $S \in [0, 1.6]$.

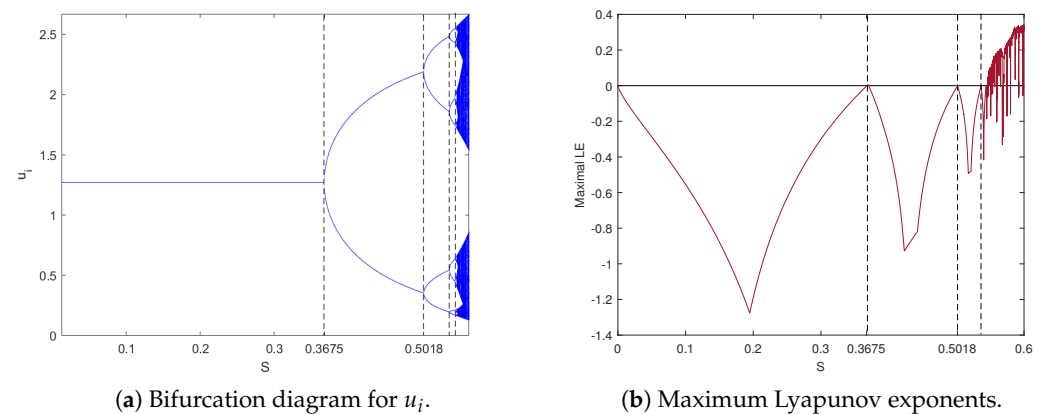


Figure 4. (a) Period–doubling bifurcation diagram of system (10). (b) Corresponding maximum Lyapunov exponents for the parameter values: $A_1 = 1.2$, $B_1 = 2.5$, $C = 2$, $p = 2$, $A_2 = 2$, $B_2 = 0.21$, $M = 0.3$, $\beta = 0.75$, and $S \in [0, 0.6]$.

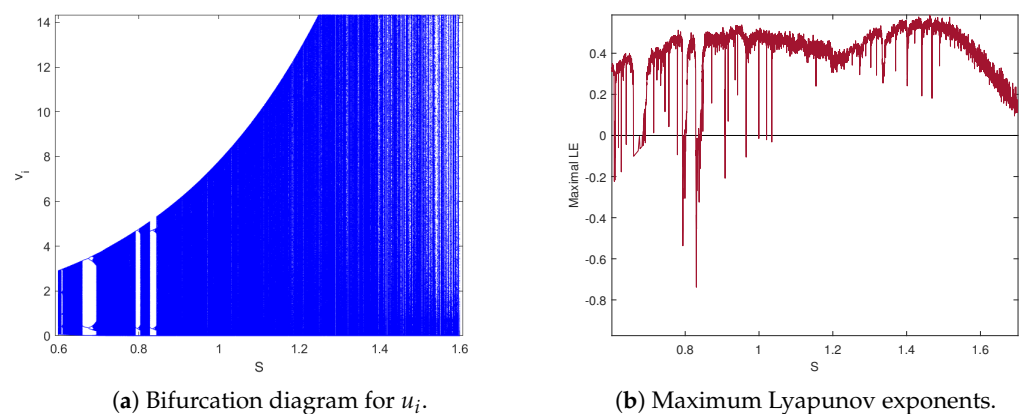


Figure 5. (a) Period–doubling bifurcation diagram of system (10). (b) Corresponding maximum Lyapunov exponents for the parameter values: $A_1 = 1.2$, $B_1 = 2.5$, $C = 2$, $p = 2$, $A_2 = 2$, $B_2 = 0.21$, $M = 0.3$, $\beta = 0.75$, and $S \in [0.6, 1.6]$.

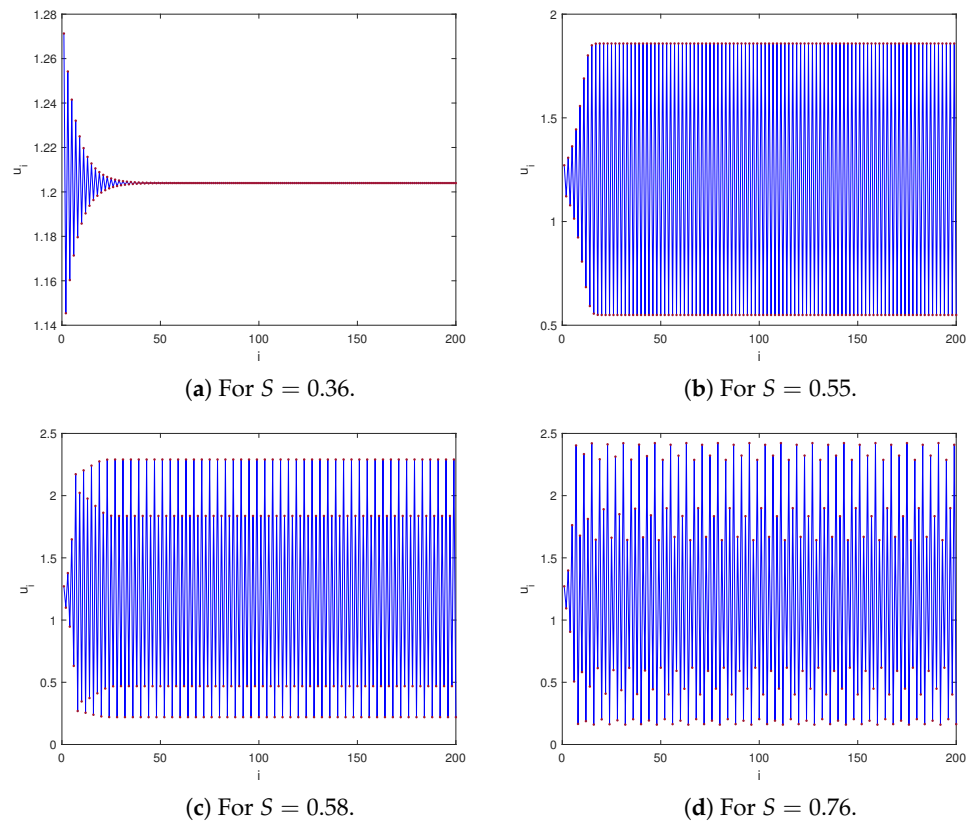


Figure 6. Time series for the parameter values: $A_1 = 1.2$, $B_1 = 2.5$, $C = 2$, $p = 2$, $A_2 = 2$, $B_2 = 0.21$, $M = 0.3$, and $\beta = 0.75$.

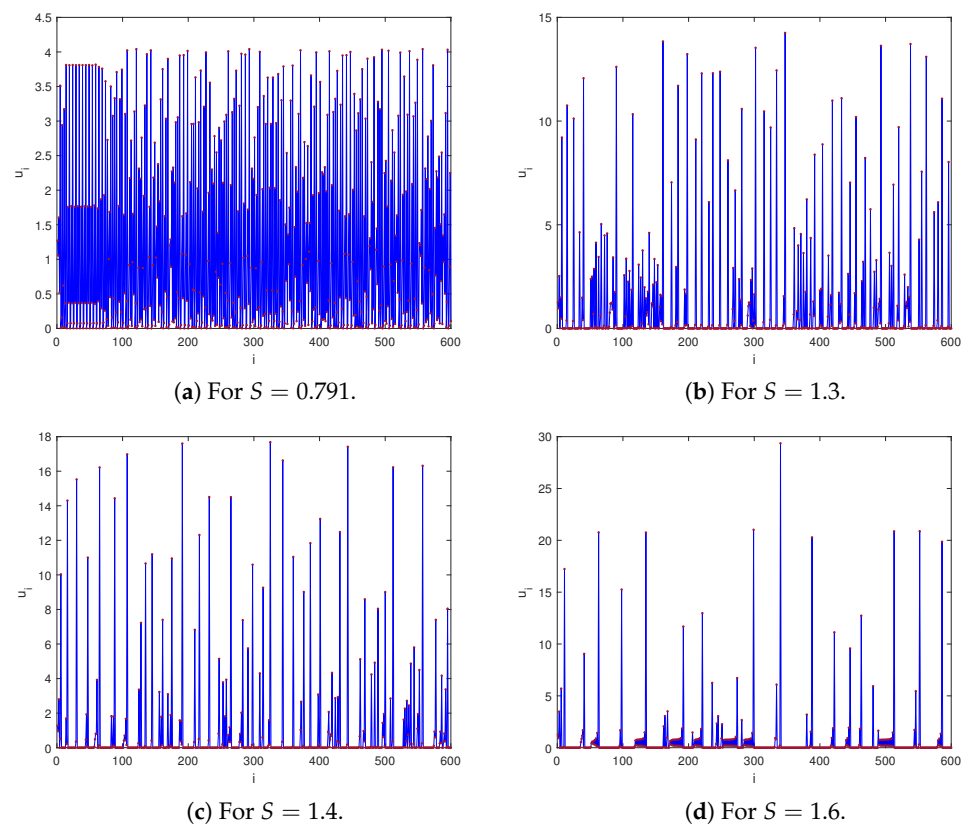


Figure 7. Time series for the parameter values: $A_1 = 1.2$, $B_1 = 2.5$, $C = 2$, $p = 2$, $A_2 = 2$, $B_2 = 0.21$, $M = 0.3$, and $\beta = 0.75$.

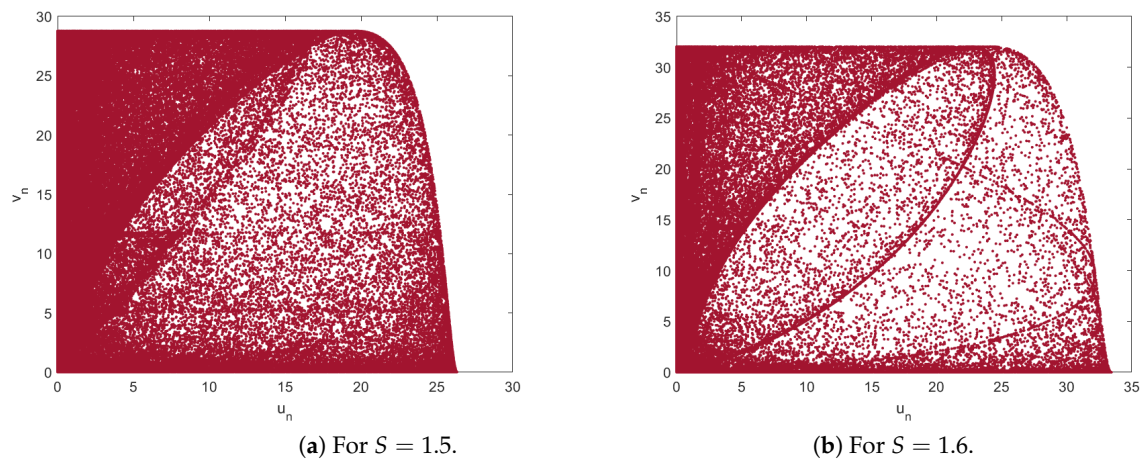


Figure 8. Phase portraits for the parameter values: $A_1 = 1.2$, $B_1 = 2.5$, $C = 2$, $p = 2$, $A_2 = 2$, $B_2 = 0.21$, $M = 0.3$, and $\beta = 0.75$.

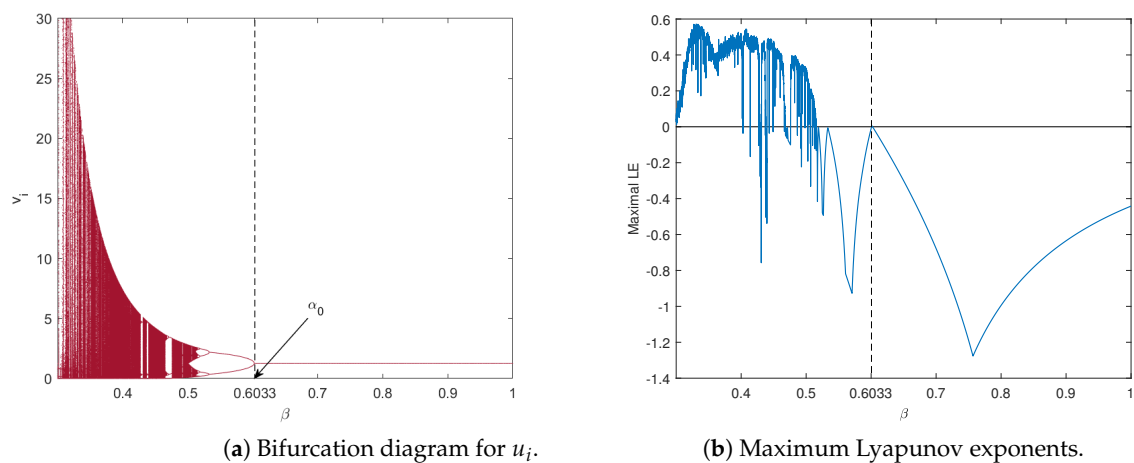


Figure 9. (a) Subcritical period-doubling bifurcation diagram of system (10). (b) Corresponding maximum Lyapunov exponents for the parameter values: $A_1 = 1.2$, $B_1 = 2.5$, $C = 2$, $p = 2$, $A_2 = 2$, $B_2 = 0.21$, $M = 0.3$, $S = 0.2$, and $\beta \in [0, 1]$.

In Example 2, we show how well the hybrid control method controls chaotic behaviors and stabilizes system (10) under the parameter values

$$A_1 = 1.2, B_1 = 2.5, C_1 = 2, p = 2, A_2 = 2, B_2 = 0.21, M = 0.3, \beta = 0.75 \text{ and } S = 1.34,$$

and the initial conditions $(u_0, v_0) = (1.3, 8.3)$. Using these values produces a period-doubling bifurcation, which causes system (10) to lose the stability of its fixed points. As can be observed in Figure 1b,d, we emphasize that when the hybrid control approach is employed, the chaotic behavior vanishes (see Figure 1a) and the fixed point P_3 becomes stable. Here, we compare the outcomes of the traditional integer-order model investigated by Wu et al. [21] with those of the suggested fractional-order commensal symbiosis model with the Allee effect. Despite both models employing similar biological bases, their dynamical behaviors differ dramatically because of the presence of fractional-order derivatives in the current study. In terms of equilibria existence and stability, both models have a trivial equilibrium, boundary equilibria, and a single positive equilibrium. Techniques like Jacobian matrix analysis were used in [21] to conclusively demonstrate that the positive equilibrium is stable. In comparison, the fractional-order model has a more sophisticated stability structure, where the stability is obtained using some conditions on the eigenvalues. The presence of bifurcation phenomena highly distinguishes the two models. Wu et al. [21]

did not present a bifurcation or oscillatory behavior in the model they analyzed. However, period-doubling bifurcation appears in the present fractional-order model. In particular, while the classical model offers helpful observations about the stability and the persistence of commensal symbiosis systems with the Allee effect, it remains restricted to very simple dynamical behavior. The fractional-order model expands previous findings by integrating new effects and highlighting the presence of various dynamical phenomena, such as stability flipping and period-doubling bifurcations.

8. Conclusions

We mention the key findings in this section. The piecewise-constant approximation technique has provided effective results for the discretization of model (1). We have found that model (10) has four fixed points, namely a trivial fixed point, a predator-free fixed point, a prey-free fixed point, and a coexistence fixed point. These points become stable under some conditions provided in Theorems 1–3. Furthermore, we have noted that if $\mathcal{L}_1 \neq 0$ and $\mathcal{L}_2 \neq 0$, then system (10) encounters a period-doubling bifurcation at the positive fixed point P_3 , when the parameter h varies within a small neighborhood of $\mathcal{B}_1$. However, when $\mathcal{L}_2 > 0$, then model (10) bifurcates from the fixed point P_3 to a two-periodic stable orbit. The controllability of system (16) has been successfully shown in Lemma 3. We then discussed some numerical examples from which we noticed that the point $P_3 = (1.2713, 9.5238)$ loses its stability and system (10) experiences a period-doubling bifurcation under some parameter values together with the initial conditions $(u_0, v_0) = (0.1513, 0.4362)$. In terms of biology, this system has encountered a period-doubling bifurcation when it started to alternate between two different dynamic stages rather than repeating the same state. This normally indicates the initiation of complicated or chaotic behavior.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A

In this section, we show the values of the coefficients presented in system (14).

$$\begin{aligned}
 a_{11} &= \left(1 - \frac{(S_1 + \bar{S})^\beta}{\beta} H_1\right), \quad a_{12} = \frac{(S_1 + \bar{S})^\beta}{\beta} H_2, \quad a_{21} = 0, \quad a_{22} = \left(1 - \frac{(S_1 + \bar{S})^\beta}{\beta} H_3\right), \\
 a_{13} &= \left(u_i B_1^2 \left(\frac{S^\beta}{\beta}\right)^2 - 2B_1 \left(\frac{S^\beta}{\beta}\right)\right), \quad a_{14} = \frac{CS^\beta}{\beta} \left(\frac{pv_i^{p-1}}{(1+v_i^p)^2}\right) \left(1 - u_i B_1 \frac{S^\beta}{\beta}\right), \\
 a_{15} &= u_i \left(C \left(\frac{S^\beta}{\beta}\right) \left(\frac{p(p-1)v_i^{p-2}}{(1+v_i^p)^2} - \frac{2p^2 v_i^{2p-2}}{(1+v_i^p)^3}\right) + C^2 \left(\frac{S^\beta}{\beta}\right)^2 \left(\frac{pv_i^{p-1}}{(1+v_i^p)^2}\right)^2\right), \\
 a_{16} &= \left(3B_1^2 \left(\frac{S^\beta}{\beta}\right)^2 - u_i B_1^3 \left(\frac{S^\beta}{\beta}\right)^3\right), \quad a_{17} = C \left(\frac{S^\beta}{\beta}\right)^2 \left(\frac{pv_i^{p-1}}{(1+v_i^p)^2}\right) \left(-2B_1 + u_i B_1^2 \left(\frac{S^\beta}{\beta}\right)\right), \\
 a_{18} &= \left(1 - \frac{u_i B_1 S^\beta}{\beta}\right) \left(\frac{CS^\beta}{\beta} \left(\frac{p(p-1)v_i^{p-2}}{(1+v_i^p)^2} - \frac{2p^2 v_i^{2p-2}}{(1+v_i^p)^3}\right) + \frac{CS^{\beta^2}}{\beta} \left(\frac{pv_i^{p-1}}{(1+v_i^p)^2}\right)^2\right), \\
 b_{11} &= S^{\beta-1} \left(\left(A_1 - B_1 u_i + \frac{Cv_i^p}{1+v_i^p}\right) \left(1 - u_i B_1 \frac{S^\beta}{\beta}\right) - B_1\right),
 \end{aligned}$$

$$\begin{aligned}
b_{12} &= u_i C \left(\frac{p v_i^{p-1}}{(1+v_i^p)^2} \right) S^{\beta-1} \left(1 + \left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right) \frac{S^\beta}{\beta} \right), \\
b_{13} &= u_i \left(\left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right)^2 S^{2\beta-2} + \left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right) (\beta-1) S^{\beta-2} \right), \\
b_{14} &= S^{\beta-1} \left(-\frac{2B_1 S^\beta}{\beta} \left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right) + \frac{2B_1^2 S^\beta}{\beta} + u_i \left(\frac{B_1 S^\beta}{\beta} \right)^2 \left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right) - u_i \left(\frac{B_1 S^\beta}{\beta} \right)^3 \right), \\
b_{15} &= u_i S^{\beta-1} \left(C \left(\frac{p(p-1)v_i^{p-2}}{(1+v_i^p)^2} - \frac{2p^2 v_i^{2p-2}}{(1+v_i^p)^3} \right) + \frac{2C^2 S^\beta}{\beta} \left(\frac{p v_i^{p-1}}{(1+v_i^p)^2} \right) + \frac{C^2 S^\beta}{\beta} \left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right) \right), \\
b_{17} &= u_i \left(\left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right)^2 S^{3\beta-3} + 3 \left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right) (\beta-1) S^{2\beta-3} + (\beta-1)(\beta-2) S^{\beta-3} \right), \\
b_{18} &= \left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right) \left(\left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right) S^{2\beta-2} \left(1 - \frac{u_i B_1 S^\beta}{\beta} \right) - 2B_1 S^{2\beta-2} + (\beta-1) S^{\beta-2} \right), \\
b_{19} &= u_i C \phi'(v_i) \left(\left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right)^2 S^{2\beta-2} + 2 \left(A_1 - B_1 u_i + \frac{C v_i^p}{1+v_i^p} \right) (\beta-1) S^{\beta-2} \right), \\
a_{19} &= u_i \left(C \frac{S^\beta}{\beta} \phi'''(v_i) + 3C^2 \left(\frac{S^\beta}{\beta} \right)^2 \left(\frac{p v_i^{p-1}}{(1+v_i^p)^2} \right) \left(\frac{p(p-1)v_i^{p-2}}{(1+v_i^p)^2} - \frac{2p^2 v_i^{2p-2}}{(1+v_i^p)^3} \right) + C^3 \left(\frac{S^\beta}{\beta} \right)^3 \left(\frac{p v_i^{p-1}}{(1+v_i^p)^2} \right)^3 \right),
\end{aligned}$$

where

$$\begin{aligned}
\phi'''(v_i) &= \frac{p(p-1)(p-2)v_i^{p-3}}{(1+v_i^p)^2} - \frac{6p^2(p-1)v_i^{2p-3}}{(1+v_i^p)^3} + \frac{6p^3v_i^{3p-3}}{(1+v_i^p)^4}, \\
a_{23} &= a_{24} = a_{21} = a_{26} = a_{27} = b_{24} = b_{26} = 0, \\
a_{25} &= \left(\frac{2S^\beta}{\beta} \frac{A_2 M - 2B_2 M v_i - B_2 v_i^2}{(M+v_i)^2} + \frac{v_i S^\beta - 2B_2 M(M+v_i)^2 - 2(M+v_i)(A_2 M - 2B_2 M v_i - B_2 v_i^2)}{\beta (M+v_i)^4} \right. \\
&\quad \left. + v_i \left(\frac{S^\beta}{\beta} \frac{A_2 M - 2B_2 M v_i - B_2 v_i^2}{(M+v_i)^2} \right)^2 \right), \quad b_{23} = \left(\left(\frac{v_i(A_2 - B_2 v_i) S^{\beta-1}}{M+v_i} \right)^2 + \frac{v_i(A_2 - B_2 v_i)(\beta-1) S^{\beta-2}}{M+v_i} \right), \\
b_{22} &= \left(\left(1 + v_i \left(\frac{S^\beta}{\beta} \frac{A_2 M - 2B_2 M v_i - B_2 v_i^2}{(M+v_i)^2} \right) \right) \frac{v_i(A_2 - B_2 v_i) S^{\beta-1}}{M+v_i} + \left(v_i S^{\beta-1} \frac{A_2 M - 2B_2 M v_i - B_2 v_i^2}{(M+v_i)^2} \right) \right), \\
a_{23} &= \left(6 \left(\frac{S^\beta}{\beta} \frac{A_2 M - 2B_2 M v_i - B_2 v_i^2}{(M+v_i)^2} \right)^2 + 4v_i \left(\frac{S^\beta}{\beta} \frac{A_2 M - 2B_2 M v_i - B_2 v_i^2}{(M+v_i)^2} \right) \right. \\
&\quad \left(\frac{S^\beta - 2B_2 M(M+v_i)^2 - 2(M+v_i)(A_2 M - 2B_2 M v_i - B_2 v_i^2)}{\beta (M+v_i)^4} \right) + v_i \left(\frac{S^\beta}{\beta} \frac{A_2 M - 2B_2 M v_i - B_2 v_i^2}{(M+v_i)^2} \right)^3 \\
&\quad + 3 \left(\frac{S^\beta - 2B_2 M(M+v_i)^2 - 2(M+v_i)(A_2 M - 2B_2 M v_i - B_2 v_i^2)}{\beta (M+v_i)^4} \right) \\
&\quad \left. + v_i \left(\frac{S^\beta}{\beta} \frac{6(A_2 M - 2B_2 M v_i - B_2 v_i^2)(M+v_i)^2 + 6B_2 M(M+v_i)^3}{(M+v_i)^6} \right) \right),
\end{aligned}$$

$$\begin{aligned}
b_{25} &= \left(\left(\frac{2S^\beta}{\beta} \frac{A_2M - 2B_2Mv_i - B_2v_i^2}{(M+v_i)^2} + v_i \left(\frac{S^\beta - 2B_2M(M+v_i)^2 - 2(M+v_i)(A_2M - 2B_2Mv_i - B_2v_i^2)}{(M+v_i)^4} \right) \right. \right. \\
&\quad \left. \left. + v_i \left(\frac{S^\beta}{\beta} \frac{A_2M - 2B_2Mv_i - B_2v_i^2}{(M+v_i)^2} \right)^2 \right) \left(\frac{v_i(A_2 - B_2v_i)S^{\beta-1}}{M+v_i} \right) + 2v_i \left(\frac{S^\beta}{\beta} \frac{A_2M - 2B_2Mv_i - B_2v_i^2}{(M+v_i)^2} \right) \right), \\
b_{27} &= v_i \left(\left(\frac{v_i(A_2 - B_2v_i)S^{\beta-1}}{M+v_i} \right)^3 + 3 \left(\frac{v_i(A_2 - B_2v_i)S^{\beta-1}}{M+v_i} \right) \left(\frac{v_i(A_2 - B_2v_i)(\beta-1)S^{\beta-2}}{M+v_i} \right) \right. \\
&\quad \left. + \frac{v_i(A_2 - B_2v_i)(\beta-1)(\beta-2)S^{\beta-3}}{M+v_i} \right), \\
a_{29} &= \left(6 \left(\frac{S^\beta}{\beta} \frac{A_2M - 2B_2Mv_i - B_2v_i^2}{(M+v_i)^2} \right)^2 + 3 \left(\frac{S^\beta}{\beta} \frac{A_2M - 2B_2Mv_i - B_2v_i^2}{(M+v_i)^2} \right)^3 + \left(\frac{A_2M - 2B_2Mv_i - B_2v_i^2}{(M+v_i)^2} \right) \right. \\
&\quad \left. \frac{4v_iS^\beta}{\beta} \left(\frac{S^\beta - 2B_2M(M+v_i)^2 - 2(M+v_i)(A_2M - 2B_2Mv_i - B_2v_i^2)}{(M+v_i)^4} \right) \right).
\end{aligned}$$

References

1. Bronstein, J.L. *Mutualism*; Oxford University Press: Oxford, UK, 2015.
2. Chen, F.; Chen, Y.; Li, Z.; Chen, L. Note on the persistence and stability property of a commensalism model with Michaelis–Menten harvesting and Holling type II commensalistic benefit. *Appl. Math. Lett.* **2022**, *134*, 108381. [\[CrossRef\]](#)
3. Tarasov, V.E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2011.
4. Liu, X.; Yue, Q. Stability property of the boundary equilibria of a symbiotic model of commensalism and parasitism with harvesting in commensal populations. *AIMS Math.* **2022**, *7*, 18793–18808. [\[CrossRef\]](#)
5. Xu, L.; Xue, Y.; Lin, Q.; Lei, C. Global Attractivity of Symbiotic Model of Commensalism in Four Populations with Michaelis–Menten Type Harvesting in the First Commensal Populations. *Axioms* **2022**, *11*, 337. [\[CrossRef\]](#)
6. Chen, F.; Chong, Y.; Lin, S. Global stability of a commensal symbiosis model with Holling II functional response and feedback controls. *WSEAS Trans. Syst. Control* **2022**, *17*, 279–286. [\[CrossRef\]](#)
7. Lin, Q. Allee effect increasing the final density of the species subject to the Allee effect in a Lotka–Volterra commensal symbiosis model. *Adv. Differ. Equ.* **2018**, *2018*, 196. [\[CrossRef\]](#)
8. Lei, C. Dynamic behaviors of a Holling type commensal symbiosis model with the first species subject to Allee effect. *Commun. Math. Biol. Neurosci.* **2019**, *2019*, 3.
9. Işık, S. Stability and period-doubling Bifurcation in a modified commensal symbiosis model with Allee effect. *Erzincan Univ. J. Sci. Technol.* **2022**, *15*, 310–324. [\[CrossRef\]](#)
10. Zhong, J.; Chen, L.; Chen, F. Stability and bifurcation in a two-patch commensal symbiosis model with nonlinear dispersal and additive Allee effect. *Int. J. Biomath.* **2024**, 2450099. [\[CrossRef\]](#)
11. Khalil, R.; Al Horani, M.; Yousef, A.; Sababheh, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* **2014**, *264*, 65–70. [\[CrossRef\]](#)
12. Abdeljawad, T. On conformable fractional calculus. *J. Comput. Appl. Math.* **2014**, *279*, 57–66. [\[CrossRef\]](#)
13. Berkal, M.; Navarro, J.F.; Hamada, M.Y.; Semmar, B. Qualitative behavior for a discretized conformable fractional-order predator–prey model. *J. Appl. Math. Comput.* **2025**, *71*, 4815–4837. [\[CrossRef\]](#)
14. Berkal, M.; Almatrafi, M.B. Bifurcation and Stability of Two-Dimensional Activator–Inhibitor Model with Fractional-Order Derivative. *Fractal Fract.* **2023**, *7*, 344. [\[CrossRef\]](#)
15. Berkal, M.; Navarro, J.F. Qualitative behavior of a two-dimensional discrete-time prey–predator model. *Comput. Math. Methods* **2021**, *3*, e1193. [\[CrossRef\]](#)
16. Berkal, M.; Navarro, J.F. Qualitative study of a second order difference equation. *Turk. J. Math.* **2023**, *47*, 516–527. [\[CrossRef\]](#)
17. Kuznetsov, Y.A.; Kuznetsov, I.A.; Kuznetsov, Y. *Elements of Applied Bifurcation Theory*; Springer: New York, NY, USA, 1998; Volume 112.
18. Wiggins, S. *Introduction to Applied Nonlinear Dynamical System and Chaos*; Springer: New York, NY, USA, 2003.
19. Ahmed, R.; Tahir, N.; Ali Shah, N. An analysis of the stability and bifurcation of a discrete-time predator–prey model with the slow–fast effect on the predator. *Chaos Interdiscip. J. Nonlinear Sci.* **2024**, *34*, 033127. [\[CrossRef\]](#) [\[PubMed\]](#)

20. Din, Q.; Zulfiqar, M.A. Qualitative behavior of a discrete predator-prey system under fear effects. *Z. Naturforsch. A* **2022**, *77*, 1023–1043. [[CrossRef](#)]
21. Wu, R.; Li, L.; Lin, Q. A Holling type commensal symbiosis model involving Allee effect. *Commun. Math. Biol. Neurosci.* **2018**, *2018*, 6. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.