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Sandwich Results for Holomorphic Functions Related to an Integral Operator

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Abstract

In this paper, we introduce a new logarithmic integral operator that unifies differentiation and fractional integration within the complex domain. The present work addresses this gap by applying the proposed operator to analytic functions represented by alternating power series. The method demonstrates that the coefficients can be reorganized in a controlled manner without affecting convergence or analytic behavior. Using this framework, we derive third-order differential subordination and superordination results, which naturally lead to corresponding sandwich-type results. The findings confirm that the introduced operator offers an effective analytical tool for studying distortion, growth, and mapping properties of analytic functions, with promising potential for future applications in fluid mechanics.

Keywords: holomorphic functions; logarithmic integral operator; fractional calculus; differential subordination; sandwich-type results

MSC: 30C45

1. Introduction

The differential conditions within analytic function theory have undergone significant development over many decades. Among the earliest contributions in this field was a 1974 publication by Miller and colleagues [1], which focused on Bazilevic functions. Their work established the fundamental principle that, for any real number ς and an analytic function p defined on the unit disk $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, specific properties can be derived under certain differential conditions

$$\Re\left(p(\zeta) + \varsigma \frac{\zeta p'(\zeta)}{p(\zeta)}\right) > 0 \qquad \text{implies that} \qquad \Re\, p(\zeta) > 0.$$

This lemma is considered a framework for deducing the properties of a function based on a given set of its derivatives. At the time, the range of differential implications available within analytic function theory was very limited. More generally, in complex analysis, many results allow the use of differential constraints to reveal the general behavior of functions. A well-known example in this field is the Noshiro–Warsawski theorem ([2]

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p. 47), which shows that whenever the condition $\operatorname{Re} f'(\zeta) > 0$ for all $\zeta \in \mathbb{U}$ holds for an analytic function f, then f is necessarily a one-to-one function within $\mathbb{U}$. Over time, similar new questions have emerged, expanding the scope of these differential conditions and increasing their applicability. In 1981, Miller and Mocanu [3] formalized the concept of differential subordination, which enabled the development of a systematic methodology for analyzing this type of inequality and exploring its implications for the geometric properties of analytic functions.

In the analysis of differential equations, a central problem concerns the extent to which information about the derivatives of an analytic function determines the behavior of the function itself. Let Ω and Λ be subsets of $\mathbb{C}$. Suppose that ξ is analytic in $\mathbb{U}$ and that $D[\xi]$ is well defined for a given differential operator D. A natural question is to identify suitable conditions on D, Ω, and Λ ensuring the implication

$$D[\xi](\mathbb{U}) \subset \Omega \quad \Longrightarrow \quad \xi(\mathbb{U}) \subset \Lambda.$$

Earlier investigations [4–6] established analytic conditions involving a function

$$\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C},$$

together with the regions Ω and Λ, which guarantee the third-order differential implication

$$\{\psi(\xi(\zeta), \zeta\xi'(\zeta), \zeta^2\xi''(\zeta), \zeta^3\xi^{(3)}(\zeta); \zeta) : \zeta \in \mathbb{U}\} \subset \Omega \quad \Longrightarrow \quad \xi(\mathbb{U}) \subset \Lambda.$$

We begin by recalling several classes of functions that will be used throughout this paper. Let $\mathcal{O} = \mathcal{O}(\mathbb{U})$ denote the family of analytic functions in $\mathbb{U}$. For $n \in \mathbb{Z}^+$ and $a \in \mathbb{C}$, define

$$\mathcal{O}[a, n] = \left\{ f \in \mathcal{O} : f(\zeta) = a + a_n\zeta^n + a_{n+1}\zeta^{n+1} + \cdots \right\}.$$

In particular, we write $\mathcal{O}[0, n] = \mathcal{O}_n$ and $\mathcal{O}[1, n] = \mathcal{O}'_n$. We also consider the class $\mathcal{A}$ of normalized analytic functions represented by

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n\zeta^n, \qquad \zeta \in \mathbb{U}. \tag{1}$$

For functions $f, g \in \mathcal{O}$, the function f is said to be subordinate to g, denoted $f(\zeta) \prec g(\zeta)$, if there exists a Schwarz function $u : \mathbb{U} \to \mathbb{U}$ with $u(0) = 0$ such that $f(\zeta) = g(u(\zeta))$, $\zeta \in \mathbb{U}$. If g is analytic and injective, then this is equivalent to the conditions $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (see [7–9]). When Ω and Λ are simply connected, there exist conformal mappings $\chi : \mathbb{U} \to \Lambda$ and $h : \mathbb{U} \to \Omega$ such that $\chi(0) = a$ and $h(0) = \psi(a, 0, 0, 0; 0)$. In this setting, the implication above can be rewritten in terms of subordination as

$$\psi\Big(\xi(\zeta), \zeta\xi'(\zeta), \zeta^2\xi''(\zeta), \zeta^3\xi^{(3)}(\zeta); \zeta\Big) \prec h(\zeta) \quad \Longrightarrow \quad \xi(\zeta) \prec \chi(\zeta).$$

Any function χ satisfying this implication is called a *dominant*. If a dominant $\hat{\chi}$ satisfies $\hat{\chi}(\zeta) \prec \chi(\zeta)$ for every other dominant χ, then $\hat{\chi}$ is referred to as the *best dominant* .

As a counterpart to differential subordination, Tang et al. introduced the notion of third-order differential superordination in 2014 [10], focusing on the inclusion

$$\Omega \subset \left\{ \psi(\xi(\zeta), \zeta\xi'(\zeta), \zeta^2\xi''(\zeta), \zeta^3\xi^{(3)}(\zeta); \zeta) : \zeta \in \mathbb{U} \right\}.$$

Their work established several implications for meromorphic p-valent functions associated with operators involving the Liu–Srivastava operator. Subsequent studies [11,12] analyzed the influence of generalized Bessel functions on differential subordination and superordination, with additional contributions appearing in [13–18].

In recent years, and as a result of the significant developments in the works [19–24], many studies have focused on investigating the concept of differential subordination of analytic functions within the framework of generalized fractional calculus. Building on this research momentum, this work presents new formulas and structures in fractional calculus, paving the way for the development of third-order results related to differential subordination.

To examine the analytic behavior of the proposed integrative operator, it was initially applied to an analytic function expressed as an alternating power series. The results obtained show that this operator systematically reshapes the function's coefficients while preserving its analytic properties within defined intervals. Subsequently, the operator was applied to a set of well-known analytic functions expressed as power series extensions, including the functions e^{ζ}, $sin\zeta$, $cos\zeta$, $sinh\zeta$, and $cosh\zeta$.

The analysis revealed that the proposed operator induces a regular and precise change in the function's parameters, ensuring that the function remains convergent and analytic within the desired domain. The operator also demonstrates a clear ability to generate measurable changes in growth behavior and mapping properties at the composite level. These results indicate that the developed integration operator offers an effective analytical tool for understanding the distortion, growth, and mapping properties of analytic functions within the framework of composite analysis.

Definition 1. *For the analytic function $f(\zeta)$ in $\mathbb{U}$, with parameters $\varsigma, \varrho \in \mathbb{C}$, $\Re(\varrho) > 0$, $\varsigma \in \mathbb{N}$, and $\epsilon \geq 2$, the integral operator $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} : \mathcal{A} \longrightarrow \mathcal{A}$, related to the double logarithmic kernel, is stated by*

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta) = \frac{\left(\ln(\ln(\epsilon))\right)^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)} \int_0^{\infty} v^{\varsigma-1} (\ln(\epsilon))^{-v} f(\zeta v^{\varrho})\, dv,$$

where $\Gamma(\cdot)$ indicates the gamma function.

Integral transformations are among the generality significant analytical tools in applied mathematics, as they can transform complex differential equations into simpler and more manageable algebraic forms, thus facilitating the attainment of precise analytical solutions (see [25]). The Laplace transform is one of the oldest and most widely used of these transformations; it was introduced by Pierre-Simon Laplace and is based on the exponential nucleus e^{-st}. This transformation has proven highly effective in studying linear systems and analyzing their behavior (see [26]).

The Mellin transform is an intermediate structure between the Laplace and Fourier transforms, relying on the nucleus t^{s-1}, and has been widely used in complex analysis and analytic number theory [27]. The Aboodh transform [28], on the other hand, was developed as a modern extension of the Laplace transform, with both the nucleus and the domain reformulated to accommodate a wider range of differential and integral equations, making it a more flexible and suitable tool for modern applications.

Remark 1 (Relationship Between the Proposed Transform and Classical Transforms). *It is easy to assume that the proposed transformation $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}$ is a general case of many classical transformations. In particular*

1. $\mathbb{I}^{\varsigma,\varrho}_{1} f(\zeta) = \frac{(\ln 1)^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)} (\mathcal{M} f_{\zeta,\varrho})(\varsigma) = 0$, *where $\mathcal{M}$ is Mellin transform and $f_{\zeta,\varrho}(v) := f(\zeta v^{\varrho})$;*

2. $\mathbb{I}^{1,1}_{e^s} f(\zeta) = s^2 \mathcal{L}(f_\zeta)(s)$, *where* $\mathcal{L}$ *is Laplace transform and* $f_\zeta(v) := f(\zeta v)$;
3. *From [28] the Aboodh transform* A *satisfy* $\mathcal{L} := sA$, *then* $\mathbb{I}^{1,1}_{e^s} f(\zeta) = s^3 A(f_\zeta)(s)$, *where* A *is Aboodh transform and* $f_\zeta(v) := f(\zeta v)$.

Consequently, the operator offers an analytical framework covering classical cases, including analytic functions.

The integral operator introduced in Definition 1 encompasses several well-known operators as special or limiting cases under appropriate choices of the parameters. In particular, when the logarithmic kernel is removed and suitable parameter reductions are applied, the operator reduces to classical fractional integral operators studied in the literature, including those associated with the Srivastava–Owa and Noor frameworks.

Moreover, by fixing the logarithmic scaling and restricting the order parameters, connections with Tremblay-type fractional derivative operators can be established. However, unlike these classical operators, the presence of the logarithmic kernel introduces an additional degree of flexibility that allows a systematic rearrangement of coefficients in alternating power series while preserving analyticity and convergence.

Theorem 1. *Let* $f \in \mathcal{A}$ *be analytic in the unit disc* $\mathbb{U}$. *Then, for parameters* $\varsigma \in \mathbb{N}$, $\varrho \in \mathbb{C}$ *with* $\Re(\varrho) > 0$, *and* $\epsilon \geq 2$, *the integral operator*

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta) = \frac{\big(\ln(\ln(\epsilon))\big)^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)} \int_0^\infty v^{\varsigma-1} (\ln(\epsilon))^{-v} f(\zeta v^\varrho)\, dv$$

is well-defined and convergent for all $\zeta \in \mathbb{U}$.

Proof. Since f is analytic in $\mathbb{U}$, it is bounded on any closed disc $\overline{\mathbb{U}_r} = \{\zeta : |\zeta| \leq r < 1\}$. Hence, there exists a constant $M > 0$ such that

$$|f(\zeta)| \leq M \quad \text{for all } |\zeta| \leq r.$$

Now, consider the integrand of the operator:

$$\left| v^{\varsigma-1} (\ln \epsilon)^{-v} f(\zeta v^\varrho) \right| \leq M v^{\varsigma-1} (\ln \epsilon)^{-v}.$$

Observe that

$$(\ln \epsilon)^{-v} = e^{-v \ln(\ln \epsilon)},$$

so the integral becomes

$$\int_0^\infty v^{\varsigma-1} (\ln \epsilon)^{-v}\, dv = \int_0^\infty v^{\varsigma-1} e^{-v \ln(\ln \epsilon)}\, dv.$$

Next, we apply the standard linear change of variable:

$$u = v \ln(\ln \epsilon) \quad \Longrightarrow \quad v = \frac{u}{\ln(\ln \epsilon)}, \quad dv = \frac{du}{\ln(\ln \epsilon)}.$$

Substituting gives

$$\int_0^\infty \left(\frac{u}{\ln(\ln \epsilon)} \right)^{\varsigma-1} e^{-u} \frac{du}{\ln(\ln \epsilon)} = (\ln(\ln \epsilon))^{-\varsigma} \int_0^\infty u^{\varsigma-1} e^{-u}\, du = (\ln(\ln \epsilon))^{-\varsigma} \Gamma(\varsigma),$$

where $\Gamma(\varsigma)$ is the gamma function, which is finite for $\varsigma > 0$.

Therefore, by the comparison test, the integral defining $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta)$ converges absolutely for all $\zeta \in \mathbb{U}$, and the operator is well-defined. □

Example 1. ***(1)*** *Let f of the form (1). Then By applying the operator $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}$ to $f(\zeta)$, we obtain*

$$\begin{aligned}\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta) &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)} \int_0^\infty v^{\varsigma-1} (\ln(\epsilon))^{-v} \left[\zeta v^{\varrho} + \sum_{n=2}^{\infty} a_n \zeta^n v^{n\varrho}\right] dv \\ &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)} \left[\zeta \int_0^\infty v^{\varsigma+\varrho-1} e^{-v\ln(\ln(\epsilon))} dv + \sum_{n=2}^{\infty} a_n \zeta^n \int_0^\infty v^{\varsigma+n\varrho-1} e^{-v\ln(\ln(\epsilon))} dv\right].\end{aligned} \tag{2}$$

Let $c = v\ln(\ln(\epsilon))$, so that $v = \dfrac{c}{\ln(\ln(\epsilon))}$ and $dv = \dfrac{dc}{\ln(\ln(\epsilon))}$. Under this transformation, Equation (2) becomes

$$\begin{aligned}\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta) = \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)} [&\frac{\zeta}{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}} \int_0^\infty c^{\varsigma+\varrho-1} e^{-c} dc \\ &+ \sum_{n=2}^{\infty} \frac{a_n \zeta^n}{(\ln(\ln(\epsilon)))^{\varsigma+n\varrho}} \int_0^\infty c^{\varsigma+n\varrho-1} e^{-c} dc].\end{aligned}$$

Using the definition of the Gamma function, we have

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\Gamma(\varsigma+n\varrho)}{\Gamma(\varsigma+\varrho)\,(\ln(\ln(\epsilon)))^{\varrho(n-1)}} a_n \zeta^n.$$

(2) *For $f_1(\zeta) = e^{\zeta}$, where*

$$e^{\zeta} = \sum_{n=0}^{\infty} \frac{\zeta^n}{n!},$$

we have

$$\begin{aligned}\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} e^{\zeta} &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)} \int_0^\infty v^{\varsigma-1} (\ln(\epsilon))^{-v} \sum_{n=0}^{\infty} \frac{\zeta^n v^{n\varrho}}{n!} dv \\ &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)} \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} \int_0^\infty v^{\varsigma+n\varrho-1} e^{-v\ln(\ln(\epsilon))}\, dv.\end{aligned}$$

Substituting $c = v\ln(\ln(\epsilon))$, we obtain

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} e^{\zeta} = \sum_{n=0}^{\infty} \frac{\Gamma(\varsigma+n\varrho)}{\Gamma(\varsigma+\varrho)\,(\ln(\ln(\epsilon)))^{\varrho(n-1)}\, n!} \zeta^n. \tag{3}$$

Now, if we take $\varsigma = 1$, $\varrho = 1$, and $\epsilon = e^e$, then from (3) we obtain

$$\begin{aligned}\mathbb{I}^{1,1}_{\ln(e^e)} e^{\zeta} &= \sum_{n=0}^{\infty} \frac{\Gamma(1+n)}{\Gamma(2)\,(\ln(\ln(e^e)))^{1\cdot(n-1)}\, n!} \zeta^n \\ &= \sum_{n=0}^{\infty} \zeta^n = 1 + z + \zeta^2 + \zeta^3 + \cdots = \frac{1}{1-\zeta}, \qquad |\zeta| < 1.\end{aligned}$$

(3) *Let $f_2(\zeta) = \sin\zeta$, where*

$$\sin\zeta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \zeta^{2n+1}.$$

Then, the application of the operator $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}$ *to* $f(\zeta)$ *yields*

$$\begin{aligned}\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}\sin\zeta &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)}\int_0^\infty v^{\varsigma-1}(\ln(\epsilon))^{-v}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\zeta^{2n+1}v^{(2n+1)\varrho}\,dv\\ &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\zeta^{2n+1}\int_0^\infty v^{\varsigma+2n\varrho+\varrho-1}e^{-v(\ln(\ln(\epsilon)))}\,dv.\end{aligned} \tag{4}$$

Let $c = v\ln(\ln(\epsilon))$, *which implies* $v = \dfrac{c}{\ln(\ln(\epsilon))}$ *and* $dv = \dfrac{dc}{\ln(\ln(\epsilon))}$. *Hence, Equation* (4) *becomes*

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}\sin\zeta = \sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma(\varsigma+\varrho(2n+1))}{\Gamma(\varsigma+\varrho)\,(\ln(\ln(\epsilon)))^{2n\varrho}\,(2n+1)!}\,\zeta^{2n+1}.$$

Now, if $\varsigma = 1$, $\varrho = 1$, *and* $\epsilon = e^e$, *we obtain*

$$\begin{aligned}\mathbb{I}^{1,1}_{\ln(e^e)}\sin\zeta &= \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(2n+2)}{\Gamma(2)\,(2n+1)!}\,\zeta^{2n+1}\\ &= \zeta-\zeta^3+\zeta^5-\zeta^7+\cdots = \zeta(1-\zeta^2+\zeta^4-\zeta^6+\cdots) = \frac{\zeta}{1+\zeta^2},\quad |\zeta|<1.\end{aligned}$$

(4) *Let* $f_3(\zeta) = \cos\zeta$, *where*

$$\cos\zeta = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\zeta^{2n}.$$

Then, by applying the operator $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}$ *to* $f(\zeta)$, *we obtain*

$$\begin{aligned}\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}\cos\zeta &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)}\int_0^\infty v^{\varsigma-1}(\ln(\epsilon))^{-v}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\zeta^{2n}v^{2n\varrho}\,dv\\ &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\zeta^{2n}\int_0^\infty v^{\varsigma+2n\varrho-1}e^{-v(\ln(\ln(\epsilon)))}\,dv.\end{aligned} \tag{5}$$

Let $c = v\ln(\ln(\epsilon))$, *hence* $v = \dfrac{c}{\ln(\ln(\epsilon))}$ *and* $dv = \dfrac{dc}{\ln(\ln(\epsilon))}$. *Then Equation* (5) *becomes*

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}\cos\zeta = \sum_{n=0}^{\infty}\frac{(-1)^n\,\Gamma(\varsigma+2n\varrho)}{\Gamma(\varsigma+\varrho)\,(\ln(\ln(\epsilon)))^{\varrho(2n-1)}(2n)!}\,\zeta^{2n}.$$

If $\varsigma = 1$, $\varrho = 1$, *and* $\epsilon = e^e$, *then*

$$\begin{aligned}\mathbb{I}^{1,1}_{\ln(e^e)}\cos\zeta &= \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(2n+1)}{\Gamma(2)(2n)!}\,\zeta^{2n}\\ &= 1-\zeta^2+\zeta^4-\zeta^6+\cdots = \frac{1}{1+\zeta^2},\qquad |\zeta|<1.\end{aligned}$$

(5) *Let* $f_4(\zeta) = \sinh\zeta$, *where*

$$\sinh\zeta = \sum_{n=0}^{\infty}\frac{\zeta^{2n+1}}{(2n+1)!}.$$

Then, applying $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}$ *to* $f(\zeta)$ *gives*

$$\begin{aligned}\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}\sinh\zeta &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)}\int_0^\infty v^{\varsigma-1}(\ln(\epsilon))^{-v}\sum_{n=0}^\infty\frac{\zeta^{2n+1}}{(2n+1)!}v^{(2n+1)\varrho}\,dv\\ &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)}\sum_{n=0}^\infty\frac{\zeta^{2n+1}}{(2n+1)!}\int_0^\infty v^{\varsigma+2n\varrho+\varrho-1}e^{-v(\ln(\ln(\epsilon)))}\,dv.\end{aligned}$$

Using the substitution $c = v\ln(\ln(\epsilon))$, *we obtain the following.*

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}\sinh\zeta = \sum_{n=0}^\infty\frac{\Gamma(\varsigma+\varrho(2n+1))}{\Gamma(\varsigma+\varrho)\,(\ln(\ln(\epsilon)))^{2n\varrho}(2n+1)!}\,\zeta^{2n+1}.$$

Now, for $\varsigma = 1$, $\varrho = 1$, *and* $\epsilon = e^e$, *we have*

$$\begin{aligned}\mathbb{I}^{1,1}_{\ln(e^e)}\sinh\zeta &= \sum_{n=0}^\infty\frac{\Gamma(2n+2)}{\Gamma(2)(2n+1)!}\,\zeta^{2n+1}\\ &= \zeta+\zeta^3+\zeta^5+\zeta^7+\cdots = \zeta(1+\zeta^2+\zeta^4+\zeta^6+\cdots) = \frac{\zeta}{1-\zeta^2},\ |\zeta|<1.\end{aligned}$$

(6) *Let* $f_5(\zeta) = \cosh\zeta$, *where*

$$\cosh\zeta = \sum_{n=0}^\infty\frac{\zeta^{2n}}{(2n)!}.$$

Then, applying the operator $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}$ *to* $f(\zeta)$, *we obtain*

$$\begin{aligned}\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}\cosh\zeta &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)}\int_0^\infty v^{\varsigma-1}(\ln(\epsilon))^{-v}\sum_{n=0}^\infty\frac{\zeta^{2n}}{(2n)!}v^{2n\varrho}\,dv\\ &= \frac{(\ln(\ln(\epsilon)))^{\varsigma+\varrho}}{\Gamma(\varsigma+\varrho)}\sum_{n=0}^\infty\frac{\zeta^{2n}}{(2n)!}\int_0^\infty v^{\varsigma+2n\varrho-1}e^{-v(\ln(\ln(\epsilon)))}\,dv.\end{aligned}\tag{6}$$

Let $c = v\ln(\ln(\epsilon))$, *hence* $v = \dfrac{c}{\ln(\ln(\epsilon))}$ *and* $dv = \dfrac{dc}{\ln(\ln(\epsilon))}$. *Then, Equation* (6) *becomes*

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}\cosh\zeta = \sum_{n=0}^\infty\frac{\Gamma(\varsigma+2n\varrho)}{\Gamma(\varsigma+\varrho)\,(\ln(\ln(\epsilon)))^{\varrho(2n-1)}(2n)!}\,\zeta^{2n}.$$

If $\varsigma = 1$, $\varrho = 1$, *and* $\epsilon = e^e$, *then*

$$\begin{aligned}\mathbb{I}^{1,1}_{\ln(e^e)}\cosh\zeta &= \sum_{n=0}^\infty\frac{\Gamma(2n+1)}{\Gamma(2)(2n)!}\,\zeta^{2n}\\ &= 1+\zeta^2+\zeta^4+\zeta^6+\cdots = \frac{1}{1-\zeta^2},\qquad |\zeta|<1.\end{aligned}$$

We can readily verify that

$$\zeta\left(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)\right)' = \left(\frac{\varsigma}{\varrho}+1\right)\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta) - \frac{\varsigma}{\varrho}\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta).\tag{7}$$

To establish the theoretical results presented in the following sections, we begin by introducing the fundamental definitions and two auxiliary lemmas that will be used throughout the subsequent analysis.

Definition 2 ([4]). *Let $\overline{\mathbb{U}}$ denote the closure of the open unit disk $\mathbb{U}$ and let*

$$E(\chi) = \left\{\mathcal{G} \in \partial\mathbb{U} : \lim_{\zeta\to\mathcal{G}} \chi(\zeta) = \infty\right\}.$$

The class $\mathcal{Q}$ consists of all analytic and univalent functions on $\overline{\mathbb{U}} \setminus E(\chi)$ such that $\chi'(\mathcal{G}) \neq 0$ for every $\mathcal{G} \in \partial\mathbb{U} \setminus E(\chi)$. Moreover, $\mathcal{Q}(a)$ denotes the subclass of $\mathcal{Q}$ for which $\chi(0) = a$. In particular, we set $\mathcal{Q}_0 := \mathcal{Q}(0)$ and $\mathcal{Q}_1 := \mathcal{Q}(1)$.

Definition 3 ([4]). *Let $\Omega \subset \mathbb{C}$, $\chi \in \mathcal{Q}_a$, and $n = 2, 3, \ldots$ The class of admissible functions $\Psi_n[\Omega, \chi]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ that satisfy the following condition: $\psi(r, s, t, u; \zeta) \notin \Omega$ whenever*

$$r = \chi(\mathcal{G}), \quad s = m\mathcal{G}\chi'(\mathcal{G}), \quad \Re\Big(\frac{t}{s} + 1\Big) \geq m\,\Re\Big(1 + \frac{\mathcal{G}\chi''(\mathcal{G})}{\chi'(\mathcal{G})}\Big),$$

$$\Re\Big(\frac{u}{s}\Big) \geq m^2\,\Re\Big(\frac{\mathcal{G}^2\chi'''(\mathcal{G})}{\chi'(\mathcal{G})}\Big),$$

where $\zeta \in \mathbb{U}$, $\mathcal{G} \in \partial\mathbb{U} \setminus E(\chi)$, and $m \geq n$.

Lemma 1 ([4], Theorem 1, p. 449). *Let $\xi \in \mathcal{O}[a, n]$ with $n \geq 2$, and let $\chi \in \mathcal{Q}_a$ satisfy*

$$\Re\Big(\frac{\mathcal{G}\chi''(\mathcal{G})}{\chi'(\mathcal{G})}\Big) \geq 0, \qquad \Big|\frac{\zeta\,\xi'(\zeta)}{\chi'(\mathcal{G})}\Big| \leq m,$$

where $\zeta \in \mathbb{U}$, $\mathcal{G} \in \partial\mathbb{U} \setminus E(\chi)$, and $m \geq n$. If $\Omega \subset \mathbb{C}$, $\psi \in \Psi_n[\Omega, \chi]$, and

$$\Big\{\psi\Big(\xi(\zeta), \zeta\xi'(\zeta), \zeta^2\xi''(\zeta), \zeta^3\xi'''(\zeta); \zeta\Big) : \zeta \in \mathbb{U}\Big\} \subset \Omega,$$

then

$$\xi(\zeta) \prec \chi(\zeta).$$

Definition 4 ([10]). *Let $\Omega \subset \mathbb{C}$, $\chi \in \mathcal{O}[a, n]$ with $\chi'(\zeta) \neq 0$, and $n = 2, 3, \ldots$ The class of admissible functions $\Psi'_n[\Omega, \chi]$ consists of functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \longrightarrow \mathbb{C}$ satisfying $\psi(r, s, t, u; \mathcal{G}) \in \Omega$ whenever*

$$r = \chi(\zeta), \quad s = \frac{\zeta\chi'(\zeta)}{m}, \quad \Re\Big(\frac{t}{s} + 1\Big) \leq \frac{1}{m}\Re\Big(1 + \frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\Big),$$

$$\Re\Big(\frac{u}{s}\Big) \leq \frac{1}{m^2}\Re\Big(\frac{\zeta^2\chi'''(\zeta)}{\chi'(\zeta)}\Big),$$

where $\zeta \in \mathbb{U}$, $\mathcal{G} \in \partial\mathbb{U}$ and $m \geq n$.

Lemma 2 ([10]). *Let $\xi \in \mathcal{O}[a, n]$ with $n \geq 2$, and let $\psi \in \Psi'_n[\Omega, \chi]$. Suppose that*

$$\psi\Big(\xi(\zeta), \zeta\xi'(\zeta), \zeta^2\xi''(\zeta), \zeta^3\xi'''(\zeta); \zeta\Big)$$

is univalent in $\mathbb{U}$, and that $\xi \in \mathcal{Q}_a$ satisfies

$$\Re\Big(\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\Big) \geq 0, \qquad \Big|\frac{\zeta\,\xi'(\zeta)}{\chi'(\zeta)}\Big| \leq m,$$

where $\zeta \in \mathbb{U}$, $\mathcal{G} \in \partial\mathbb{U}$, *and* $m \geq n$. *If* $\Omega \subset \mathbb{C}$ *and*

$$\Omega \subset \left\{ \psi\left(\xi(\zeta), \zeta\xi'(\zeta), \zeta^2\xi''(\zeta), \zeta^3\xi'''(\zeta); \zeta\right) : \zeta \in \mathbb{U} \right\},$$

then

$$\chi(\zeta) \prec \xi(\zeta).$$

The motivation behind this study stems from the investigation of integral operators in complex analysis, particularly in deriving subordination and superordination results. Although numerous classical transforms and operators, such as Mellin, Noor, and Srivastava–Owa operators, have been extensively studied, there remains a significant gap in the literature regarding operators that combine classical transforms with a logarithmic kernel. This gap has motivated us to develop new results, including novel sandwich-type theorems.

While a wide variety of integral and fractional operators, such as the Srivastava–Owa, Tremblay, Noor, and Liu–Srivastava operators, have been extensively studied in the context of differential subordination and superordination, there is a significant gap in the literature concerning operators involving a logarithmic kernel within a generalized fractional framework. Therefore, the operator introduced in this paper should be viewed not as an isolated construct but as a structured generalization that combines logarithmic scaling with fractional integration.

The fundamental novelty of this work lies not only in the definition of the operator but also in the novel third-order differential subordination, superordination, and sandwich-type results obtained within this generalized framework. Specifically, the proposed framework enables the derivation of results, particularly in the third-order case, that cannot be directly obtained from classical operators under the same acceptability conditions. Consequently, this work expands existing theories by providing a broader analytic environment in which new subclasses of analytic functions can be studied.

In other words, this study aims to achieve the following objectives:

1. To explore new operators.
2. To present new results concerning third-order differential subordination.
3. To enhance the theoretical understanding of these results and their role in advancing future research in this field.

2. Third-Order Differential Subordination and Superordination Involving $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}$

In this section, we establish several new results concerning third-order differential subordination and superordination associated with the integral operator $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}$. These results extend the theoretical framework introduced in the previous sections and provide useful tools for analyzing analytic functions within the unit disk.

Parameter Convention. For the remainder of this section, we fix the parameters as follows: $\varsigma, \varrho \in \mathbb{C}$ with $\Re(\varrho) > 0$, $\varsigma \in \mathbb{N}$, and $\epsilon \geq 2$, so that these values do not need to be repeated in each statement or proof.

Definition 5. *Let* $\Omega \subset \mathbb{C}$ *and* $\chi \in \mathcal{Q}_0 \cap \mathcal{O}_0$. *The class* $\mathcal{M}_1[\Omega, \chi]$ *of admissible functions consists of those functions* $\varphi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C}$ *that satisfy*

$$\varphi(a, b, c, d; \zeta) \notin \Omega,$$

whenever the following conditions hold:

$$a = \chi(\mathcal{G}), \qquad b = \frac{-\varsigma\chi(\mathcal{G}) + \varrho m\,\mathcal{G}\chi'(\mathcal{G})}{\varsigma+\varrho},$$

$$\Re\left(\frac{a\varsigma(1+\varsigma)+\varsigma a+(\varsigma+\varrho)\big(-b+c(1+\varsigma+\varrho)\big)}{\varrho(-\varsigma a+b(\varsigma+\varrho))}\right) \geq m\,\Re\Big(1+\frac{\mathcal{G}\chi''(\mathcal{G})}{\chi'(\mathcal{G})}\Big),$$

$$\begin{aligned}\Re\Bigg[&\frac{1}{\varrho^2(-\varsigma a+b(\varsigma+\varrho))}\Big(a\varsigma(1+\varsigma)(2+\varsigma)+d(\varsigma+\varrho)(1+\varsigma+\varrho)(2+\varsigma+\varrho)\\&+(2-\varsigma^2+(3+\varsigma)\varrho+\varrho^2)(\varsigma a-b(\varsigma+\varrho))\\&-(3+\varsigma+3\varrho)(a\varsigma(1+\varsigma)+c(\varsigma+\varrho)(1+\varsigma+\varrho)+(1+\varrho)(\varsigma a-b(\varsigma+\varrho)))\Big)\Bigg]\\&\geq m^2\,\Re\Big(\frac{\mathcal{G}^2\chi'''(\mathcal{G})}{\chi'(\mathcal{G})}\Big)\end{aligned}$$

where $\zeta \in \mathbb{U}$, $\mathcal{G} \in \partial\mathbb{U} \setminus E(\chi)$, *and* $m \geq 2$.

Theorem 2. *Let* $\Omega \subset \mathbb{C}$, $\varphi \in \mathcal{M}_1[\Omega, \chi]$, $f \in \mathcal{A}$, *and* $\chi \in \mathcal{Q}_0$ *satisfy*

$$\Re\Big(\frac{\mathcal{G}\chi''(\mathcal{G})}{\chi'(\mathcal{G})}\Big) \geq 0 \quad \textit{and} \quad \Big|\zeta\Big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta)\Big)'\Big| \leq m|\chi'(\mathcal{G})|, \tag{8}$$

for $\zeta \in \mathbb{U}$, $\mathcal{G} \in \partial\mathbb{U} \setminus E(\chi)$, *and* $m \geq 2$. *If*

$$\Big\{\varphi\Big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta), \mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)} f(\zeta), \mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)} f(\zeta), \mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)} f(\zeta); \zeta\Big) : \zeta \in \mathbb{U}\Big\} \subset \Omega, \tag{9}$$

then

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta) \prec \chi(\zeta), \quad \zeta \in \mathbb{U}.$$

Proof. Define the analytic function $\xi(\zeta)$ in $\mathbb{U}$ by

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta) = \xi(\zeta). \tag{10}$$

Differentiating (10) and applying (7), we obtain

$$\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)} f(\zeta) = \frac{-\varsigma\,\xi(\zeta) + \varrho\,\zeta\,\xi'(\zeta)}{\varsigma+\varrho}. \tag{11}$$

Differentiating (11) and using (7) again, we get

$$\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)} f(\zeta) = \frac{-\varsigma(1+\varsigma)\,\xi(\zeta) + \varrho(1+\varrho)\,\zeta\,\xi'(\zeta) + \varrho^2\,\zeta^2\,\xi''(\zeta)}{(\varsigma+\varrho)(1+\varsigma+\varrho)}. \tag{12}$$

Once more, differentiating (12) and applying (7), we find

$$\begin{aligned}\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)} f(\zeta) = \frac{1}{(\varsigma+\varrho)(1+\varsigma+\varrho)(2+\varsigma+\varrho)}\Big[&-\varsigma(2+3\varsigma+\varsigma^2)\,\xi(\zeta)\\&+\varrho(2-\varsigma^2+3\varrho+\varsigma\varrho+\varrho^2)\,\zeta\,\xi'(\zeta)+\varrho^2(3+\varsigma+3\varrho)\,\zeta^2\,\xi''(\zeta)+\varrho^3\,\zeta^3\,\xi'''(\zeta)\Big].\end{aligned} \tag{13}$$

Define the transforms from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\begin{aligned} a &= r, \\ b &= \frac{-\varsigma r + \varrho s}{\varsigma + \varrho}, \\ c &= \frac{-\varsigma(1+\varsigma)r + \varrho(1+\varrho)s + \varrho^2 t}{(\varsigma+\varrho)(1+\varsigma+\varrho)}, \\ d &= \frac{-\varsigma(2+3\varsigma+\varsigma^2)r + \varrho(2-\varsigma^2+3\varrho+\varsigma\varrho+\varrho^2)s + \varrho^2(3+\varsigma+3\varrho)t + \varrho^3 u}{(\varsigma+\varrho)(1+\varsigma+\varrho)(2+\varsigma+\varrho)}. \end{aligned} \tag{14}$$

We then define $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ by

$$\begin{aligned} &\psi(r,s,t,u;\zeta) = \varphi(a,b,c,d;\zeta) \\ &= \varphi\Bigg(r, \frac{-\varsigma r + \varrho s}{\varsigma+\varrho}, \frac{-\varsigma(1+\varsigma)r + \varrho(1+\varrho)s + \varrho^2 t}{(\varsigma+\varrho)(1+\varsigma+\varrho)}, \\ &\frac{-\varsigma(2+3\varsigma+\varsigma^2)r + \varrho(2-\varsigma^2+3\varrho+\varsigma\varrho+\varrho^2)s + \varrho^2(3+\varsigma+3\varrho)t + \varrho^3 u}{(\varsigma+\varrho)(1+\varsigma+\varrho)(2+\varsigma+\varrho)}; \zeta\Bigg). \end{aligned} \tag{15}$$

In view of (10)–(14), it follows from (15) that

$$\begin{aligned} &\psi\Big(\xi(\zeta), \zeta\xi'(\zeta), \zeta^2\xi''(\zeta), \zeta^3\xi'''(\zeta); \zeta\Big) \\ &= \varphi\Big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta), \mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)} f(\zeta), \mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)} f(\zeta), \mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)} f(\zeta); \zeta\Big). \end{aligned}$$

Therefore, from (9), we have

$$\psi(\xi(\zeta), \zeta\xi'(\zeta), \zeta^2\xi''(\zeta), \zeta^3\xi'''(\zeta); \zeta) \in \Omega.$$

Furthermore,

$$\frac{t}{s} + 1 = \frac{a\varsigma(1+\varsigma) + \varsigma a + (\varsigma+\varrho)(-b + c(1+\varsigma+\varrho))}{\varrho(-\varsigma a + b(\varsigma+\varrho))},$$

and

$$\begin{aligned} \frac{u}{s} = \frac{1}{\varrho^2(-\varsigma a + b(\varsigma+\varrho))}\Big(&a\varsigma(1+\varsigma)(2+\varsigma) + d(\varsigma+\varrho)(1+\varsigma+\varrho)(2+\varsigma+\varrho) \\ &+ (2-\varsigma^2+(3+\varsigma)\varrho+\varrho^2)(\varsigma a - b(\varsigma+\varrho)) - (3+\varsigma+3\varrho)\big(a\varsigma(1+\varsigma) \\ &+ c(\varsigma+\varrho)(1+\varsigma+\varrho) + (1+\varrho)(\varsigma a - b(\varsigma+\varrho))\big)\Big). \end{aligned}$$

Consequently, the condition for $\varphi \in \mathcal{M}_1[\Omega, \chi]$ in Definition 5 is equivalent to the condition for $\psi \in \Psi_2[\Omega, \chi]$ in Definition 3. Therefore, in view of (8) and by Lemma 1, we obtain

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta) \prec \chi(\zeta),$$

which evidently completes the proof of our theorem. □

Briefly, we note that if $\Omega \subsetneq \mathbb{C}$ is a simply connected region, then there exists a conformal mapping h such that $h(\mathbb{U}) = \Omega$, where h maps $\mathbb{U}$ onto Ω. In this case, we denote $\mathcal{M}_1[h(\mathbb{U}), \chi]$ by $\mathcal{M}_1[h, \chi]$. The following corollary is an immediate consequence of Theorem 1.

Corollary 1. *Let $\varphi \in \mathcal{M}_1[h,\chi]$, $f \in \mathcal{A}$, and $\chi \in \mathcal{Q}_0$ satisfy (8) and*

$$\varphi\Big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta),\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta),\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta),\mathbb{I}^{(\varsigma+3,\varrho)}_{\ln(\epsilon)}f(\zeta);\zeta\Big) \prec h(\zeta).$$

Then

$$\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta) \prec \chi(\zeta).$$

Putting $\chi(\zeta) = P\zeta$, $P > 0$, in Definition 5, the class $\mathcal{M}_1[\Omega,\chi]$, now denoted by $\mathcal{M}_1[\Omega,P]$, is described below.

Definition 6. *Let $\Omega \subseteq \mathbb{C}$ and $P > 0$. The class $\mathcal{M}_1[\Omega,P]$ of admissible functions consists of those functions $\varphi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy*

$$\varphi\Bigg(Pe^{i\theta},\ \frac{(-\varsigma+\varrho m)Pe^{i\theta}}{\varsigma+\varrho},\ \frac{\varrho^2e^{-i\theta}L+\varrho(1+\varrho)mP-\varsigma(1+\varsigma)P}{e^{-i\theta}(\varsigma+\varrho)(1+\varsigma+\varrho)},$$
$$\frac{\varrho^3Ne^{-i\theta}+\varrho^2(3+\varsigma+3\varrho)Le^{-i\theta}+\varrho(2-\varsigma^2+3\varrho+\varsigma\varrho+\varrho^2)mP-\varsigma(2+3\varsigma+\varsigma^2)P}{e^{-i\theta}(\varsigma+\varrho)(1+\varsigma+\varrho)(2+\varsigma+\varrho)}\Bigg) \notin \Omega.$$

whenever $\theta \in \mathbb{R}$, $m \geq 2$, $\Re(Le^{-i\theta}) \geq m(m-1)P$, and $\Re(Ne^{-i\theta}) \geq 0$.

Corollary 2. *Let $\varphi \in \mathcal{M}_1[\Omega,P]$ and $f \in \mathcal{A}$ satisfy*

$$\Big|\zeta\Big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)\Big)'\Big| \leq mP, \quad (\zeta \in \mathbb{U},\ m \geq 2,\ P > 0),$$

and

$$\varphi\Big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta),\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta),\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta),\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)}f(\zeta);\zeta\Big) \in \Omega.$$

Then

$$\Big|\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)\Big| < P.$$

Corollary 3. *Let $\varphi \in \mathcal{M}_1[\Omega,P]$ and $f \in \mathcal{A}$ satisfy*

$$\Big|\zeta\big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)\big)'\Big| \leq mP \qquad (\zeta \in \mathbb{U},\ m \geq 2,\ P > 0),$$

and

$$\Big|\varphi\big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta),\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta),\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta),\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)}f(\zeta);\zeta\big)\Big| < P.$$

Then

$$\Big|\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)\Big| < P \qquad (\zeta \in \mathbb{U}).$$

Example 2. *Let $f \in \mathcal{A}$ satisfy*

$$\Big|\zeta\big(\mathbb{I}^{\varsigma,1}_{\ln(\epsilon)}f(\zeta)\big)'\Big| \leq 2P.$$

If

$$\Big|\big(\mathbb{I}^{\varsigma,1}_{\ln(\epsilon)}f(\zeta)\big)^2+\varsigma\,\mathbb{I}^{\varsigma,1}_{\ln(\epsilon)}f(\zeta)+(\varsigma+1)\,\mathbb{I}^{\varsigma+1,1}_{\ln(\epsilon)}f(\zeta)\Big| < 2P,$$

then

$$\mathbb{I}^{\varsigma,1}_{\ln(\epsilon)}f(\zeta) \prec P\zeta \qquad (\zeta \in \mathbb{U}).$$

Proof. Let $\varphi(a,b,c,d;\zeta) = a^2+\varsigma a+(\varsigma+1)b$ and $\Omega = h(\mathbb{U})$, where $h(\zeta) = 2P\zeta$, $P>0$. To apply Corollary 2, it suffices to show that $\varphi \in \mathcal{M}_1[h(\mathbb{U}),P]$. It is therefore sufficient to verify that

$$\begin{aligned}|\varphi(a,b,c,d;\zeta)| &= \left|a^2+\varsigma a+(\varsigma+1)b\right| \\ &= \left|(Pe^{i\theta})^2+\varsigma Pe^{i\theta}+(\varsigma+1)\left(\frac{2-\varsigma}{\varsigma+1}\right)Pe^{i\theta}\right| \\ &= \left|P^2e^{i\theta}+\varsigma P+(2-\varsigma)P\right| \\ &= \left|P^2e^{i\theta}+2P\right| \\ &\geq 2P,\end{aligned}$$

where $\Re(e^{-i\theta}N) \geq 0$, $\Re(e^{-i\theta}L) \geq 2P$, and $0 \leq \theta < 2\pi$. □

Definition 7. *Let $\Omega \subseteq \mathbb{C}$ and $\chi \in \mathcal{Q}_1 \cap \mathcal{O}_1'$. The class $\mathcal{M}_2[\Omega,\chi]$ of admissible functions consists of those functions $\varphi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy*

$$\varphi(a,b,c,d;\zeta) \notin \Omega,$$

such that

$$a = \chi(\mathcal{G}), \quad b = \chi(\mathcal{G}) + \frac{\varrho m \mathcal{G}\chi'(\mathcal{G})}{\varsigma+\varrho},$$

$$\Re\left(\frac{-a(\varsigma+\varrho)-c(1+\varsigma+\varrho)+b(1+2\varsigma+2\varrho)}{(a-b)\varrho}\right) \geq m\,\Re\left(\frac{\mathcal{G}\chi''(\mathcal{G})}{\chi'(\mathcal{G})}+1\right),$$

and

$$\begin{aligned}&\Re\left[\frac{1}{(-a+b)\varrho^2(\varsigma+\varrho)}\left((1+\varsigma+\varrho)(2+\varsigma+\varrho)d(\varsigma+\varrho)-a\Big(\varsigma^3+\varsigma[2+3\varrho(2+\varrho)]\right.\right.\\ &+3\varsigma^2(1+\varrho)+\varrho[(1+\varrho)(2+\varrho)]\Big)+\frac{(a-b)(\varsigma+\varrho)\varrho}{\varrho}\left(2+3\varsigma^2+9\varrho+7\varrho^2+\varsigma(6+9\varrho)\right)\\ &\left.\left.-3(1+\varsigma+2\varrho)\varrho^2\frac{(\varsigma+\varrho)\big(c(1+\varsigma+\varrho)+a(\varsigma+2\varrho)-b(1+2\varsigma+3\varrho)\big)}{\varrho^2}\right)\right]\\ &\geq m^2\,\Re\left(\frac{\mathcal{G}^2\chi'''(\mathcal{G})}{\chi'(\mathcal{G})}\right)\end{aligned}$$

where $\zeta \in \mathbb{U}$, $\mathcal{G} \in \partial\mathbb{U} \setminus E(\chi)$, and $m \geq 2$.

Theorem 3. *Let $\varphi \in \mathcal{M}_2[\Omega,\chi]$, $f \in \mathcal{A}$, and $\chi \in \mathcal{Q}_1$ satisfy*

$$\Re\left(\frac{\mathcal{G}\,\chi''(\mathcal{G})}{\chi'(\mathcal{G})}\right) \geq 0 \quad \textit{and} \quad \left|\frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta)-\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}\right| \leq \left|\frac{m\varrho\,\mathcal{G}\,\chi'(\mathcal{G})}{\varsigma+\varrho}\right|, \tag{16}$$

for $\zeta \in \mathbb{U}$, $\mathcal{G} \in \partial\mathbb{U} \setminus E(\chi)$, and $m \geq 2$. If

$$\left\{\varphi\left(\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta},\frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta},\frac{\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta},\frac{\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta};\zeta\right) : \zeta \in \mathbb{U}\right\} \subset \Omega, \tag{17}$$

then

$$\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta} \prec \chi(\zeta) \qquad (\zeta \in \mathbb{U}).$$

Proof. Define the analytic function $\xi(\zeta)$ in $\mathbb{U}$ by

$$\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta} = \xi(\zeta). \tag{18}$$

In view of (7) and (18), we obtain

$$\frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta} = \xi(\zeta) + \frac{\varrho\zeta\xi'(\zeta)}{\varsigma+\varrho}. \tag{19}$$

Differentiating (19) with respect to ζ and using (7), we get

$$\frac{\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta} = \xi(\zeta) + \frac{\varrho(1+2\varsigma+3\varrho)\zeta\xi'(\zeta) + \varrho^2\zeta^2\xi''(\zeta)}{(\varsigma+\varrho)(1+\varsigma+\varrho)}. \tag{20}$$

Differentiating (20) and applying (7), we obtain

$$\begin{aligned}\frac{\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta} = {} & \frac{1}{(\varsigma+\varrho)(1+\varsigma+\varrho)(2+\varsigma+\varrho)}\Big(\big(\varsigma^3+3\varsigma^2(1+\varrho)+\varrho(2+3\varrho+\varrho^2)\\ & +\varsigma(2+6\varrho+3\varrho^2)\big)\,\xi(\zeta)+\varrho\big(2+3\varsigma^2+9\varrho+7\varrho^2+\varsigma(6+9\varrho)\big)\,\zeta\xi'(\zeta)\\ & +3(1+\varsigma+2\varrho)\varrho^2\,\zeta^2\xi''(\zeta)+\varrho^3\,\zeta^3\xi^{(3)}(\zeta)\Big).\end{aligned} \tag{21}$$

Next, define the transforms from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\begin{aligned} a &= r,\\ b &= r+\frac{\varrho s}{\varsigma+\varrho},\\ c &= r+\frac{\varrho(1+2\varsigma+3\varrho)\,s+\varrho^2 t}{(\varsigma+\varrho)(1+\varsigma+\varrho)},\\ d &= \frac{1}{(\varsigma+\varrho)(1+\varsigma+\varrho)(2+\varsigma+\varrho)}\Big(\big(\varsigma^3+3\varsigma^2(1+\varrho)+\varrho(2+3\varrho+\varrho^2)\\ &\quad +\varsigma(2+6\varrho+3\varrho^2)\big)\,r+\varrho\big(2+3\varsigma^2+9\varrho+7\varrho^2+\varsigma(6+9\varrho)\big)\,s\\ &\quad +3(1+\varsigma+2\varrho)\varrho^2\,t+\varrho^3\,u\Big).\end{aligned} \tag{22}$$

Let the function $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ be defined by

$$\psi(r,s,t,u;\zeta) = \varphi(a,b,c,d;\zeta). \tag{23}$$

In view of (18)–(22), it follows from (23) that

$$\begin{aligned}&\psi\Big(\xi(\zeta),\zeta\xi'(\zeta),\zeta^2\xi''(\zeta),\zeta^3\xi^{(3)}(\zeta);\zeta\Big)\\ &= \varphi\left(\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta};\zeta\right).\end{aligned}$$

Therefore, condition (17) is equivalent to

$$\psi\Big(\xi(\zeta),\zeta\xi'(\zeta),\zeta^2\xi''(\zeta),\zeta^3\xi^{(3)}(\zeta);\zeta\Big) \in \Omega.$$

Furthermore, one readily computes

$$\frac{t}{s}+1=\frac{-a(\varsigma+\varrho)-c(1+\varsigma+\varrho)+b(1+2\varsigma+2\varrho)}{(a-b)\varrho}.$$

and

$$\begin{aligned}\frac{u}{s}=&\frac{1}{(-a+b)\,\varrho^2(\varsigma+\varrho)}\Bigg((1+\varsigma+\varrho)(2+\varsigma+\varrho)d(\varsigma+\varrho)\\&-a\Big(\varsigma^3+\varsigma[2+3\varrho(2+\varrho)]+3\varsigma^2[1+\varrho]+\varrho[(1+\varrho)(2+\varrho)]\Big)\\&+\frac{(a-b)(\varsigma+\varrho)\varrho[2+3\varsigma^2+9\varrho+7\varrho^2+\varsigma[6+9\varrho]]}{\varrho}\\&-3(1+\varsigma+2\varrho)\varrho^2\frac{(\varsigma+\varrho)(c(1+\varsigma+\varrho)+a(\varsigma+2\varrho)-b(1+2\varsigma+3\varrho))}{\varrho^2}\Bigg)\end{aligned}$$

Hence, the condition for $\varphi\in\mathcal{M}_2[\Omega,\chi]$ in Definition 7 is equivalent to the condition for $\psi\in\Psi_2[\Omega,\chi]$ in Definition 3. Therefore, in view of (16), from Lemma 1, we obtain

$$\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}\prec\chi(\zeta),$$

which completes the proof. □

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then there exists a conformal mapping $h:\mathbb{U}\to\Omega$ such that $h(\mathbb{U})=\Omega$. In this case, we denote $\mathcal{M}_2[h(\mathbb{U}),\chi]$ by $\mathcal{M}_2[h,\chi]$.

The following corollary is an immediate consequence of Theorem 2:

Corollary 4. *Let $\varphi\in\mathcal{M}_2[h,\chi]$, $f\in\mathcal{A}$, and $\chi\in\mathcal{Q}_1$ satisfy (16) and*

$$\varphi\left(\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta},\frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta},\frac{\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta},\frac{\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta};\zeta\right)\prec h(\zeta).$$

Then

$$\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}\prec\chi(\zeta).$$

Putting $\chi(\zeta)=1+P\zeta$, $P>0$ in Definition 7, the class $\mathcal{M}_2[\Omega,\chi]$, denoted now by $\mathcal{M}_2[\Omega,P]$, is described below.

Definition 8. *Let $\Omega\subset\mathbb{C}$ and $P>0$. The class $\mathcal{M}_2[\Omega,P]$ of admissible functions consists of those functions $\varphi:\mathbb{C}^4\times\mathbb{U}\to\mathbb{C}$ that satisfy*

$$\begin{aligned}&\varphi\Bigg(Pe^{i\theta},\Big(1+\frac{\varrho m}{\varsigma+\varrho}\Big)Pe^{i\theta},P+\frac{\varrho(1+2\varsigma+3\varrho)mP+\varrho^2Le^{-i\theta}}{e^{-i\theta}(\varsigma+\varrho)(1+\varsigma+\varrho)},\frac{1}{(\varsigma+\varrho)(1+\varsigma+\varrho)(2+\varsigma+\varrho)}\\&\Big((\varsigma^3+3\varsigma^2(1+\varrho)+\varrho(2+3\varrho+\varrho^2)+\varsigma(2+6\varrho+3\varrho^2))Pe^{i\theta}\\&+\varrho(2+3\varsigma^2+9\varrho+7\varrho^2+\varsigma(6+9\varrho))mPe^{i\theta}+3(1+\varsigma+2\varrho)\varrho^2L+\varrho^3N\Big);\zeta\Bigg)\notin\Omega,\end{aligned}$$

whenever $\theta\in\mathbb{R}$, $m\geq2$, $\Re(Le^{-i\theta})\geq m(m-1)P$, and $\Re(Ne^{-i\theta})\geq0$.

Corollary 5. *Let $\varphi \in \mathcal{M}_2[\Omega, P]$ and $f \in \mathcal{A}$ satisfy*

$$\Re\left(\frac{\mathcal{G}\chi''(\mathcal{G})}{\chi'(\mathcal{G})}\right) \geq 0, \quad \left|\frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)} f(\zeta) - \mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}\right| \leq \left|\frac{m\varrho\chi'(\mathcal{G})}{\varsigma+\varrho}\right|, \quad (\zeta \in \mathbb{U}, m \geq 2, P > 0),$$

and

$$\varphi\left(\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}; \zeta\right) \in \Omega.$$

Then

$$\left|\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta} - 1\right| < P.$$

Choose $\Omega = \{\zeta \in \mathbb{C} : |\zeta - 1| < P\}$, then $\mathcal{M}_2[\Omega, P] = \mathcal{M}_2[P]$, and Corollary 5 takes the following form:

Corollary 6. *Let $\varphi \in \mathcal{M}_2[P]$ and $f \in \mathcal{A}$ satisfy*

$$\left|\frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)} f(\zeta) - \mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}\right| \leq \left|\frac{m\varrho\chi'(\mathcal{G})}{\varsigma+\varrho}\right|, \quad (\zeta \in \mathbb{U}, m \geq 2, P > 0),$$

and

$$\left|\varphi\left(\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta}; \zeta\right) - 1\right| < P.$$

Then

$$\left|\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)} f(\zeta)}{\zeta} - 1\right| < P.$$

The remaining concepts and results in this section will be similar to those already discussed, but instead of subordination, we will examine superordination, as they are two associated concepts. Accordingly, we will begin with the definition below.

Definition 9. *Let $\Omega \subset \mathbb{C}$ and $\chi \in \mathcal{O}_0$ with $\chi'(\zeta) \neq 0$. The class $\mathcal{M}'_1[\Omega, \chi]$ of admissible functions consists of those functions $\varphi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy*

$$\varphi(a, b, c, d; \mathcal{G}) \in \Omega,$$

such that

$$a = \chi(\zeta), \quad b = \frac{-\varsigma\chi(\zeta) + \varrho m\zeta\chi'(\zeta)}{\varsigma+\varrho},$$

$$\Re\left(\frac{a\varsigma(1+\varsigma) + \varsigma a + (\varsigma+\varrho)(-b + c(1+\varsigma+\varrho))}{\varrho(-\varsigma a + b(\varsigma+\varrho))}\right) \leq \frac{1}{m}\Re\left(\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)} + 1\right),$$

and

$$\Re\left[\frac{1}{\varrho^2(-\varsigma a+b(\varsigma+\varrho))}\Big(a\varsigma(1+\varsigma)(2+\varsigma)+d(\varsigma+\varrho)(1+\varsigma+\varrho)(2+\varsigma+\varrho)\right.$$
$$+(2-\varsigma^2+(3+\varsigma)\varrho+\varrho^2)(\varsigma a-b(\varsigma+\varrho))$$
$$\left.-(3+\varsigma+3\varrho)(a\varsigma(1+\varsigma)+c(\varsigma+\varrho)(1+\varsigma+\varrho)+(1+\varrho)(\varsigma a-b(\varsigma+\varrho)))\Big)\right]$$
$$\geq \frac{1}{m^2}\,\Re\Big(\frac{\zeta^2\chi'''(\zeta)}{\chi'(\zeta)}\Big)$$

where $\zeta \in \mathbb{U}$, $\mathcal{G} \in \partial\mathbb{U}$, *and* $m \geq 2$.

Theorem 4. *Let* $\Omega \subset \mathbb{C}$ *and* $\varphi \in \mathcal{M}_1'[\Omega,\chi]$, $f \in \mathcal{A}$, $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta) \in \mathcal{Q}_0$, *and* $\chi \in \mathcal{O}_0$ *with* $\chi'(\zeta) \neq 0$ *satisfy the following conditions:*

$$\Re\Big(\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\Big) \geq 0 \quad \textit{and} \quad \Big|\zeta\Big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)\Big)'\Big| \leq m|\chi'(\zeta)|, \tag{24}$$

and $\varphi(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)}f(\zeta);\zeta)$ *is univalent in* $\mathbb{U}$. *Then*

$$\Omega \subset \Big\{\varphi(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)}f(\zeta);\zeta) : \zeta \in \mathbb{U}\Big\}, \tag{25}$$

implies that

$$\chi(\zeta) \prec \mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta), \quad \zeta \in \mathbb{U}.$$

Proof. Let the function $\xi(\zeta)$ be defined by (10) and ψ by (15). Since $\varphi \in \mathcal{M}_1'[\Omega,\chi]$, from (24) and the condition (25), we have

$$\Omega \subset \Big\{\psi(\xi(\zeta), \zeta\xi'(\zeta), \zeta^2\xi''(\zeta), \zeta^3\xi'''(\zeta);\zeta) : \zeta \in \mathbb{U}\Big\}.$$

The admissibility condition for $\varphi \in \mathcal{M}_1'[\Omega,\chi]$ in Definition 8 is equivalent to the admissibility condition for $\varphi \in \Psi_n[\Omega,\chi]$ as given in Definition 4 with $n = 2$.

Thus, by (25) and applying Lemma 2, the proof is complete. □

Briefly, we note that if $\Omega \subsetneq \mathbb{C}$ is a simply connected region, then there exists a conformal mapping h such that $h(\mathbb{U}) = \Omega$, where h maps $\mathbb{U}$ onto Ω. In this case, we denote $\mathcal{M}_1'[h(\mathbb{U}),\chi]$ by $\mathcal{M}_1'[h,\chi]$.

The following corollary is an immediate consequence of Theorem 3.

Corollary 7. *Let* $\varphi \in \mathcal{M}_1'[h,\chi]$, $f \in \mathcal{A}$, $\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta) \in \mathcal{Q}_0$, *and* $\chi \in \mathcal{O}_0$ *with* $\chi'(\zeta) \neq 0$ *satisfy condition* (24), *and assume that*

$$\varphi\Big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)}f(\zeta);\zeta\Big)$$

is univalent in $\mathbb{U}$. *Then*

$$h(\zeta) \prec \varphi\Big(\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta), \mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)}f(\zeta);\zeta\Big)$$

implies that

$$\chi(\zeta) \prec \mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta), \quad \zeta \in \mathbb{U}.$$

Definition 10. *Let $\Omega \subseteq \mathbb{C}$ and $\chi \in \mathcal{O}_1'$ with $\chi'(\zeta) \neq 0$. The class $\mathcal{M}_2'[\Omega, \chi]$ of admissible functions consists of those functions $\varphi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy*

$$\varphi(a, b, c, d; \mathcal{G}) \in \Omega,$$

such that

$$a = \chi(\zeta), \quad b = \chi(\zeta) + \frac{\varrho m \zeta \chi'(\zeta)}{\varsigma + \varrho},$$

and

$$\Re\left(\frac{-a(\varsigma+\varrho) - c(1+\varsigma+\varrho) + b(1+2\varsigma+2\varrho)}{(a-b)\varrho}\right) \leq \frac{1}{m}\Re\left(\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)} + 1\right),$$

and

$$\begin{aligned}
&\Re\Bigg[\frac{1}{(-a+b)\varrho^2(\varsigma+\varrho)}\Big((1+\varsigma+\varrho)(2+\varsigma+\varrho)d(\varsigma+\varrho) - a\big(\varsigma^3 + \varsigma[2+3\varrho(2+\varrho)] + 3\varsigma^2(1+\varrho)\\
&+ \varrho[(1+\varrho)(2+\varrho)]\big) + \frac{(a-b)(\varsigma+\varrho)\varrho[2+3\varsigma^2+9\varrho+7\varrho^2+\varsigma(6+9\varrho)]}{\varrho}\\
&- 3(1+\varsigma+2\varrho)\varrho^2\frac{(\varsigma+\varrho)\big(c(1+\varsigma+\varrho)+a(\varsigma+2\varrho)-b(1+2\varsigma+3\varrho)\big)}{\varrho^2}\Big)\Bigg]\\
&\leq \frac{1}{m^2}\Re\left(\frac{\zeta^2\chi'''(\zeta)}{\chi'(\zeta)}\right),
\end{aligned}$$

where $\zeta \in \mathbb{U}$, $\mathcal{G} \in \partial\mathbb{U}$, and $m \geq 2$.

Theorem 5. *Let $\Omega \subset \mathbb{C}$ and $\varphi \in \mathcal{M}_2'[\Omega, \chi]$, $f \in \mathcal{A}$, $\frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta} \in \mathcal{Q}_1$, and $\chi \in \mathcal{O}_1'$ with $\chi'(\zeta) \neq 0$ satisfy the following conditions:*

$$\begin{aligned}
\Re\left(\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\right) &\geq 0,\\
\left|\frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+1,\varrho} f(\zeta) - \mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta}\right| &\leq \left|\frac{m\varrho\chi'(\zeta)}{\varsigma+\varrho}\right|.
\end{aligned} \tag{26}$$

If

$$\varphi\left(\frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+1,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+2,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+3,\varrho} f(\zeta)}{\zeta}; \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subseteq \left\{\varphi\left(\frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+1,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+2,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+3,\varrho} f(\zeta)}{\zeta}; \zeta\right) : \zeta \in \mathbb{U}\right\}, \tag{27}$$

which leads to

$$\chi(\zeta) \prec \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta}, \quad (\zeta \in \mathbb{U}).$$

Proof. Let the function $\xi(\zeta)$ be defined by (18) and ψ by (23). Since $\varphi \in \mathcal{M}_1'[\Omega, \chi]$, relations (26) and (27) yield

$$\Omega \subseteq \left\{\psi\Big(\xi(\zeta), \zeta\xi'(\zeta), \zeta^2\xi''(\zeta), \zeta^3\xi'''(\zeta); \zeta\Big) : \zeta \in \mathbb{U}\right\}.$$

The admissibility condition for $\varphi \in \mathcal{M}_2^{'}[\Omega, \chi]$ in Definition 10 is equivalent to the admissibility condition for $\varphi \in \Psi_n[\Omega, \chi]$ as given in Definition 4 with $n = 2$. Thus, by (27) and applying Lemma 2, the proof is complete. □

Briefly, we note that if $\Omega \subsetneq \mathbb{C}$ is a simply connected region, then there exists a conformal mapping h such that $h(\mathbb{U}) = \Omega$, where h maps $\mathbb{U}$ onto Ω. In this case, we denote $\mathcal{M}_2^{'}[h(\mathbb{U}), \chi]$ by $\mathcal{M}_2^{'}[h, \chi]$. The following corollary is an immediate consequence of Theorem 5.

Corollary 8. *Let $\Omega \subset \mathbb{C}$ and $\varphi \in \mathcal{M}_2^{'}[h, \chi]$, $f \in \mathcal{A}$, $\dfrac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta} \in \mathcal{Q}_1$, and $\chi \in \mathcal{O}_1^{'}$ with $\chi^{'}(\zeta) \neq 0$ satisfy the following conditions:*

$$\begin{aligned} \Re\left(\frac{\zeta \chi^{''}(\zeta)}{\chi^{'}(\zeta)}\right) &\geq 0, \\ \left|\frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+1,\varrho} f(\zeta) - \mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta}\right| &\leq \left|\frac{m\varrho\chi^{'}(\zeta)}{\varsigma + \varrho}\right|. \end{aligned} \tag{28}$$

If

$$\varphi\left(\frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+1,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+2,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+3,\varrho} f(\zeta)}{\zeta}; \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h(\zeta) \prec \varphi\left(\frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+1,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+2,\varrho} f(\zeta)}{\zeta}, \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma+3,\varrho} f(\zeta)}{\zeta}; \zeta\right)$$

leads to

$$\chi(\zeta) \prec \frac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta}, \quad (\zeta \in \mathbb{U}).$$

3. Sandwich-Type Theorems Involving $\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho}$

In this section, combining Corollaries 1 and 7 (Corollaries 4 and 8, respectively), we obtain the following sandwich-type results.

Corollary 9. *Let h_1 and χ_1 be analytic functions in $\mathbb{U}$, h_2 be a univalent function in $\mathbb{U}$, χ_2 and $\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)$ belong to $\mathcal{Q}_0 \cap \mathcal{O}_0$, and $\varphi \in \mathcal{M}_1^{'}[h_1, \chi_1] \cap \mathcal{M}_1[h_2, \chi_2]$.*
If $f \in \mathcal{A}$,

$$\varphi\left(\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta), \mathbb{I}_{\ln(\epsilon)}^{\varsigma+1,\varrho} f(\zeta), \mathbb{I}_{\ln(\epsilon)}^{\varsigma+2,\varrho} f(\zeta), \mathbb{I}_{\ln(\epsilon)}^{\varsigma+3,\varrho} f(\zeta); \zeta\right)$$

is univalent in $\mathbb{U}$ and the conditions (8) and (24) hold, then

$$h_1(\zeta) \prec \varphi\left(\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta), \mathbb{I}_{\ln(\epsilon)}^{\varsigma+1,\varrho} f(\zeta), \mathbb{I}_{\ln(\epsilon)}^{\varsigma+2,\varrho} f(\zeta), \mathbb{I}_{\ln(\epsilon)}^{\varsigma+3,\varrho} f(\zeta); \zeta\right) \prec h_2(\zeta),$$

which leads to

$$\chi_1(\zeta) \prec \mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta) \prec \chi_2(\zeta), \quad (\zeta \in \mathbb{U}).$$

Corollary 10. *Let h_1 and χ_1 be analytic functions in $\mathbb{U}$, h_2 be a univalent function in $\mathbb{U}$, χ_2 and $\dfrac{\mathbb{I}_{\ln(\epsilon)}^{\varsigma,\varrho} f(\zeta)}{\zeta}$ belong to $\mathcal{Q}_1 \cap \mathcal{O}_1^{'}$, and $\varphi \in \mathcal{M}_2^{'}[h_1, \chi_1] \cap \mathcal{M}_2[h_2, \chi_2]$.*

If $f \in \mathcal{A}$ *and*

$$\varphi\left(\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}; \zeta\right)$$

is univalent in $\mathbb{U}$ *and the conditions (16) and (28) hold, then*

$$h_1(\zeta) \prec \varphi\left(\frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+1,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+2,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}, \frac{\mathbb{I}^{\varsigma+3,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta}; \zeta\right) \prec h_2(\zeta),$$

which leads to

$$\chi_1(\zeta) \prec \frac{\mathbb{I}^{\varsigma,\varrho}_{\ln(\epsilon)}f(\zeta)}{\zeta} \prec \chi_2(\zeta), \quad (\zeta \in \mathbb{U}).$$

4. Conclusions

In this work, we introduced a logarithmic integral operator that merges the concepts of differentiation and fractional integration within the complex domain. Unlike previous approaches, this operator allows a systematic rearrangement of coefficients in alternating power series without affecting convergence or the analytic nature of the functions. Employing this framework, we established third-order differential subordination and superordination results, which naturally extend to sandwich-type theorems. The presented findings demonstrate the versatility of the operator in analyzing the growth, distortion, and mapping properties of analytic functions. This approach opens avenues for further research and potential applications in applied mathematics, including fluid mechanics and other fields requiring complex function analysis.

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References

1. Miller, S.S.; Mocanu, P.T.; Reade, M.O. Bazilevic functions and generalized convexity. *Rev. Roum. Math. Pures Appl.* **1974**, *19*, 213–224.
2. Duren, P.L. *Univalent Functions*; Springer Verlag: New York, NY, USA; Berlin/Germany; Tokyo, Japan, 1983.
3. Miller, S.S.; Mocanu, P.T. Differential subordination and univalent functions. *Mich. Math. J.* **1981**, *28*, 157–171. [CrossRef]
4. Antonino, J.A.; Miller, S.S. Third-Order Differential Inequalities and Subordinations in the Complex Plane. *Complex Var. Elliptic Equ.* **2011**, *56*, 439–454. [CrossRef]

5. Goldstein, M.; Hall, R.R.; Sheil-Small, T.; Smith, H.L. Convexity Preservation of Inverse Euler Operators and a Problem of S. Miller. *Bull. Lond. Math. Soc.* **1982**, *14*, 537–541. [CrossRef]
6. Ponnusamy, S.; Juneja, O.P. Third-order differential inequalities in the complex plane. In *Current Topics in Analytical Functions Theory*; World Scientific Publishing Co. Pte. Ltd.: Singapore, 1992; pp. 274–290.
7. Bulboacă, T. *Differential Subordinations and Superordinations: Recent Results*; Casa Cărţii de Ştiinţă: Cluj-Napoca, Romania, 2005.
8. Miller, S.S.; Mocanu, P.T. *Differential Subordinations: Theory and Applications*; CRC Press: Boca Raton, FL, USA, 2000.
9. Miller, S.S.; Mocanu, P.T. Subordinants of differential superordinations. *Complex Var. Theory Appl.* **2003**, *48*, 815–826. [CrossRef]
10. Tang, H.; Srivastava, H.M.; Li, S.-H.; Ma, L.-N. Third-order differential subordination and superordination results for meromorphically multivalent functions associated with the Liu–Srivastava operator. In *Abstract and Applied Analysis*; Hindawi Publishing Corporation: London, UK, 2014; p. 792175.
11. Tang, H.; Deniz, E. Third-order differential subordination results for analytic functions involving the generalized Bessel functions. *Acta Math. Sci.* **2014**, *34*, 1707–1719. [CrossRef]
12. Tang, H.; Srivastava, H.M.; Deniz, E.; Li, S.-H. Third-order differential superordination involving the generalized Bessel functions. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 1669–1688. [CrossRef]
13. Ali, R.M.; Ravichandran, V.; Seenivasagan, N. Differential subordination and superordination of analytic functions defined by the Dziok–Srivastava linear operator. *J. Frankl. Inst.* **2010**, *347*, 1762–1781. [CrossRef]
14. Barciz, Á.; Kant, S.; Prajapat, J. Differential subordination and superordination results associated with the Wright function. *Bull. Iran. Math. Soc.* **2016**, *42*, 1459–1477.
15. Cho, N.E.; Nishiwaki, J.; Owa, S.; Srivastava, H.M. Subordination and superordination for multivalent functions associated with a class of fractional differintegral operators. *Integral Transform. Spec. Funct.* **2010**, *21*, 243–258. [CrossRef]
16. Hadi, S.H.; Darus, M.; Ghanim, F.; Alb Lupaş, A. Sandwich-type theorems for a family of non-Bazilevič functions involving a q-analog integral operator. *Mathematics* **2023**, *11*, 2479. [CrossRef]
17. Morais, J.; Zayed, H.M.; Srivastava, R. Third-order differential subordinations for multivalent functions in the theory of source–sink dynamics. *Math. Methods Appl. Sci.* **2021**, *44*, 11269–11287. [CrossRef]
18. Tayyah, A.S.; Atshan, W.G. Differential subordination and superordination results for p-valent analytic functions associated with (r, k)-Srivastava fractional integral calculus. *MethodsX* **2024**, *13*, 103079. [CrossRef]
19. Zayed, H.M.; Bulboacă, T. Sandwich results for higher order fractional derivative operators. *Mat. Stud.* **2018**, *49*, 52–66. [CrossRef]
20. El-Ityan, M.; Cătaş, A.; Hammad, S.; El-Deeb, S. Third Order Differential Subordination Results Associated with Tremblay Fractional Derivative Operator for p-Valent Analytic Functions. *Fractal Fract.* **2026**, *10*, 103. [CrossRef]
21. Hadi, S.H.; Darus, M.; Alamri, B.; Altınkaya, Ş.; Alatawi, A. On classes of ζ-uniformly q-analogue of analytic functions with some subordination results. *Appl. Math. Sci. Eng.* **2024**, *32*, 12803.. [CrossRef]
22. Mostafa, A.O.; Aouf, M.K.; Zayed, H.M.; Bulboacă, T. Multivalent functions associated with Srivastava–Saigo–Owa fractional differintegral operator. *Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Math.* **2018**, *112*, 1409–1429. [CrossRef]
23. Tayyah, A.S.; Atshan, W.G.; Oros, G.I. Third-order differential subordination results for meromorphic functions associated with the inverse of the Legendre chi function via the Mittag-Leffler identity. *Mathematics* **2025**, *13*, 2089. [CrossRef]
24. Tayyah, A.S.; Atshan, W.G.; Yalçın, S. Third-Order Differential Subordination and Superordination Results for p-Valent Analytic Function Involving Fractional Derivative Operator. *Math. Methods Appl. Sci.* **2026**, *49*, 67–77. [CrossRef]
25. Debnath, L.; Bhatta, D. *Integral Transforms and Their Applications*, 3rd ed.; Chapman & Hall/CRC: Boca Raton, FL, USA, 2014.
26. Doetsch, G. *Introduction to the Theory and Application of the Laplace Transformation*; Springer: Berlin, Germany, 1974.
27. Flajolet, P.; Gourdon, X.; Dumas, P. Mellin transforms and asymptotics: Harmonic sums. *Theor. Comput. Sci.* **1995**, *10*, 3–58. [CrossRef]
28. Aboodh, K.S. The new integral transform "Aboodh Transform". *Glob. J. Pure Appl. Math.* **2013**, *9*, 35–43.

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