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Erdélyi-Type Integrals for F_K Function and Their q -Analogues

Liang-Jia Guo and Min-Jie Luo *

School of Mathematics and Statistics, Donghua University, Shanghai 201620, China; guoliangjia77@outlook.com

* Correspondence: mathwinnie@live.com or mathwinnie@dhu.edu.cn

Abstract

In this paper, we revisit the recent result of Luo, Xu, and Raina on an Erdélyi-type integral for Saran's three-variable hypergeometric function F_K . We provide a new proof of this integral and derive an attractive new integral related to Appell's function F_2 . A further extension on the L -variable F_K function, which appears in physics, is also discussed. Furthermore, we prove various q -Erdélyi-type integrals for the q -analogue of the F_K -function. An interesting discrete analogue is also included. We also provide a valuable compilation of the sources for known Erdélyi-type integrals of many different hypergeometric functions.

Keywords: Erdélyi-type integral; fractional integration by parts; hypergeometric function; q -analogue; Saran's function

MSC: 33C05; 33C65; 33C70; 33D15; 33D70

1. Introduction

1.1. Background and Motivation

The Gauss hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; z \right] := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \quad (|z| < 1),$$

where $\alpha, \beta \in \mathbb{C}$ and $\gamma \notin \mathbb{Z}_{\leq 0} := \{0, -1, -2, \dots\}$ (see [1], p. 56 and [2], p. 384). It was shown by Euler that [1] (p. 59, Equation (10))

$${}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; z \right] = \begin{cases} \int_0^1 (1-zt)^{-\alpha} d\mu_{\beta, \gamma-\beta}(t) & (\Re(\gamma) > \Re(\beta) > 0), \\ \int_0^1 (1-zt)^{-\beta} d\mu_{\alpha, \gamma-\alpha}(t) & (\Re(\gamma) > \Re(\alpha) > 0). \end{cases} \quad (1)$$

$${}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; z \right] = \begin{cases} \int_0^1 (1-zt)^{-\alpha} d\mu_{\beta, \gamma-\beta}(t) & (\Re(\gamma) > \Re(\beta) > 0), \\ \int_0^1 (1-zt)^{-\beta} d\mu_{\alpha, \gamma-\alpha}(t) & (\Re(\gamma) > \Re(\alpha) > 0). \end{cases} \quad (2)$$

Here and throughout the paper, we let $\mu_{\alpha, \beta}$ be the *Dirichlet measure* defined by [3] (p. 64, Definition 4.4-1)

$$d\mu_{\alpha, \beta}(t) := m_{\alpha, \beta}(t) dt, \quad (3)$$

where

$$m_{\alpha, \beta}(t) := \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1} (1-t)^{\beta-1} \quad (\min\{\Re(\alpha), \Re(\beta)\} > 0).$$

Obviously, $\int_0^1 d\mu_{\alpha, \beta}(t) = 1$. In general, integrals (1) and (2) cannot be directly transformed into each other, but Riemann [4] indicated that both could be derived from a multiple



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integral. According to the existing literature, Schellenberg [5], Wirtinger [6], Poole [7] and Erdélyi [8] studied this problem. We are particularly interested in Erdélyi's and Poole's methods.

Erdélyi's method (see [8], p. 272–273) is to substitute

$$(1 - zt)^{-\alpha} = \int_0^1 (1 - zts)^{-\gamma} d\mu_{\alpha, \gamma - \alpha}(s) \quad (\Re(\gamma) > \Re(\alpha) > 0)$$

into the first integral (1) to obtain the double integral

$$\int_0^1 \int_0^1 (1 - zts)^{-\gamma} d\mu_{\beta, \gamma - \beta}(t) d\mu_{\alpha, \gamma - \alpha}(s), \quad (4)$$

which is symmetric in α and β . If we integrate with respect to t first, then we obtain the second integral (2). Note that after introducing the new variables $x_1 = s/(1 - s)$ and $x_2 = t/(1 - t)$ and making the substitution $z \rightarrow 1 - z$ in Erdélyi's Formula (4), we obtain the double integral

$$\int_0^\infty \int_0^\infty \frac{x_1^\alpha x_2^\beta}{(1 + x_1 + x_2 + zx_1 x_2)^\gamma} \frac{dx_1 dx_2}{x_1 x_2} = \frac{\Gamma(\beta)\Gamma(\gamma - \beta)\Gamma(\alpha)\Gamma(\gamma - \alpha)}{(\Gamma(\gamma))^2} \cdot {}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; 1 - z \right]. \quad (5)$$

The integral on the left-hand side of (5) is also known as the *Euler–Mellin integral* (see [9,10]).

Poole's method [7] is to apply the integration by parts to the contour integral representation of ${}_2F_1$, which is very delightful. After reading Poole's paper, Erdélyi [11] realized that Poole's method can be generalized by using the fractional derivatives. By inventing the method of *fractional integration by parts*, Erdélyi not only gave a simple description of Poole's derivation but also reproduced the Bateman integral

$${}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; z \right] = \int_0^1 {}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \lambda \end{matrix}; zx \right] d\mu_{\lambda, \gamma - \lambda}(x) \quad (\Re(\gamma) > \Re(\lambda) > 0). \quad (6)$$

In [12], Erdélyi further developed his method and proved the following important integrals for ${}_2F_1$:

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; z \right] &= \int_0^1 (1 - zx)^{-\alpha'} {}_2F_1 \left[\begin{matrix} \alpha - \alpha', \beta \\ \lambda \end{matrix}; zx \right] \\ &\quad \cdot {}_2F_1 \left[\begin{matrix} \alpha', \beta - \lambda \\ \gamma - \lambda \end{matrix}; \frac{(1 - x)z}{1 - xz} \right] d\mu_{\lambda, \gamma - \lambda}(x) \quad (\Re(\gamma) > \Re(\lambda) > 0), \end{aligned} \quad (7)$$

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; z \right] &= \int_0^1 (1 - zx)^{\lambda - \alpha - \beta} {}_2F_1 \left[\begin{matrix} \lambda - \alpha, \lambda - \beta \\ \eta \end{matrix}; zx \right] \\ &\quad \cdot {}_2F_1 \left[\begin{matrix} \alpha + \beta - \lambda, \lambda - \eta \\ \gamma - \eta \end{matrix}; \frac{(1 - x)z}{1 - xz} \right] d\mu_{\eta, \gamma - \eta}(x) \quad (\Re(\gamma) > \Re(\eta) > 0), \end{aligned} \quad (8)$$

$$\begin{aligned} {}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; z \right] &= \int_0^1 {}_3F_2 \left[\begin{matrix} \alpha, \beta, \eta \\ \lambda, \nu \end{matrix}; zx \right] d\mu_{\eta - \lambda, \gamma - \lambda, \gamma - \lambda + \eta - \nu, \nu}(x) \\ &\quad (\min\{\Re(\nu), \Re(\lambda), \Re(\gamma - \lambda + \eta - \nu)\} > 0). \end{aligned} \quad (9)$$

In (9), $\mu_{\alpha, \beta, \gamma, \eta}$ is the *hypergeometric measure* defined by [13] (p. 6, Equation (11))

$$d\mu_{\alpha, \beta, \gamma, \eta}(t) := m_{\alpha, \beta, \gamma, \eta}(t) dt, \quad (10)$$

where $\min\{\Re(\eta), \Re(\gamma), \Re(\eta + \gamma - \alpha - \beta)\} > 0$ and

$$m_{\alpha,\beta,\gamma,\eta}(t) := \frac{\Gamma(\eta + \gamma - \alpha)\Gamma(\eta + \gamma - \beta)}{\Gamma(\eta)\Gamma(\gamma)\Gamma(\eta + \gamma - \alpha - \beta)} t^{\eta-1} (1-t)^{\gamma-1} {}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; 1-t \right].$$

We have $\int_0^1 d\mu_{\alpha,\beta,\gamma,\eta}(t) = 1$ and $m_{0,\beta,\gamma,\eta}(t) = m_{\alpha,0,\gamma,\eta}(t) = m_{\eta,\gamma}(t)$.

It is particularly noteworthy that Erdélyi's integrals are not only of computational interest but also have many important and profound applications. Gasper [14] has shown that (8) provides a natural way to the formulas of Dirichlet–Mehler type for the Jacobi polynomials and the generalized Legendre functions. Virchenko and Fedotova also frequently used the integrals (7) and (8) in their study of the generalized Legendre functions (see [15–17]). Sprinkhuizen-Kuyper [18] used (8) to prove a compositional property for her fractional integral operator $I_v^{\mu,\lambda}$. The integral (8) also plays an important role in the proof of product formula for biangle polynomials [19]. The proofs of the compositional properties of Saigo's fractional integral operators $I^{\alpha,\beta,\eta}$ and $J^{\alpha,\beta,\eta}$ are based on integrals (7) and (8) [20]. Raina [21] showed that the integral (7) can be used to construct a solution to a certain Abel-type integral equation involving the Appell hypergeometric function F_3 in the kernel.

We refer to integrals that have a form similar to integrals (7)–(9) as *Erdélyi-type integrals*. More precisely, for a given hypergeometric function, its Erdélyi-type integrals can generally express it as an integral of the product of two functions of the same type (see [13], p. 2).

Many Erdélyi-type integrals for various kinds of univariate and multivariate hypergeometric functions have been discovered using the technique of Erdélyi's fractional integration by parts. For example, Luo and Raina [22] recently extended (8) to a special class of generalized hypergeometric functions by using the theory of Miller and Paris and then found their applications in the theory of certain generalized fractional integral operator [23]. The table in Appendix A lists all hypergeometric functions for which Erdélyi-type integrals and some of their q - and discrete analogues have been established.

1.2. The Plan of the Work

Very recently, there has been a sustained growth in research interest regarding the three-variable hypergeometric function F_K introduced by Saran [24] and many interesting results have been achieved (see [13,25–28]). The F_K -function is defined by

$$F_K[\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x, y, z] := \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_{n+p} (\beta_1)_{m+p} (\beta_2)_n}{(\gamma_1)_m (\gamma_2)_n (\gamma_3)_p m! n! p!} x^m y^n z^p \quad (11)$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha_2)_p (\beta_1)_p}{p! (\gamma_3)_p} {}_2F_1 \left[\begin{matrix} \beta_1 + p, \alpha_1 \\ \gamma_1 \end{matrix}; x \right] {}_2F_1 \left[\begin{matrix} \alpha_2 + p, \beta_2 \\ \gamma_2 \end{matrix}; y \right] z^p, \quad (12)$$

where $(x, y, z) \in \mathbb{D}_K := \{(x, y, z) \in \mathbb{C}^3 : |x| < 1, |y| < 1, |z| < (1 - |x|)(1 - |y|)\}$.

In 2022, Luo et al. [13] discovered an elegant Erdélyi-type integral for Saran's F_K -function.

Theorem 1 ([13]). *Let $\Re(\alpha_1 + \eta_1) > \Re(\lambda_1) > 0$, $\Re(\beta_2 + \mu_2) > \Re(\lambda_2) > 0$ and $\Re(\gamma_3) > \Re(\beta_1) > 0$. Then we have*

$$\begin{aligned}
F_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1 + \eta_1, \beta_2 + \mu_2, \gamma_3; x, y, z] &= \int_0^1 \int_0^1 \int_0^1 (1-ux)^{-\lambda_3} (1-vy)^{-\eta_2} \\
&\cdot F_K[\alpha_1, \alpha_2 - \eta_2, \alpha_2 - \eta_2, \beta_1 - \lambda_3, \beta_2, \beta_1 - \lambda_3; \alpha_1 - \lambda_1 + \eta_1, \beta_2 - \lambda_2 + \mu_2, \beta_1 - \lambda_3; ux, vy, wz] \\
&\cdot F_K\left[\lambda_1 - \eta_1, \eta_2, \eta_2, \lambda_3, \lambda_2 - \mu_2, \lambda_3; \lambda_1, \lambda_2, \lambda_3; \frac{(1-u)x}{1-ux}, \frac{(1-v)y}{1-vy}, \frac{wz}{(1-ux)(1-vy)}\right] \\
&\cdot d\mu_{\alpha_1 - \lambda_1 + \eta_1, \lambda_1}(u) d\mu_{\beta_2 - \lambda_2 + \mu_2, \lambda_2}(v) d\mu_{\beta_1, \gamma_3 - \beta_1}(w),
\end{aligned} \tag{13}$$

where $(x, y, z) \in \mathbb{D}_K$ and $d\mu_{\alpha, \beta}(t)$ is defined in (3).

However, the initial proof of integral (13) was relatively difficult. In fact, Luo et al. first established a very general integral involving the Srivastava–Daoust function through fractional integration by parts, and then reduced it to (13) by specializing the parameters. In this paper, we shall provide a direct and simple proof of Theorem 1 in Section 3.1. Then in Section 3.2, we shall derive a curious integral representation for Appell function F_2 from Theorem 1. A further extension, which is related to Belitsky’s generalization of F_K [29] (p. 69), is considered in Section 3.3.

Section 4 devotes to the various q -Erdélyi-type integrals for q -analogue of F_K . Throughout this paper, we assume that $0 < q < 1$. For $a \in \mathbb{C}$, we define the q -shifted factorials as follows

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j) \quad (n \in \mathbb{Z}_{\geq 1})$$

and

$$(a; q)_\infty = \lim_{n \rightarrow +\infty} (a; q)_n = \prod_{j=0}^{\infty} (1 - aq^j).$$

The multiple q -shifted factorials are defined by $(a_1, \dots, a_m; q)_n = (a_1; q)_n \cdots (a_m; q)_n$, where n could be an integer or $+\infty$. Let $f : \mathbb{C}^k \rightarrow \mathbb{C}$ be a function. The k -dimensional q -integral can be defined by (see, for example, [30], p. 1 and [31], p. 241)

$$\int_{[0,1]^k} f(\mathbf{x}) d_q \mathbf{x} = (1-q)^k \sum_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^k} f(q^{\mathbf{n}}) q^{\langle \mathbf{n} \rangle}, \tag{14}$$

where $\mathbf{x} = (x_1, \dots, x_k)$, $\sum_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^k} \equiv \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty}$ and $\langle \mathbf{n} \rangle = n_1 + \cdots + n_k$.

The q -gamma function $\Gamma_q(x)$ is defined by [32] (p. 20, Equation (1.10.1))

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}$$

and the q -beta function is defined by [32] (p. 22, Equation (1.10.13))

$$B_q(x, y) = \frac{\Gamma_q(x) \Gamma_q(y)}{\Gamma_q(x+y)} = (1-q) \frac{(q, q^{x+y}; q)_\infty}{(q^x, q^y; q)_\infty}.$$

Note that $\lim_{q \rightarrow 1^-} \Gamma_q(x) = \Gamma(x)$ and $\lim_{q \rightarrow 1^-} B_q(x, y) = B(x, y)$. The q -integral representation for $B_q(x, y)$ is given by [32] (p. 23, Equation (1.11.7))

$$B_q(x, y) = \int_0^1 t^{x-1} \frac{(tq; q)_\infty}{(tq^y; q)_\infty} d_q t, \quad \Re(x) > 0, y \notin \mathbb{Z}_{\leq 0}.$$

The q -analogue of F_K can be defined by (see [33,34])

$$\Phi_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; q, x, y, z] := \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1; q)_m (\alpha_2; q)_{n+p} (\beta_1; q)_{m+p} (\beta_2; q)_n}{(\gamma_1, q; q)_m (\gamma_2, q; q)_n (\gamma_3, q; q)_p} x^m y^n z^p \quad (15)$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha_2, \beta_1; q)_p}{(\gamma_3, q; q)_p} {}_2\phi_1 \left[\begin{matrix} \beta_1 q^p, \alpha_1 \\ \gamma_1 \end{matrix}; q, x \right] {}_2\phi_1 \left[\begin{matrix} \alpha_2 q^p, \beta_2 \\ \gamma_2 \end{matrix}; q, y \right] z^p, \quad (16)$$

where $|x| < 1$, $|y| < 1$, $|z| < 1$ and ${}_r\phi_s$ is the basic hypergeometric series defined by [32] (p. 4, Equation (1.2.22))

$${}_r\phi_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right] := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{1+s-r} z^n. \quad (17)$$

For simplicity, we write

$$\begin{aligned} \tilde{\Phi}_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; q, x, y, z] \\ = \Phi_K[q^{\alpha_1}, q^{\alpha_2}, q^{\alpha_2}, q^{\beta_1}, q^{\beta_2}, q^{\beta_1}; q^{\gamma_1}, q^{\gamma_2}, q^{\gamma_3}; q, x, y, z] \end{aligned} \quad (18)$$

and

$${}_r\tilde{\phi}_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right] = {}_r\phi_s \left[\begin{matrix} q^{a_1}, \dots, q^{a_r} \\ q^{b_1}, \dots, q^{b_s} \end{matrix}; q, z \right].$$

The *only* q -integral representation (of Bateman type) for Φ_K is given by Ernst [33] (p. 10, Theorem 5):

$$\begin{aligned} \tilde{\Phi}_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; q, x, y, z] \\ = \int_{[0,1]^3} \tilde{\Phi}_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; v_1, v_2, v_3; q, xu, yv, zw] \\ \cdot d\mu_{v_1, \gamma_1 - v_1}(u; q) d\mu_{v_2, \gamma_2 - v_2}(v; q) d\mu_{v_3, \gamma_3 - v_3}(w; q). \end{aligned} \quad (19)$$

Here and in what follows, we define the q -analogue of the Dirichlet measure (3) as follows:

$$d\mu_{\alpha, \beta}(t; q) := m_{\alpha, \beta}(t; q) d_q t, \quad (20)$$

where

$$m_{\alpha, \beta}(t; q) := \frac{\Gamma_q(\alpha + \beta)}{\Gamma_q(\alpha)\Gamma_q(\beta)} t^{\alpha-1} \frac{(tq; q)_{\infty}}{(tq^{\beta}; q)_{\infty}}, \quad \min\{\Re(\alpha), \Re(\beta)\} > 0.$$

Obviously, we have $\int_0^1 d\mu_{\alpha, \beta}(t; q) = 1$ and $\lim_{q \rightarrow 1^-} m_{\alpha, \beta}(t; q) = m_{\alpha, \beta}(t)$.

Actually, in Section 4, we shall first prove a Joshi–Vyas type theorem and derive some useful corollaries from it. Then a discrete analogue for one of the corollaries is obtained. Finally, we present a q -analogue of Theorem 1.

In the next section, we will further introduce some results related to q -series.

2. Preliminaries

Gasper [35] discussed the q -analogues of Erdélyi's integrals (7)–(9) and provided their discrete analogues. The integral (7) has the following q -analogue [35] (p. 4, Equation (1.13))

$$\begin{aligned}
{}_2\tilde{\phi}_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; q, x \right] &= \int_0^1 \frac{(xtq^{\alpha'}; q)_{\infty}}{(xt; q)_{\infty}} {}_2\tilde{\phi}_1 \left[\begin{matrix} \alpha - \alpha', \beta \\ \lambda \end{matrix}; q, xtq^{\alpha'} \right] \\
&\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{\alpha'}, q^{\beta-\lambda}, t^{-1} \\ q^{\gamma-\lambda}, q/(xt) \end{matrix}; q, q \right] d\mu_{\lambda, \gamma-\lambda}(t; q),
\end{aligned} \quad (21)$$

where $\Re(\gamma) > \Re(\lambda) > 0$ and $d\mu_{\alpha, \beta}(t; q)$ is defined in (20). Before stating the q -analogue of (9), we have to first introduce the q -analogue of the hypergeometric measure (10):

$$d\mu_{\alpha, \beta, \gamma, \eta}(t; q) := m_{\alpha, \beta, \gamma, \eta}(t; q) d_q t, \quad (22)$$

where $\min\{\Re(\eta), \Re(\gamma), \Re(\eta + \gamma - \alpha - \beta)\} > 0$ and

$$\begin{aligned}
m_{\alpha, \beta, \gamma, \eta}(t; q) &= \frac{\Gamma_q(\eta + \gamma - \alpha) \Gamma_q(\eta + \gamma - \beta)}{\Gamma_q(\eta) \Gamma_q(\gamma) \Gamma_q(\eta + \gamma - \alpha - \beta)} \\
&\quad \cdot t^{\eta-1} \frac{(tq; q)_{\infty}}{(tq^{\gamma}; q)_{\infty}} {}_3\phi_1 \left[\begin{matrix} q^{\alpha}, q^{\beta}, t^{-1} \\ q^{\gamma} \end{matrix}; q, tq^{\gamma-\alpha-\beta} \right].
\end{aligned} \quad (23)$$

We have $\int_0^1 d\mu_{\eta-\lambda, \gamma-\lambda, \gamma-\lambda+\eta-\nu, \nu}(t; q) = 1$ and $\lim_{q \rightarrow 1^-} m_{\alpha, \beta, \gamma, \eta}(t; q) = m_{\alpha, \beta, \gamma, \eta}(t)$. Then the q -analogue of (9) is given by [35] (p. 4, Equation (1.14))

$${}_2\tilde{\phi}_1 \left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; q, x \right] = \int_0^1 {}_3\tilde{\phi}_2 \left[\begin{matrix} \alpha, \beta, \eta \\ \lambda, \nu \end{matrix}; q, xt \right] d\mu_{\eta-\lambda, \gamma-\lambda, \gamma-\lambda+\eta-\nu, \nu}(t; q), \quad (24)$$

where $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma + \eta - \lambda - \nu)\} > 0$. Joshi and Vyas [36] also derived some nice q -Erdélyi-type integrals, one of which is discussed in Section 4.

In the same paper [35], Gasper also found the following discrete q -analogue of (24):

$$\begin{aligned}
{}_3\phi_2 \left[\begin{matrix} \alpha, \beta, q^{-n} \\ \gamma, \delta \end{matrix}; q, q \right] &= \frac{(q, \lambda; q)_n}{(\gamma, \mu; q)_n} \sum_{k=0}^n \frac{(\nu; q)_k (\gamma\mu/(\lambda\nu); q)_{n-k}}{(q; q)_k (q; q)_{n-k}} \nu^{n-k} \\
&\quad \cdot {}_3\phi_2 \left[\begin{matrix} \mu/\lambda, \gamma/\lambda, q^{k-n} \\ \gamma\mu/(\lambda\nu), q^{1-n}/\lambda \end{matrix}; q, q^{1-k}/\nu \right] {}_4\phi_3 \left[\begin{matrix} \alpha, \beta, \mu, q^{-k} \\ \lambda, \nu, \delta \end{matrix}; q, q \right].
\end{aligned} \quad (25)$$

Finally, we also need the following q -hypergeometric function of three variables (see [37], p. 96 and [38], p. 35):

$$\begin{aligned}
\phi^{(3)}[x, y, z] &\equiv \phi^{(3)} \left[\begin{matrix} (a) :: (b); (b'); (b'') : (c); (c'); (c'') \\ (e) :: (g); (g'); (g'') : (h); (h'); (h'') \end{matrix}; q, x, y, z \right] \\
&:= \sum_{m, n, p=0}^{\infty} \frac{\prod_{r=1}^A (a_r; q)_{m+n+p}}{\prod_{r=1}^E (e_r; q)_{m+n+p}} \cdot \frac{\prod_{r=1}^B (b_r; q)_{m+n} \prod_{r=1}^{B'} (b'_r; q)_{n+p} \prod_{r=1}^{B''} (b''_r; q)_{p+m}}{\prod_{r=1}^G (g_r; q)_{m+n} \prod_{r=1}^{G'} (g'_r; q)_{n+p} \prod_{r=1}^{G''} (g''_r; q)_{p+m}} \\
&\quad \cdot \frac{\prod_{r=1}^C (c_r; q)_m \prod_{r=1}^{C'} (c'_r; q)_n \prod_{r=1}^{C''} (c''_r; q)_p}{\prod_{r=1}^H (h_r; q)_m \prod_{r=1}^{H'} (h'_r; q)_n \prod_{r=1}^{H''} (h''_r; q)_p} \cdot \frac{x^m}{(q; q)_m} \frac{y^n}{(q; q)_n} \frac{z^p}{(q; q)_p},
\end{aligned} \quad (26)$$

where $|x| < 1$, $|y| < 1$ and $|z| < 1$.

3. Theorem 1 Revisited

3.1. An Alternative Proof of Theorem 1

For convenience, let I denote the triple integral in (13). By (12), the first F_K -function in the integrand can be expanded as

$$F_K[\alpha_1, \alpha_2 - \eta_2, \alpha_2 - \eta_2, \beta_1 - \lambda_3, \beta_2, \beta_1 - \lambda_3; \alpha_1 - \lambda_1 + \eta_1, \beta_2 - \lambda_2 + \mu_2, \beta_1 - \lambda_3; ux, vy, wz] \\ = \sum_{m=0}^{\infty} \frac{(\alpha_2 - \eta_2)_m}{m!} {}_2F_1 \left[\begin{matrix} \beta_1 - \lambda_3 + m, \alpha_1 \\ \alpha_1 - \lambda_1 + \eta_1 \end{matrix}; ux \right] {}_2F_1 \left[\begin{matrix} \alpha_2 - \eta_2 + m, \beta_2 \\ \beta_2 - \lambda_2 + \mu_2 \end{matrix}; vy \right] (wz)^m. \quad (27)$$

Since $\mathbb{D}_K$ is a Reinhardt domain, we have immediately $(ux, vy, wz) \in \mathbb{D}_K$ ($u, v, w \in (0, 1)$) and thus the region of convergence of the series in (27) is clear. For the second F_K -function in the integrand, we have

$$F_K \left[\lambda_1 - \eta_1, \eta_2, \eta_2, \lambda_3, \lambda_2 - \mu_2, \lambda_3; \lambda_1, \lambda_2, \lambda_3; \frac{(1-u)x}{1-ux}, \frac{(1-v)y}{1-vy}, \frac{wz}{(1-ux)(1-vy)} \right] \\ = \sum_{n=0}^{\infty} \frac{(\eta_2)_n}{n!} {}_2F_1 \left[\begin{matrix} \lambda_3 + n, \lambda_1 - \eta_1 \\ \lambda_1 \end{matrix}; \frac{(1-u)x}{1-ux} \right] \\ \cdot {}_2F_1 \left[\begin{matrix} \eta_2 + n, \lambda_2 - \mu_2 \\ \lambda_2 \end{matrix}; \frac{(1-v)y}{1-vy} \right] \cdot \left[\frac{wz}{(1-ux)(1-vy)} \right]^n. \quad (28)$$

Note that

$$\left| \frac{(1-u)x}{1-ux} \right| < \frac{1-u}{1-u|x|} < 1, \quad \left| \frac{(1-v)y}{1-vy} \right| < \frac{1-v}{1-v|y|} < 1$$

and

$$\frac{\left| \frac{wz}{(1-ux)(1-vy)} \right|}{\left(1 - \left| \frac{(1-u)x}{1-ux} \right| \right) \left(1 - \left| \frac{(1-v)y}{1-vy} \right| \right)} = \frac{w|z|}{(|1-ux| - (1-u)|x|)(|1-vy| - (1-v)|y|)} \\ \leq \frac{|z|}{(1-|x|)(1-|y|)} < 1,$$

namely,

$$\left(\frac{(1-u)x}{1-ux}, \frac{(1-v)y}{1-vy}, \frac{wz}{(1-ux)(1-vy)} \right) \in \mathbb{D}_K.$$

Thus the region of convergence of the series in (28) is also fully characterized. In the subsequent calculations, we may safely interchange the order of integration and summation, as well as the order of summation, by Fubini's theorem.

Now we have

$$I = \sum_{m=0}^{\infty} \frac{(\alpha_2 - \eta_2)_m}{m!} z^m \sum_{n=0}^{\infty} \frac{(\eta_2)_n}{n!} z^n \int_0^1 w^{m+n} d\mu_{\beta_1, \gamma_3 - \beta_1}(w) \\ \cdot \int_0^1 (1-ux)^{-\lambda_3-n} {}_2F_1 \left[\begin{matrix} \beta_1 - \lambda_3 + m, \alpha_1 \\ \alpha_1 - \lambda_1 + \eta_1 \end{matrix}; ux \right] \\ \cdot {}_2F_1 \left[\begin{matrix} \lambda_3 + n, \lambda_1 - \eta_1 \\ \lambda_1 \end{matrix}; \frac{(1-u)x}{1-ux} \right] d\mu_{\alpha_1 - \lambda_1 + \eta_1, \lambda_1}(u) \\ \cdot \int_0^1 (1-vy)^{-\eta_2-n} {}_2F_1 \left[\begin{matrix} \alpha_2 - \eta_2 + m, \beta_2 \\ \beta_2 - \lambda_2 + \mu_2 \end{matrix}; vy \right] \\ \cdot {}_2F_1 \left[\begin{matrix} \eta_2 + n, \lambda_2 - \mu_2 \\ \lambda_2 \end{matrix}; \frac{(1-v)y}{1-vy} \right] d\mu_{\beta_2 - \lambda_2 + \mu_2, \lambda_2}(v). \quad (29)$$

Note that

$$\int_0^1 w^{m+n} d\mu_{\beta_1, \gamma_3 - \beta_1}(w) = \frac{(\beta_1)_{m+n}}{(\gamma_3)_{m+n}}, \quad (30)$$

where $\Re(\gamma_3) > \Re(\beta_1) > 0$. By letting $\alpha' \rightarrow \lambda_3 + n$, $\beta \rightarrow \alpha_1$, $\lambda \rightarrow \alpha_1 - \lambda_1 + \eta_1$, $\alpha \rightarrow \beta_1 + m + n$ and $\gamma \rightarrow \alpha_1 + \eta_1$ in the Erdélyi integral (7), we obtain

$$\begin{aligned} & \int_0^1 (1-ux)^{-\lambda_3-n} {}_2F_1 \left[\begin{matrix} \beta_1 - \lambda_3 + m, \alpha_1 \\ \alpha_1 - \lambda_1 + \eta_1 \end{matrix}; ux \right] \\ & \cdot {}_2F_1 \left[\begin{matrix} \lambda_3 + n, \lambda_1 - \eta_1 \\ \lambda_1 \end{matrix}; \frac{(1-u)x}{1-ux} \right] d\mu_{\alpha_1 - \lambda_1 + \eta_1, \lambda_1}(u) = {}_2F_1 \left[\begin{matrix} \beta_1 + m + n, \alpha_1 \\ \alpha_1 + \eta_1 \end{matrix}; x \right], \end{aligned} \quad (31)$$

where $\Re(\alpha_1 + \eta_1) > \Re(\lambda_1) > 0$. By letting $\alpha' \rightarrow \eta_2 + n$, $\beta \rightarrow \beta_2$, $\lambda \rightarrow \beta_2 - \lambda_2 + \mu_2$, $\alpha \rightarrow \alpha_2 + m + n$ and $\gamma \rightarrow \beta_2 + \mu_2$ in the Erdélyi integral (7), we obtain

$$\begin{aligned} & \int_0^1 (1-vy)^{-\eta_2-n} {}_2F_1 \left[\begin{matrix} \alpha_2 - \eta_2 + m, \beta_2 \\ \beta_2 - \lambda_2 + \mu_2 \end{matrix}; vy \right] \\ & \cdot {}_2F_1 \left[\begin{matrix} \eta_2 + n, \lambda_2 - \mu_2 \\ \lambda_2 \end{matrix}; \frac{(1-v)y}{1-vy} \right] d\mu_{\beta_2 - \lambda_2 + \mu_2, \lambda_2}(v) = {}_2F_1 \left[\begin{matrix} \alpha_2 + m + n, \beta_2 \\ \beta_2 + \mu_2 \end{matrix}; y \right], \end{aligned} \quad (32)$$

where $\Re(\beta_2 + \mu_2) > \Re(\lambda_2) > 0$. With the help of (30)–(32), (29) can be simplified to

$$I = \sum_{m,n=0}^{\infty} \frac{(\beta_1)_{m+n} (\alpha_2 - \eta_2)_m (\eta_2)_n}{(\gamma_3)_{m+n} m! n!} {}_2F_1 \left[\begin{matrix} \beta_1 + m + n, \alpha_1 \\ \alpha_1 + \eta_1 \end{matrix}; x \right] {}_2F_1 \left[\begin{matrix} \alpha_2 + m + n, \beta_2 \\ \beta_2 + \mu_2 \end{matrix}; y \right] z^{m+n}. \quad (33)$$

To simplify the double series in (33), we first use the familiar series transformation formula [39] (p. 56, Lemma 10)

$$\sum_{m,n=0}^{\infty} \Omega(n, m) = \sum_{m=0}^{\infty} \sum_{n=0}^m \Omega(n, m-n) \quad (34)$$

to obtain

$$I = \sum_{m=0}^{\infty} \frac{(\beta_1)_m}{(\gamma_3)_m} {}_2F_1 \left[\begin{matrix} \beta_1 + m, \alpha_1 \\ \alpha_1 + \eta_1 \end{matrix}; x \right] {}_2F_1 \left[\begin{matrix} \alpha_2 + m, \beta_2 \\ \beta_2 + \mu_2 \end{matrix}; y \right] z^m \sum_{n=0}^m \frac{(\alpha_2 - \eta_2)_{m-n} (\eta_2)_n}{(m-n)! n!}. \quad (35)$$

Then the use of the Chu–Vandermonde identity

$$(a+b)_m = \sum_{n=0}^m \binom{m}{n} (a)_{m-n} (b)_n$$

gives

$$\sum_{n=0}^m \frac{(\alpha_2 - \eta_2)_{m-n} (\eta_2)_n}{(m-n)! n!} = \frac{(\alpha_2)_m}{m!}. \quad (36)$$

The result (13) now follows by substituting (36) into (35).

Remark 1. Joshi and Vyas [40] have shown that the Erdélyi integrals (7)–(9) can be derived by using the series manipulation technique. Thus, inspired by the proof presented in this section, we can also write down a proof using only the series manipulation technique. However, such a proof is quite complicated, so we choose to omit it here.

3.2. A Curious Integral Representation for F_2

In this subsection, we derive a neat and interesting integral from Theorem 1. To the best of our knowledge, this integral has not been explicitly mentioned in the existing literature.

Theorem 2. Let $\Re(c_1) > \Re(d_1) > 0$ and $\Re(c_2) > \Re(b_2) > 0$. Then we have

$$F_2[a_1, b_1, b_2; c_1, c_2; y, z] = \int_0^1 \int_0^1 (1 - wz - vy)^{-a_1} {}_2F_1 \left[\begin{matrix} a_1 - a_2, c_1 - b_1 - d_1 \\ c_1 - d_1 \end{matrix}; \frac{vy}{vy + wz - 1} \right] \\ \cdot {}_2F_1 \left[\begin{matrix} a_2, b_1 + d_1 - c_1 \\ d_1 \end{matrix}; \frac{(1-v)y}{1-vy-wz} \right] d\mu_{c_1-d_1, d_1}(v) d\mu_{b_2, c_2-b_2}(w). \quad (37)$$

Proof. Letting $x = 0$ in (13), it reduces to

$$F_2[\alpha_2, \beta_2, \beta_1; \beta_2 + \mu_2, \gamma_3; y, z] \\ = \int_0^1 \int_0^1 (1 - vy)^{-\eta_2} F_2[\alpha_2 - \eta_2, \beta_2, \beta_1 - \lambda_3; \beta_2 - \lambda_2 + \mu_2, \beta_1 - \lambda_3; vy, wz] \\ \cdot F_2 \left[\begin{matrix} \eta_2, \lambda_2 - \mu_2, \lambda_3 \\ \lambda_2, \lambda_3 \end{matrix}; \frac{(1-v)y}{1-vy}, \frac{wz}{1-vy} \right] d\mu_{\beta_2-\lambda_2+\mu_2, \lambda_2}(v) d\mu_{\beta_1, \gamma_3-\beta_1}(w). \quad (38)$$

Next, letting $\alpha_2 \rightarrow a_1$, $\beta_2 \rightarrow b_1$, $\beta_1 \rightarrow b_2$, $\mu_2 \rightarrow c_1 - b_1$, $\gamma_3 \rightarrow c_2$, $\lambda_2 \rightarrow d_1$, $\lambda_3 \rightarrow d_2$ and $\eta_2 \rightarrow a_2$ in (38) gives

$$F_2[a_1, b_1, b_2; c_1, c_2; y, z] \\ = \int_0^1 \int_0^1 (1 - vy)^{-a_2} F_2[a_1 - a_2, b_1, b_2 - d_2; c_1 - d_1, b_2 - d_2; vy, wz] \\ \cdot F_2 \left[\begin{matrix} a_2, d_1 - c_1 + b_1, d_2 \\ d_1, d_2 \end{matrix}; \frac{(1-v)y}{1-vy}, \frac{wz}{1-vy} \right] d\mu_{c_1-d_1, d_1}(v) d\mu_{b_2, c_2-b_2}(w). \quad (39)$$

Then, by using [41] (p. 306, Equation (109))

$$F_2[a, b, b'; c, b'; y, z] = (1 - z)^{-a} {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix}; \frac{y}{1-z} \right], \quad (40)$$

we have

$$(1 - vy)^{-a_2} F_2 \left[\begin{matrix} a_2, d_1 - c_1 + b_1, d_2 \\ d_1, d_2 \end{matrix}; \frac{(1-v)y}{1-vy}, \frac{wz}{1-vy} \right] \\ = (1 - vy - wz)^{-a_2} {}_2F_1 \left[\begin{matrix} a_2, b_1 + d_1 - c_1 \\ d_1 \end{matrix}; \frac{(1-v)y}{1-vy-wz} \right]. \quad (41)$$

Note that d_2 is merely an auxiliary parameter introduced for the transformation step and does not appear in our final result. By using (40) and the Pfaff transformation [41] (p. 300, Equation (79)), we have

$$F_2[a_1 - a_2, b_1, b_2 - d_2; c_1 - d_1, b_2 - d_2; vy, wz] \\ = (1 - wz)^{-a_1+a_2} {}_2F_1 \left[\begin{matrix} a_1 - a_2, b_1 \\ c_1 - d_1 \end{matrix}; \frac{vy}{1-wz} \right] \\ = (1 - wz - vy)^{-a_1+a_2} {}_2F_1 \left[\begin{matrix} a_1 - a_2, c_1 - b_1 - d_1 \\ c_1 - d_1 \end{matrix}; \frac{vy}{vy + wz - 1} \right]. \quad (42)$$

Substituting (41) and (42) into (39) leads us to (37). $\square$

Remark 2. We can prove (39) directly by using the method described in Section 3.1. In addition, the integral (37) can also be derived from Manocha's integral [42] (p. 239, Equation (13)):

$$F_2[a, b, c; d, e; y, z] = \int_0^1 \int_0^1 (1 - vy - wz)^{-a'} F_2[a - a', b, c; \lambda, \eta; yv, zw] \cdot F_2\left[a', b - \lambda, c - \eta; d - \lambda, e - \eta; \frac{(1-v)y}{1-vy-wz}, \frac{(1-w)z}{1-vy-wz}\right] d\mu_{\lambda, d-\lambda}(v) d\mu_{\eta, e-\eta}(w), \quad (43)$$

where $\Re(d) > \Re(\lambda) > 0$ and $\Re(e) > \Re(\eta) > 0$. In fact, if we set $\eta \rightarrow c$ in (43) and then use the two transformation formulas as in the proof above, we obtain

$$F_2[a, b, c; d, e; y, z] = \int_0^1 \int_0^1 (1 - vy - wz)^{-a} {}_2F_1\left[\begin{matrix} a - a', \lambda - b \\ \lambda \end{matrix}; \frac{vy}{vy + wz - 1}\right] \cdot {}_2F_1\left[\begin{matrix} a', b - \lambda \\ d - \lambda \end{matrix}; \frac{(1-v)y}{1-vy-wz}\right] d\mu_{\lambda, d-\lambda}(v) d\mu_{c, e-c}(w). \quad (44)$$

Letting further $\lambda \rightarrow c_1 - d_1$, $b \rightarrow b_1$, $a \rightarrow a_1$, $a' \rightarrow a_2$, $c \rightarrow b_2$, $d \rightarrow c_1$ and $e \rightarrow c_2$ in (44) leads us again to the integral (37).

3.3. A Further Extension of Theorem 1

As usual, the discrete convolution product of two sequences $\mathbf{a} := a(m, n)$ and $\mathbf{b} := b(m, n)$ is defined by

$$(\mathbf{a} \star \mathbf{b})(m, n) := \sum_{i=0}^m \sum_{j=0}^n a(m-i, n-j) b(i, j).$$

Let us consider the function of the following form:

$$\mathcal{F}^{\mathbf{a}} \left[\begin{matrix} \alpha_1, \beta_1 : \alpha_2, \beta_2 \\ \gamma_1 : \gamma_2 \end{matrix}; x_1, x_2, x_3, x_4 \right] := \sum_{m, n=0}^{\infty} a(m, n) {}_2F_1 \left[\begin{matrix} \alpha_1 + m, \beta_1 \\ \gamma_1 \end{matrix}; x_1 \right] \cdot {}_2F_1 \left[\begin{matrix} \alpha_2 + n, \beta_2 \\ \gamma_2 \end{matrix}; x_2 \right] x_3^m x_4^n. \quad (45)$$

Functions of this type appear frequently in many situations and have properties similar to Saran's F_K -function.

If we take

$$a(m, n) = \begin{cases} \frac{(\alpha_1)_n (\alpha_2)_n}{n! (\gamma_3)_n}, & m = n, \\ 0, & m \neq n, \end{cases}$$

then the function (45) reduces to Saran's function:

$$F_K[\beta_1, \alpha_2, \alpha_2, \alpha_1, \beta_2, \alpha_1; \gamma_1, \gamma_2, \gamma_3; x_1, x_2, x_3 x_4].$$

Another significant example originates in physics. In 2018, Belitsky [29] (p. 69) generalized Saran's F_K -function of three variables to L variables, i.e.,

$$F_K \left[\begin{matrix} a_1, b_1, \dots, b_{L-1}, a_2 \\ c_1, \dots, c_L \end{matrix}; z_1, \dots, z_L \right] := \sum_{n_1, \dots, n_L=0}^{\infty} \frac{(a_1)_{n_1} (b_1)_{n_1+n_2} \dots (b_{L-1})_{n_{L-1}+n_L} (a_2)_{n_L}}{(c_1)_{n_1} \dots (c_L)_{n_L}} \frac{z_1^{n_1}}{n_1!} \dots \frac{z_L^{n_L}}{n_L!}. \quad (46)$$

Then Rosenhaus [43] employed the function (46) in his study of n -point conformal blocks in the comb channel. More recently, Ferrando et al. [44] used a *rescaled version* of this function in evaluating the conformal integral associated with the n -point conformal partial wave in the comb channel. Taking $L = 4$ in (46), we obtain

$$\begin{aligned} & F_K \left[\begin{matrix} a_1, b_1, b_2, b_3, a_2 \\ c_1, c_2, c_3, c_4 \end{matrix}; z_1, z_2, z_3, z_4 \right] \\ &= \sum_{n_1, n_2, n_3, n_4=0}^{\infty} \frac{(a_1)_{n_1} (b_1)_{n_1+n_2} (b_2)_{n_2+n_3} (b_3)_{n_3+n_4} (a_2)_{n_4}}{(c_1)_{n_1} (c_2)_{n_2} (c_3)_{n_3} (c_4)_{n_4}} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4}}{n_1! n_2! n_3! n_4!} \\ &= \sum_{n_2, n_3=0}^{\infty} \frac{(b_1)_{n_2} (b_2)_{n_2+n_3} (b_3)_{n_3}}{(c_2)_{n_2} (c_3)_{n_3} n_2! n_3!} {}_2F_1 \left[\begin{matrix} a_1, b_1 + n_2 \\ c_1 \end{matrix}; z_1 \right] {}_2F_1 \left[\begin{matrix} a_2, b_3 + n_3 \\ c_4 \end{matrix}; z_4 \right] z_2^{n_2} z_3^{n_3}, \quad (47) \end{aligned}$$

which also has the form of (45). The region of convergence of (47) is determined by

$$|z_1| < 1, |z_4| < 1 \quad \text{and} \quad \frac{|z_2|}{1 - |z_1|} + \frac{|z_3|}{1 - |z_4|} < 1.$$

Here it is worth mentioning that Khichi has introduced the multivariate hypergeometric functions $H_B^{(n)}$ ([41], p. 308, Equation (123)), which appears quite similar to the function (46), although they are different in nature.

By using the method from Section 3.1, we have the following theorem.

Theorem 3. Let $\Re(\tau_j) > \Re(\gamma_j) > 0$ ($j = 1, \dots, 4$). Then we have

$$\begin{aligned} & \mathcal{F}^c \left[\begin{matrix} \alpha_1 + \lambda_1, \beta_1 : \alpha_2 + \lambda_2, \beta_2 \\ \tau_1 : \tau_2 \end{matrix}; x_1, x_2, x_3, x_4 \right] = \int_0^1 \int_0^1 \int_0^1 \int_0^1 (1 - u_1 x_1)^{-\lambda_1} (1 - u_2 x_2)^{-\lambda_2} \\ & \cdot \mathcal{F}^a \left[\begin{matrix} \alpha_1, \beta_1 : \alpha_2, \beta_2 \\ \gamma_1 : \gamma_2 \end{matrix}; x_1 u_1, x_2 u_2, x_3 u_3, x_4 u_4 \right] \\ & \cdot \mathcal{F}^b \left[\begin{matrix} \lambda_1, \beta_1 - \gamma_1 : \lambda_2, \beta_2 - \gamma_2 \\ \tau_1 - \gamma_1 : \tau_2 - \gamma_2 \end{matrix}; \frac{(1 - u_1) x_1}{1 - u_1 x_1}, \frac{(1 - u_2) x_2}{1 - u_2 x_2}, \frac{u_3 x_3}{1 - u_1 x_1}, \frac{u_4 x_4}{1 - u_2 x_2} \right] \\ & \cdot \prod_{j=1}^4 d\mu_{\gamma_j, \tau_j - \gamma_j}(u_j), \quad (48) \end{aligned}$$

where

$$\mathbf{c}(m, n) = \frac{(\gamma_3)_m (\gamma_4)_n}{(\tau_3)_m (\tau_4)_n} (\mathbf{a} \star \mathbf{b})(m, n). \quad (49)$$

Proof. Since the proof is very similar to the one given in Section 3.1, we omit the details and retain only key steps. Let us denote the right-hand side of (48) by I . Then, by using (45), we have

$$\begin{aligned} I &= \sum_{m_1, m_2=0}^{\infty} \sum_{n_1, n_2=0}^{\infty} a(m_1, m_2) b(n_1, n_2) x_3^{m_1+n_1} x_4^{m_2+n_2} \\ & \cdot \int_0^1 (1 - u_1 x_1)^{-\lambda_1 - n_1} {}_2F_1 \left[\begin{matrix} \alpha_1 + m_1, \beta_1 \\ \gamma_1 \end{matrix}; x_1 u_1 \right] \\ & \cdot {}_2F_1 \left[\begin{matrix} \lambda_1 + n_1, \beta_1 - \gamma_1 \\ \tau_1 - \gamma_1 \end{matrix}; \frac{(1 - u_1) x_1}{1 - u_1 x_1} \right] d\mu_{\gamma_1, \tau_1 - \gamma_1}(u_1) \end{aligned}$$

$$\begin{aligned}
& \cdot \int_0^1 (1 - u_2 x_2)^{-\lambda_2 - n_2} {}_2F_1 \left[\begin{matrix} \alpha_2 + m_2, \beta_2 \\ \gamma_2 \end{matrix}; x_2 u_2 \right] \\
& \cdot {}_2F_1 \left[\begin{matrix} \lambda_2 + n_2, \beta_2 - \gamma_2 \\ \tau_2 - \gamma_2 \end{matrix}; \frac{(1 - u_2)x_2}{1 - u_2 x_2} \right] d\mu_{\gamma_2, \tau_2 - \gamma_2}(u_2) \\
& \cdot \int_0^1 u_3^{m_1 + n_1} d\mu_{\gamma_3, \tau_3 - \gamma_3}(u_3) \int_0^1 u_4^{m_2 + n_2} d\mu_{\gamma_4, \tau_4 - \gamma_4}(u_4). \tag{50}
\end{aligned}$$

By using (7) and (30), we obtain

$$\begin{aligned}
I = & \sum_{m_1, m_2=0}^{\infty} \sum_{n_1, n_2=0}^{\infty} a(m_1, m_2) b(n_1, n_2) x_3^{m_1 + n_1} x_4^{m_2 + n_2} \frac{(\gamma_3)_{m_1 + n_1}}{(\tau_3)_{m_1 + n_1}} \frac{(\gamma_4)_{m_2 + n_2}}{(\tau_4)_{m_2 + n_2}} \\
& \cdot {}_2F_1 \left[\begin{matrix} \alpha_1 + \lambda_1 + m_1 + n_1, \beta_1 \\ \tau_1 \end{matrix}; x_1 \right] {}_2F_1 \left[\begin{matrix} \alpha_2 + \lambda_2 + m_2 + n_2, \beta_2 \\ \tau_2 \end{matrix}; x_2 \right].
\end{aligned}$$

Next, interchanging the order of summation and using the series transformation Formula (35), as done previously in Section 3.1, we obtain

$$\begin{aligned}
I = & \sum_{m_1, n_1=0}^{\infty} \sum_{m_2, n_2=0}^{\infty} a(m_1, m_2) b(n_1, n_2) x_3^{m_1 + n_1} x_4^{m_2 + n_2} \frac{(\gamma_3)_{m_1 + n_1}}{(\tau_3)_{m_1 + n_1}} \frac{(\gamma_4)_{m_2 + n_2}}{(\tau_4)_{m_2 + n_2}} \\
& \cdot {}_2F_1 \left[\begin{matrix} \alpha_1 + \lambda_1 + m_1 + n_1, \beta_1 \\ \tau_1 \end{matrix}; x_1 \right] {}_2F_1 \left[\begin{matrix} \alpha_2 + \lambda_2 + m_2 + n_2, \beta_2 \\ \tau_2 \end{matrix}; x_2 \right] \\
= & \sum_{m_1, m_2=0}^{\infty} x_3^{m_1} x_4^{m_2} \frac{(\gamma_3)_{m_1}}{(\tau_3)_{m_1}} \frac{(\gamma_4)_{m_2}}{(\tau_4)_{m_2}} {}_2F_1 \left[\begin{matrix} \alpha_1 + \lambda_1 + m_1, \beta_1 \\ \tau_1 \end{matrix}; x_1 \right] {}_2F_1 \left[\begin{matrix} \alpha_2 + \lambda_2 + m_2, \beta_2 \\ \tau_2 \end{matrix}; x_2 \right] \\
& \cdot \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} a(m_1 - n_1, m_2 - n_2) b(n_1, n_2) \\
= & \sum_{m_1, m_2=0}^{\infty} \frac{(\gamma_3)_{m_1}}{(\tau_3)_{m_1}} \frac{(\gamma_4)_{m_2}}{(\tau_4)_{m_2}} (\mathbf{a} \star \mathbf{b})(m_1, m_2) \\
& \cdot {}_2F_1 \left[\begin{matrix} \alpha_1 + \lambda_1 + m_1, \beta_1 \\ \tau_1 \end{matrix}; x_1 \right] {}_2F_1 \left[\begin{matrix} \alpha_2 + \lambda_2 + m_2, \beta_2 \\ \tau_2 \end{matrix}; x_2 \right] x_3^{m_1} x_4^{m_2},
\end{aligned}$$

which upon using the notation (49) leads us to the desired result. $\square$

4. Some q -Erdélyi-Type Integrals

4.1. A Joshi–Vyas-Type Theorem

Inspired by the formula [40] (p. 133) of Joshi and Vyas and its useful q -analogue [36] (p. 646), we establish—using the measure in (22)—the following theorem of Joshi–Vyas type.

Theorem 4. Let $\Phi(z_1, \dots, z_k)$ be defined by

$$\Phi(z_1, \dots, z_k) := \sum_{n_1, \dots, n_k=0}^{\infty} c(n_1, \dots, n_k) z_1^{n_1} \cdots z_k^{n_k}$$

where $\{c(n_1, \dots, n_k)\}$ is a suitable sequence so that the series converges absolutely and uniformly in a certain region. Then

$$\begin{aligned} & \sum_{n_1, \dots, n_k=0}^{\infty} c(n_1, \dots, n_k) \prod_{j=1}^k \frac{(q^{\nu_j}, q^{\lambda_j}; q)_{n_j}}{(q^{\gamma_j}, q^{\eta_j}; q)_{n_j}} z_j^{n_j} \\ &= \int_{[0,1]^k} \Phi(z_1 t_1, \dots, z_k t_k) \prod_{j=1}^k d\mu_{\eta_j - \lambda_j, \gamma_j - \lambda_j, \gamma_j - \lambda_j + \eta_j - \nu_j, \nu_j}(t_j; q). \end{aligned} \quad (51)$$

Proof. The result follows by the moment formula

$$I_\ell := \int_0^1 t^\ell d\mu_{\eta - \lambda, \gamma - \lambda, \gamma - \lambda + \eta - \nu, \nu}(t; q) = \frac{(q^\nu, q^\lambda; q)_\ell}{(q^\gamma, q^\eta; q)_\ell}, \quad (52)$$

which can be proved by evaluating the q -integral directly. Here, for the sake of completeness and to facilitate readers' understanding, we briefly present the derivation of (52).

In view of (22) and (23), we have

$$I_\ell = C \int_0^1 t^{\nu + \ell - 1} \frac{(tq; q)_\infty}{(tq^{\gamma - \lambda + \eta - \nu}; q)_\infty} {}_3\phi_1 \left[\begin{matrix} q^{\eta - \lambda}, q^{\gamma - \lambda}, t^{-1} \\ q^{\gamma - \lambda + \eta - \nu} \end{matrix}; q, tq^{\lambda - \nu} \right] d_q t,$$

where

$$C := \frac{\Gamma_q(\gamma)\Gamma_q(\eta)}{\Gamma_q(\lambda)\Gamma_q(\nu)\Gamma_q(\gamma - \lambda + \eta - \nu)}.$$

Then, using (14), (17) and (34), we obtain

$$\begin{aligned} I_\ell &= C(1 - q) \sum_{k=0}^{\infty} q^{k(\nu + \ell)} \frac{(q^{1+k}; q)_\infty}{(q^{\gamma - \lambda + \eta - \nu + k}; q)_\infty} {}_3\phi_1 \left[\begin{matrix} q^{\eta - \lambda}, q^{\gamma - \lambda}, q^{-k} \\ q^{\gamma - \lambda + \eta - \nu} \end{matrix}; q, q^{\lambda - \nu + k} \right] \\ &= C(1 - q) \sum_{k=0}^{\infty} q^{k(\nu + \ell)} \frac{(q^{1+k}; q)_\infty}{(q^{\gamma - \lambda + \eta - \nu + k}; q)_\infty} \sum_{m=0}^k \frac{(q^{\eta - \lambda}, q^{\gamma - \lambda}, q^{-k}; q)_m}{(q^{\gamma - \lambda + \eta - \nu}, q; q)_m} \\ &\quad \cdot (-1)^m q^{km + (\lambda - \nu)m - \binom{m}{2}} \\ &= C(1 - q) \sum_{k,m=0}^{\infty} \frac{(q^{1+k+m}; q)_\infty}{(q^{\gamma - \lambda + \eta - \nu + k+m}; q)_\infty} \frac{(q^{\eta - \lambda}, q^{\gamma - \lambda}, q^{-k-m}; q)_m}{(q^{\gamma - \lambda + \eta - \nu}, q; q)_m} \\ &\quad \cdot (-1)^m q^{(k+m)(\nu + \ell) + (k+m)m + (\lambda - \nu)m - \binom{m}{2}}. \end{aligned}$$

By noting that $(q^{k+m+1}; q)_\infty (q^{-k-m}; q)_m = (q^{1+k}; q)_\infty (-1)^m q^{-km - \frac{1}{2}(m+1)m}$, the above expression can be further simplified to

$$\begin{aligned} I_\ell &= C(1 - q) \sum_{k,m=0}^{\infty} \frac{(q^{1+k}; q)_\infty}{(q^{\gamma - \lambda + \eta - \nu + k+m}; q)_\infty} \frac{(q^{\eta - \lambda}, q^{\gamma - \lambda}; q)_m}{(q^{\gamma - \lambda + \eta - \nu}, q; q)_m} q^{k\ell + m\ell + k\nu + \lambda m} \\ &= C(1 - q) \sum_{m=0}^{\infty} q^{m(\ell + \lambda)} \frac{(q^{\eta - \lambda}, q^{\gamma - \lambda}; q)_m}{(q^{\gamma - \lambda + \eta - \nu}, q; q)_m} \sum_{k=0}^{\infty} q^{k(\ell + \nu)} \frac{(q^{1+k}; q)_\infty}{(q^{\gamma - \lambda + \eta - \nu + k+m}; q)_\infty}, \end{aligned} \quad (53)$$

where the inner sum can be evaluated via the q -beta function as

$$\begin{aligned} \sum_{k=0}^{\infty} q^{k(\ell + \nu)} \frac{(q^{1+k}; q)_\infty}{(q^{\gamma - \lambda + \eta - \nu + k+m}; q)_\infty} &= \frac{1}{1 - q} \int_0^1 t^{\ell + \nu - 1} \frac{(tq; q)_\infty}{(tq^{m + \gamma + \eta - \lambda - \nu}; q)_\infty} d_q t \\ &= \frac{(q, q^{\gamma + \eta - \lambda + m + \ell}; q)_\infty}{(q^{\nu + \ell}, q^{\gamma + \eta - \lambda - \nu + m}; q)_\infty}. \end{aligned} \quad (54)$$

Substituting (54) into (53) gives

$$\begin{aligned} I_\ell &= C(1-q) \frac{(q, q^{\gamma+\eta-\lambda+\ell}; q)_\infty}{(q^{v+\ell}, q^{\gamma+\eta-\lambda-v}; q)_\infty} \sum_{m=0}^{\infty} \frac{(q^{\eta-\lambda}, q^{\gamma-\lambda}; q)_m}{(q^{\gamma+\eta-\lambda+\ell}, q; q)_m} q^{m(\ell+\lambda)} \\ &= C(1-q) \frac{(q, q^{\gamma+\eta-\lambda+\ell}; q)_\infty}{(q^{v+\ell}, q^{\gamma+\eta-\lambda-v}; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q^{\eta-\lambda}, q^{\gamma-\lambda} \\ q^{\gamma+\eta-\lambda+\ell} \end{matrix}; q, q^{\ell+\lambda} \right]. \end{aligned} \quad (55)$$

The ${}_2\phi_1$ in (55) can be easily evaluated by using the q -Gauss sum [32] (p. 354, Equation (II.8)) and the result is

$$I_\ell = C(1-q) \frac{(q^{\gamma+\ell}, q^{\eta+\ell}, q; q)_\infty}{(q^{v+\ell}, q^{\lambda+\ell}, q^{\gamma+\eta-\lambda-v}; q)_\infty} = \frac{(q^v, q^\lambda; q)_\ell}{(q^\gamma, q^\eta; q)_\ell}.$$

This completes the proof. $\square$

Remark 3. When $k = 1$, the Formula (51) reduces to Joshi and Vyas' result

$$\sum_{n=0}^{\infty} c(n) \frac{(q^v, q^\lambda; q)_n}{(q^\gamma, q^\eta; q)_n} z^n = \int_0^1 \Phi(z t) d\mu_{\eta-\lambda, \gamma-\lambda, \gamma-\lambda+\eta-v, v}(t; q), \quad (56)$$

where $\Phi(z) = \sum_{n=0}^{\infty} c(n) z^n$. We observe that if we take

$$c(n) = \begin{cases} 1, & n = \ell, \\ 0, & n \neq \ell \end{cases}$$

in (56), then $\Phi(z)$ reduces to z^ℓ and thus (51) reduces to the moment Formula (52). The Formula (51) can also be considered as Ernst's q -analogue for Exton's result (see [45], p. 189, Theorem 3.1).

Letting $k = 3$ and

$$c(m, n, p) = (q^{\alpha_2}; q)_{n+p} (q^{\beta_1}; q)_{p+m} \frac{(q^{\alpha_1}, q^{\eta_1}; q)_m (q^{\beta_2}, q^{\eta_2}; q)_n (q^{\eta_3}; q)_p}{(q^{v_1}, q^{\lambda_1}, q; q)_m (q^{v_2}, q^{\lambda_2}, q; q)_n (q^{\lambda_3}, q^{\eta_3}, q; q)_p}$$

in Theorem 4 gives the following corollary.

Corollary 1. When $\min\{\Re(\gamma_i + \eta_i - \lambda_i - v_i), \Re(\lambda_i), \Re(v_i)\} > 0$ ($i = 1, 2, 3$), we have

$$\begin{aligned} &\tilde{\Phi}_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1, \gamma_1, \gamma_2, \gamma_3; q, x, y, z] \\ &= \int_{[0,1]^3} \tilde{\phi}^{(3)} \left[\begin{matrix} - :: -; \alpha_2; \beta_1 : \alpha_1, \eta_1; \beta_2, \eta_2; \eta_3 \\ - :: -; -; - : v_1, \lambda_1; v_2, \lambda_2; v_3, \lambda_3 \end{matrix}; q, xt_1, yt_2, zt_3 \right] \\ &\quad \cdot \prod_{j=1}^3 d\mu_{\eta_j-\lambda_j, \gamma_j-\lambda_j, \gamma_j-\lambda_j+\eta_j-v_j, v_j}(t_j; q), \end{aligned} \quad (57)$$

where $\phi^{(3)}$ is defined by (26).

Remark 4. This integral (57) is clearly a q -analogue of the integral by Luo et al. [13] (Theorem 1). In particular, when $x = 0$, (57) reduces to

$$\begin{aligned} \tilde{\Phi}_2[\alpha_2, \beta_2, \beta_1; \gamma_2, \gamma_3; q, y, z] &= \int_{[0,1]^2} \tilde{\Phi}_{0:2:2}^{1:2:2} \left[\begin{matrix} \alpha_2 : \beta_2, \eta_2; \beta_1, \eta_3 \\ - : v_2, \lambda_2; v_3, \lambda_3 \end{matrix}; q, yt_2, zt_3 \right] \\ &\quad \cdot \prod_{j=2}^3 d\mu_{\eta_j-\lambda_j, \gamma_j-\lambda_j, \gamma_j-\lambda_j+\eta_j-v_j, v_j}(t_j; q), \end{aligned} \quad (58)$$

which is the q -analogue of Koschmieder's integral representation for F_2 ([13], Corollary 1). Here, Φ_2 and $\Phi_{0:2;2}^{1:2;2}$ denote the q -Appell function [32] (p. 283, Equation (10.2.6)) and the q -Kampé de Fériet function [41] (p. 349), respectively.

Letting $\lambda_1 = \alpha_1$, $\lambda_2 = \beta_2$ and $\lambda_3 = \eta_3$ in (57), we obtain the following corollary.

Corollary 2. When $\min\{\Re(\gamma_1 + \eta_1 - \alpha_1 - \nu_1), \Re(\alpha_1), \Re(\nu_1)\} > 0$, $\min\{\Re(\gamma_2 + \eta_2 - \beta_2 - \nu_2), \Re(\beta_2), \Re(\nu_2)\} > 0$ and $\Re(\gamma_3) > \Re(\nu_3) > 0$, we have

$$\begin{aligned} & \tilde{\Phi}_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1, \gamma_1, \gamma_2, \gamma_3; q, x, y, z] \\ &= \int_{[0,1]^3} \tilde{\Phi}_K[\eta_1, \alpha_2, \alpha_2, \beta_1, \eta_2, \beta_1; \nu_1, \nu_2, \nu_3; q, xt_1, yt_2, zt_3] \\ & \cdot d\mu_{\eta_1 - \alpha_1, \gamma_1 - \alpha_1, \gamma_1 - \alpha_1 + \eta_1 - \nu_1, \nu_1}(t_1; q) d\mu_{\eta_2 - \beta_2, \gamma_2 - \beta_2, \gamma_2 - \beta_2 + \eta_2 - \nu_2, \nu_2}(t_2; q) d\mu_{\nu_3, \gamma_3 - \nu_3}(t_3; q). \end{aligned} \quad (59)$$

Remark 5. This integral is the q -analogue of the integral by Luo and Raina [28] (p. 14, Theorem 4.1).

In the next subsection, we shall show that Corollary 2 has a nice discrete analogue.

4.2. A Discrete q -Analogue of Corollary 2

Based on Gasper's Formula (25), we obtain the following results.

Theorem 5.

$$\begin{aligned} & \phi^{(3)} \left[\begin{matrix} - :: -; q^{\alpha_2}; q^{\beta_1} : q^{\alpha_1}, q^{-r}; q^{\beta_2}, q^{-s}; q^{-t} \\ - :: -; -; - : q^{\gamma_1}, \delta_1; q^{\gamma_2}, \delta_2; q^{\gamma_3}, \delta_3 \end{matrix} ; q, q, q, q \right] \\ &= \sum_{i=0}^r \sum_{j=0}^s \sum_{k=0}^t w_1(i, r; q) w_2(j, s; q) w_3(k, t; q) \\ & \cdot \phi^{(3)} \left[\begin{matrix} - :: -; q^{\alpha_2}; q^{\beta_1} : q^{\lambda_1}, q^{-i}; q^{\lambda_2}, q^{-j}; q^{-k} \\ - :: -; -; - : q^{\mu_1}, \delta_1; q^{\mu_2}, \delta_2; q^{\mu_3}, \delta_3 \end{matrix} ; q, q, q, q \right], \end{aligned} \quad (60)$$

where

$$\begin{aligned} w_1(i, r; q) &:= \frac{(q^{\alpha_1}, q; q)_r}{(q^{\gamma_1}, q^{\lambda_1}; q)_r} \frac{(q^{\gamma_1 + \lambda_1 - \alpha_1 - \mu_1}; q)_{r-i}}{(q; q)_{r-i}} \frac{(q^{\mu_1}; q)_i}{(q; q)_i} \\ & \cdot {}_3\phi_2 \left[\begin{matrix} q^{\lambda_1 - \alpha_1}, q^{\gamma_1 - \alpha_1}, q^{i-r} \\ q^{\gamma_1 + \lambda_1 - \alpha_1 - \mu_1}, q^{1-r-\alpha_1} \end{matrix} ; q, q^{1-i-\mu_1} \right] q^{(r-i)\mu_1}, \\ w_2(j, s; q) &:= \frac{(q^{\beta_2}, q; q)_s}{(q^{\gamma_2}, q^{\lambda_2}; q)_s} \frac{(q^{\gamma_2 + \lambda_2 - \beta_2 - \mu_2}; q)_{s-j}}{(q; q)_{s-j}} \frac{(q^{\mu_2}; q)_j}{(q; q)_j} \\ & \cdot {}_3\phi_2 \left[\begin{matrix} q^{\lambda_2 - \beta_2}, q^{\gamma_2 - \beta_2}, q^{j-s} \\ q^{\gamma_2 + \lambda_2 - \beta_2 - \mu_2}, q^{1-s-\beta_2} \end{matrix} ; q, q^{1-j-\mu_2} \right] q^{(s-j)\mu_2} \end{aligned}$$

and

$$w_3(k, t; q) := \frac{(q; q)_t}{(q^{\gamma_3}; q)_t} \frac{(q^{\gamma_3 - \mu_3}; q)_{t-k}}{(q; q)_{t-k}} \frac{(q^{\mu_3}; q)_k}{(q; q)_k} q^{(t-k)\mu_3}.$$

Proof. For convenience, we denote the right-hand side of (60) by S . By (26), it is not difficult to see that

$$\begin{aligned} & \phi^{(3)} \left[\begin{matrix} - :: -; q^{\alpha_2}; q^{\beta_1} : q^{\lambda_1}, q^{-i}; q^{\lambda_2}, q^{-j}; q^{-k} \\ - :: -; -; - : q^{\mu_1}, \delta_1; q^{\mu_2}, \delta_2; q^{\mu_3}, \delta_3 \end{matrix} ; q, q, q, q \right] \\ &= \sum_{p=0}^k \frac{(q^{\alpha_2}, q^{\beta_1}, q^{-k}; q)_p}{(q^{\mu_3}, \delta_3, q; q)_p} q^p {}_3\phi_2 \left[\begin{matrix} q^{\beta_1+p}, q^{\lambda_1}, q^{-i} \\ q^{\mu_1}, \delta_1 \end{matrix} ; q, q \right] {}_3\phi_2 \left[\begin{matrix} q^{\alpha_2+p}, q^{\lambda_2}, q^{-j} \\ q^{\mu_2}, \delta_2 \end{matrix} ; q, q \right]. \end{aligned}$$

Then

$$\begin{aligned} S &= \sum_{i=0}^r \sum_{j=0}^s \sum_{k=0}^t w_1(i, r; q) w_2(j, s; q) w_3(k, t; q) \sum_{p=0}^k \frac{(q^{\alpha_2}, q^{\beta_1}, q^{-k}; q)_p}{(q^{\mu_3}, \delta_3, q; q)_p} q^p \\ &\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{\beta_1+p}, q^{\lambda_1}, q^{-i} \\ q^{\mu_1}, \delta_1 \end{matrix} ; q, q \right] {}_3\phi_2 \left[\begin{matrix} q^{\alpha_2+p}, q^{\lambda_2}, q^{-j} \\ q^{\mu_2}, \delta_2 \end{matrix} ; q, q \right] \\ &= \sum_{k=0}^t w_3(k, t; q) \sum_{p=0}^k \frac{(q^{\alpha_2}, q^{\beta_1}, q^{-k}; q)_p}{(q^{\mu_3}, \delta_3, q; q)_p} q^p \\ &\quad \sum_{i=0}^r w_1(i, r; q) {}_3\phi_2 \left[\begin{matrix} q^{\beta_1+p}, q^{\lambda_1}, q^{-i} \\ q^{\mu_1}, \delta_1 \end{matrix} ; q, q \right] \\ &\quad \sum_{j=0}^s w_2(j, s; q) {}_3\phi_2 \left[\begin{matrix} q^{\alpha_2+p}, q^{\lambda_2}, q^{-j} \\ q^{\mu_2}, \delta_2 \end{matrix} ; q, q \right]. \end{aligned} \quad (61)$$

Letting $\alpha \rightarrow q^{\alpha_1}$, $\beta \rightarrow q^{\beta_1+p}$, $\gamma \rightarrow q^{\gamma_1}$, $\delta \rightarrow \delta_1$, $\mu \rightarrow q^{\lambda_1}$, $\lambda \rightarrow q^{\alpha_1}$ and $\nu \rightarrow q^{\mu_1}$ in (25), we have

$$\begin{aligned} \sum_{i=0}^r w_1(i, r; q) {}_3\phi_2 \left[\begin{matrix} q^{\beta_1+p}, q^{\lambda_1}, q^{-i} \\ q^{\mu_1}, \delta_1 \end{matrix} ; q, q \right] &= {}_3\phi_2 \left[\begin{matrix} q^{\alpha_1}, q^{\beta_1+p}, q^{-r} \\ q^{\gamma_1}, \delta_1 \end{matrix} ; q, q \right] \\ &= \sum_{i=0}^r \frac{(q^{\alpha_1}, q^{\beta_1}, q^{-r}; q)_i}{(q^{\gamma_1}, \delta_1, q; q)_i} \frac{(q^{\beta_1+i}; q)_p}{(q^{\beta_1}; q)_p} q^i. \end{aligned} \quad (62)$$

Letting $\alpha \rightarrow q^{\alpha_2+p}$, $\beta \rightarrow q^{\beta_2}$, $\gamma \rightarrow q^{\gamma_2}$, $\delta \rightarrow \delta_2$, $\mu \rightarrow q^{\lambda_2}$, $\lambda \rightarrow q^{\beta_2}$ and $\nu \rightarrow q^{\mu_2}$ in (25), we obtain

$$\begin{aligned} \sum_{j=0}^s w_2(j, s; q) {}_3\phi_2 \left[\begin{matrix} q^{\alpha_2+p}, q^{\lambda_2}, q^{-j} \\ q^{\mu_2}, \delta_2 \end{matrix} ; q, q \right] &= {}_3\phi_2 \left[\begin{matrix} q^{\alpha_2+p}, q^{\beta_2}, q^{-s} \\ q^{\gamma_2}, \delta_2 \end{matrix} ; q, q \right] \\ &= \sum_{j=0}^s \frac{(q^{\alpha_2}, q^{\beta_2}, q^{-s}; q)_j}{(q^{\gamma_2}, \delta_2, q; q)_j} \frac{(q^{\alpha_2+j}; q)_p}{(q^{\alpha_2}; q)_p} q^j. \end{aligned} \quad (63)$$

Now, substituting (62) and (63) into (61) gives

$$\begin{aligned} S &= \sum_{i=0}^r \frac{(q^{\alpha_1}, q^{\beta_1}, q^{-r}; q)_i}{(q^{\gamma_1}, \delta_1, q; q)_i} q^i \sum_{j=0}^s \frac{(q^{\alpha_2}, q^{\beta_2}, q^{-s}; q)_j}{(q^{\gamma_2}, \delta_2, q; q)_j} q^j \\ &\quad \sum_{k=0}^t w_3(k, t; q) \sum_{p=0}^k \frac{(q^{\alpha_2+j}, q^{\beta_1+i}, q^{-k}; q)_p}{(q^{\mu_3}, \delta_3, q; q)_p} q^p. \end{aligned} \quad (64)$$

Letting $\alpha \rightarrow q^{\alpha_2+j}$, $\beta \rightarrow q^{\beta_1+i}$, $\gamma \rightarrow q^{\gamma_3}$, $\delta \rightarrow \delta_3$, $\mu = \lambda$ and $\nu \rightarrow q^{\mu_3}$ in (25) gives

$$\sum_{k=0}^t w_3(k, t; q) \sum_{p=0}^k \frac{(q^{\alpha_2+j}, q^{\beta_1+i}, q^{-k}; q)_p}{(q^{\mu_3}, \delta_3, q; q)_p} q^p = {}_3\phi_2 \left[\begin{matrix} q^{\alpha_2+j}, q^{\beta_1+i}, q^{-t} \\ q^{\gamma_3}, \delta_3 \end{matrix} ; q, q \right]. \quad (65)$$

Substituting (65) into (64), we obtain

$$\begin{aligned} S &= \sum_{i=0}^r \frac{(q^{\alpha_1}, q^{\beta_1}, q^{-r}; q)_i}{(q^{\gamma_1}, \delta_1, q; q)_i} q^i \sum_{j=0}^s \frac{(q^{\alpha_2}, q^{\beta_2}, q^{-s}; q)_j}{(q^{\gamma_2}, \delta_2, q; q)_j} q^j \sum_{k=0}^t \frac{(q^{\alpha_2+j}, q^{\beta_1+i}, q^{-t}; q)_k}{(q^{\gamma_3}, \delta_3, q; q)_k} q^k \\ &= \sum_{i=0}^r \sum_{j=0}^s \sum_{k=0}^t (q^{\alpha_2}; q)_{j+k} (q^{\beta_1}; q)_{k+i} \frac{(q^{\alpha_1}, q^{-r}; q)_i}{(q^{\gamma_1}, \delta_1, q; q)_i} \frac{(q^{\beta_2}, q^{-s}; q)_j}{(q^{\gamma_2}, \delta_2, q; q)_j} \frac{(q^{-t}; q)_k}{(q^{\gamma_3}, \delta_3, q; q)_k} q^{i+j+k}. \end{aligned}$$

Interpreting the above triple sum as the $\phi^{(3)}$ -function leads us to the Formula (60). $\square$

By letting $i \rightarrow r - i$, $j \rightarrow s - j$, $k \rightarrow t - k$, $\delta_1 \rightarrow q^{1-r}/x$, $\delta_2 \rightarrow q^{1-s}/y$, $\delta_3 \rightarrow q^{1-t}/z$ in (60), we have

$$\begin{aligned} &\phi^{(3)} \left[\begin{array}{c} - :: -; q^{\alpha_2}; q^{\beta_1} : q^{\alpha_1}, q^{-r}; q^{\beta_2}, q^{-s}; q^{-t} \\ - :: -; -; - : q^{\gamma_1}, q^{1-r}/x; q^{\gamma_2}, q^{1-s}/y; q^{\gamma_3}, q^{1-t}/z; q, q, q, q \end{array} \right] \\ &= \sum_{i=0}^r \sum_{j=0}^s \sum_{k=0}^t w_1(r-i, r; q) w_2(s-j, s; q) w_3(t-k, t; q) \\ &\quad \cdot \phi^{(3)} \left[\begin{array}{c} - :: -; q^{\alpha_2}; q^{\beta_1} : q^{\lambda_1}, q^{-i}; q^{\lambda_2}, q^{-j}; q^{-k} \\ - :: -; -; - : q^{\mu_1}, q^{1-r}/x; q^{\mu_2}, q^{1-s}/y; q^{\mu_3}, q^{1-t}/z; q, q, q, q \end{array} \right]. \end{aligned} \quad (66)$$

Note that

$$\begin{aligned} \lim_{r \rightarrow +\infty} w_1(r-i, r; q) &= \frac{(q^{\alpha_1}, q^{\mu_1}; q)_{\infty}}{(q^{\gamma_1}, q^{\lambda_1}; q)_{\infty}} \frac{(q^{\gamma_1+\lambda_1-\alpha_1-\mu_1}; q)_i}{(q; q)_i} q^{i\mu_1} \\ &\quad \cdot {}_3\phi_1 \left[\begin{array}{c} q^{\lambda_1-\alpha_1}, q^{\gamma_1-\alpha_1}, q^{-i} \\ q^{\gamma_1+\lambda_1-\alpha_1-\mu_1} \end{array} ; q, q^{\alpha_1-\mu_1+i} \right], \end{aligned} \quad (67)$$

$$\begin{aligned} \lim_{s \rightarrow +\infty} w_2(s-j, s; q) &= \frac{(q^{\beta_2}, q^{\mu_2}; q)_{\infty}}{(q^{\gamma_2}, q^{\lambda_2}; q)_{\infty}} \frac{(q^{\gamma_2+\lambda_2-\beta_2-\mu_2}; q)_j}{(q; q)_j} q^{j\mu_2} \\ &\quad \cdot {}_3\phi_1 \left[\begin{array}{c} q^{\lambda_2-\beta_2}, q^{\gamma_2-\beta_2}, q^{-j} \\ q^{\gamma_2+\lambda_2-\beta_2-\mu_2} \end{array} ; q, q^{\beta_2-\mu_2+j} \right] \end{aligned} \quad (68)$$

and

$$\lim_{t \rightarrow +\infty} w_3(t-k, t; q) := \frac{(q^{\mu_3}; q)_{\infty}}{(q^{\gamma_3}; q)_{\infty}} \frac{(q^{\gamma_3-\mu_3}; q)_k}{(q; q)_k} q^{k\mu_3}. \quad (69)$$

Letting $r, s, t \rightarrow +\infty$ in (66) and making use of (67)–(69), we obtain

$$\begin{aligned} &\tilde{\Phi}_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; q, x, y, z] \\ &= \frac{(q^{\alpha_1}, q^{\mu_1}, q^{\gamma_1-\lambda_1-\alpha_1-\mu_1}; q)_{\infty}}{(q^{\gamma_1}, q^{\lambda_1}, q; q)_{\infty}} \frac{(q^{\beta_2}, q^{\mu_2}, q^{\gamma_2+\lambda_2-\beta_2-\mu_2}; q)_{\infty}}{(q^{\gamma_2}, q^{\lambda_2}, q; q)_{\infty}} \frac{(q^{\mu_3}, q^{\gamma_3-\mu_3}; q)_{\infty}}{(q^{\gamma_3}, q; q)_{\infty}} \\ &\quad \cdot \sum_{i,j,k=0}^{\infty} \frac{(q^{1+i}; q)_{\infty}}{(q^{\gamma_1+\lambda_1-\alpha_1-\mu_1+i}; q)_{\infty}} \frac{(q^{1+j}; q)_{\infty}}{(q^{\gamma_2+\lambda_2-\beta_2-\mu_2+j}; q)_{\infty}} \frac{(q^{1+k}; q)_{\infty}}{(q^{\gamma_3-\mu_3+k}; q)_{\infty}} q^{i\mu_1+j\mu_2+k\mu_3} \\ &\quad \cdot {}_3\phi_1 \left[\begin{array}{c} q^{\lambda_1-\alpha_1}, q^{\gamma_1-\alpha_1}, q^{-i} \\ q^{\gamma_1+\lambda_1-\alpha_1-\mu_1} \end{array} ; q, q^{\alpha_1-\mu_1+i} \right] {}_3\phi_1 \left[\begin{array}{c} q^{\lambda_2-\beta_2}, q^{\gamma_2-\beta_2}, q^{-j} \\ q^{\gamma_2+\lambda_2-\beta_2-\mu_2} \end{array} ; q, q^{\beta_2-\mu_2+j} \right] \\ &\quad \cdot \tilde{\Phi}_K[\lambda_1, \alpha_2, \alpha_2, \beta_1, \lambda_2, \beta_1; \mu_1, \mu_2, \mu_3; q, xq^i, yq^j, zq^k]. \end{aligned}$$

Using definition (14) of the multiple q -integral, the expression can be further reduced to (59).

4.3. A Q -Analogue of Theorem 1

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a function of one variable. We define the q -shift operator $\mathbb{E}_q$ by $\mathbb{E}_q f(x) = f(qx)$ ($0 < q < 1$) [46] (Definition 1.3). Inductively, for any $n \in \mathbb{Z}_{\geq 1}$,

$$\mathbb{E}_q^n f(x) = f(q^n x)$$

and in particular, $\mathbb{E}_q^0 f(x) = f(x)$. In addition, $\lim_{q \rightarrow 1^-} \mathbb{E}_q^n f(x) = f(x)$. For k -dimensional case, we use the notation $\mathbb{E}_{q, x_i}$ to denote the operator acting on the i -th variable of f , namely,

$$\mathbb{E}_{q, x_i} f(x_1, \dots, x_i, \dots, x_k) = f(x_1, \dots, qx_i, \dots, x_k), \quad i = 1, \dots, k.$$

We shall frequently use the formula

$$\mathbb{E}_{q, x}^n \mathbb{E}_{q, y}^m f(x, y) = f(q^n x, q^m y), \quad (n, m) \in \mathbb{Z}_{\geq 0}^2. \quad (70)$$

If $\mathcal{E}(x) = \sum_{n=0}^{\infty} a_n x^n$ ($|x| < R$) is an analytic functions, then

$$\mathcal{E}(t \mathbb{E}_{q, x}) f(x) = \sum_{n=0}^{\infty} a_n t^n f(q^n x), \quad |t| < R,$$

where $f(x)$ is also assumed to be bounded (or, in many situations, analytic) in a neighborhood of 0. We have

$$\lim_{q \rightarrow 1^-} \mathcal{E}(t \mathbb{E}_{q, x}) f(x) = \mathcal{E}(t) f(x).$$

The limit helps overcome the difficulty in deriving the q -analogue of Theorem 1.

Theorem 6. Let $\Re(\alpha_1 + \eta_1) > \Re(\lambda_1) > 0$, $\Re(\beta_2 + \mu_2) > \Re(\lambda_2) > 0$ and $\Re(\gamma_3) > \Re(\beta_1) > 0$. Then we have

$$\begin{aligned} & \tilde{\Phi}_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1 + \eta_1, \beta_2 + \mu_2, \gamma_3; q, x, y, z] = \int_{[0,1]^3} \frac{(uxq^{\lambda_3}, vyq^{\eta_2}; q)_{\infty}}{(ux, vy; q)_{\infty}} \\ & \cdot \phi^{(3)} \left[\begin{array}{c} - :: -; q^{\eta_2}; q^{\lambda_3} : q^{\lambda_1 - \eta_1}, u^{-1}; q^{\lambda_2 - \mu_2}, v^{-1}; - \\ - :: -; -; - : q^{\lambda_1}, q/(ux); q^{\lambda_2}, q/(vy); uxq^{\lambda_3}, vyq^{\eta_2}, q^{\lambda_3}; \\ q, q, q, wzq^{\alpha_2 - \eta_2} \mathbb{E}_{q, x} \mathbb{E}_{q, y} \end{array} \right] \\ & \cdot \tilde{\Phi}_K[\alpha_1, \alpha_2 - \eta_2, \alpha_2 - \eta_2, \beta_1 - \lambda_3, \beta_2, \beta_1 - \lambda_3; \alpha_1 - \lambda_1 + \eta_1, \beta_2 - \lambda_2 + \mu_2, \beta_1 - \lambda_3; \\ & q, uxq^{\lambda_3}, vyq^{\eta_2}, wz] d\mu_{\alpha_1 - \lambda_1 + \eta_1, \lambda_1}(u; q) d\mu_{\beta_2 - \lambda_2 + \mu_2, \lambda_2}(v; q) d\mu_{\beta_1, \gamma_3 - \beta_1}(w; q). \end{aligned} \quad (71)$$

Proof. We use the method from Section 3.1. Let I denote the triple q -integral in (71). By the definition (26) of the $\phi^{(3)}$ -function, we have

$$\begin{aligned} & \phi^{(3)} \left[\begin{array}{c} - :: -; q^{\eta_2}; q^{\lambda_3} : q^{\lambda_1 - \eta_1}, u^{-1}; q^{\lambda_2 - \mu_2}, v^{-1}; - \\ - :: -; -; - : q^{\lambda_1}, q/(ux); q^{\lambda_2}, q/(vy); uxq^{\lambda_3}, vyq^{\eta_2}, q^{\lambda_3}; \\ q, q, q, wzq^{\alpha_2 - \eta_2} \mathbb{E}_{q, x} \mathbb{E}_{q, y} \end{array} \right] \\ & = \sum_{k=0}^{\infty} \frac{(q^{\eta_2}; q)_k}{(uxq^{\lambda_3}, vyq^{\eta_2}; q)_k} {}_3\phi_2 \left[\begin{array}{c} q^{\lambda_3 + k}, q^{\lambda_1 - \eta_1}, u^{-1} \\ q^{\lambda_1}, q/(ux) \end{array} ; q, q \right] \\ & \cdot {}_3\phi_2 \left[\begin{array}{c} q^{\eta_2 + k}, q^{\lambda_2 - \mu_2}, v^{-1} \\ q^{\lambda_2}, q/(vy) \end{array} ; q, q \right] \frac{(wzq^{\alpha_2 - \eta_2} \mathbb{E}_{q, x} \mathbb{E}_{q, y})^k}{(q; q)_k}. \end{aligned}$$

Then, in view of (70) and (16), we have

$$\begin{aligned}
I &= \int_{[0,1]^3} \frac{(uxq^{\lambda_3}, vyq^{\eta_2}; q)_{\infty}}{(ux, vy; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^{\eta_2}; q)_k}{(uxq^{\lambda_3}, vyq^{\eta_2}; q)_k} \\
&\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{\lambda_3+k}, q^{\lambda_1-\eta_1}, u^{-1} \\ q^{\lambda_1}, q/(ux) \end{matrix}; q, q \right] {}_3\phi_2 \left[\begin{matrix} q^{\eta_2+k}, q^{\lambda_2-\mu_2}, v^{-1} \\ q^{\lambda_2}, q/(vy) \end{matrix}; q, q \right] \frac{(wzq^{\alpha_2-\eta_2})^k}{(q; q)_k} \\
&\quad \cdot \tilde{\Phi}_K [\alpha_1, \alpha_2 - \eta_2, \alpha_2 - \eta_2, \beta_1 - \lambda_3, \beta_2, \beta_1 - \lambda_3; \alpha_1 - \lambda_1 + \eta_1, \beta_2 - \lambda_2 + \mu_2, \beta_1 - \lambda_3; \\
&\quad \quad uxq^{k+\lambda_3}, vyq^{k+\eta_2}, wz] d\mu_{\alpha_1-\lambda_1+\eta_1, \lambda_1}(u; q) d\mu_{\beta_2-\lambda_2+\mu_2, \lambda_2}(v; q) d\mu_{\beta_1, \gamma_3-\beta_1}(w; q) \\
&= \int_{[0,1]^3} \sum_{k=0}^{\infty} \frac{(q^{\eta_2}; q)_k}{(q; q)_k} \frac{(uxq^{k+\lambda_3}, vyq^{k+\eta_2}; q)_{\infty}}{(ux, vy; q)_{\infty}} (wzq^{\alpha_2-\eta_2})^k \\
&\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{\lambda_3+k}, q^{\lambda_1-\eta_1}, u^{-1} \\ q^{\lambda_1}, q/(ux) \end{matrix}; q, q \right] {}_3\phi_2 \left[\begin{matrix} q^{\eta_2+k}, q^{\lambda_2-\mu_2}, v^{-1} \\ q^{\lambda_2}, q/(vy) \end{matrix}; q, q \right] \\
&\quad \cdot \sum_{\ell=0}^{\infty} \frac{(q^{\alpha_2-\eta_2}; q)_{\ell}}{(q; q)_{\ell}} {}_2\tilde{\Phi}_1 \left[\begin{matrix} \beta_1 - \lambda_3 + \ell, \alpha_1 \\ \alpha_1 - \lambda_1 + \eta_1 \end{matrix}; q, uxq^{k+\lambda_3} \right] {}_2\tilde{\Phi}_1 \left[\begin{matrix} \alpha_2 - \eta_2 + \ell, \beta_2 \\ \beta_2 - \lambda_2 + \mu_2 \end{matrix}; q, vyq^{k+\eta_2} \right] \\
&\quad \cdot (wz)^{\ell} d\mu_{\alpha_1-\lambda_1+\eta_1, \lambda_1}(u; q) d\mu_{\beta_2-\lambda_2+\mu_2, \lambda_2}(v; q) d\mu_{\beta_1, \gamma_3-\beta_1}(w; q) \\
&= \sum_{k=0}^{\infty} \frac{(q^{\eta_2}; q)_k}{(q; q)_k} (zq^{\alpha_2-\eta_2})^k \sum_{\ell=0}^{\infty} \frac{(q^{\alpha_2-\eta_2}; q)_{\ell}}{(q; q)_{\ell}} z^{\ell} \int_0^1 w^{k+\ell} d\mu_{\beta_1, \gamma_3-\beta_1}(w; q) \\
&\quad \cdot \int_0^1 \frac{(uxq^{\lambda_3+k}; q)_{\infty}}{(ux; q)_{\infty}} {}_2\tilde{\Phi}_1 \left[\begin{matrix} \beta_1 - \lambda_3 + \ell, \alpha_1 \\ \alpha_1 - \lambda_1 + \eta_1 \end{matrix}; q, uxq^{\lambda_3+k} \right] \\
&\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{\lambda_3+k}, q^{\lambda_1-\eta_1}, u^{-1} \\ q^{\lambda_1}, q/(ux) \end{matrix}; q, q \right] d\mu_{\alpha_1-\lambda_1+\eta_1, \lambda_1}(u; q) \\
&\quad \cdot \int_0^1 \frac{(vyq^{\eta_2+k}; q)_{\infty}}{(vy; q)_{\infty}} {}_2\tilde{\Phi}_1 \left[\begin{matrix} \alpha_2 - \eta_2 + \ell, \beta_2 \\ \beta_2 - \lambda_2 + \mu_2 \end{matrix}; q, vyq^{\eta_2+k} \right] \\
&\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{\eta_2+k}, q^{\lambda_2-\mu_2}, v^{-1} \\ q^{\lambda_2}, q/(vy) \end{matrix}; q, q \right] d\mu_{\beta_2-\lambda_2+\mu_2, \lambda_2}(v; q). \tag{72}
\end{aligned}$$

By letting $\alpha \rightarrow \beta_1 + k + \ell$, $\beta \rightarrow \alpha_1$, $\gamma \rightarrow \alpha_1 + \eta_1$, $\lambda \rightarrow \alpha_1 - \lambda_1 + \eta_1$ and $\alpha' \rightarrow \lambda_3 + k$ in (21), we have

$$\begin{aligned}
&\int_0^1 \frac{(uxq^{\lambda_3+k}; q)_{\infty}}{(ux; q)_{\infty}} {}_2\tilde{\Phi}_1 \left[\begin{matrix} \beta_1 - \lambda_3 + \ell, \alpha_1 \\ \alpha_1 - \lambda_1 + \eta_1 \end{matrix}; q, uxq^{\lambda_3+k} \right] \\
&\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{\lambda_3+k}, q^{\lambda_1-\eta_1}, u^{-1} \\ q^{\lambda_1}, q/(ux) \end{matrix}; q, q \right] d\mu_{\alpha_1-\lambda_1+\eta_1, \lambda_1}(u; q) = {}_2\tilde{\Phi}_1 \left[\begin{matrix} \beta_1 + k + \ell, \alpha_1 \\ \alpha_1 + \eta_1 \end{matrix}; q, x \right], \tag{73}
\end{aligned}$$

where $\Re(\alpha_1 + \eta_1) > \Re(\lambda_1) > 0$. Similarly, let $\alpha \rightarrow \alpha_2 + k + \ell$, $\beta \rightarrow \beta_2$, $\gamma \rightarrow \beta_2 + \mu_2$, $\lambda = \beta_2 - \lambda_2 + \mu_2$ and $\alpha' \rightarrow \eta_2 + k$ in (21), then

$$\begin{aligned}
&\int_0^1 \frac{(vyq^{\eta_2+k}; q)_{\infty}}{(vy; q)_{\infty}} {}_2\tilde{\Phi}_1 \left[\begin{matrix} \alpha_2 - \eta_2 + \ell, \beta_2 \\ \beta_2 - \lambda_2 + \mu_2 \end{matrix}; q, vyq^{\eta_2+k} \right] \\
&\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{\eta_2+k}, q^{\lambda_2-\mu_2}, v^{-1} \\ q^{\lambda_2}, q/(vy) \end{matrix}; q, q \right] d\mu_{\beta_2-\lambda_2+\mu_2, \lambda_2}(v; q) = {}_2\tilde{\Phi}_1 \left[\begin{matrix} \alpha_2 + k + \ell, \beta_2 \\ \beta_2 + \mu_2 \end{matrix}; q, y \right], \tag{74}
\end{aligned}$$

where $\Re(\beta_2 + \mu_2) > \Re(\lambda_2) > 0$. Substituting (73) and (74) in (72) gives

$$\begin{aligned}
 I &= \sum_{k=0}^{\infty} \frac{(q^{\eta_2}; q)_k}{(q; q)_k} (q^{\alpha_2 - \eta_2})^k \sum_{\ell=0}^{\infty} \frac{(q^{\alpha_2 - \eta_2}; q)_{\ell}}{(q; q)_{\ell}} \frac{(q^{\beta_1}; q)_{k+\ell}}{(q^{\gamma_3}; q)_{k+\ell}} z^{k+\ell} \\
 &\quad \cdot {}_2\tilde{\phi}_1 \left[\begin{matrix} \beta_1 + k + \ell, \alpha_1 \\ \alpha_1 + \eta_1 \end{matrix}; q, x \right] {}_2\tilde{\phi}_1 \left[\begin{matrix} \alpha_2 + k + \ell, \beta_2 \\ \beta_2 + \mu_2 \end{matrix}; q, y \right] \\
 &= \sum_{k=0}^{\infty} \frac{(q^{\beta_1}; q)_k}{(q^{\gamma_3}; q)_k} z^k {}_2\tilde{\phi}_1 \left[\begin{matrix} \beta_1 + k, \alpha_1 \\ \alpha_1 + \eta_1 \end{matrix}; q, x \right] {}_2\tilde{\phi}_1 \left[\begin{matrix} \alpha_2 + k, \beta_2 \\ \beta_2 + \mu_2 \end{matrix}; q, y \right] \\
 &\quad \cdot \sum_{\ell=0}^k \frac{(q^{\eta_2}; q)_{k-\ell}}{(q; q)_{k-\ell}} \frac{(q^{\alpha_2 - \eta_2}; q)_{\ell}}{(q; q)_{\ell}} (q^{\alpha_2 - \eta_2})^{k-\ell} \\
 &= \sum_{k=0}^{\infty} \frac{(q^{\alpha_2}, q^{\beta_1}; q)_k}{(q^{\gamma_3}, q; q)_k} z^k {}_2\tilde{\phi}_1 \left[\begin{matrix} \beta_1 + k, \alpha_1 \\ \alpha_1 + \eta_1 \end{matrix}; q, x \right] {}_2\tilde{\phi}_1 \left[\begin{matrix} \alpha_2 + k, \beta_2 \\ \beta_2 + \mu_2 \end{matrix}; q, y \right]. \quad (75)
 \end{aligned}$$

By interpreting the last series in (75) as the q -Saran function $\tilde{\Phi}_K$, we obtain the desired result (71).

Note that in the above calculations, the interchange of order of integration and summation, as well as the order of summation, is also justified by Fubini's theorem. $\square$

Remark 6. The Formula (71) obtained here involves the q -shift operators $\mathbb{E}_{q,x}$ and $\mathbb{E}_{q,y}$, which makes it appears slightly different from the familiar integrals associated with ${}_2\phi_1$. It would, therefore, be interesting to explore whether alternative q -Erdélyi-type integrals for Φ_K can be derived through a different approach. We conjecture that the use of q -fractional integration by parts [47] (p. 159, Theorem 5.14) may offer a viable path for addressing this problem.

Finally, we show that in some cases, the integrand in (6) can be simplified so that the operators $\mathbb{E}_{q,x}$ and $\mathbb{E}_{q,y}$ do not appear any more.

Corollary 3. When $\min\{\Re(\alpha_1), \Re(\eta_1), \Re(\beta_2), \Re(\mu_2)\} > 0$ and $\Re(\gamma_3) > \Re(\beta_1) > 0$, we have

$$\begin{aligned}
 \tilde{\Phi}_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1 + \eta_1, \beta_2 + \mu_2, \gamma_3; q, x, y, z] &= \int_{[0,1]^3} \frac{(uxq^{\beta_1}, vyq^{\alpha_2}; q)_{\infty}}{(ux, vy; q)_{\infty}} \\
 &\quad \cdot {}_3\phi_2 \left[\begin{matrix} q^{\alpha_2}, 0, 0 \\ uxq^{\beta_1}, vyq^{\alpha_2} \end{matrix}; q, wz \right] d\mu_{\alpha_1, \eta_1}(u; q) d\mu_{\beta_2, \mu_2}(v; q) d\mu_{\beta_1, \gamma_3 - \beta_1}(w; q). \quad (76)
 \end{aligned}$$

Proof. By letting $\lambda_1 \rightarrow \eta_1$ and $\lambda_2 \rightarrow \mu_2$ in (71), we obtain

$$\begin{aligned}
 \tilde{\Phi}_K[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \alpha_1 + \eta_1, \beta_2 + \mu_2, \gamma_3; q, x, y, z] &= \int_{[0,1]^3} \frac{(uxq^{\lambda_3}, vyq^{\eta_2}; q)_{\infty}}{(ux, vy; q)_{\infty}} \\
 &\quad \cdot \mathcal{K}(q; x, y, z; u, v, w) d\mu_{\alpha_1, \eta_1}(u; q) d\mu_{\beta_2, \mu_2}(v; q) d\mu_{\beta_1, \gamma_3 - \beta_1}(w; q), \quad (77)
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{K}(q; x, y, z; u, v, w) &:= {}_3\phi_2 \left[\begin{matrix} q^{\eta_2}, 0, 0 \\ uxq^{\lambda_3}, vyq^{\eta_2} \end{matrix}; q, wzq^{\alpha_2 - \eta_2} \mathbb{E}_{q,x} \mathbb{E}_{q,y} \right] \\
 &\quad \frac{(uxq^{\beta_1}, vyq^{\alpha_2}; q)_{\infty}}{(uxq^{\lambda_3}, vyq^{\eta_2}; q)_{\infty}} {}_3\phi_2 \left[\begin{matrix} q^{\alpha_2 - \eta_2}, 0, 0 \\ uxq^{\beta_1}, vyq^{\alpha_2} \end{matrix}; q, wz \right].
 \end{aligned}$$

To evaluate $\mathcal{K}(q; x, y, z; u, v, w)$, we first note that

$$\begin{aligned}\mathcal{K}(q; x, y, z; u, v, w) &= \sum_{k=0}^{\infty} \frac{(q^{\eta_2}; q)_k (wzq^{\alpha_2 - \eta_2})^k (uxq^{\beta_1 + k}, vyq^{\alpha_2 + k}; q)_{\infty}}{(uxq^{\lambda_3}, vyq^{\eta_2}, q; q)_k (uxq^{\lambda_3 + k}, vyq^{\eta_2 + k}; q)_{\infty}} \\ &\quad \cdot \sum_{p=0}^{\infty} \frac{(q^{\alpha_2 - \eta_2}; q)_p (wz)^p}{(uxq^{\beta_1 + k}, vyq^{\alpha_2 + k}, q; q)_p} \\ &= \frac{(uxq^{\beta_1}, vyq^{\alpha_2}; q)_{\infty}}{(uxq^{\lambda_3}, vyq^{\eta_2}; q)_{\infty}} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(q^{\eta_2}; q)_k (q^{\alpha_2 - \eta_2}; q)_p q^{k(\alpha_2 - \eta_2)} (wz)^{k+p}}{(uxq^{\beta_1}, vyq^{\alpha_2}; q)_{k+p} (q; q)_k (q; q)_p} \\ &= \frac{(uxq^{\beta_1}, vyq^{\alpha_2}; q)_{\infty}}{(uxq^{\lambda_3}, vyq^{\eta_2}; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(wz)^k}{(uxq^{\beta_1}, vyq^{\alpha_2}, q; q)_k} \\ &\quad \cdot \sum_{p=0}^k \begin{bmatrix} k \\ p \end{bmatrix}_q (q^{\eta_2}; q)_{k-p} (q^{\alpha_2 - \eta_2}; q)_p q^{(k-p)(\alpha_2 - \eta_2)},\end{aligned}$$

where $\begin{bmatrix} k \\ p \end{bmatrix}_q$ denotes the q -binomial coefficient [32] (p. 24). Then the use of the q -binomial theorem [32] (p. 25) gives

$$\begin{aligned}\mathcal{K}(q; x, y, z; u, v, w) &= \frac{(uxq^{\beta_1}, vyq^{\alpha_2}; q)_{\infty}}{(uxq^{\lambda_3}, vyq^{\eta_2}; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^{\alpha_2}; q)_k (wz)^k}{(uxq^{\beta_1}, vyq^{\alpha_2}, q; q)_k} \\ &= \frac{(uxq^{\beta_1}, vyq^{\alpha_2}; q)_{\infty}}{(uxq^{\lambda_3}, vyq^{\eta_2}; q)_{\infty}} {}_3\phi_2 \left[\begin{matrix} q^{\alpha_2}, 0, 0 \\ uxq^{\beta_1}, vyq^{\alpha_2}; q, wz \end{matrix} \right].\end{aligned}\quad (78)$$

Substituting (78) into (77) we obtain (76). $\square$

5. Conclusions

In this paper, we continue the recent work of Luo, Xu, and Raina on Erdélyi-type integrals for Saran's function F_K , providing a new proof of one of their main results (Theorem 1). The new proof is based on the direct series manipulation technique and the well-known integral (7) (without resorting to fractional integration by parts and the Srivastava–Daoust function), which actually offers a new approach to deriving formulas. Building on this, we obtain an important generalization, which is applicable to the L -variable F_K function (46) that arises in physics. Furthermore, some interesting q -analogues and discrete analogues are also discussed.

By reviewing the literature on Erdélyi-type integrals of various hypergeometric functions over the past century, this paper compiles the table in Appendix A, which reflects the overall landscape of research in this direction. As already clarified in Section 1, whether out of computational interest or due to their usefulness, it is highly valuable to continue exploring Erdélyi-type integrals for other hypergeometric functions not yet included in this table.

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Appendix A

Type of Function		Reference
1 variable	${}_2F_1$	Erdélyi (1939) [11,12] Joshi-Vyas (2003) [40]
	${}_pF_q$	Joshi-Vyas (2003) [40] Luo-Raina (2017) [22]
	q -analogues	Gasper (2000) [35] Joshi-Vyas (2006) [36] Vyas (2024) [48]
	discrete analogues	Vyas-Bhatnagar-Fatawat-Suthar-Purohit (2022) [49] Bhatnagar-Vyas (2022) [50]
	others	Laine (1982) [51] Virchenko-Rumiantseva (2008) [52] Virchenko-Ovcharenko (2011) [53]
n variables ($n \geq 2$)	Appell	F_1 Manocha (1967) [42] Mittal (1977) [54]
		F_2 Koschmieder (1947) [55] Manocha (1967) [42] Mittal (1977) [54]
		F_3 Mittal (1977) [54]
		F_4 Manocha (1965) [56] Mittal (1977) [54]
		F_A Manocha-Sharma (1969) [57] Chandel (1971) [58]
		F_B Chandel (1971) [58]
		F_C Chandel (1971) [58]
		F_D Koschmieder (1962) [59] Manocha-Sharma (1969) [57] Chandel (1971) [58] Khudozhnikov (2003) [60]
	Saran	F_M Manocha-Sharma (1969) [57]
		F_K Luo-Raina (2021) [28] Luo-Xu-Raina (2022) [13]
	q -analogues	<i>the present paper</i>
	others	Volkodavov-Nikolaev (1993) [61]

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