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Solution of Time Fractional SIQR Epidemic System and Research with Respect to the Fractional Order

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Abstract

This article deals with the global existence and uniqueness of solutions to a fractional-order SIQR epidemic model, alongside its intricate chaotic and complex dynamics as functions of the fractional order. The well-posedness of the model solutions, including global existence, uniqueness, and positivity, is established by constructing appropriate Lyapunov functions. The local and global stability analyses are conducted for both the disease-free and endemic equilibria of the model. An asymptotic solution of the system in the form of series is derived by the Laplace–Adomian decomposition method (L–ADM), and its convergence is rigorously proved. Subsequently, numerical analysis determines and interprets the optimal truncation order of this asymptotic solution. Numerical simulations are performed based on the asymptotic solution, and the dynamics and chaos of the dynamic system with respect to the fractional order are analyzed and illustrated in terms of the maximum Lyapunov exponent and structural complexity. Finally, a local sensitivity analysis is conducted for each state variable with respect to the model parameters.

Keywords: well-posedness; fractional order; dynamic stability; asymptotic solution; convergence analysis; numerical simulation; chaos and complexity analysis; sensitivity index

1. Introduction

Fractional compartment models have been used to simulate the diffusion of drugs in the human organ system [1,2]; moreover, fractional derivative models have been used to study the spread of in factious diseases such as dengue fever, H1N1 and HIV/AIDS [3–5], and they have been used in the exploration of spatial dynamics in non-local epidemic models [6], etc. With the outbreak of COVID-19, there has been a surge in research on the model construction, numerical simulation and prediction of the epidemic with a large number of fractional-order dynamical systems (see [7–10], for instance, and [11–14] and the references therein).

However, these fractional-order epidemic models are usually obtained by directly replacing integer-order derivatives with fractional-order derivatives; while this may be mathematically interesting, it may lack physical meaning. If fractional derivatives are not incorporated into the model, it may be difficult to explain the units of the model parameters and may violate the law of mass conservation. Therefore, in recent years, there have been various attempts to address these difficulties, and we only refer to the most successful theoretical work given by Angstmann et al. [15–17]. Such fractional-order epidemic systems based on the stochastic processes are in line with physical laws. However, the mathematical derivation is a complicated process involving in Laplace transform and

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its inverse transform, and the governing equations are always expressed in complex forms, and so the stability analysis and computation of the system become difficult in general cases. In the literature [18], Wang and Li streamlined the modeling process by directly selecting a power-law survival function to derive a fractional-order SAIR epidemic model, which features a more concise expression and simpler computational solution.

Considering the significant role of quarantine measures in the transmission of the COVID-19 pandemic, influenza A (H1N1), dengue fever and Nipah virus, this study establishes a fractional-order SIQR epidemic model

$$\begin{cases} \tau^{\alpha-1}{}_0^C D_t^\alpha S(t) = \delta S(t)\left(1-\frac{S(t)}{N}\right) - \frac{\beta S(t) I(t)}{1+\kappa N}, \\ \tau^{\alpha-1}{}_0^C D_t^\alpha I(t) = \frac{\beta S(t) I(t)}{1+\kappa N} - (\mu+\gamma_1+d) I(t), \\ \tau^{\alpha-1}{}_0^C D_t^\alpha Q(t) = \mu I(t) - (\gamma_2+d) Q(t), \\ \tau^{\alpha-1}{}_0^C D_t^\alpha R(t) = \gamma_1 I(t) + \gamma_2 Q(t) - d R(t), \end{cases} \tag{1}$$

Here, $S = S(t)$, $I = I(t)$, $Q = Q(t)$, and $R = R(t)$ represent the number of susceptible individuals, confirmed infectors, quarantine carriers, and recovered individuals, respectively; τ is the time-scale factor, and ${}_0^C D_t^\alpha$ denotes the Caputo fractional operator on $t > 0$ with the order $\alpha \in (0, 1]$, which is defined via [19].

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(s)(t-s)^{-\alpha}, t > 0. \tag{2}$$

In addition, δ denotes the growth rate, and N denotes the environmental carrying capacity, β is the contact rate between susceptible individuals and asymptomatic patients, μ represents the isolation rate of confirmed patients, γ_1 denotes the recovery rate among confirmed patients, γ_2 signifies the recovery rate among quarantine patients, κ is the coefficient for nonlinear contact rate, and d stands for the mortality rate. All the above parameters are assumed to take non-negative values. The initial value condition for the model (1) is given as:

$$S(0) = S_0, I(0) = I_0, Q(0) = Q_0, R(0) = R_0, \tag{3}$$

where S_0, I_0, Q_0 and R_0 are non-negative values and bounded. Thus, we get a forward problem composed of Equation (1) with the initial condition (3), which is a nonlinear fractional-order dynamical system, named as FSIQR epidemic model.

For the well-posedness of the FSIQR, we not only prove the existence, uniqueness, and positivity of the solutions, but also extend the existence and uniqueness to a global scale. Regarding the study of the existence and uniqueness of solutions for fractional-order nonlinear equations, the monograph [20] by Jin and Diethelm's monograph [21] provide a systematic exposition. The authors of [22] investigate the existence and uniqueness of analytic solutions for Caputo nonlinear fractional differential equations with multi-order. Study [23] studies the conditions for the existence and uniqueness of solutions to time-delay fractional-order nonlinear differential equations. Concerning research on the positivity-preserving property of solutions for fractional-order nonlinear equations, the authors of [23] present a comparison theorem; the authors of [24] present a maximum principle for the generalized time fractional diffusion equation. Furthermore, the authors of [25] also analyze the existence, uniqueness, and positivity-preserving property of solutions for a Caputo fractional-order SIS model.

Based on the basic reproduction number, the local and global stability of the disease-free equilibrium point and the endemic equilibrium point of the model have been discussed in a large number studies in the literature (see [26–29]). However, its dynamic behavior

seems to be independent of the fractional order. In order to reveal the relationship between the fractional-order dynamic system and the order of the fractional derivative, we need to obtain explicit solutions for nonlinear fractional-order dynamic systems. There are various methods to obtain semi-analytical or asymptotic solutions for nonlinear systems, such as the Homotopy Perturbation Method (HPM [30]), the Variational Iteration Method (VIM [31]), the Homotopy Analysis Method (HAM [32]), and the Adomian decomposition method (ADM [33,34]).

In fact, the Adomian decomposition method (ADM) and the Laplace–ADM method have been utilized widely in solving of nonlinear differential equations over the past four decades, as seen in works such as [35–38]. However, the convergence and computability of the asymptotic solution, and the existence of an exact solution to the forward problem have not been insufficiently studied. In the early work given by Adomian and Cherruault (see [33,34,39,40]), some convergence results in abstract forms were given under strong assumptions on the nonlinear operator and the series in the solution. Recently, Kumar and their collaborators gave reviews on the ADM method and related semi-analytical methods for solving of nonlinear differential models [41,42], but the assumptions on the convergence of the series and the existence of a solution remain. In this work, we will not only prove the convergence of the solution for fractional-order system (1), but also derive computable formulas for asymptotic solutions with optimal truncation through numerical analysis. We will employ the Laplace–Adomian decomposition method (L–ADM) to obtain asymptotic solutions for the FSIQR system and perform numerical simulations to investigate dynamic behaviors of the fractional-order system with respect to the fractional order.

Based on the series formula of asymptotic solution, we will further investigate the chaos and complexity of the FSIQR epidemic system, as conducted in [18,43–45]. However, our primary objective is to explore the relationship between dynamical behaviors and the order of fractional-order systems, which distinguishes this work from previous work. The maximum Lyapunov exponent serves as a crucial indicator for measuring the chaos intensity in dynamical systems. Stronger chaos corresponds to higher system complexity, so we additionally measure the C_0 complexity for each compartment [46], with numerical simulations supporting theoretical analysis and computation. Finally, we perform local sensitivity analysis [47] of state variables concerning parameters, further examining the role of parameters—including fractional orders—in the system dynamics.

The rest of this study is organized as follows.

In Section 2, the well-posedness of the model solution is analyzed. The existence, uniqueness, and positivity of the solution are discussed, and its existence time is extended to infinity. In Section 3, dynamics and stability of the system are given based on the method of characteristic root and Lyapunov function. In Section 4, the asymptotic solution is deduced by the Lapace-ADM. The convergence of the asymptotic solution to the exact solution is proved, and the computability is illustrated by numerical analysis. In Section 5, the C_0 complexity of the system with respect to the order is computed based on the asymptotic solution, and the chaotic dynamics of the fractional-order epidemic system is analyzed and illustrated. In Section 6, by using local sensitivity index analysis, we discuss the role of each parameter in the model during the spread of infectious diseases. The discussion is given in Section 7 and the conclusions are given in Section 8.

2. The Well-Posedness of the Solution

2.1. Preliminaries

Firstly, we provide the reader with some useful definitions and describe the properties of fractional derivatives.

The Riemann–Liouville integral operator of order $\alpha \in (0,1]$ is defined as [19]

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha} f(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function, and the Laplace transform formula for the Caputo fractional derivative

$$L\left\{ {}_0^C D_t^{\alpha} f(t) \right\} = p^{\alpha} F(p) - p^{\alpha-1} f(0),$$

where $F(p) = L\{f(t)\}$, $\alpha \in (0,1]$.

Prior to discussing the well-posedness of solutions, we need to conduct some preparatory work. We present the following three lemmas, which are necessary for establishing the existence, uniqueness, and positivity of solutions to model (1) under initial condition (3). Lemma 1 is adapted from monograph [20] and the work in [19], while Lemma 2 and Lemma 3 reference works [24,48], respectively.

Lemma 1 ([19,20]). *Consider a system of nonlinear fractional differential equations with Caputo fractional derivative*

$$\begin{cases} {}_0^C D_t^{\alpha} y(t) = F(t,y), t > 0, \\ y(0) = y_0, \end{cases} \tag{4}$$

where $y \in G$ is a real-valued vector function, and G is an open bounded set in R^n ($n \geq 1$), ${}_0^C D_t^{\alpha}$, $0 < \alpha < 1$, is the Caputo fractional operator on $t > 0$. Let $F \in C([0,\infty) \times G)$ be continuous, and locally Lipschitz in $y \in G$. Then, for any $y_0 \in G$, there exists a $t_\varepsilon > 0$, such that the initial problem (4) has a unique solution $y \in C([0, t_\varepsilon))$.

Lemma 2 ([24]). *Let $f \in W_t^1((0,T)) \cap C([0,T])$ attain its maximum (respectively, minimum) over the interval $[0,T]$ at the point $t_0 \in (0,T]$. Then, the Caputo fractional derivative of the function f is non-negative (respectively, non-positive) at the point t_0 for any $\alpha \in (0,1]$:*

$$\left({}_0^C D_t^{\alpha} f \right)(t_0) \geq 0 (respectively, \left({}_0^C D_t^{\alpha} f \right)(t_0) \leq 0).$$

Here $W_t^1((0,T)) = \left\{ f \in C^1((0,T]) : f' \in L^1((0,T)) \right\}$ and $L^1((0,T))$ is the set of functions Lebesgue integrable in (0,T).

Lemma 3 ([48]). *Suppose $\varphi \in [0,T_1] \cap C^1(0,T_1]$ for some $T_1 > 0$. Then, for $0 < \alpha \leq 1$ and any C^1 convex function $E(\cdot)$, for all $t \in (0,T_1]$, we have*

$${}_0^C D_t^{\alpha} E(\varphi)(t) \leq E'(\varphi(t)) {}_0^C D_t^{\alpha} \varphi(t) \tag{5}$$

2.2. Well-Posedness of Solution

By Lemmas 1–3, it is not difficult to get the following assertion

Theorem 1. *Assuming that the model parameters τ, δ, κ, β, μ, γ_1, γ, d and N (as given in system (1)) are bounded and non-negative, then for any initial value $(S(0), I(0), Q(0), R(0)) \in R_+^4$, system (1) has a unique global positive solution for all $t \geq 0$.*

Proof of Theorem 1. We first prove the global existence and uniqueness of the solution of (1). Denote $X(t) = [S(t), I(t), Q(t), R(t)]^T$ and the FAIQR model (1) can be rewritten as:

$${}_0^C D_t^{\alpha} X(t) = F(X)(t), \tag{6}$$

with initial condition $X(0) = [S_0, I_0, Q_0, R_0]^T$. Where $F(X) = [f_1, f_2, f_3, f_4]^T$, and

$$\begin{cases} f_1(X) = \delta S(t)\left(1 - \frac{S(t)}{N}\right) - \frac{\beta S(t) I(t)}{1+\kappa N}, \\ f_2(X) = \frac{\beta S(t) I(t)}{1+\kappa N} - (\mu + \gamma_1 + d) I(t), \\ f_3(X) = \mu I(t) - (\gamma_2 + d) Q(t), \\ f_4(X) = \gamma_1 I(t) + \gamma_2 Q(t) - d R(t). \end{cases}$$

It is not difficult to deduce that there exists a constant $L > 0$, such that $\forall X_1, X_2$ and $t > 0$, there holds

$$\|F(X_1) - F(X_2)\|_\infty \leq L \|X_1 - X_2\|_\infty, \tag{7}$$

where

$$L = \max\{3\delta + 2\beta, 2(\beta + \mu + \gamma_1 + \gamma_2) + d\}. \tag{8}$$

Therefore, the function $F(X)$ satisfies the locally Lipschitz condition. By Lemma 1, model (1) has a unique continuous solution on $[0, t_\varepsilon)$, where t_ε denotes the blow-up time of the solution. To prove that this solution is global, it suffices to show that $t = t_\varepsilon$.

Assume that m_0 is sufficiently large such that both $S(0)$ and $I(0)$ lie in the interval $\left[\frac{1}{m_0}, m_0\right]$. For any integer $m \leq m_0$, define:

$$t_m = \inf\left\{ t \in [0, t_\varepsilon) : S(t) \notin \left(\frac{1}{m}, m\right) or I(t) \notin \left(\frac{1}{m}, m\right) \right\},$$

Clearly, as $m \to \infty$, t_m is increasing with respect to m. Let $t_\infty = \lim\limits_{m \to 0} t_m$, then $t_\infty \leq t_\varepsilon$. If we can prove that $t_\infty = \infty$, then $t_\varepsilon = \infty$, and thus $(S(t), I(t)) \in R_+^2$ for all $t \leq 0$. Otherwise, there exists a constant $M > 0$, such that $t_\infty \leq M$. Then, there exists an integer $m_1 \geq m_0$, such that $t_m \leq M$ for all $m \geq m_1$.

Define a non-negative C^2 function $V : R_+^2 \to R_+$

$$V(S, I) = \left(S - a - a \ln \frac{S}{a}\right) + (I - 1 - \ln I), \tag{9}$$

where a is a positive constant. Since $x - a - a \ln \frac{x}{a} \geq 0$, the function $V(S, I)$ is non-negative. According to Lemma 3, we have:

$$\begin{aligned} {}_0^C D_t^\alpha V \leq & (1 - \frac{a}{S}) {}_0^C D_t^\alpha S + (1 - \frac{1}{I}) {}_0^C D_t^\alpha I, \\ = & S\delta - \frac{\delta S^2}{N} - a\delta + \frac{a\delta S}{N} + \frac{a\beta I}{1 + \kappa N} \\ & - (\mu + \gamma_1 + d) I - \frac{\beta S}{1 + \kappa N} + (\mu + \gamma_1 + d). \end{aligned}$$

Let $a = \frac{1}{\beta}(\mu + \gamma_1 + d)(1 + \kappa N)$, then we have

$$\begin{aligned} {}_0^C D_t^\alpha V \leq & -\frac{\delta}{N} S^2 + (\delta + \frac{a\delta}{N} - \frac{\beta}{1 + \kappa N}) S + (\mu + \gamma_1 + d) - a\delta, \\ \leq & \left(\frac{N\delta(1 + \kappa N) + a\delta(1 + \kappa N) - N\beta}{2\delta(1 + \kappa N)}\right)^2 + (\mu + \gamma_1 + d) = K, \end{aligned}$$

where K is a positive constant, we have

$${}_0^C D_t^\alpha V \leq K. \tag{10}$$

Applying the Riemann–Liouville fractional integral of order α to both sides of Equation (10) over the interval $[0, M]$, we get:

$$V(S(M), I(M)) \leq V(S(0), I(0)) + \frac{KM^{\alpha}}{\Gamma(1+\alpha)}. \tag{11}$$

Let $\Omega_m = \{t_m \leq M\}, m \geq m_1$. Note that at least one of $S(t_m)$ or $I(t_m)$ equals m or $\frac{1}{m}$, and therefore, $V(S(t_m), I(t_m))$ is not less than $m - 1 - \ln m$ or $\frac{1}{m} - 1 - \ln \frac{1}{m}$. Consequently

$$V(S(0), I(0)) + \frac{KM^{\alpha}}{\Gamma(1+\alpha)} \geq Max\{m - 1 - \ln m; \frac{1}{m} - 1 - \ln \frac{1}{m}\},$$

let $m \to \infty$, we have

$$V(S(0), I(0)) + \frac{KM^{\alpha}}{\Gamma(1+\alpha)} > \infty, \tag{12}$$

which is contradiction. Therefore, $t_{\infty} = \infty$, meaning that $S(t)$ and I(t) do not blow up in a finite way.

For the third equation in model (1), $Q(t)$ can be expressed as:

$$Q(t) = Q(0)e^{-\frac{\gamma_2+d}{\Gamma(1+\alpha)}t^{\alpha}} + \frac{\mu\alpha}{\Gamma(1+\alpha)} \int_0^t e^{\frac{\gamma_2+d}{\Gamma(1+\alpha)}(s^{\alpha}-t^{\alpha})}(t-s)^{\alpha-1} I(s)ds,$$

Since $I(t)$ is a global solution, $Q(t)$ is also a global solution. Similarly, $R(t)$ can be shown to be a global solution. Next, we will prove the positivity of the solutions.

Firstly, according to the initial value conditions (3), we know when t close 0,

$$\lim_{t \to 0^+} (S(t), I(t), Q(t), R(t)) = (S_0, I_0, Q_0, R_0) \geq 0,$$

then $(S(t), I(t), Q(t), R(t))$ are non-negative when t close 0.

By way of contradiction, suppose that $t_1 > 0$, such that at least one of $S(t)$, $I(t)$, $Q(t)$, and $R(t)$ is non-positive. The continuity of the solution implies that there exists t_0, such that at least one of $S(t_0)$, $I(t_0)$, $Q(t_0)$, and $R(t_0)$ is equal to zero. Without loss of generality, we assume that t_0 is the minimal time with such property.

If $S(t_0) = 0$, then ${}_0^C D_t^{\alpha} S(t_0) = 0$, implies that $S(t_0) = C$, $C \geq 0$ is constant; this is a contradiction. If $S(t_0) > 0, I(t_0) = 0$, then ${}_0^C D_t^{\alpha} S(t_0) = \tau^{1-\alpha}\delta S(t_0)\left(1 - \frac{S(t_0)}{N}\right) > 0$, which implies that there exists $\varepsilon > 0$, such that $S(t)$ is strictly monotone increasing in interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. Let $t_2 \in (t_0 - \varepsilon, t_0)$, then $S(t_2) < S(t_0) = 0$, which implies that there exists $t_3 \in (0, t_2)$, satisfied $S(t_3) = 0$; this is a contradiction. Thus, $S(t) \geq 0$. Similarly, we have $I \leq (t) \geq 0$, $Q(t) \geq 0$ and $R(t) \geq 0$.

So, system (1) has a unique global positive solution for all $t \geq 0$. □

3. Qualitative Behaviors of the Model

3.1. Local Stability Analysis of the Equilibria

The model (1) has a disease-free equilibrium $E_0 = (N, 0, 0, 0)$ and an endemic equilibrium $E_+ = (S_*; A_*; I_*; R_*)$, where

$$\begin{cases} S_* = \frac{N}{R_0}, \\ I_* = \frac{\delta}{\beta}\left(1 - \frac{1}{R_0}\right)(1 + \kappa N), \\ Q_* = \frac{\mu\delta}{\beta}\left(1 - \frac{1}{R_0}\right)\frac{1+\kappa}{\gamma_2+d}, \\ R_* = \frac{\delta}{\beta}\left(\gamma_1 + \frac{\mu\gamma_2}{\gamma_2+d}\right)\left(1 - \frac{1}{R_0}\right)(1 + \kappa N), \end{cases} \tag{13}$$

and R_0 is the basic reproductive number,

$$R_0 = \frac{\beta N}{(1 + \kappa N)(\mu + \gamma_1 + d)}. \tag{14}$$

For the sake of studying the local stability of model (1), we introduce the Lemma 4 (see [21]).

Lemma 4 ([21]). *Consider the n-dimensional fractional differential equation system* $\left({}_0^C D_t^\alpha X\right)(t) = AX(t)$, *where A is an arbitrary constant* $n \times n$ *matrix and* $\alpha \in (0,1)$.

(i) *The solution* $X(t) = 0$ *of the system is asymptotically stable if and only if all eigenvalues* $\lambda_i (i = 1, 2, \cdots, n)$ *of the matrix A satisfy* $|\arg(\lambda_i)| > \alpha\pi/2$.

(ii) *The solution* $X(t) = 0$ *of the system is stable if and only if the eigenvalues satisfy* $|\arg(\lambda_i)| \geq \alpha\pi/2$ *and any eigenvalue with* $|\arg(\lambda_i)| = \alpha\pi/2$ *has a geometric multiplicity that coincides with its algebraic multiplicity (i.e., an eigenvalue that is an l-fold zero of of the characteristic polynomial has l linearly independent eigenvectors).*

We linearize the model (1) into the form

$$\begin{cases} \tau^{\alpha-1} {}_0^C D_t^\alpha S(t) = m_1 S(t) + m_2 I(t), \\ \tau^{\alpha-1} {}_0^C D_t^\alpha I(t) = n_1 S(t) + n_2 I(t), \\ \tau^{\alpha-1} {}_0^C D_t^\alpha Q(t) = \mu I(t) - (\gamma_2 + d) Q(t), \\ \tau^{\alpha-1} {}_0^C D_t^\alpha R(t) = \gamma_1 I(t) + \gamma_2 Q(t) - d R(t), \end{cases} \tag{15}$$

where

$$\begin{cases} m_1 = \delta\left(1 - \frac{2S_*}{N}\right)\frac{\beta I_*}{1+\kappa N}, m_2 = -\frac{\beta S_*}{1+\kappa N}, \\ n_1 = \frac{\beta I_*}{1+\kappa N}, n_2 = \frac{\beta S_*}{1+\kappa N} - (\eta + \gamma_1 + d). \end{cases} \tag{16}$$

Taking Laplace transform of both sides of (15), one obtains

$$\begin{cases} \lambda^\alpha X_1(\lambda) - \lambda^{\alpha-1} S_0 = \tau^{1-\alpha} m_1 X_1(\lambda) + \tau^{1-\alpha} m_2 X_2(\lambda), \\ \lambda^\alpha X_2(\lambda) - \lambda^{\alpha-1} I_0 = \tau^{1-\alpha} n_1 X_1(\lambda) + \tau^{1-\alpha} n_2 X_2(\lambda), \\ \lambda^\alpha X_3(\lambda) - \lambda^{\alpha-1} Q_0 = \tau^{1-\alpha}(\eta X_2(t) - (\gamma_2 + d) X_3(t)), \\ \lambda^\alpha X_4(\lambda) - \lambda^{\alpha-1} R_0 = \tau^{1-\alpha}(\gamma_1 X_2(t) + \gamma_2 X_3(t) - d X_4(t)), \end{cases} \tag{17}$$

where, $X_1(\lambda)$, $X_2(\lambda)$, $X_3(\lambda)$, and $X_4(\lambda)$ are Laplace transform for $S(t)$, $I(t)$, $Q(t)$, and $R(t)$, respectively. Equation (17) can be expressed in a matrix form

$$A(\lambda) \begin{pmatrix} X_1(\lambda) \\ X_2(\lambda) \\ X_3(\lambda) \\ X_4(\lambda) \end{pmatrix} = \begin{pmatrix} k_1(\lambda) \\ k_2(\lambda) \\ k_3(\lambda) \\ k_4(\lambda) \end{pmatrix}, \tag{18}$$

where

$$A(\lambda) = \begin{pmatrix} \lambda^\alpha - \tau^{1-\alpha} m_1 & -\tau^{1-\alpha} m_2 & 0 & 0 \\ -\tau^{1-\alpha} n_1 & \lambda^\alpha - \tau^{1-\alpha} n_2 & 0 & 0 \\ 0 & -\tau^{1-\alpha}\mu & \lambda^\alpha + \tau^{1-\alpha}(\gamma_2 + d) & 0 \\ 0 & -\tau^{1-\alpha}\gamma_1 & -\tau^{1-\alpha}\gamma_2 & \lambda^\alpha + \tau^{1-\alpha} d \end{pmatrix},$$

and

$$k_1(\lambda) = \lambda^{\alpha-1} S_0, k_2(\lambda) = \lambda^{\alpha-1} I_0, \\ k_3(\lambda) = \lambda^{\alpha-1} Q_0, k_4(\lambda) = \lambda^{\alpha-1} R_0.$$

Here, $A(\lambda)$ and $\det(A(\lambda))$ are the characteristic matrix and characteristic polynomial of (1), respectively. The properties of eigenvalues of characteristic equation indicate the stability of system (17). Multiplying both sides of (18) by λ, one obtains

$$A(\lambda)\begin{pmatrix} \lambda X_1(\lambda) \\ \lambda X_2(\lambda) \\ \lambda X_3(\lambda) \\ \lambda X_4(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda k_1(\lambda) \\ \lambda k_2(\lambda) \\ \lambda k_3(\lambda) \\ \lambda k_4(\lambda) \end{pmatrix}. \tag{19}$$

Therefore, if all roots of the equation det (A (λ)) = 0 lie in open left complex plane, i.e., $Re(\lambda) < 0$, then we consider (19) in $Re(\lambda) \geq 0$. In this restricted area, system (19) has a unique solution $(\lambda X_1(\lambda), \lambda X_2(\lambda), \lambda X_3(\lambda), \lambda X_4(\lambda))$, so that

$$\lim_{\lambda \to 0, \mathrm{Re}(\lambda) \geq 0} \lambda X_i(\lambda) = 0, i = 1, 2, 3, 4.$$

Based on the assumption of all roots of the equation det $(A(\lambda)) = 0$, and Final-Value Theorem (FVT) [24], one can obtain

$$\begin{cases} \lim\limits_{t \to +\infty} S(t) = \lim\limits_{\lambda \to 0, \mathrm{Re}(\lambda) \geq 0} \lambda X_1(\lambda) = 0, \\ \lim\limits_{t \to +\infty} I(t) = \lim\limits_{\lambda \to 0, \mathrm{Re}(\lambda) \geq 0} \lambda X_2(\lambda) = 0, \\ \lim\limits_{t \to +\infty} Q(t) = \lim\limits_{\lambda \to 0, \mathrm{Re}(\lambda) \geq 0} \lambda X_3(\lambda) = 0, \\ \lim\limits_{t \to +\infty} R(t) = \lim\limits_{\lambda \to 0, \mathrm{Re}(\lambda) \geq 0} \lambda X_4(\lambda) = 0, \end{cases} \tag{20}$$

which provides that the stability results for the fractional-order system (15). Therefore, we arrive at the following Theorem.

Theorem 2. *If all the roots of the characteristic equation det* $(A(\lambda)) = 0$ *have negative real parts, satisfying the condition* $|\arg \lambda_i| > \frac{\alpha\pi}{2}$, *then the positive equilibrium points of system* (1) *are asymptotically stable.*

Substituting the disease-free equilibrium point E_0 into the characteristic matrix $A(\lambda)$, we obtain the characteristic equation as follows:

$$\left(\lambda^\alpha + \tau^{1-\alpha}(\gamma_2 + d)\right)\left(\lambda^\alpha + \tau^{1-\alpha} d\right)\left(\lambda^\alpha + \tau^{1-\alpha}\delta\right)\left(\lambda^\alpha + \tau^{1-\alpha}(\eta + \gamma_1 + d)(R_0 - 1)\right) = 0 \tag{21}$$

Obviously, the characteristic Equation (21) has three negative eigenvalues, $-\tau^{1-\alpha}\delta$, $\tau^{1-\alpha}(\gamma_2 + d)$, $-\tau^{1-\alpha} d$. We know that, when $R_0 \leq 1$, the fourth eigenvalue $\tau^{1-\alpha}(\eta + \gamma_1 + d)(R_0 - 1) < 0$ is negative, and the disease-free equilibrium is stable. Conversely, when $R > 1$, $\tau^{1-\alpha}(\eta + \gamma_1 + d)(R_0 - 1) > 0$, meaning the characteristic Equation (21) has one positive eigenvalue, and the disease-free equilibrium becomes unstable. Therefore, regarding the stability of the disease-free equilibrium, we can conclude the following theorem.

Theorem 3. *The disease-free equilibrium* E_0 *is local stability when* $R_0 \leq 1$, *and it becomes unstable when* $R_0 > 1$, *where* R_0 *is the basic reproduction number given by Equation* (14).

Now, we discuss the local stability at an endemic equilibrium. Substituting the disease-free equilibrium point E_+ into the characteristic matrix $A(\lambda)$, we get

$$\left((\lambda^\alpha)^2 + \lambda^\alpha \frac{\tau^{1-\alpha}\delta}{R_0} + (\tau^{1-\alpha})^2 \frac{\delta}{R_0}(\eta + \gamma_1 + d)(R_0 - 1)\right)(\lambda^\alpha + \tau^{1-\alpha} d)(\lambda^\alpha + \tau^{1-\alpha}(\gamma_2 + d)) = 0, \tag{22}$$

Obviously, Equation (22) has two negative roots, $-\tau^{1-\alpha} d$ and $-\tau^{1-\alpha}(\gamma_2 + d)$. Therefore, in order to judge stability of the disease-free equilibrium, we only need to analyze the characteristic roots of the following equation

$$(\lambda^\alpha)^2 + \lambda^\alpha \frac{\tau^{1-\alpha}\delta}{R_0} + (\tau^{1-\alpha})^2 \frac{\delta}{R_0}(\eta + \gamma_1 + d)(R_0 - 1) = 0 \tag{23}$$

We know that $\tau^{1-\alpha}\frac{\delta}{R_0} > 0$. When $R_0 > 1$, $(\tau^{1-\alpha})^2\frac{\delta}{R_0}(\eta + \gamma_1 + d)(R_0 - 1) > 0$, and by Vieta's formulas, it follows that Equation (23) has two negative roots, implying the endemic equilibrium is locally stable. Conversely, when $R_0 < 1$, $(\tau^{1-\alpha})^2\frac{\delta}{R_0}(\eta + \gamma_1 + d)(R_0 - 1) < 0$, indicating that Equation (23) has at least one positive root, which means the endemic equilibrium is unstable. Therefore, we have established the following theorem.

Theorem 4. *The endemic equilibrium E_+ is local stability when $R_0 \geq 1$, and it is unstable when $R_0 < 1$.*

3.2. Global Stability

Now, we study the global stability of the disease-free equilibrium E_0 and the endemic equilibrium point E_+, using using Lyapunov functional. For the disease-free equilibrium E_0, we have established the following theorem.

Theorem 5. *If $S \ll N$ and $\delta S > d(I + Q + R)$, then the FSIQR model (1) is globally asymptotically stable at the disease-free equilibrium point $E_0 = (N, 0, 0, 0)$.*

Proof of Theorem 5. Let $x(t) = S(t) - N$, we define the non-negative definite function at a disease-free equilibrium point E_0 of (1) as

$$V_1(x(t), I(t), Q(t), R(t)) = \frac{1}{2}[x(t) + I(t) + Q(t) + R(t)]^2. \tag{24}$$

Clearly, $V_1(x(t), I(t), Q(t), R(t))$ is a regular function if and only if $V_1(0, 0, 0, 0) = (0, 0, 0, 0)$. Taking a fractional-order derivative on both sides, using Lemma 3, yields

$$\begin{aligned}
{}_0^C D_t^\alpha V_1 &\leq (x + I + Q + R)\,{}_0^C D_t^\alpha (x + I + Q + R),\\
&= (x + I + Q + R)(-\delta(x + N)\frac{x}{N} - d(I + Q + R)),\\
&= (-N + S + I + Q + R)(\delta(1 - \frac{S}{N})S - d(I + Q + R)).
\end{aligned}$$

Since $-N + S + I + Q + R \leq 0$, based on the assumptions: $S \ll N$ and $\delta S > d(I + Q + R)$, we know that $\delta(1 - \frac{S}{N})S - d(I + Q + R) \geq 0$, then

$${}_0^C D_t^\alpha V_1 \leq (-N + S + I + Q + R)(\delta(1 - \frac{S}{N})S - d(I + Q + R)) \leq 0.$$

Finally, the proof is completed, the FSIQR model (1) is globally asymptotically stable at the disease-free equilibrium point E_0. □

Similarly, for the endemic equilibrium point E_+, we have established the following theorem.

Theorem 6. *If $\delta \leq \frac{\beta I}{R_0(1+\kappa N)}$ and $\frac{\beta S}{1+\kappa N} > \frac{1}{\mu} + \mu + \gamma_1 + d$, the FSIQR model (1) is globally asymptotically stable at the endemic equilibrium point $E_+ = (S_*, I_*, Q_*, R_*)$.*

Proof of Theorem 6. We define the non-negative definite function at an endemic equilibrium point E_+ of (1) as

$$\begin{aligned}
V_2(S(t), I(t), Q(t), R(t)) =& (S - S_* - S_* \ln \frac{S}{S_*}) + (I - I_* - I_* \ln \frac{I}{I_*})\\
&+ (Q - Q_* - Q_* \ln \frac{Q}{Q_*}) + (R - R_* - R_* \ln \frac{R}{R_*}),
\end{aligned} \tag{25}$$

when $t \geq 0$, the function $y(t) - c_* - c_* ln\frac{y(t)}{c_*} \geq 0$, where c_* is non-negative constants. Then, $V_2(S(t), I(t), Q(t), R(t))$ is a regular function. Taking a fractional-order derivative on both sides, using Lemma 3, yields

$$\begin{aligned}
{}_0^C D_t^\alpha V_2 \leq & \frac{S-S_*}{S}\partial_t^\alpha S + \frac{I-I_*}{I}\partial_t^\alpha I + \frac{Q-Q_*}{Q}\partial_t^\alpha Q + \frac{R-R_*}{R}\partial_t^\alpha R, \\
= & -\frac{\delta}{N}\tau^{1-\alpha}(S-S_*)^2 - (S-S_*)(\delta(1-\frac{S_*}{N}) - \frac{\beta I}{1+\kappa N}) + \frac{\beta S I}{1+\kappa N} \\
& - \frac{\beta S I_*}{1+\kappa N} - (\mu+\gamma_1+d)I + (\mu+\gamma_1+d)I_* + \mu(1-\frac{Q_*}{Q})I \\
& - (\gamma_2+d)(Q-Q_*) + (1-\frac{R_*}{R})\gamma_1 I + (1-\frac{R_*}{R})\gamma_2 Q - d(R-R_*), \\
= & -\frac{\delta}{N}(S-S_*)^2 - (S-S_*)\delta(1-\frac{1}{R_0}) - \frac{\beta S_* I}{1+\kappa N} - (\mu\frac{Q_*}{Q} + \gamma_1\frac{R_*}{R})I \\
& + (\frac{1}{\mu} + \mu + \gamma_1 + d - \frac{\beta S}{1+\kappa N})I_* - \gamma_2\frac{R_*}{R}Q - d(R-R_*) - d(I+Q), \\
= & -\frac{\delta}{N}((S-S_*) + \frac{N}{2}(1-\frac{1}{R_0}))^2 + N(\delta - \frac{\beta I}{R_0(1+\kappa N)}) - (\mu\frac{Q_*}{Q} + \gamma_1\frac{R_*}{R})I \\
& + (\frac{1}{\mu} + \mu + \gamma_1 + d - \frac{\beta S}{1+\kappa N})I_* - \gamma_2\frac{R_*}{R}Q - d(R-R_*) - d(I+Q).
\end{aligned}$$

Based on the assumptions: $\delta \leq \frac{\beta I}{R_0(1+\kappa N)}$ and $\frac{\beta S}{1+\kappa N} > \frac{1}{\mu} + \mu + \gamma_1 + d$, we can get $\partial_t^\alpha V_2 \leq 0$, the FSIQR model (1) is globally asymptotically stable at the endemic equilibrium point E_+, the proof is completed. □

4. Convergence and Computability of Asymptotic Solution

4.1. The Asymptotic Solution

Firstly by performing the Laplace transform on $t > 0$ for the model (1), there holds

$$\begin{cases}
L\{S(t)\} = \frac{S_0}{p} + \tau^{1-\alpha}\frac{1}{p^\alpha}L\{\delta S(t)(1-\frac{S(t)}{N}) - \frac{\beta S(t)I(t)}{1+\kappa N}\}, \\
L\{I(t)\} = \frac{I_0}{p} + \tau^{1-\alpha}\frac{1}{p^\alpha}L\{\frac{\beta S(t)I(t)}{1+\kappa N} - (\mu+\gamma_1+d)I(t)\}, \\
L\{Q(t)\} = \frac{Q_0}{p} + \tau^{1-\alpha}\frac{1}{p^\alpha}L\{\mu I(t) - (\gamma_2+d)Q(t)\}, \\
L\{R(t)\} = \frac{R_0}{p} + \tau^{1-\alpha}\frac{1}{p^\alpha}L\{\gamma_1 I(t) + \gamma_2 Q(t) - dR(t)\}.
\end{cases} \tag{26}$$

Next, by the ADM method, the solutions of the model (1) are assumed to be expressed in the form of infinite series given below:

$$S(t) = \sum_{k=0}^{\infty} s_k, I(t) = \sum_{k=0}^{\infty} i_k, Q(t) = \sum_{k=0}^{\infty} q_k, R(t) = \sum_{k=0}^{\infty} r_k, \tag{27}$$

and we need to work out s_k, i_k, q_k and r_k for $k = 1, \cdots$, respectively.

For the terms $S^2(t)$ and $S(t)I(t)$ in (26) are decomposed in the following forms:

$$S^2(t) = \sum_{k=0}^{\infty} A_1^k, S(t)I(t) = \sum_{k=0}^{\infty} A_2^k, \tag{28}$$

where A_1^k, A_2^k, $k = 1, \cdots$, are the so-called Adomian polynomials, which are given by

$$\begin{cases}
A_1^k = s_0 s_k + s_1 s_{k-1} + \cdots + s_k s_0, \\
A_2^k = s_0 i_k + s_1 i_{k-1} + \cdots + s_k i_0.
\end{cases} \tag{29}$$

By substituting (27) and (28) with Adomian polynomials (29) into (30), we have

$$\begin{cases} \sum\limits_{k=0}^{\infty} L\{s_k\} = \frac{S_0}{p} + \tau^{1-\alpha} \sum\limits_{k=0}^{\infty} \frac{1}{p^\alpha} L\{\delta(s_k - \frac{1}{N}A_1^k) - \frac{\beta A_2^k}{1+\kappa N}\}, \\ \sum\limits_{k=0}^{\infty} L\{i_k\} = \frac{I_0}{p} + \tau^{1-\alpha} \sum\limits_{k=0}^{\infty} \frac{1}{p^\alpha} L\{\frac{\beta A_2^k}{1+\kappa N} - (\mu+\gamma_1+d)i_k\}, \\ \sum\limits_{k=0}^{\infty} L\{q_k\} = \frac{Q_0}{p} + \tau^{1-\alpha} \sum\limits_{k=0}^{\infty} \frac{1}{p^\alpha} L\{\mu i_k - (\gamma_2+d)q_k\}, \\ \sum\limits_{k=0}^{\infty} L\{r_k\} = \frac{R_0}{p} + \tau^{1-\alpha} \sum\limits_{k=0}^{\infty} \frac{1}{p^\alpha} L\{\gamma_1 i_k + \gamma_2 q_k - d r_k\}. \end{cases} \tag{30}$$

Based on (30), an iterative algorithm for the computation of each term in (27) is

$$\begin{cases} L(s_0) = \frac{S_0}{p}, \\ L\{s_{k+1}\} = \tau^{1-\alpha} \frac{1}{p^\alpha} L\{\delta(s_k - \frac{1}{N}A_1^k) - \frac{\beta A_2^k}{1+\kappa N}\}, k = 0, 1, \cdots, \\ L(i_0) = \frac{I_0}{p}, \\ L\{i_{k+1}\} = \tau^{1-\alpha} \frac{1}{p^\alpha} L\{\frac{\beta A_2^k}{1+\kappa N} - (\mu+\gamma_1+d)i_k\}, k = 0, 1, \cdots, \\ L(q_0) = \frac{Q_0}{p}, \\ L\{q_{k+1}\} = \tau^{1-\alpha} \frac{1}{p^\alpha} L\{\mu i_k - (\gamma_2+d)q_k\}, k = 0, 1, \cdots, \\ L(r_0) = \frac{R_0}{p}, \\ L\{r_{k+1}\} = \tau^{1-\alpha} \frac{1}{p^\alpha} L\{\gamma_1 i_k + \gamma_2 q_k - d r_k\}, k = 0, 1, \cdots. \end{cases} \tag{31}$$

Finally, by performing the inverse Laplace transform for (31) term by term, we get

$$s_0 = S_0, i_0 = I_0, q_0 = Q_0, r_0 = R_0 \tag{32}$$

and there holds for any $t > 0$

$$(s_1, i_1, q_1, r_1) = \frac{\tau^{1-\alpha} t^\alpha}{\Gamma(\alpha+1)} (C_{11}, C_{21}, C_{31}, C_{41}), \tag{33}$$

where

$$\begin{cases} C_{11} = \delta S_0 (1 - \frac{S_0}{N}) - \frac{\beta S_0 I_0}{1+\kappa N}, \\ C_{21} = \frac{\beta S_0 I_0}{1+\kappa N} - (\mu+\gamma_1+d) I_0, \\ C_{31} = \mu I_0 - (\gamma_2+d) Q_0, \\ C_{41} = \gamma_1 I_0 + \gamma_2 Q_0 - d R_0. \end{cases} \tag{34}$$

Similarly, we can work out s_k, i_k, q_k and r_k for $k = 2, 3, \cdots$ based on (31). For example, there holds for $k = 2$

$$(s_2, i_2, q_2, r_2) = \frac{\tau^{2-2\alpha} t^{2\alpha}}{\Gamma(2\alpha+1)} (C_{12}, C_{22}, C_{32}, C_{42}), \tag{35}$$

where

$$\begin{cases} C_{12} = \delta(C_{11} - \frac{2C_{11}S_0}{N}) - \frac{\beta(C_{11}I_0 + S_0 C_{21})}{1+\kappa N}, \\ C_{22} = \frac{\beta(C_{11}I_0 + S_0 C_{21})}{1+\kappa N} - (\mu+\gamma_1+d) C_{21}, \\ C_{32} = \mu C_{21} - (\gamma_2+d) C_{31}, \\ C_{42} = \gamma_1 C_{21} + \gamma_2 C_{31} - d C_{41}. \end{cases} \tag{36}$$

Similarly, we can get for $k = 3, 4, \cdots$

$$(s_k, i_k, q_k, r_k) = \frac{\tau^{k(1-\alpha)} t^{k\alpha}}{\Gamma(k\alpha+1)} (C_{1k}, C_{2k}, C_{3k}, C_{4k}), \tag{37}$$

where the coefficients C_{1k}, C_{2k}, C_{3k} and C_{4k} are given as in (34) and (36). Thus, we get a solution of the forward problem (1), (3) in the form of the series:

$$\begin{cases} S(t) = S_0 + \sum\limits_{k=1}^{\infty} C_{1k} \frac{\tau^{k(1-\alpha)} t^{k\alpha}}{\Gamma(k\alpha+1)}, \\ I(t) = I_0 + \sum\limits_{k=1}^{\infty} C_{2k} \frac{\tau^{k(1-\alpha)} t^{k\alpha}}{\Gamma(k\alpha+1)}, \\ Q(t) = Q_0 + \sum\limits_{k=1}^{\infty} C_{3k} \frac{\tau^{k(1-\alpha)} t^{k\alpha}}{\Gamma(k\alpha+1)}, \\ R(t) = R_0 + \sum\limits_{k=1}^{\infty} C_{4k} \frac{\tau^{k(1-\alpha)} t^{k\alpha}}{\Gamma(k\alpha+1)}. \end{cases} \tag{38}$$

4.2. Convergence of the Asymptotic Solution

Firstly we prove the convergence of the series in (38), and we need an assertion on the asymptotic property of Gamma function given in [49].

Lemma 5 ([49]). *For $\alpha \in (0, 1]$ and $k \in N$ there holds*

$$\lim_{k\to\infty} \frac{\Gamma(k\alpha)}{\Gamma(k\alpha+\alpha)} = 0.$$

Theorem 7. *Let the initial values S_0, I_0, Q_0 and R_0, and the model parameters are all positive and bounded. Then, the four infinite series in (38) are convergent for any given finite time $t > 0$, and the asymptotic solution is well-defined.*

Proof of Theorem 7. Without loss of the generality, let us consider the series in the solution of $S(t)$. Under the conditions of the theorem, there exists a positive constant $M > 0$ independent of k such that $\frac{|C_{1k+1}|}{|C_{1k}|} \leq M$. Note $s_k = C_{1k}\tau^{k-k\alpha}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}$, we have

$$\frac{s_{1k+1}}{s_k} = \frac{C_{1k+1}}{C_{1k}}\tau^{1-\alpha}\frac{t^{\alpha}\Gamma(k\alpha)}{\Gamma((k+1)\alpha+1)} = \frac{C_{1k+1}}{C_{1k}}\frac{k\tau^{1-\alpha}}{k+1}\frac{\Gamma(k\alpha)t^{\alpha}}{\Gamma(k\alpha+\alpha)}.$$

By Lemma 7 and the condition $\frac{|C_{1k+1}|}{|C_{1k}|} \leq M$, there holds for any given finite time $t \in (0, T]$:

$$\frac{|s_{k+1}|}{|s_k|} \leq M\frac{\Gamma(k\alpha)}{\Gamma(k\alpha+\alpha)} < 1, ask \to \infty,$$

Henceforth, the series in the solution of S(t) is absolute convergent. The proof is over. □

Now, we show that the asymptotic solution is just the solution of the model (1).

Theorem 8. *Under the conditions of Theorem 7, the asymptotic solution given by (38) is the solution of the model (1) with the initial values (3).*

Proof of Theorem 8. Let us take $S(t)$ as example, and we are to testify the series solution of $S(t)$ in (38) satisfies the first equation in the model (1).

By the iterative scheme (31), and adding up all terms at the both sides of (31), respectively, we have

$$\sum_{k=0}^{\infty} L\{s_k\} = \frac{S_0}{p} + \sum_{k=0}^{\infty} \tau^{1-\alpha}\frac{1}{p^{\alpha}} L\{\delta(s_k - \frac{1}{N}A_1^k) - \frac{\beta A_2^k}{1+\kappa N}\}. \tag{39}$$

By the conditions of the theorem, the series $\sum\limits_{k=0}^{\infty} s_k$ and $\sum\limits_{k=0}^{\infty} i_k$ are all convergent, and there holds

$$L\{\sum_{k=0}^{\infty} s_k\} = L\{S(t)\}. \tag{40}$$

Follows we are to prove that the series $\sum_{k=0}^{\infty} A_1^k$ and $\sum_{k=0}^{\infty} A_2^k$ are convergent to $S^2(t)$ and $S(t)I(t)$.

By the convergent of the series $\sum_{k=0}^{\infty} s_k$ there holds $s_k \to \infty$ as $k \to \infty$. Then, for sufficiently large $K \in N$, by direct computations based on (29), we can deduce that

$$\left(\sum_{k=0}^{K} s_k\right)^2 - \sum_{k=0}^{K} A_1^k = \begin{cases} O(s_n^2), K = 2n-1, \\ O(s_{n+1}), K = 2n, \end{cases} \tag{41}$$

for $n \in N$. In fact when $K = 1$, obviously, there hold

$$\left(\sum_{k=0}^{1} s_k\right)^2 - \sum_{k=0}^{1} A_1^k = s_1^2,$$

and when $K = 2$, we can get

$$\left(\sum_{k=0}^{2} s_k\right)^2 - \sum_{k=0}^{2} A_1^k = 2s_1s_2 + s_2^2 = O(s_2),$$

which shows that (41) is valid for $n = 1$.

Furthermore, when $K = 2n + 1$, we have

$$\begin{aligned}\left(\sum_{k=0}^{2n+1} s_k\right)^2 - \sum_{k=0}^{2n+1} A_1^k =& s_{n+1}^2 + \cdots + s_{2n-1}^2 + 2s_1s_{2n+1}^2 + 2s_2(s_{2n} + s_{2n+1}) + \cdots \\ &+ 2s_n(s_{n+2} + \cdots + s_{2n+1}) + \cdots + 2s_{2n-1}(s_{2n} + s_{2n+1}) \\ &+ (s_{2n} + s_{2n+1})^2 = O(s_{n+1}^2)\end{aligned}$$

and when $K = 2n + 2$, we get

$$\begin{aligned}\left(\sum_{k=0}^{2n+2} s_k\right)^2 - \sum_{k=0}^{2n+2} A_1^k =& s_{n+2}^2 + \cdots + s_2n^2 + 2s_1s_{2n+2}^2 + 2s_2(s_{2n+1} + s_{2n+2}) + \cdots \\ &+ 2s_n(s_{n+3} + \cdots + s_{2n+2}) + \cdots + 2s_{n+1}(s_{n+2} + s_{2n+2}), \\ =& O(s_{n+2})\end{aligned}$$

Thus, the estimation (41) is valid for any $K \in N$. Noting $s_n \to 0$ as $K \to \infty$, it follows that

$$\sum_{k=0}^{\infty} A_1^k = \left(\sum_{k=0}^{\infty} s_k\right)^2 = S^2(t)$$

Similarly, we have $\sum_{k=0}^{\infty} A_2^k = S(t)I(t)$. So, noting to (40), the identity (39) is transformed to:

$$\begin{aligned}\tau^{\alpha-1}(p^\alpha L\{S(t)\} - p^{\alpha-1}S_0) &= L\left\{\sum_{k=0}^{\infty}(\delta s_k - \frac{\delta}{N}A_1^k - \frac{\beta A_2^k}{1+\kappa N})\right\} \\ &= L\{\delta S - \frac{\delta}{N}S^2 - \frac{\beta SI}{1+\kappa N}\}\end{aligned} \tag{42}$$

By performing the inverse Laplace transform on the both sides of (42) follows that the first equation on $S(t)$ in the model (1) is valid.

Similarly, the other three equations in (1) are valid, too. Therefore, the asymptotic solution given by (38) is just the solution of the model (1), with the initial values of (3). □

4.3. Computable Solution and Simulations

It is noted that the truncated dimension of an asymptotic solution by the ADM method is always small on concrete numerical examples, but there seems to be little theoretical analysis in the known literature. Here, we aim to demonstrate this point by numerical analysis, by which a computable formula for the asymptotic solution is obtained.

For any given time, $T > 0$, the period $(0, T]$ is divided into some small time periods of $(t_j, t_{j+1}]$ for $j = 0, 1, \cdots, J-1$ and $t_0 = 0$, $t_J = T$, and the value corresponding to the time of t_{j+1} is worked out from the value of t_j successively based on (38).

Let $h = t_{j+1} - t_j$ be the uniform time step, and $K \geq 1$ be the truncated dimension of the asymptotic solution. Then, a computable asymptotic solution of the system (1) at the time $t_{j=1}, j = 0, 1, \cdots$, can be expressed by:

$$\begin{cases} S^K(t_{j+1}) = C_{10}^j + \sum\limits_{k=1}^{K} C_{1k}^j \frac{\tau^{k(1-\alpha)} h^{k\alpha}}{\Gamma(k\alpha+1)}, \\ I^K(t_{j+1}) = C_{20}^j + \sum\limits_{k=1}^{K} C_{2k}^j \frac{\tau^{k(1-\alpha)} h^{k\alpha}}{\Gamma(k\alpha+1)}, \\ Q^K(t_{j+1}) = C_{30}^j + \sum\limits_{k=1}^{K} C_{3k}^j \frac{\tau^{k(1-\alpha)} h^{k\alpha}}{\Gamma(k\alpha+1)}, \\ R^K(t_{j+1}) = C_{40}^j + \sum\limits_{k=1}^{K} C_{4k}^j \frac{\tau^{k(1-\alpha)} h^{k\alpha}}{\Gamma(k\alpha+1)}, \end{cases} \tag{43}$$

where $C_{ik}^j, i = 1, 2, 3, 4, k = 0, 1, \cdots$, are the coefficients at the time $t_j, j = 0, 1, \cdots$, which are determined by the initial values and the model parameters similarly as given in (34) and (36), and here $C_{10}^0 = S(0) = S_0$, $C_{20}^0 = I(0) = I_0$, $C_{30}^0 = Q(0) = Q_0$, $C_{40}^0 = R(0) = R_0$.

In what follows, we also take $S(t)$ as an example; it is idealistic to determine an optimal truncated dimension K, such that for $\forall \varepsilon > 0$, there holds

$$|S(t_{j+1}) - S^K(t_{j+1})| \leq \varepsilon \tag{44}$$

On concrete calculations, then, $\forall \varepsilon > 0$, it is sufficient to determine a dimension $K \geq 1$, such that

$$|S^K(t_{j+1}) - S^{K-1}(t_{j+1})| = |C_{1,K}| \frac{\tau^{K(1-\alpha)} h^{K\alpha}}{\Gamma(K\alpha+1)} \leq \varepsilon \tag{45}$$

Therefore, an optimal truncated dimension K can be solved from (45) for given precision ε in theory. However, it is very difficult to work out the coefficient C_{1K}^j for an unknown K in practice. Fortunately we can give a reverse analysis by specific computations based on (45).

Example 1. *Considering the transmission patterns of infectious diseases, without loss of generality, let the model parameters be* $N = 1000$, $\kappa = 0.01$, $\delta = 0.5$, $\beta = 0.1$, $\gamma_1 = 0.05$, $\gamma_2 = 0.2$, $\mu = 0.2$ *and* $d = 0.001$. *Without loss of generality, take the fractional order* $\alpha = 0.6$, *the time-scale factor* $\tau = 0.7$, *and the final time* $T = 60$, *and the initial values are given as* $S_0 = 100$, $I_0 = 30$, $Q_0 = 20$ *and* $R_0 = 0$, *and the time step* $h = 0.2$.

By the above parameters, the truncated errors of the asymptotic solutions with the truncated dimension K can be estimated, which are listed in Table 1.

From Table 1, it can be seen that the maximal error of the four asymptotic solutions between $K = 4$ and $K = 5$ is less than $\frac{1}{2} \times 10^{-4}$. Further computations show that the truncated errors become smaller rapidly as the truncated dimension goes to large. Therefore, for the precision $\varepsilon = \frac{1}{2} \times 10^{-4}$, we can say that an optimal truncated dimension is $K = 4$

for the asymptotic solution of the forward problem, and the solution is expressed by (43) with $K = 4$ in the following computations and analysis.

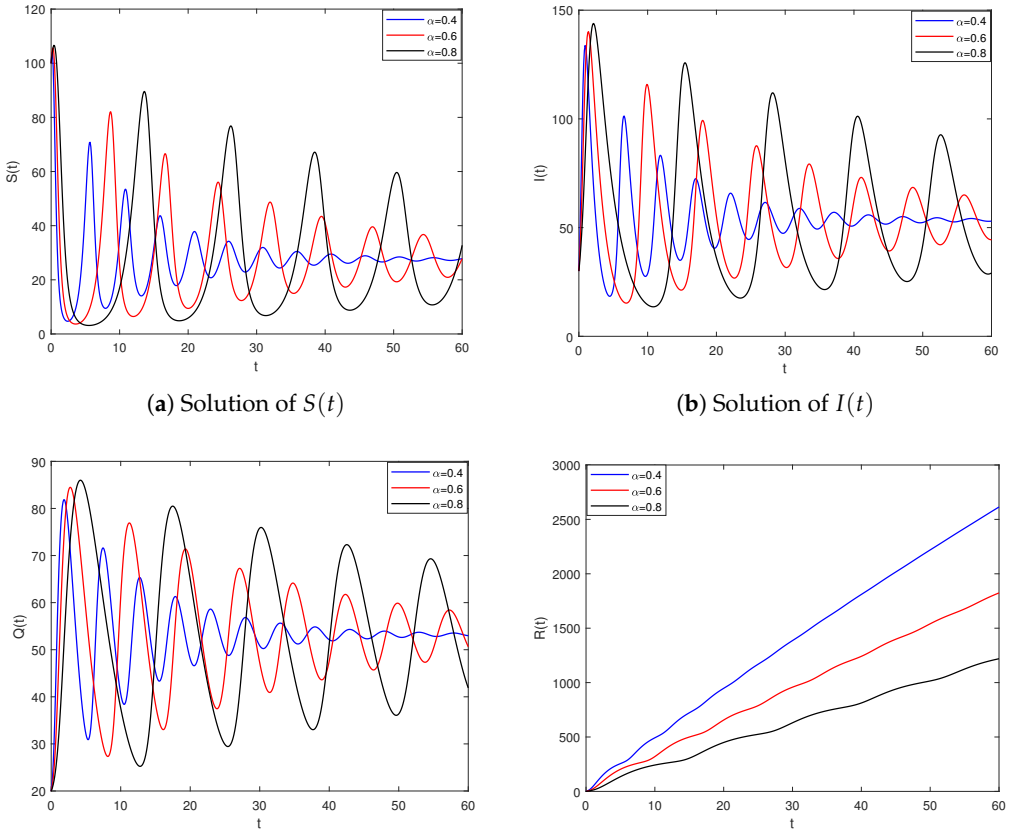
Table 1. The truncated error estimates of the asymptotic solutions.

K	$\lvert S^K - S^{K-1}\rvert$	$\lvert I^K - I^{K-1}\rvert$	$\lvert Q^K - Q^{K-1}\rvert$	$\lvert R^K - R^{K-1}\rvert$
1	0.8930	0.6523	0.2830	3.8944
2	0.0364	0.1045	0.0502	0.0075
3	0.0083	0.0012	0.0023	0.0033
4	1.39×10^{-4}	5.20×10^{-4}	1.32×10^{-4}	7.55×10^{-5}
5	3.76×10^{-5}	2.58×10^{-5}	1.27×10^{-5}	8.59×10^{-6}

By the computable asymptotic solution, we can investigate the impact of the fractional orders on the fractional-order system.

Example 2. *Furthermore, take the time $T = 60$, and the time step $h = 0.1$, and the other parameters are the same as those used in Example 1. Numerical solutions of $S(t)$, $I(t)$, $Q(t)$ and $R(t)$ with $\alpha = 0.4$ (blue line), $\alpha = 0.6$ (red line) and $\alpha = 0.8$ (black line) are plotted in Figure 1, respectively.*

From Figure 1, we can see that the outbreak of the infectious disease is a quasi-periodic process. The peak time of S, I and I are successively delayed, which coincides with the transmission law of the infectious disease, and as infectious disease outbreaks recur, the amplitude of epidemic waves gradually diminishes, eventually stabilizing. Moreover, we also find that the fractional order is related with the outbreak cycle of the infectious disease, and the larger of the fractional order, the longer the outbreak cycle of the infectious disease.



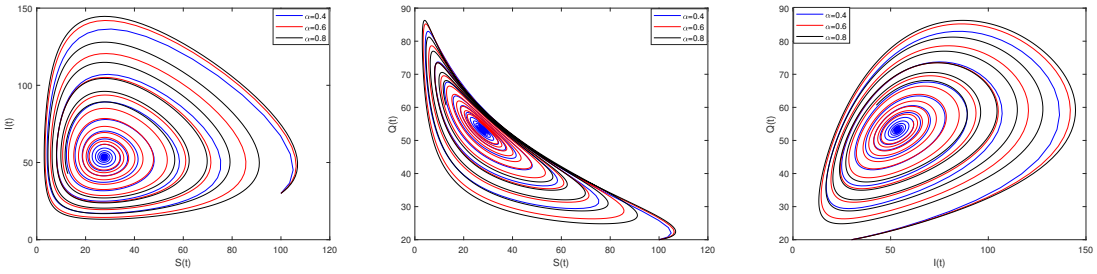
(**a**) Solution of $S(t)$ (**b**) Solution of $I(t)$

(**c**) Solution of $Q(t)$ (**d**) Solution of $R(t)$

Figure 1. Solution of $S(t)$, $I(t)$, $Q(t)$ and $R(t)$ with $\alpha = 0.4, 0.6, 0.8$ in Example 2.

Example 3. *By taking the same parameters as in Example 2, the phase portraits of SI, SQ and QI with the factional orders* $\alpha = 0.4, 0.6, 0.8$ *are plotted in Figure 2, and the phase diagrams of SAI with the fractional orders* $\alpha = 0.25, 0.35, 0.50$, *and* $\alpha = 0.65, 0.80, 0.95$ *are plotted in Figure 3, respectively.*

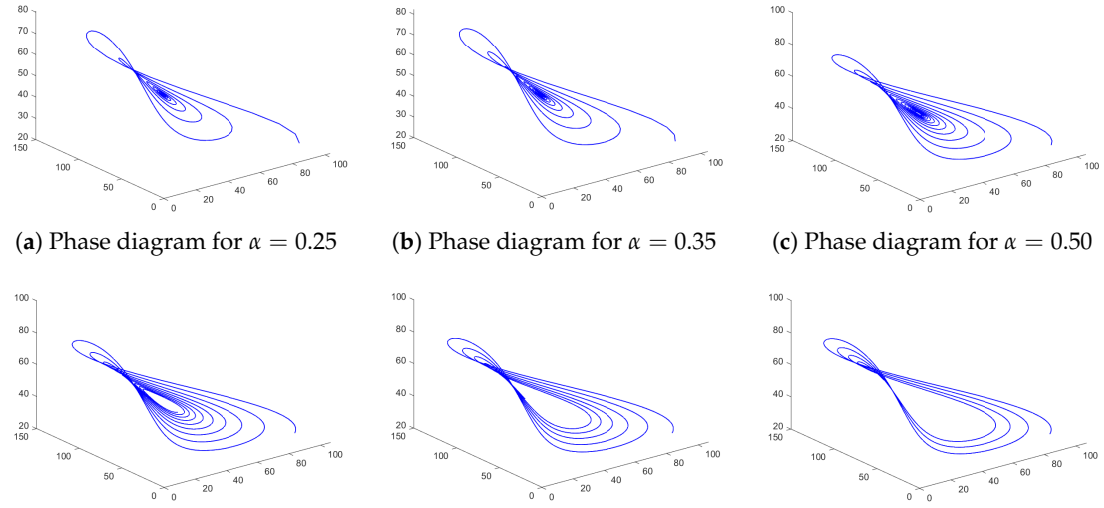
From Figure 2a–c, we can see that the orbits form an attractor resembling a limit cycle cone, which has the property of transition from period-doubling bifurcation to weak chaos through a certain law, and the smaller the factional order, the stronger the attraction. That is, the spread of infectious diseases obeys certain regulations rather than being completely random.



(**a**) *SI* phase diagram (**b**) *SQ* phase diagram (**c**) *QI* phase diagram

Figure 2. The phase diagrams with $\alpha = 0.4, 0.6, 0.8$ in Example 3.

From Figure 3a–f, we can see that, as the fractional order α varies, the SAI phase portrait oscillates between its maximum value and two minimum values, forming two stable limit-cycle conical attractors. Moreover, with an increase in α, the attraction strength of one attractor strengthens while that of the other weakens. Overall, as the fractional order increases, the oscillations diminish, indicating that the behavior of fractional-order dynamical systems is far richer than that of integer-order dynamical systems.



(**a**) Phase diagram for $\alpha = 0.25$ (**b**) Phase diagram for $\alpha = 0.35$ (**c**) Phase diagram for $\alpha = 0.50$

(**d**) Phase diagram for $\alpha = 0.65$ (**e**) Phase diagram for $\alpha = 0.80$ (**f**) Phase diagram for $\alpha = 0.95$

Figure 3. The phase diagrams of SIQ with fractional orders in Example 3.

5. Complexity and Chaos

By the numerical simulations in Section 4.3, we can see that the orbits form an attractor resembling a limit cycle cone, which has the property of transitioning from period-doubling bifurcation to weak chaos through a certain law. In what follows we are to investigate the relationship between the fractional order and the chaotic behavior of the system in terms of

Lyapunov exponents and C_0 complexity based on the asymptotic solution of the system, further illustrating the role of the fractional order in the fractional-order dynamic system.

The maximum Lyapunov exponent is a standard criteria to verify existence of chaos for a dynamical system. The maximum Lyapunov exponent of the system (1) is plotted in Figure 4. It can be observed that the positivity of the maximum Lyapunov exponent confirms the chaotic trait with $\alpha \in (0.15, 1)$.

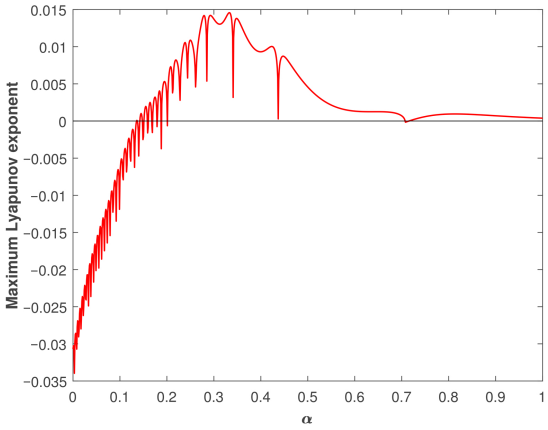


Figure 4. The maximum Lyapunov exponent with the fractional order.

It is known that, the higher chaoticity of the system, the more complex the dynamic behavior of the dynamical system is. Here, we use the C_0 Algorithm 1 measure the structural complexity of the state variables S, I, and Q, which are plotted in Figure 5a–c, respectively. We introduce the C_0 complexity Algorithm 1 for completeness of this research [46].

From Figure 6a–c, it can be seen that the structural complexity of the state variables firstly increases and then decreases with the fractional order, which is basically consistent with that of the maximum Lyapunov exponent illustrated in Figure 4. This also implies that the dynamical system (1) has weak chaotic dynamic behavior.

Algorithm 1 C_0 Algorithm

Step 1. Assume that $\{X(k), k = 0, 1, \cdots, N-1\}$ with the mean square value

$$G_N = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2. \tag{46}$$

By introducing an auxiliary parameter r, let

$$\tilde{X}(k) = \begin{cases} X(k), |X(k)|^2 > rG_N, \\ 0, |X(k)|^2 \leq rG_N. \end{cases} \tag{47}$$

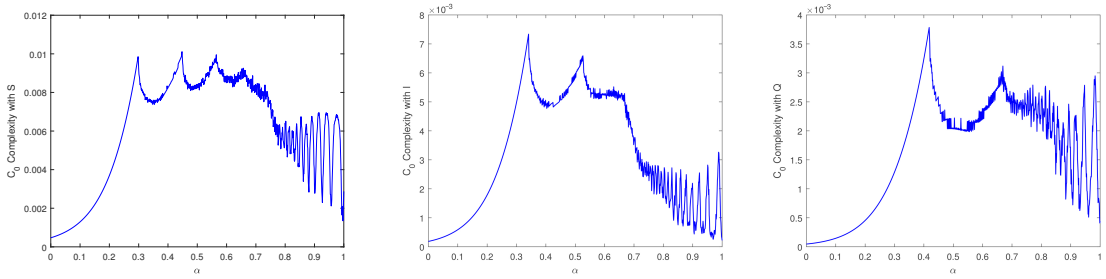
Step 2. Perform the inverse Fourier transform on the sequence $\tilde{X}(k)$, we have

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e^{j\frac{2\pi}{N}nk}, \tag{48}$$

where $n = 0, 1, \cdots, N-1$.

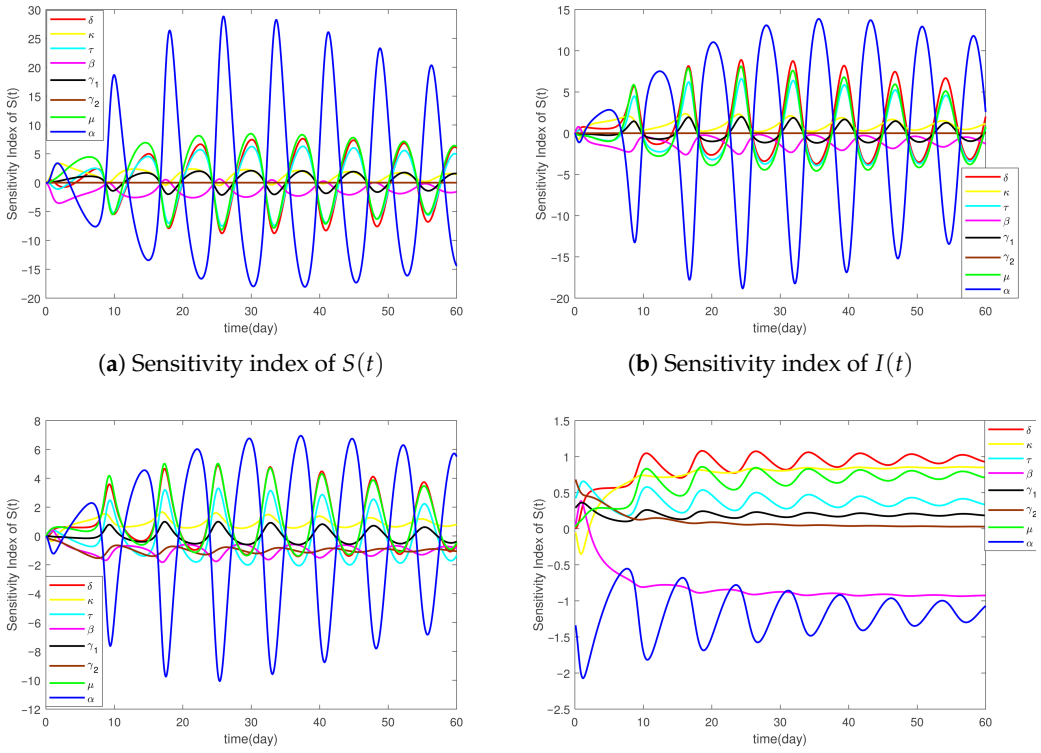
Step 3. Calculate the C_0 complexity by the expression:

$$C_0(r, N) = \sum_{n=0}^{N-1} |x(n) - \tilde{x}(n)|^2 / \sum_{n=0}^{N-1} |x(n)|^2. \tag{49}$$



(**a**) Complexity of S (**b**) Complexity of I (**c**) Complexity of Q

Figure 5. The C_0 complexity of S, I and Q with the fractional order.



(**a**) Sensitivity index of $S(t)$ (**b**) Sensitivity index of $I(t)$

(**c**) Sensitivity index of $Q(t)$ (**d**) Sensitivity index of $R(t)$

Figure 6. Sensitivity index with time in Example 4.

6. Sensitivity Analysis

When determining how to better reduce human mortality and morbidity caused by diseases, it is necessary to understand the relative importance of different factors contributing to disease transmission and epidemic spread. This understanding can be achieved through sensitivity index calculated from computational models. These sensitivity indices indicate the relative importance of each parameter in disease transmission and epidemic spread. Sensitivity index measure the relative change in state variables when parameters are altered. Assuming Y represents the original output value of a state variable, X_i denotes the original value of parameter i, and ΔY signifies the change in the output result, and ΔX_i represents the change in parameter i, then the local sensitivity index for a parameter can be defined as:

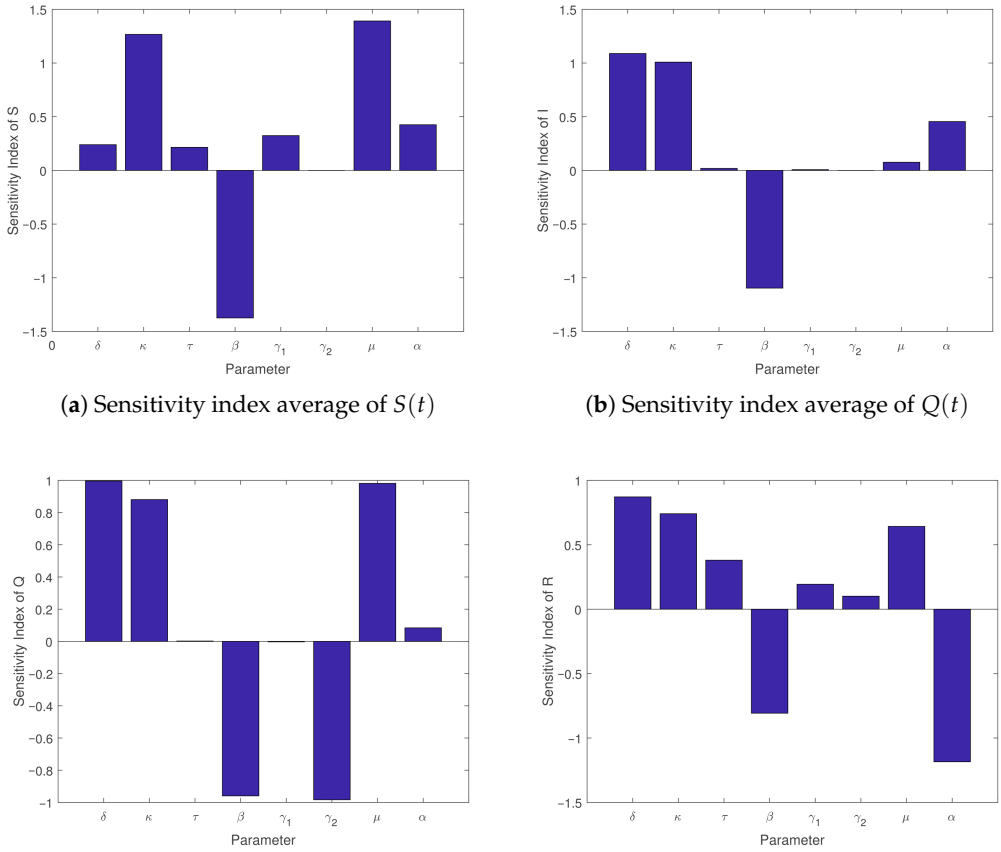
$$S_i = \frac{\Delta Y / Y}{\Delta X_i / X_i}, \tag{50}$$

where $Y = [S(t), I(t), Q(t), R(t)]$ and $X_i = [\delta, \kappa, \tau, \beta, \gamma_1, \gamma_2, \mu, \alpha]$.

Example 4. *By taking the same parameters as in Example 2, and with the factional orders* $\alpha = 0.6$, $\Delta X_i = \theta X_i$, *where* $\theta = 0.05$. *Plot the sensitivity index of each state variable with respect to each parameter* X_i *over time in Figure 6, and plot the average sensitivity index in Figure 7.*

From Figure 6a–d, we can see that the sensitivity index of the fractional order α exhibits significant fluctuations and exerts the greatest influence on each state variable. This further demonstrates that the inclusion of the fractional order greatly enriches the dynamical behavior of the system, conferring substantial advantages over integer-order systems in terms of dynamic characteristics. Additionally, it is evident that the presence of the time-scale factor also plays a notable role in the system, which provides further justification for its introduction into fractional-order epidemic models.

From the average sensitivity index in Figure 7a–d, we observe that the transmission rate β and the quarantine rate μ exert the most significant influence on the system. Appropriately increasing the quarantine rate can help reduce the transmission rate, thereby protecting susceptible individuals from infection. This also underscores the necessity of implementing quarantine measures in infectious disease prevention and control. Furthermore, an effective approach is medical intervention, such as enhancing the recovery rate, which contributes to mitigating and controlling the spread of the epidemic.



(**a**) Sensitivity index average of $S(t)$ (**b**) Sensitivity index average of $Q(t)$

(**c**) Sensitivity index average of $I(t)$ (**d**) Sensitivity index average of $R(t)$

Figure 7. Sensitivity index average in Example 4.

7. Discussion

The highlights and findings of this study are given below:

Firstly, for the well-posedness of the FSIQR model, we not only prove the existence, uniqueness, and positivity of the solutions, but also extend the existence and uniqueness to a global scale. Secondly, we employed the L–ADM to solve the model. Through theoretical

analysis, we proved the convergence of the asymptotic solution and derived a computable formula for its truncation dimension. These are important in mathematics for utilization of the ADM algorithm. Thirdly, the system complexity and chaos induced by fractional-order derivatives were analyzed, and the role of the fractional order in epidemic dynamics was investigated. To our knowledge, research on the role of the fractional order in dynamics is scarce in related fields.

This study primarily focuses on theoretical analysis and numerical simulation, which imposes limitations on research that integrates actual infectious disease data. Although the exploration of the role of fractional orders in the model presents notable strengths, relevant studies still require further in-depth investigation. In light of the existing limitations, we aim to conduct further research on the model and the role of fractional orders by incorporating actual infectious disease data in future work.

8. Conclusions

This article investigates a fractional-order SIQR epidemic model. The existence, uniqueness, and positivity of the solution to the model are proved, and extend the existence and uniqueness to a global scale. By employing the characteristic root method and constructing appropriate Lyapunov functions, both the local and global stability of the equilibrium points of the model are established. We give a proof of the convergence of the ADM algorithm for a nonlinear fractional epidemic system, and give a computable expression of the asymptotic solution. This study utilizes the maximum Lyapunov exponent and C_0 complexity to reveal the relationship between the fractional order and chaos based on the asymptotic solution. We find that the complexity and the chaotic behaviors of the fractional-order system weaken as the fractional order becomes large, which is the main contribution of this study. Finally, the relative roles of parameters in the model are analyzed using local sensitivity indices, further illustrating the advantages of the fractional-order model over its integer-order counterpart and providing a reference basis for infectious disease prevention and control measures.

Author Contributions: Conceptualization, P.L., Z.W. and G.L.; methodology, P.L., Z.W. and G.L.; validation, Z.W.; formal analysis, P.L., Z.W. and G.L.; investigation, P.L.; writing—original draft preparation, P.L.; writing—review and editing, P.L.; visualization, P.L.; funding acquisition, P.L. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: Data are contained within the article.

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Abbreviations

The following abbreviations are used in this manuscript:

FSIQR	fractional-order SIQR epidemic model
L–ADM	Laplace–Adomian decomposition method.
ADM	Adomian decomposition method

HPM	Homotopy Perturbation Method
VIM	Variational Iteration Method
HAM	Homotopy Analysis Method

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