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Nonlinear Fractal Interpolation Functions Induced by General Integral Contractions

Taoufik Moulahi [1] **and Najmeddine Attia** [2,*]

[1] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, P.O. Box 173, Al-Kharj 11942, Saudi Arabia; t.moulahi@psau.edu.sa
[2] Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia
* Correspondence: nattia@kfu.edu.sa

Abstract

The development of fractal set theory has been strongly driven by the introduction of new classes of fractal sets, among which the attractors of iterated function systems (IFSs) play a central role. In this work, we study a generalization of the classical IFS framework leading to the construction of fractal interpolation functions (FIFs) in which the standard linear ordinate scaling is replaced by a nonlinear contraction. This modification gives rise to a new family of FIFs associated with contractions of the general integral type, offering a flexible and robust approach for the approximation of experimental and irregular data. Furthermore, we introduce a class of generalized iterated function systems defined by mappings acting on product spaces of the form $f : \mathbb{X}^m \to \mathbb{X}$, with $m \in \mathbb{N}^*$. We prove the existence and uniqueness of the corresponding attractor, thereby extending several classical results from the theory of IFS and fractal interpolation.

Keywords: iterated function systems; fractal interpolation function; nonlinear contraction

1. Introduction and Preliminaries

Fractal interpolation functions (FIFs) arise in the framework of self-similar structures generated by iterated function systems (IFSs). They provide a method for interpolating a finite data set by means of a continuous function whose graph possesses fractal properties. Instead of constructing smooth interpolants, this method relies on the existence of an invariant compact set generated by a suitable family of mappings. Within a complete metric setting, such an invariant set is obtained as the fixed point of an operator induced by an IFS, and its graphical representation yields the desired interpolating function. This approach extends classical interpolation and is particularly adapted to situations where irregular or self-referential structures are prescribed. We briefly recall below the basic underlying framework.

Let $(\mathbb{X}, d)$ be a complete metric space, and denote by $\mathcal{H}(\mathbb{X})$ the set of all nonempty compact subsets of $\mathbb{X}$. For two sets $A, B \in \mathcal{H}(\mathbb{X})$, define the excess of A over B by

$$D(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y),$$

and introduce the Hausdorff distance as

$$d_H(A, B) = \max\{D(A, B), D(B, A)\}.$$

It is well known that $(\mathcal{H}(\mathbb{X}), d_H)$ is complete and becomes compact whenever the underlying space $\mathbb{X}$ is compact [1]. Fix an integer $N \geq 1$ and let $J = \{1, 2, \ldots, N\}$. Consider

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a finite family of continuous mappings $w_n : \mathbb{X} \to \mathbb{X}$, $n \in J$. A map w_n is called a contraction in the sense of Banach if there exists a constant $\alpha \in [0, 1)$ such that

$$d\big(w_n(x), w_n(y)\big) \leq \alpha\, d(x, y), \qquad \forall\, x, y \in \mathbb{X}. \tag{1}$$

These mappings induce an operator $W : \mathcal{H}(\mathbb{X}) \to \mathcal{H}(\mathbb{X})$ defined by

$$W(B) = \bigcup_{n=1}^{N} w_n(B), \qquad B \in \mathcal{H}(\mathbb{X}), \tag{2}$$

where $w_n(B) = \{w_n(x) : x \in B\}$. The collection $\mathbb{I} = \{\mathbb{X}, w_1, \ldots, w_N\}$ is referred to as an iterated function system. A compact set $G \in \mathcal{H}(\mathbb{X})$ satisfying the invariance relation $W(G) = G$ is called an attractor of the system. It is known that every IFS admits at least one attractor [2]. When all the mappings w_n satisfy the contraction condition (1), the system is said to be hyperbolic. In this situation, the operator W itself becomes a contraction on $(\mathcal{H}(\mathbb{X}), d_H)$, that is,

$$d_H\big(W(A), W(B)\big) \leq c\, d_H(A, B), \qquad \forall\, A, B \in \mathcal{H}(\mathbb{X}),$$

for some constant $c \in (0, 1)$. As a consequence, the attractor is unique and can be obtained as the limit

$$G = \lim_{k \to \infty} W^k(B),$$

for any initial compact set $B \in \mathcal{H}(\mathbb{X})$, where W^k denotes the k-fold iteration of W [2]. The theory of IFSs was developed by Barnsley and Hutchinson [1,3] in the setting of compact subsets of $\mathbb{R}^n$. This framework has since been extended to general metric spaces and to more general classes of contractive mappings. These developments have emphasized the central role played by IFSs in both theoretical investigations and practical applications [4–7].

The existence of an attractor for an iterated function system follows from fixed point principles. Within this context, fractal interpolation functions naturally appear as fixed points of the Read–Bajraktarević operator acting on appropriate spaces of continuous functions. This approach was introduced by Barnsley in 1986 [2] as a method for interpolating finite data sets, and it was shown to yield interpolation schemes that are particularly well suited for modeling complex phenomena encountered in physical and applied sciences. Since then, fractal interpolation has evolved into a flexible and powerful tool with numerous applications. Various families of fractal interpolation functions have been proposed over the years by exploiting different extensions of the classical Banach fixed point theorem. These include constructions based on Rakotch- and Matkowski-type contractions [4,8], as well as approaches relying on Geraghty-type contractive conditions [9]. Alongside these constructions, many essential properties of FIFs, such as their fractal dimension, analytical behavior, sensitivity to perturbations, smoothness, and stability, have been studied in depth [10,11].

More recently, a further generalization was proposed in [12], where fractal interpolation functions were constructed using mappings satisfying contractive conditions of integral type. To recall this framework, let Φ denote the class of functions $\varphi : [0, \infty) \to [0, \infty)$ that are non-negative, Lebesgue integrable, locally summable on $[0, \infty)$, and satisfy

$$\int_0^{\epsilon} \varphi(t)\, dt > 0, \qquad \forall\, \epsilon > 0.$$

A mapping $f : \mathbb{X} \to \mathbb{X}$ is said to be a contraction of an integral type, or a Branciari contraction, if there exist $\varphi \in \Phi$ and a constant $c \in (0, 1)$ such that

$$\int_0^{d(f(x), f(y))} \varphi(t)\, dt \leq c \int_0^{d(x,y)} \varphi(t)\, dt, \qquad \forall\, x, y \in \mathbb{X}. \tag{3}$$

Any mapping satisfying (3) is necessarily continuous and admits a unique fixed point $a \in \mathbb{X}$, with the property that

$$\lim_{k\to\infty} f^k(x) = a, \qquad \forall\, x \in \mathbb{X},$$

as established in [13]. By taking $\varphi \equiv 1$, one recovers the classical Banach contraction principle [14,15], showing that Branciari's condition genuinely extends the standard fixed point framework. It is also known that contractions of the integral type belong to several other well-studied classes of contractive mappings. In particular, every Branciari contraction is a Meir–Keeler contraction [16]. More generally, it has been shown that integral-type contractive conditions can be equivalently rewritten without the explicit use of integrals ([17] Theorem 2). As a consequence, any Branciari contraction f can be interpreted as a Browder contraction [18], that is,

$$d\big(f(x), f(y)\big) \le \bar{\varphi}\big(d(x,y)\big), \qquad \forall\, x, y \in \mathbb{X},$$

for some nondecreasing, right upper semicontinuous function $\bar{\varphi} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\bar{\varphi}(t) < t$ for all $t > 0$.

Following the introduction of contractions of the integral type by Branciari [13], fixed point theory has witnessed a growing interest in contractive conditions expressed through integral inequalities. Numerous extensions of Branciari's original result have since been proposed. For instance, in [19,20], the authors investigated mappings satisfying more general inequalities of the form

$$\int_0^{d(f(x),f(y))} \varphi(t)\,dt \le c \int_0^{M(x,y)} \varphi(t)\,dt, \qquad \forall\, x, y \in \mathbb{X},$$

or

$$\int_0^{d(f(x),f(y))} \varphi(t)\,dt \le \alpha(d(x,y)) \int_0^{d(x,y)} \varphi(t)\,dt, \qquad \forall\, x, y \in \mathbb{X},$$

where $c \in (0,1)$, $\varphi \in \Phi$, and $\alpha : \mathbb{R}_+ \to [0,1)$ satisfies $\limsup_{s\to t} \alpha(s) < 1$ for every $t > 0$. In this setting, the auxiliary function M is defined by

$$M(x,y) = \max\Big\{d(x,y),\ d(x,f(x)),\ d(y,f(y)),\ \frac{1}{2}\big(d(x,f(y)) + d(y,f(x))\big)\Big\}, \qquad x, y \in \mathbb{X}.$$

Motivated by these developments, and inspired by the recent results in [12], we propose a new framework for the construction of fractal interpolation functions based on contractive conditions of the general integral type. The main results of this paper extend and unify several existing fixed point theorems, including ([21] Theorem 3.1), ([22] Theorem 3.4), ([12] Theorem 5), and the classical attractor theorem for IFS ([2] Theorem 1). To this end, we introduce the following classes of control functions. Let $\mathcal{A}$ be the set of functions $h : \mathbb{R}_+ \to \mathbb{R}_+$ strictly increasing and consider the sets

$$\Phi_1 = \Big\{\phi \in \mathcal{A} \text{ right upper semicontinuous},\ \phi(0) = 0,\ \phi(t) < t\ \forall\, t > 0\Big\},$$

and

$$\Phi_2 = \Big\{\psi \in \mathcal{A}\ \text{ continuous at } 0 \text{ and left continuous on } (0,\infty), \psi(0) = 0,$$
$$\lim_{n\to\infty} t_n = 0 \text{ whenever } \lim_{n\to\infty} \psi(t_n) = 0\Big\}.$$

Throughout the paper, we denote by $\mathbb{N}^* = \{1, 2, 3, \dots\}$ the set of positive integers. Let $m \in \mathbb{N}^*$ and consider a mapping $f : \mathbb{X}^m \to \mathbb{X}$, where the product space $\mathbb{X}^m$ is endowed with the metric

$$d_m\big((x_1,\dots,x_m),(y_1,\dots,y_m)\big) = \max\{d(x_1,y_1),\dots,d(x_m,y_m)\}. \tag{4}$$

We say that f satisfies a contraction condition of the general integral type (CGIT) if there exists a triple $(\varphi,\phi,\psi) \in \Phi \times \Phi_1 \times \Phi_2$ such that, for every $j \in \{1,\dots,m\}$,

$$\int_0^{\psi(d(f(u),f(v)))} \varphi(t)\,dt \le \phi\left(\int_0^{\psi(d(x_j,y_j))} \varphi(t)\,dt\right), \tag{5}$$

for all $u = (x_1,\dots,x_m)$ and $v = (y_1,\dots,y_m)$ in $\mathbb{X}^m$.

In the particular case $m = 1$, any self-mapping $f : \mathbb{X} \to \mathbb{X}$ satisfying (5) admits a unique fixed point $a \in \mathbb{X}$, and the iterates of f converge to a for every initial point $x_0 \in \mathbb{X}$, that is,

$$\lim_{k\to\infty} f^k(x_0) = a.$$

This result, established in ([21] Theorem 3.1), may be regarded as a unified generalization of both Banach's contraction principle and Branciari's integral-type fixed point theorem [13,14].

The introduction of contractions of the general integral type is motivated by the limitations of classical contractive frameworks in fractal interpolation. Banach contractions impose a uniform linear condition, which excludes many nonlinear mappings arising in practical applications. Although Branciari contractions [13] extend the Banach framework, they still require a uniform constant $c < 1$, and thus cannot handle mappings that are asymptotically noncontractive at small scales [17,19]. Such situations naturally appear in fractal interpolation, where nonlinear or saturating behaviors are frequently observed [2,10]. The use of contractions of the general integral type overcomes these limitations by allowing a functional form of contraction, thereby providing a broader and more flexible framework consistent with modern fixed point theory [16,18]. To illustrate this framework, we examine a simple one-dimensional example. Let $\mathbb{X} = [0,1]$ equipped with the usual Euclidean distance and consider the mapping

$$f(x) = \frac{2x}{2+x}.$$

A direct observation shows that this mapping does not satisfy the Banach contraction principle. Indeed, one computes

$$\sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|} = \sup_{x\neq y} \frac{4}{(2+x)(2+y)} = 1,$$

which excludes the existence of a Lipschitz constant strictly smaller than one. Moreover, since

$$f'(x) = \frac{4}{(2+x)^2}, \qquad \text{and in particular } f'(0) = 1,$$

we deduce that the contraction behavior degenerates near the origin. This observation also rules out the possibility for f to be a Branciari contraction. Indeed, for any admissible $\varphi \in \Phi$, one obtains

$$\lim_{x,y\to 0} \frac{\displaystyle\int_0^{|f(x)-f(y)|} \varphi(t)\,dt}{\displaystyle\int_0^{|x-y|} \varphi(t)\,dt} = 1,$$

and hence no constant $c \in (0,1)$ can satisfy the integral-type inequality of Branciari. Nevertheless, the mapping f can be shown to fulfill a more general integral contractive condition. To this end, define

$$\begin{cases} \varphi(t) = t^{1/t-2}(1-\ln t), & t>0, \quad \varphi(0)=0, \\ \phi(t) = s\,t, & s \in (0,1), \\ \psi(t) = \gamma t, & \gamma \in (0,1). \end{cases}$$

With these choices, inequality (5) is satisfied. More precisely, a straightforward computation shows that

$$\begin{aligned} (5) &\iff |\gamma f(x) - \gamma f(y)|^{|\gamma f(x)-\gamma f(y)|^{-1}} \le s\,|\gamma x - \gamma y|^{|\gamma x-\gamma y|^{-1}} \\ &\iff \Big|\frac{\gamma(x-y)}{(1+x)(1+y)}\Big|^{(1+x)(1+y)/|\gamma(x-y)|} \le s\,|\gamma(x-y)|^{|\gamma(x-y)|^{-1}}. \end{aligned}$$

Since

$$\Big|\frac{1}{(1+x)(1+y)}\Big|^{1/|\gamma(x-y)|} \le 1 \quad \text{for all } x,y \in [0,1],$$

and using the elementary estimate

$$2|x-y| \le (1+x)(1+y),$$

we conclude that (5) holds for $s = \gamma/2$. This example shows that the CGIT condition strictly extends both Banach and Branciari contractions.

In this work, we exploit this generalized contractive structure to study a class of generalized IFSs (GIFS) generated by mappings of the form $f : \mathbb{X}^m \to \mathbb{X}$, with $m \ge 1$, satisfying a contraction condition of the general integral type. In Section 2, we prove that such a GIFS admits a unique attractor G. In the special case $m = 1$, we show in Section 3 that the attractor coincides with the graph of a continuous FIF associated with a prescribed data set (see Theorem 2). Finally, we consider a perturbed and establish a sufficient condition ensuring the preservation of continuity, thereby extending the earlier results in [23,24].

2. GIFS Generated by Contractions of General Integral Type

Let $(\mathbb{X}, d)$ be a complete metric space and let $m \in \mathbb{N}^*$. Consider a finite family of mappings

$$f_n : \mathbb{X}^m \longrightarrow \mathbb{X}, \qquad n \in J,$$

each satisfying a contraction condition of the general integral type (CGIT). The collection

$$\mathbb{I} = \{\mathbb{X}^m; f_1, f_2, \ldots, f_N\}$$

as a GIFS. To this system we associate a set-valued operator

$$F : \mathcal{H}(\mathbb{X})^m \longrightarrow \mathcal{H}(\mathbb{X}),$$

defined for any $(B_1, \ldots, B_m) \in \mathcal{H}(\mathbb{X})^m$ by

$$F(B_1, \ldots, B_m) = \bigcup_{n=1}^{N} f_n(B_1, \ldots, B_m). \tag{6}$$

A compact set $G \in \mathcal{H}(\mathbb{X})$ is said to be an attractor of the GIFS if it satisfies the invariance relation

$$F(G, G, \ldots, G) = G.$$

Before addressing the existence of such an attractor, we establish a fundamental fixed point property. More precisely, we prove that any mapping $f : \mathbb{X}^m \to \mathbb{X}$ fulfilling a contraction condition of general integral type admits a unique diagonal fixed point, namely a point $x \in \mathbb{X}$ such that

$$f(x, x, \ldots, x) = x.$$

This result is stated in Proposition 1 and extends earlier fixed point theorems obtained in ([21] Theorem 3.1) and ([22] Theorem 3.4). This result will be used to prove that the associated GIFS admits a unique attractor (Theorem 1).

Proposition 1. *Let $(\mathbb{X}, d)$ be a complete metric space and let $m \in \mathbb{N}^*$. If the mapping $f : \mathbb{X}^m \to \mathbb{X}$ satisfies the contraction condition of general integral type (5), then f admits a unique fixed point.*

Proof. Define a self-mapping $T : \mathbb{X} \to \mathbb{X}$ by

$$T(x) = f(x, \ldots, x), \qquad x \in \mathbb{X}.$$

When $m > 1$, applying condition (5) with $u = (x, \ldots, x)$ and $v = (y, \ldots, y)$ yields

$$\int_0^{\psi\left(d(T(x),T(y))\right)} \varphi(t)\, dt \leq \phi\Big(\int_0^{\psi\left(d(x,y)\right)} \varphi(t)\, dt \Big), \qquad \forall\, x, y \in \mathbb{X},$$

for some $(\varphi, \phi, \psi) \in \Phi \times \Phi_1 \times \Phi_2$. Thus, the mapping T itself satisfies a contraction condition of general integral type. By the fixed point theorem of ([21] Theorem 3.1), T admits a unique fixed point $a \in \mathbb{X}$. It follows immediately that $f(a, \ldots, a) = a$, and uniqueness is inherited from that of T. □

Example 1. *In this example we take $m = 2$ and consider the metric space*

$$\mathbb{X} = [0, 1] \cup \{4\}$$

equipped with the Euclidean distance. We define a mapping $f : \mathbb{X}^2 \to \mathbb{X}$ by

$$f(x, y) = \begin{cases} \dfrac{x}{3}, & \text{if } x, y \in [0, 1], \\ \dfrac{1}{3}\dfrac{y}{1+y}, & \text{if } x = 4 \text{ and } y \neq 4, \\ 0, & \text{if } y = 4, \end{cases}$$

together with the auxiliary functions

$$\phi(t) = \begin{cases} \dfrac{t}{3}, & 0 \leq t \leq 1, \\ \dfrac{t^2}{1+t}, & t > 1, \end{cases} \qquad \varphi(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ e^t, & t > 1, \end{cases}$$

and

$$\psi(t) = \begin{cases} \dfrac{t}{2}, & 0 \leq t \leq 1, \\ t^2, & t > 1. \end{cases}$$

We verify that f satisfies the contraction condition of the general integral type (5). Let $u = (x, y)$ and $v = (x', y')$ be arbitrary elements of $\mathbb{X}^2$. Since the situation $y = y' = 4$ is immediate, we restrict to the case $(y, y') \neq (4, 4)$ and analyze the remaining configurations.
(i) Both points in $[0, 1]^2$**.** *In this case $f(u) = x/3$ and $f(v) = x'/3$, and hence*

$$\int_0^{\psi(d(f(u),f(v)))} \varphi(t)\,dt = \int_0^{\psi\left(\frac{1}{3}|x-x'|\right)} \varphi(t)\,dt = \frac{1}{6}|x-x'|.$$

On the other hand,

$$\frac{1}{6}|x-x'| = \phi\left(\int_0^{\frac{1}{2}|x-x'|} \varphi(t)\,dt\right) \leq \phi\left(\int_0^{\psi(d_2(u,v))} \varphi(t)\,dt\right).$$

(ii) $f(u) = \frac{1}{3}\frac{y}{1+y}$ ***and*** $f(v) = 0$. *This occurs when* $x = 4$, $y \neq 4$ *and* $y' = 4$ *(the symmetric configuration is analogous). Then,*

$$\int_0^{\psi(d(f(u),f(v)))} \varphi(t)\,dt \leq \int_0^{\psi\left(\frac{y}{3}\right)} \varphi(t)\,dt = \frac{y}{6}.$$

Since ϕ *is strictly increasing on* $(1,\infty)$ *and satisfies*

$$\phi(t) > \frac{1}{2} \qquad \text{for all } t > 1,$$

and since $|y-y'| > 2$, *we deduce*

$$\frac{y}{6} < \phi\left(e^{|y-y'|^2} + 1 - e\right) = \phi\left(\int_0^{\psi(|y-y'|)} \varphi(t)\,dt\right) \leq \phi\left(\int_0^{\psi(d_2(u,v))} \varphi(t)\,dt\right).$$

(iii) $f(u) = \frac{1}{3}\frac{y}{1+y}$ ***and*** $f(v) = \frac{1}{3}\frac{y'}{1+y'}$. *This corresponds to* $x = x' = 4$ *with* $y, y' \neq 4$, *we obtain*

$$\int_0^{\psi(d(f(u),f(v)))} \varphi(t)\,dt \leq \int_0^{\psi\left(\frac{1}{3}|y-y'|\right)} \varphi(t)\,dt = \frac{1}{6}|y-y'|.$$

As before, this quantity is controlled by the right-hand side of (5).

(iv) $f(u) = x/3$ ***and*** $f(v) = 0$. *This occurs when* $x, y \neq 4$ *and* $y' = 4$, *then which yields*

$$\int_0^{\psi(d(f(u),f(v)))} \varphi(t)\,dt = \int_0^{\psi\left(\frac{x}{3}\right)} \varphi(t)\,dt \leq \frac{1}{6}.$$

Using again the growth of ϕ *on* $(1,\infty)$, *the desired inequality follows.*

(v) $f(u) = \frac{x}{3}$ ***and*** $f(v) = \frac{1}{3}\frac{y'}{1+y'}$. *This occurs when* $x, y \neq 4$ *while* $x' = 4$ *and* $y' \neq 4$. *Then*

$$\int_0^{\psi(d(f(u),f(v)))} \varphi(t)\,dt \leq \int_0^{\psi\left(\frac{2}{3}\right)} \varphi(t)\,dt \leq 1 + e^{\frac{1}{4}|x-x'|^2} - e,$$

which is again dominated by the right-hand side of (5).

Combining all cases, we conclude that f *satisfies the contraction condition of the general integral type.*

Remark 1. *The fixed point conclusion of Proposition 1 may fail if either the positivity of* ψ *or the strict inequality* $\phi(t) < t$ *is not satisfied; see Example 2.*

Example 2. *1. This example illustrates that although condition (5) may hold when* ψ *takes negative values, it may become degenerate and fail to ensure the fixed point conclusion of Proposition 1; hence the assumption* $\psi \in \Phi_2$ *is essential. Let* $\mathbb{X} = \mathbb{R}_+$, *endowed with the Euclidean metric, and let* $m = 1$. *Define* $f : \mathbb{X} \to \mathbb{X}$ *by*

$$f(x) = x + 2,$$

and consider the functions $\varphi, \phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ *given by*

$$\varphi(t) = \alpha > 0, \qquad \phi(t) = s\,t, \qquad \psi(t) = -t,$$

where $s \in (0,1)$. *Then, for all* $x, y \in \mathbb{X}$, *we have*

$$\begin{aligned}\int_0^{\psi(d(f(x),f(y)))} \varphi(t)\,dt &= \int_0^{-|f(x)-f(y)|} \alpha\,dt = -\alpha\,|f(x)-f(y)| \\ &= -\alpha\,d(x,y) \le -s\,\alpha\,d(x,y) \\ &= \phi(-\alpha\,d(x,y)) = \phi\left(\int_0^{\psi(d(x,y))} \varphi(t)\,dt\right).\end{aligned}$$

Hence, the inequality (5) is satisfied. However, the mapping f *is a translation on* $\mathbb{R}_+$ *and therefore admits no fixed point. This example shows that allowing* ψ *to take negative values destroys the fixed point property, which justifies the assumption* $\psi \ge 0$ *in the definition of contractions of general integral type.*

2. *In this example we illustrate that the condition (5) may hold even when* $\phi(t) = t$ *(so that the strict inequality* $\phi(t) < t$ *is violated), while the fixed point conclusion of Proposition 1 fails. Let* $m = 2$ *and* $\mathbb{X} = [0,1]$, *endowed with the Euclidean metric, and define* $f : \mathbb{X}^2 \to \mathbb{X}$ *by* $f(x,y) = x$. *Consider the functions* $\varphi, \phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ *given by*

$$\varphi(t) = 1, \qquad \phi(t) = t, \qquad \psi(t) = t.$$

Consequently, we have

$$\int_0^{\psi(d(f(u),f(v)))} \varphi(t)\,dt = |x_1 - x_2| = \phi\left(\int_0^{\psi(d(x_1,x_2))} \varphi(t)\,dt\right),$$

and hence the inequality (5) is satisfied. However, any point $(x,y) \in \mathbb{X}^2$ *with* x *arbitrary satisfies* $f(x,y) = x$, *so every* $x \in [0,1]$ *is a fixed point of* f. *Therefore,* f *admits infinitely many fixed points, and the conclusion of Proposition 1 fails. This shows that the strict contractivity condition* $\phi(t) < t$ *is essential in Proposition 1.*

Theorem 1. *Let* $(\mathbb{X}, d)$ *be a complete metric space and let* $m \in \mathbb{N}^*$. *Assume that, for every* $n \in J$, *the mapping* $f_n : \mathbb{X}^m \longrightarrow \mathbb{X}$ *satisfies the contraction condition of the general integral type (5). Then the associated generalized*

$$\mathbb{I} = \{\mathbb{X}^m; f_1, f_2, \ldots, f_N\}$$

admits a unique invariant compact set $G \in \mathcal{H}(\mathbb{X})$, *which is the attractor of the system.*

Proof. Let $A = (A_1, \ldots, A_m)$ and $B = (B_1, \ldots, B_m)$ be two arbitrary elements of $\mathcal{H}(\mathbb{X})^m$. Fix an index $j \in \{1, \ldots, m\}$ and select a triple $(\varphi, \phi, \psi) \in \Phi \times \Phi_1 \times \Phi_2$ for which the general integral contractive condition (5) is satisfied by each mapping f_n. We first show that the associated set-valued operator $F : \mathcal{H}(\mathbb{X})^m \to \mathcal{H}(\mathbb{X})$ fulfills the inequality

$$\int_0^{\psi\left(d_H\left(F(A),F(B)\right)\right)} \varphi(t)\,dt \le \phi\left(\int_0^{\psi\left(d_H(A_j,B_j)\right)} \varphi(t)\,dt\right). \tag{7}$$

Once inequality (7) is proved, F satisfies the CGIT condition on $(\mathcal{H}(\mathbb{X}), d_H)$. Proposition 1 then guarantees the existence of a unique fixed point $G \in \mathcal{H}(\mathbb{X})$, which coincides with the unique attractor of the GIFS.

We now verify inequality (7). Let z be an arbitrary element of $F(A)$. By definition of F, there exist an index $n \in J$ and points $x_i \in A_i$ $(i = 1, \ldots, m)$ such that

$$z = f_n(x_1, \ldots, x_m).$$

Since B_j is compact, there exists $y_j^* \in B_j$ satisfying

$$d(x_j, y_j^*) = \inf_{y \in B_j} d(x_j, y).$$

Consequently,

$$d(x_j, y_j^*) \leq D(A_j, B_j) \leq d_H(A_j, B_j).$$

Using the monotonicity of ϕ and ψ, we obtain

$$\begin{aligned}
\int_0^{\psi\left(d(z,F(B))\right)} \varphi(t)\,dt &\leq \int_0^{\psi\left(d(z,f_n(B))\right)} \varphi(t)\,dt \\
&\leq \int_0^{\psi\left(d(f_n(x_1,\ldots,x_m),f_n(y_1^*,\ldots,y_m^*))\right)} \varphi(t)\,dt \\
&\leq \phi\left(\int_0^{\psi\left(d(x_j,y_j^*)\right)} \varphi(t)\,dt\right) \\
&\leq \phi\left(\int_0^{\psi\left(d_H(A_j,B_j)\right)} \varphi(t)\,dt\right),
\end{aligned}$$

where $y_i^* = x_i$ for $i \neq j$ and $d(z, F(B)) := \inf_{y \in F(B)} d(z, y)$ denotes the distance from the point z to the set $F(B)$ induced by the metric d on $\mathbb{X}$.

Since $F(A)$ is compact and the mapping $z \mapsto d(z, F(B))$ is continuous, taking the supremum over $z \in F(A)$ yields

$$\int_0^{\psi\left(D(F(A),F(B))\right)} \varphi(t)\,dt \leq \phi\left(\int_0^{\psi\left(d_H(A_j,B_j)\right)} \varphi(t)\,dt\right).$$

By symmetry, an analogous inequality holds with A and B interchanged, namely,

$$\int_0^{\psi\left(D(F(B),F(A))\right)} \varphi(t)\,dt \leq \phi\left(\int_0^{\psi\left(d_H(A_j,B_j)\right)} \varphi(t)\,dt\right).$$

Combining these two estimates and recalling the definition of the Hausdorff metric d_H, we obtain (7), which completes the proof. □

In the following, we relate the CGIT condition to a generalized Lipschitz framework. In particular, when the control function ϕ is linear, the CGIT condition reduces to a generalized Lipschitz-type inequality, and the associated fractal operator preserves the corresponding contractive constant. Consider the case $m = 2$ and, for a mapping $f : \mathbb{X}^2 \to \mathbb{X}$, define the associated operator $F_f : \mathcal{H}(\mathbb{X}) \times \mathcal{H}(\mathbb{X}) \to \mathcal{H}(\mathbb{X})$ as

$$F_f(B_1, B_2) = f(B_1, B_2) = \{f(x_1, x_2) : x_1 \in B_1,\ x_2 \in B_2\}. \tag{8}$$

Let $(\varphi, \psi) \in \Phi \times \Phi_2$. We introduce the following Lipschitz-type constant associated with f:

$$\mathcal{L}ip(f) = \sup_{j \in \{1,2\}} \sup_{\substack{u=(x_1,x_2),\, v=(y_1,y_2) \\ u \neq v}} \frac{\displaystyle\int_0^{\psi(d(f(u),f(v)))} \varphi(t)\,dt}{\displaystyle\int_0^{\psi(d(x_j,y_j))} \varphi(t)\,dt}, \tag{9}$$

whenever the denominator is nonzero. Similarly, we define the Lipschitz-type constant associated with the set-valued operator $F_f : \mathcal{H}(\mathbb{X})^2 \to \mathcal{H}(\mathbb{X})$ by

$$\mathcal{L}ip(F_f) = \sup_{j\in\{1,2\}} \sup_{\substack{A=(A_1,A_2),\, B=(B_1,B_2)\\ A\neq B}} \frac{\displaystyle\int_0^{\psi\left(d_H(F_f(A),F_f(B))\right)} \varphi(t)\,dt}{\displaystyle\int_0^{\psi\left(d_H(A_j,B_j)\right)} \varphi(t)\,dt}. \tag{10}$$

Proposition 2. *Consider the mapping $f : \mathbb{X}^2 \to \mathbb{X}$ such that $\mathcal{L}ip(f) < +\infty$. Then the operator F_f defined in (8) satisfies*

$$\mathcal{L}ip(F_f) \leq \mathcal{L}ip(f).$$

In particular, if we assume that $\mathcal{L}ip(f) < 1$, then the mapping F_f is a contraction of general integral type with $\phi(t) = \mathcal{L}ip(f)\, t$.

Proof. Let $A = (A_1, A_2)$ and $B = (B_1, B_2)$ be elements of $\mathcal{H}(\mathbb{X})^2$, and fix $j \in \{1,2\}$. Let $z \in F_f(A_1, A_2)$. Then there exists $(x_1, x_2) \in A_1 \times A_2$ such that

$$z = f(x_1, x_2).$$

Since B_j is compact, there exists $y_j^* \in B_j$ satisfying

$$d(x_j, y_j^*) = \inf_{y\in B_j} d(x_j, y).$$

Define $x_i^* = x_i$ for $i \neq j$. Then $(x_1^*, x_2^*) \in B_1 \times B_2$ and

$$d(x_j, y_j^*) \leq D(A_j, B_j) \leq d_H(A_j, B_j).$$

Using the monotonicity of ψ and the definition of $\mathcal{L}ip(f)$, we obtain

$$\begin{aligned}\int_0^{\psi(d(z,F_f(B_1,B_2)))} \varphi(t)\,dt &\leq \int_0^{\psi(d(f(x_1,x_2),f(x_1^*,x_2^*)))} \varphi(t)\,dt\\ &\leq \mathcal{L}ip(f) \int_0^{\psi(d(x_j,y_j^*))} \varphi(t)\,dt\\ &\leq \mathcal{L}ip(f) \int_0^{\psi(d_H(A_j,B_j))} \varphi(t)\,dt,\end{aligned}$$

where

$$d(z, F_f(B_1, B_2)) := \inf_{y\in F_f(B_1,B_2)} d(z, y)$$

denotes the distance from the point $z \in \mathbb{X}$ to the set $F_f(B_1, B_2)$ with respect to the metric d on $\mathbb{X}$. Since $F_f(A)$ is compact and the map $z \mapsto d(z, F_f(B))$ is continuous, taking the supremum over $z \in F_f(A)$ yields

$$\int_0^{\psi(D(F_f(A),F_f(B)))} \varphi(t)\,dt \leq \mathcal{L}ip(f) \int_0^{\psi(d_H(A_j,B_j))} \varphi(t)\,dt.$$

By symmetry, the same estimate holds with A and B interchanged. Hence,

$$\int_0^{\psi(d_H(F_f(A),F_f(B)))} \varphi(t)\,dt \leq \mathcal{L}ip(f) \int_0^{\psi(d_H(A_j,B_j))} \varphi(t)\,dt,$$

which proves the claim. This proves that $Lip(F_f) \leq Lip(f)$. Now consider the IFS $\mathbb{I} = \left\{\mathbb{X}^2, f_1, f_2, \ldots, f_N\right\}$ and the operator F defined by (6). Then

$$\mathcal{L}ip(F) \leq \max_n \mathcal{L}ip(f_n).$$

Indeed, let $c_F := \max_n Lip(f_n)$. Then

$$\begin{aligned} d_H(F(A), F(B)) &= d_H(\bigcup_{n=1}^{N} f_n(A_1, A_2), \bigcup_{n=1}^{N} f_n(B_1, B_2)) \\ &\leq \max_n d_H(f_n(A_1, A_2), f_n(B_1, B_2)) \\ &\leq c_F \max_n (d_H(A_1, B_1), d_H(A_2, B_2)). \end{aligned}$$

□

3. Fractal Interpolation Functions Under General Integral-Type Contractions

Let $\Delta = \{x_0 < x_1 < \cdots < x_N\}$ be a partition of a compact real interval $I = [x_0, x_N]$. We denote by $\mathcal{C}(I)$ the Banach space of real-valued continuous functions defined on I, endowed with the uniform norm. Throughout this section, we consider a finite data set

$$\{(x_n, y_n) \in I \times \mathbb{R} : n = 0, 1, \ldots, N\},$$

where the ordinate values satisfy $y_n \in [a, b]$ for some real numbers $a < b$. The ambient space $I \times [a, b]$ is viewed as a compact metric space equipped with the metric defined in (4).

3.1. Construction of a Fractal Interpolation Function

For each $n \in J = \{1, 2, \ldots, N\}$, let $I_n = [x_{n-1}, x_n]$. We associate to every subinterval I_n a contractive homeomorphism $L_n : I \to I_n$ satisfying

$$\begin{aligned} &L_n(x_0) = x_{n-1}, \qquad L_n(x_N) = x_n, \\ &|L_n(c_1) - L_n(c_2)| \leq l\,|c_1 - c_2|, \qquad \forall\, c_1, c_2 \in I, \end{aligned} \tag{11}$$

for some constant $0 \leq l < 1$. The family $\{L_n\}_{n \in J}$ induces a self-similar decomposition of the interval I that is consistent with the prescribed interpolation nodes. We next introduce a collection of continuous mappings $F_n : I \times [a, b] \to \mathbb{R}$ satisfying the endpoint conditions

$$F_n(x_0, y_0) = y_{n-1}, \qquad F_n(x_N, y_N) = y_n, \tag{12}$$

which guarantee that the interpolation constraints are met at each partition point. In addition, we impose a nonlinear contractive condition of the general integral type on the second variable. More precisely, we assume that

$$\int_0^{\psi\left(|F_n(x,y) - F_n(x,y')|\right)} \varphi(t)\,dt \leq \phi\left(\int_0^{\psi(|y-y'|)} \varphi(t)\,dt\right), \qquad x \in I,\ y, y' \in [a, b], \tag{13}$$

for some triple $(\varphi, \phi, \psi) \in \Phi \times \Phi_1 \times \Phi_2$. Condition (13) provides a nonlinear control of the vertical variation of the maps F_n and replaces the classical affine scaling assumption commonly used in standard fractal interpolation schemes. As such, it allows the construction of a broader class of fractal interpolation functions governed by contractions of the general integral type.

Example 3. *Consider the following system:*

$$\begin{cases} L_n(x) = a_n x + k_n, \\ F_n(x, y) = \dfrac{y}{1 + y} + q_n(x), \end{cases}$$

where the real constants a_n, k_n and the function q_n are chosen so that conditions (11) and (12) are satisfied. This class of functions introduces a nonlinear dependence on the vertical variable y, while preserving the interpolation constraints. Let

$$\phi(t) = \frac{t}{1+t}, \qquad \psi(t) = t, \qquad \varphi(t) = 2t.$$

Then, for all $x \in I$ and $y, y' \in [a,b] \subset [0,+\infty)$, we compute

$$\begin{aligned}\int_0^{\psi(|F_n(x,y)-F_n(x,y')|)} \varphi(t)\,dt &= \int_0^{|F_n(x,y)-F_n(x,y')|} 2t\,dt \\ &= \left(\frac{y}{1+y} - \frac{y'}{1+y'}\right)^2 = \frac{(y-y')^2}{(1+y)^2(1+y')^2} \\ &\le \frac{(y-y')^2}{(1+|y-y'|)^2} = \phi\left(\int_0^{\psi(|y-y'|)} \varphi(t)\,dt\right).\end{aligned}$$

Hence, condition (13) is fulfilled for this choice of (φ, ϕ, ψ). On the other hand, if one replaces ϕ by a linear function of the form $\phi(t) = c\,t$ with $0 < c < 1$, then (13) fails. Indeed, this would require

$$\frac{1}{(1+y)^2(1+y')^2} < c < 1, \qquad \forall\, y \neq y' \geq 0,$$

which is clearly impossible. This example therefore illustrates that the nonlinear choice of ϕ is essential and cannot, in general, be reduced to a linear contraction.

Now, we consider the function $w_n : I \times [a,b] \to I_n \times \mathbb{R}$ defined by

$$w_n(x,y) = \big(L_n(x), F_n(x,y)\big),$$

for all $n \in J$ and let $\mathbb{I} : \{I \times [a,b]\,,\ w_n\,,\ n = 1,2,\ldots,N\}$ be the associated IFS. Our main result in this section is the following.

Theorem 2. *Suppose that the functions F_n satisfy the integral-type contractive condition (13). Then, the $\mathbb{I}$ introduced above admits a unique attractor G. Moreover, this attractor is the graph of a continuous function $f : I \to [a,b]$ interpolating the prescribed data set, that is, $f(x_n) = y_n$ for all $n = 0,1,\ldots,N$.*

Condition (13) allows the construction of a broad class of interpolation schemes adapted to various approximation problems, thereby enhancing the flexibility of the fractal interpolation approach. The function f obtained in Theorem 2 is referred to as the fractal interpolation function (FIF) associated with the system $\mathbb{I}$. To formalize the fixed point framework, we introduce the closed subset

$$\mathcal{G} = \Big\{f \in \mathcal{C}(I)\ :\ f(I) \subset [a,b],\ f(x_0) = y_0,\ f(x_N) = y_N\Big\},$$

which naturally encodes the interpolation constraints and is endowed with the uniform norm. Given the families $\{L_n\}_{n=1}^N$ and $\{F_n\}_{n=1}^N$, we define the Read–Bajraktarević operator $\mathsf{T} : \mathcal{G} \to \mathcal{G}$ by

$$(\mathsf{T}f)(x) = F_n\big(L_n^{-1}(x),\, f(L_n^{-1}(x))\big), \qquad x \in I_n,\ n = 1,\ldots,N. \tag{14}$$

Lemma 1. *Assume that each mapping $F_n : I \times [a,b] \to \mathbb{R}$ satisfies (13). Then:*

1. *The operator T is a contraction of the general integral type on $\mathcal{G}$;*
2. *The unique fixed point of T interpolates the data set Δ.*

Proof. 1. Fix $n \in \{1, \dots, N\}$ and let $x \in I_n$. For arbitrary $f, g \in \mathcal{G}$, applying (13) with $y = f(L_n^{-1}(x))$ and $y' = g(L_n^{-1}(x))$ yields

$$\int_0^{\psi\left(|F_n(L_n^{-1}(x), f(L_n^{-1}(x))) - F_n(L_n^{-1}(x), g(L_n^{-1}(x)))|\right)} \varphi(t)\, dt \le \phi\left(\int_0^{\psi\left(|f(L_n^{-1}(x)) - g(L_n^{-1}(x))|\right)} \varphi(t)\, dt\right).$$

Since $|f(L_n^{-1}(x)) - g(L_n^{-1}(x))| \le \|f - g\|_\infty$ and both ϕ and ψ are increasing, we obtain

$$\int_0^{\psi\left(|(\mathsf{T}f)(x) - (\mathsf{T}g)(x)|\right)} \varphi(t)\, dt \le \phi\left(\int_0^{\psi(\|f-g\|_\infty)} \varphi(t)\, dt\right).$$

Taking the supremum over all $x \in I$, and since ψ is left continuous, leads to

$$\int_0^{\psi\left(\|\mathsf{T}f - \mathsf{T}g\|_\infty\right)} \varphi(t)\, dt \le \phi\left(\int_0^{\psi(\|f-g\|_\infty)} \varphi(t)\, dt\right),$$

which shows that T is a contraction of the general integraltype.

2. By the previous step and the completeness of $\mathcal{G}$, the operator T admits a unique fixed point $f \in \mathcal{G}$. By construction, f satisfies the functional Equation (14). Denoting by $G_f = \{(x, f(x)) : x \in I\}$ the graph of f, we compute

$$\begin{aligned} \bigcup_{n=1}^{N} w_n(G_f) &= \bigcup_{n=1}^{N} \{(L_n(x), F_n(x, f(x))) : x \in I\} \\ &= \bigcup_{n=1}^{N} \{(x, F_n(L_n^{-1}(x), f(L_n^{-1}(x)))) : x \in I_n\} \\ &= \bigcup_{n=1}^{N} \{(x, f(x)) : x \in I_n\} = \{(x, f(x)) : x \in I\}. \end{aligned}$$

Hence, G_f is invariant under the and coincides with its attractor. Finally, the interpolation property follows directly from the boundary conditions imposed on the functions F_n. Indeed, for $i = 0, 1, \dots, N-1$, we have $f(x_i) = y_i$ and $f(x_N) = y_N$. Therefore, the unique fixed point f of the operator T is a FIF whose graph interpolates the prescribed data set Δ and coincides with the attractor of the corresponding IFS.

□

Example 4. *Let* $\Delta := \Big\{(0,0), (1/4,1), (1/2,-1), (2/3,1/3), (1,0)\Big\}$ *be a prescribed interpolation data set. We consider the following:*

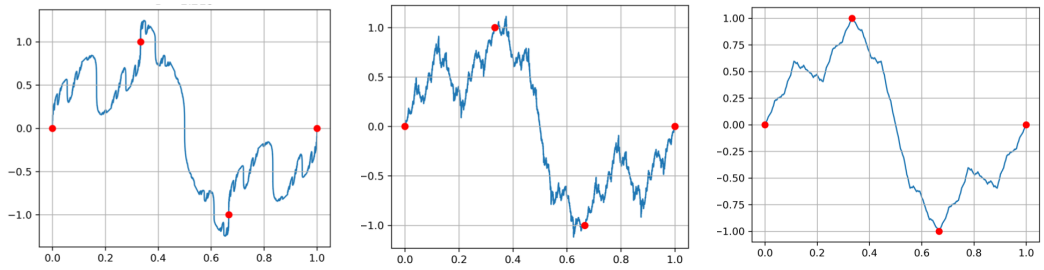
$$\begin{cases} L_1(x) = \frac{1}{4}x, \\ L_2(x) = \frac{1}{4}x + \frac{1}{4}, \\ L_3(x) = \frac{1}{6}x + \frac{1}{2}, \\ L_4(x) = \frac{1}{3}x + \frac{2}{3}, \end{cases} \qquad \begin{cases} F_1(x,y) = g_1(y) + x, \\ F_2(x,y) = g_2(y) - 2x + 1, \\ F_3(x,y) = g_3(y) + \frac{4}{3}x - 1, \\ F_4(x,y) = g_4(y) - \frac{1}{3}x + \frac{1}{3}. \end{cases} \tag{15}$$

When the functions g_n coincide, we simply denote them by a single function g. Figure 1 illustrates the corresponding FIFs obtained for different choices of g. The choice of the nonlinear function g affects the geometric regularity of the corresponding FIF. The geometric complexity is measured by the box-counting dimension

$$D = \lim_{r \to 0} \frac{\log \mathcal{N}_r}{-\log r},$$

where $\mathcal{N}_r$ denotes the minimal number of $r \times r$ squares required to cover the graph of the FIF. The box dimension is a widely used and effective tool for describing the geometric irregularity of fractal curves. In particular, the transition between smoother and more irregular FIFs is strongly governed by the choice of the function g.

We now compare this situation with the classical affine case introduced by Barnsley [2], where each F_n is a Banach contraction with respect to the second variable. Specifically, we take $g_n(y) = \alpha_n y$ with parameters $\alpha_n \in (-1, 1)$, $n = 1, 2, 3, 4$. The constants α_n are referred to as vertical scaling factors, and the corresponding FIF is usually denoted by f^α, with $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ (Figure 2).

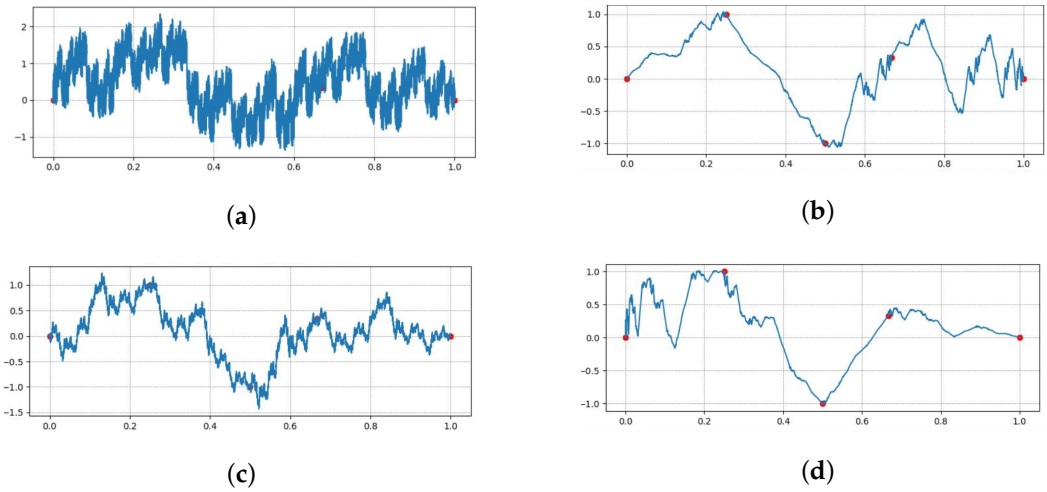


(a) **(b)** **(c)**

Figure 1. FIFs associated with system (15) for three different nonlinear vertical transformations g. **(a)** $g(y) = \dfrac{y}{2 + \frac{2}{5}y}$; **(b)** $g(y) = \dfrac{1}{2}\dfrac{\frac{3}{4}y}{1 + \frac{3}{4}y^2}$; **(c)** $g(y) = \dfrac{1}{5}\dfrac{\frac{3}{4}y}{1 + \frac{3}{4}y^2}$.

Since the interpolation points are uniformly distributed and are not collinear, one can apply the results of [25] to conclude that the box-counting dimension of the function f^α is given by

$$D = 1 + \frac{\log\left(\sum_{n=1}^{4} |\alpha_n|\right)}{\log(4)}, \qquad \text{whenever } \sum_{n=1}^{4} |\alpha_n| > 1. \tag{16}$$



(a) **(b)**

(c) **(d)**

Figure 2. Affine fractal interpolation functions associated with system (14) for different vertical scaling parameters α_n. **(a)** $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.75$, $D = 1.789$; **(b)** $\alpha_1 = 0.15$, $\alpha_2 = -0.1$, $\alpha_3 = -0.35$, $\alpha_4 = 0.65$, $D = 1.196$; **(c)** $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -0.5$, $D = 1.492$; **(d)** $\alpha_1 = 0.65$, $\alpha_2 = -0.3$, $\alpha_3 = -0.05$, $\alpha_4 = 0.15$, $D = 1.164$.

A natural question arising from this comparison is whether, in the general nonlinear setting, the box dimension of a FIF depends only on $\|g\|_\infty$. The examples displayed in Figure 1 suggest that this is not the case. Indeed, despite the fact that the norms of the functions involved are proportional, the resulting box dimensions exhibit distinct and non-monotonic behaviors. This indicates that the box dimension depends on finer properties of g than its uniform norm.

3.2. Proof of Theorem 2

Consider the $\{I \times [a,b],\ w_n,\ n \in J\}$ and assume that each mapping w_n satisfies a CGIT condition on the compact metric space $I \times [a,b]$. Under these assumptions, the associated Hutchinson operator satisfies the same CGIT condition, which ensures the existence of a unique invariant set.

Proposition 3. *Let $(\varphi, \phi, \psi) \in \Phi \times \Phi_1 \times \Phi_2$ and assume that, for every $n \in J$,*

$$\int_0^{\psi\left(d(w_n(u),w_n(v))\right)} \varphi(t)\,dt \leq \phi\left(\int_0^{\psi(d_m(u,v))} \varphi(t)\,dt\right), \qquad \forall\, u, v \in I \times [a,b].$$

Then the $\{I \times [a,b],\ w_n,\ n \in J\}$ admits a unique attractor $G \in \mathcal{H}(I \times [a,b])$.

Remark 2. *The conclusion of Proposition 3 remains valid if the mappings w_n are defined on an arbitrary compact metric space instead of $I \times [a,b]$.*

The proof of Theorem 2 relies on the next two lemmas. Together, they show that the attractor obtained above is the graph of a continuous function and that it is uniquely determined. Continuity follows from Lemma 1, while uniqueness is a direct consequence of the fact that the union of two attractors is again an attractor. Let $\widetilde{I}$ denote the projection of G onto the horizontal axis, that is,

$$\widetilde{I} = \{x \in I :\ \exists\, y \in [a,b] \text{ such that } (x,y) \in G\}.$$

Lemma 2. *The projection of the attractor satisfies $\widetilde{I} = I$.*

Proof. Since $G = \bigcup_{n=1}^{N} w_n(G)$, projecting onto the first coordinate yields

$$\widetilde{I} = \bigcup_{n=1}^{N} L_n(\widetilde{I}).$$

The family $\{L_n\}_{n \in J}$ defines a hyperbolic IFS on I whose unique attractor is the whole interval I. Hence $\widetilde{I} = I$. □

Lemma 3. *Under the assumptions of Theorem 2, the following assertions hold:*

1. *The attractor G is the graph of a function $f : I \to [a,b]$;*
2. *The attractor G is unique.*

Proof. 1. Since $\widetilde{I} = I$, for every $x \in I$ there exists at least one $y \in [a,b]$ such that $(x,y) \in G$. We show that this ordinate is unique.
For $n = 0, \ldots, N$, define

$$\mathsf{X}_n = \{(x,y) \in G :\ x = x_n\}.$$

Case $x = x_0$. Because $w_n(\mathsf{X}_0) \cap \mathsf{X}_0 = \varnothing$ for all $n \neq 1$, it follows that $w_1(\mathsf{X}_0) = \mathsf{X}_0$. Moreover, the CGIT condition implies that w_1 admits a unique fixed point on the compact set X_0. Hence, $\mathsf{X}_0 = \{(x_0, y_0)\}$.
Case $x = x_N$. A similar argument yields $\mathsf{X}_N = \{(x_N, y_N)\}$.
Case $x = x_n$, $1 \leq n \leq N-1$. Using the self-similar structure of G, we obtain $\mathsf{X}_n = w_{n+1}(\mathsf{X}_0) \cup w_n(\mathsf{X}_N) = \{(x_n, y_n)\}$.

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Case $x \in I \setminus \{x_0, \dots, x_N\}$. Assume that $(x, y_x), (x, y'_x) \in G$. Let $\xi \in I$ be a point where the maximum of $|y_x - y'_x|$ is attained. There exists $n \in J$ such that $\xi \in (x_{n-1}, x_n)$ and points $u = (L_n^{-1}(\xi), t)$, $v = (L_n^{-1}(\xi), t')$ in G satisfying

$$y_\xi = F_n(L_n^{-1}(\xi), t), \qquad y'_\xi = F_n(L_n^{-1}(\xi), t').$$

Applying the CGIT condition and using $\phi(t) < t$ for $t > 0$, we obtain $y_\xi = y'_\xi$, and hence $y_x = y'_x$. This proves that G is the graph of a function f.

2. Let $A \in \mathcal{H}(I \times [a, b])$ and denote by $A_I = \{x \in I : (x, y) \in A\}$ its projection onto I. Define

$$L(B) = \bigcup_{n=1}^{N} L_n(B), \qquad L^m(B) = L(L^{m-1}(B)).$$

Since $\{L_n\}_{n=1}^N$ is a hyperbolic IFS with attractor I, we have $d_H(L^m(A_I), I) \to 0$. Let f be the unique fixed point of the Read–Bajraktarević operator and set

$$T_m = \{(x, f(x)) : x \in L^m(A_I)\}.$$

By the uniform continuity of f, one obtains $d_H(T_m, G) \to 0$. Define

$$\gamma_m = \sup \Big\{ \int_0^{\psi(d((x,f(x)),(x,y)))} \varphi(t)\, dt : (x, y) \in W^m(A) \Big\}.$$

Using the self-referential equation satisfied by f and the CGIT condition, we obtain $\gamma_m \le \phi(\gamma_{m-1})$. Since $\phi(t) < t$ for $t > 0$, it follows that $\gamma_m \to 0$. Consequently, $d_H(W^m(A), G) \to 0$, which proves that G is the unique attractor.

□

Example 5. *Let* $x_0 < x_1 < \cdots < x_N$ *be a subdivision of the compact interval* $I = [x_0, x_N]$ *and consider the nonlinear function* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by* $g(y) = \frac{y}{1+y}$. *Let* K *be a compact subset of the domain of* g, *and assume that a data set*

$$\{(x_n, y_n) \in I \times K : n = 0, 1, \dots, N\}$$

is prescribed. Denote by $\mathcal{C}(I, K)$ *the Banach space of continuous functions from* I *into* K, *endowed with the uniform norm. Let* $b : \mathcal{C}(I, K) \to \mathcal{C}(I, K)$ *be a bounded linear operator which is not the identity and satisfies the endpoint invariance condition*

$$b(f)(x_0) = f(x_0), \qquad b(f)(x_N) = f(x_N), \qquad \forall f \in \mathcal{C}(I, K). \tag{17}$$

Our objective is to construct a new FIF, denoted by $\hat{f}$, *which approximates the fractal interpolant* f^α *introduced in Example 4. More precisely, we shall prove that there exists a constant* $\gamma > 0$ *such that*

$$\|\hat{f} - f^\alpha\|_\infty \le \frac{1}{\gamma(\gamma + 2)} \|b(f^\alpha) - f^\alpha\|_\infty. \tag{18}$$

To this end, we consider the IFS generated by the mappings L_n *defined in (15) together with the functions* $F_n : I \times K \to \mathbb{R}$ *given by*

$$F_n(x, y) = g(y) + f^\alpha(L_n(x)) - g(b(f^\alpha)(x)), \qquad n = 1, \dots, N. \tag{19}$$

One verifies that conditions (11) and (12) are satisfied. Moreover, the contractive condition of the general integral type (13) holds with the choice

$$\begin{cases} \varphi(t) = t^{1/t-2}(1-\ln t), \quad t>0, \quad \varphi(0)=0, \\ \phi(t) = \dfrac{t}{2}, \\ \psi(t) = t. \end{cases}$$

Consequently, the associated IFS admits a unique attractor, which is the graph of a continuous function $\hat{f} : I \to K$ interpolating the given data, that is,

$$\hat{f}(x_n) = y_n, \qquad n = 0, 1, \ldots, N.$$

Furthermore, $\hat{f}$ is characterized as the unique fixed point of the corresponding Read–Bajraktarević operator T, and therefore satisfies the functional equation

$$\hat{f}(x) = g\Big(\hat{f} \circ L_n^{-1}\Big)(x) + f^{\alpha}(x) - g\Big(b(f^{\alpha}) \circ L_n^{-1}\Big)(x), \qquad x \in I_n. \tag{20}$$

Since the function g is continuously differentiable on $\mathbb{R}_+$, there exists a constant $\gamma > 0$ such that

$$\begin{aligned} \|\hat{f} - f^{\alpha}\|_{\infty} &\leq \|g \circ \hat{f} - g \circ f^{\alpha}\|_{\infty} + \|g \circ b(f^{\alpha}) - g \circ f^{\alpha}\|_{\infty} \\ &\leq \frac{1}{(1+\gamma)^2}\Big(\|\hat{f} - f^{\alpha}\|_{\infty} + \|b(f^{\alpha}) - f^{\alpha}\|_{\infty}\Big). \end{aligned}$$

Rearranging the above inequality yields the estimate (18), which completes the construction. This estimating provides a stability bound for the constructed fractal interpolant. It shows that the difference between $\hat{f}$ and f^{α} is controlled by the deviation of $b(f^{\alpha})$ from f^{α}. In particular, small bounded perturbations of the identity operator produce only small changes in the resulting fractal interpolant, ensuring robustness of the construction.

3.3. Continuity Preservation Under Perturbations of the IFS

Fractal interpolation schemes are known to be sensitive to modifications of their defining mappings. In particular, perturbations of the vertical component may destroy the matching conditions at interpolation nodes, leading to a loss of global continuity of the attractor. It is therefore natural to investigate under which conditions continuity is preserved when the IFS is perturbed. In this subsection, we establish explicit criteria ensuring that nonlinear vertical perturbations do not affect the continuity of the resulting fractal interpolant.

We consider first a reference acting on $I \times [a, b]$, defined by the mappings

$$\begin{cases} L_n(x) = a_n x + e_n, \\ F_n(x, y) = g_n(y) + q_n(x), \end{cases} \qquad n = 1, \ldots, N, \tag{21}$$

where the coefficients a_n, e_n satisfy the geometric constraints (11), and where each function F_n is continuous and fulfills the interpolation and contractive assumptions (12) and (13). We now introduce a perturbed version of this system by modifying only the vertical component. More precisely, we define a new family of maps

$$\begin{cases} L_n(x) = a_n x + e_n, \\ \widetilde{F}_n(x, y) = \widetilde{g}_n(y) + q_n(x) + \lambda_n(x), \end{cases} \qquad n = 1, \ldots, N, \tag{22}$$

where each $\widetilde{g}_n$ satisfies a contraction condition of general integral type, namely (5), and where $\lambda_n : I \to \mathbb{R}$ represents an additional perturbation term. To quantify the deviation between the original and perturbed systems, we introduce the auxiliary functions

$$\delta_n(y) = \widetilde{g}_n(y) - g_n(y), \qquad \gamma_n = \lambda_n(x_N) - \lambda_n(x_0),$$

defined for all $y \in [a, b]$ and $n = 1, \ldots, N$. The perturbed system is said to satisfy the continuity matching condition if the vertical components coincide at adjacent interpolation nodes, that is, if

$$\widetilde{F}_n(x_N, y_N) = \widetilde{F}_{n+1}(x_0, y_0), \qquad n = 1, \ldots, N-1. \tag{23}$$

In the classical affine setting where $g_n(y) = \alpha_n y$ and $\widetilde{g}_n(y) = \alpha'_n y$, this condition was analyzed in [24], where constant perturbations λ_n were sufficient. More recently, extensions allowing spatially dependent scaling factors were investigated in [23,26]. In the present nonlinear framework, we assume that the perturbation terms satisfy, for every $1 \leq n < N$, the recursive relation

$$\lambda_{n+1}(x_0) = \sum_{k=1}^{n} \delta_k(y_N) - \sum_{k=2}^{n+1} \delta_k(y_0) + \sum_{k=1}^{n} \gamma_k + \lambda_1(x_0). \tag{24}$$

Theorem 3. *Under assumption (24), the perturbed IFS defined by (22) satisfies the continuity matching condition (23).*

Proof. Fix an index $n \in \{1, \ldots, N-1\}$. The continuity requirement (23) is equivalent to the identity

$$\widetilde{g}_n(y_N) + q_n(x_N) + \lambda_n(x_N) = \widetilde{g}_{n+1}(y_0) + q_{n+1}(x_0) + \lambda_{n+1}(x_0).$$

Using the fact that the original system satisfies the interpolation conditions (12), the terms involving q_n cancel, yielding

$$\lambda_{n+1}(x_0) = \widetilde{g}_n(y_N) - \widetilde{g}_{n+1}(y_0) + g_{n+1}(y_0) - g_n(y_N) + \lambda_n(x_N).$$

Rewriting this identity in terms of δ_n and γ_n, we obtain

$$\lambda_{n+1}(x_0) = \delta_n(y_N) - \delta_{n+1}(y_0) + \gamma_n + \lambda_n(x_0).$$

Iterating this relation backward from n down to 1 leads exactly to Equation (24), which proves that the continuity condition (23) is satisfied for all $n = 1, \ldots, N-1$. □

Example 6. *In this example, we consider the sets of data points*

$$\Delta := \Big\{(0,0), (1/3, 1), (2/3, -1), (1, 0)\Big\},$$

Let us and we consider the IFS defined by (21) where

$$L_n(x) = \frac{x}{3} + \frac{n-1}{N}, \quad q_1(x) = x, \quad q_2(x) = 1 - 2x, \quad q_3(x) = x - 1 \text{ and } g_n(y) = -0.5y.$$

Now, we consider the system (22) where $\widetilde{g}(y) = \frac{y}{1+\beta y}$ *with* $\beta > 0$. *Since* $y_N = y_0 = 0$ *then*

$$\lambda_{n+1}(x_0) = \delta_1(0) - \delta_{n+1}(0) + \sum_{k=1}^{n} \gamma_k = \sum_{k=1}^{n} \lambda_k(1) - \lambda_k(0).$$

So buy choosing $\lambda_n(x) = x + n - 1$, *the system (22) satisfies the continuous condition.*

4. Conclusions

Given a finite collection of interpolation nodes $\Delta = \{(x_i, y_i)\}_{i=0}^{N}$, FIFs constitute a flexible framework for modeling data exhibiting irregular or non-smooth behavior. Such functions naturally arise as invariant sets generated by appropriate s. In this work, we establish new existence and uniqueness results for the attractors associated with IFSs governed by contraction conditions of the general integral type. Within this framework, we introduce a new class of nonlinear FIFs, which are characterized as fixed points of the Read–Bajraktarević operator. More precisely, the interpolating function f satisfies the self-referential equation

$$f(x) = F_n\big(L_n^{-1}(x),\, f(L_n^{-1}(x))\big), \qquad x \in I_n,$$

thereby revealing the recursive structure underlying the proposed generalized interpolation scheme. This framework extends the classical affine theory and provides greater flexibility for modeling complex and irregular data. The results obtained in this work naturally lead to several directions for further investigation.

1. Consider the generated by the mappings

$$W_n(x, y) = \big(a_n(x) + e_n,\ \alpha_n(x, y) + q_n(x)\big), \qquad n = 1, \ldots, N,$$

under assumption (13). A natural continuation of the present work is to investigate quantitative error estimates arising from small perturbations of the nonlinear vertical component $\alpha_n(x, y)$. The particular case where $\alpha_n(x, y) = \beta_n(x)\, y$ has already been addressed in several contributions, including [23,27]. In this linear setting, fractal interpolation systems have been extensively studied; see, for instance [10,26,28,29]. It is well known that the collection of vertical scaling factors plays a central role in determining both the geometric structure and the analytical properties of the associated FIF. In particular, the smoothness and regularity of FIFs can be characterized in terms of these scaling functions [27,29], which makes them a powerful tool for accurately modeling highly irregular or rough curves. Extending such a sensitivity and regularity analysis to the more general nonlinear and integral-type dependence encoded in $\alpha_n(x, y)$ constitutes an interesting and largely open problem, and is expected to reveal new phenomena beyond those observed in the linear case.
2. Recent studies on fractal interpolation functions generated by generalized IFSs have shown that extending the classical Banach framework through alternative contractive mechanisms leads to richer and more flexible models. In particular, Suzuki-type contractions [30], given by

$$\frac{1}{2}\, d(x, Tx) < d(x, y) \quad \Longrightarrow \quad d(Tx, Ty) \le c\, d(x, y), \qquad \forall\, x, y \in X,$$

for some constant $0 < c < 1$, provide a conditional extension of Banach's principle that has proved effective in fixed point theory. Since Suzuki-type conditions and generalized integral-type contractions with a ψ-transformation are, in general, independent, a natural and nontrivial perspective for future research is to investigate FIFs whose associated operator T satisfies a generalized integral contraction under a Suzuki-type conditional assumption. More precisely, one may aim to establish an analogue of Theorem 2 under the hypothesis that, for all $x, y \in X$,

$$\frac{1}{2}\, d(x, Tx) < d(x, y) \quad \Longrightarrow \quad \int_0^{\psi(d(Tx,Ty))} \varphi(t)\, dt \le \phi\left(\int_0^{\psi(d(x,y))} \varphi(t)\, dt\right),$$

for some $(\varphi, \phi, \psi) \in \Phi \times \Phi_1 \times \Phi_2$.

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