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Lagrange Adjoint System for Linear Fractional Systems with Distributed Delays

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Abstract

In the present work we introduce an appropriate definition of the notion of a Lagrange (formal) adjoint system for homogeneous systems with incommensurate-order derivatives in Caputo's sense and distributed delays. Under natural conditions, the existence and uniqueness of the adjoint fundamental matrix for these systems are proven, and an explicit relation between the resolvent of the formal adjoint system and the adjoint fundamental matrix is established. A generalization of the well-known Halanay relation is also introduced and proven. Furthermore, an explicit relation between the fundamental matrix and its adjoint fundamental matrix is established.

Keywords: distributed delay; linear fractional system; Lagrange adjoint system; Caputo fractional derivative; fractional derivative in Caputo's sense

MSC: 34A08; 34A12; 34D20

1. Introduction

Nowadays, the theory of fractional differential equations is the subject of intensive research. This is due to the fact that they have proved to be an important tool in modelling real-world processes, due to the possibility of more accurate representation using the memory contained in the definition of fractional derivatives. Comprehensive information on the theory of fractional calculus and differential equations can be found in the excellent monographs of Kilbas et al. [1], Podlubny [2], and Hilfer [3]. In Jiao et al. [4], fractional differential equations with distributed order are considered. For the theory and applications of impulsive fractional differential equations we refer to Stamova and Stamov [5]. For a good overview of the modern scope of investigations we recommend [6–8], and for practical applications in modern optimization we recommend [9], in physics [10,11], and in neural networks [12].

In the present work, we consider linear systems with incommensurate-order fractional derivatives in the sense of Caputo and distributed delays. The reason we have studied distributed delay systems is that this type of delay, according to Riesz's theorem, includes as special cases all types of delays (constant, variable, etc.) and therefore the results we have obtained will be applicable to all types of delays.



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Inspired by the integer case (by integer case we mean delayed systems with first-order derivatives), here we consider a Lagrange (formal) adjoint system for a homogeneous delayed linear fractional system. The main challenge of conducting these investigations in the fractional case is the presence of two major obstacles of principal character that must be overcome. The first is the loss of key properties of the first-order derivative, in terms of its locality and symmetry, as well as the important Leibniz rule. The second is the non-determinism, caused by the fact that, in general, a shift in the lower terminal of the fractional derivatives leads to different results, which is in contrast with the case of deterministic systems described by integer-order differential equations. For example, the study of backward continuation of the solutions of delayed linear fractional systems and their corresponding Lagrange (formal) adjoint systems leads to the necessity to investigate the properties of fractional differential equations with a deviating argument of the advanced type. Moreover, even in the integer case, the continuation problem has features not found in ordinary differential equations, as shown by Myshkis [13]. For a good historical overview of integer (first)-order systems with delays and the themes studied in the present article, we recommend the monographs of Halanay [14], Hale et al. [15], and Kolmanovskii et al. [16].

The explanations above explain the absence of results concerning formal adjoint systems for fractional equations or systems with delays. According to our knowledge, Ref. [17] is the first paper containing such results for delayed fractional systems, mainly related to the backward continuation of the solutions.

The article is organized as follows. Section 2 includes an overview of the definitions and basic properties of the left- and right-side Riemann–Liouville fractional integrals and derivatives, as well as of the Caputo left- and right-side fractional derivatives. The same section also presents the statement of the problem and some notation and results used in the article. In Section 3, following [17], we introduce a formal (Lagrange) adjoint system for the homogeneous system and establish that the final problem (FP) for the adjoint system possesses a unique solution called the adjoint fundamental matrix. An explicit relation between the resolvent of the formal adjoint system and the adjoint fundamental matrix is established too. In Section 4, a generalization for the fractional case of the well-known relation established by A. Halanay in [14] for delayed systems with integer (first)-order derivatives is also introduced and proven. Furthermore, in the same section, an explicit relation between the fundamental matrix and its adjoint fundamental matrix is also established. Some examples are presented in Section 5. Section 6 is devoted to comments and conclusions.

2. Preliminaries and Problem Statement

In this section, for readers' convenience, we present the notation used and recall the definitions of Riemann–Liouville and Caputo fractional derivatives, with some of their required properties, following the books [1,2]. Moreover, we also present some auxiliary results that are used in the exposition below.

With $L_1^{loc}(\mathbb{R}, \mathbb{R})$, we denote the linear spaces of all locally Lebesgue-integrable functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and by $BL_1^{loc}(\mathbb{R}, \mathbb{R}) \subset L_1^{loc}(\mathbb{R}, \mathbb{R})$ its subspace of all locally bounded functions.

For arbitrary $a, b \in \mathbb{R}$, $q \in (0, 1]$ and any $g \in L_1^{loc}(\mathbb{R}, \mathbb{R})$, we define the left- and right-sided fractional integral operators, Riemann–Liouville, and Caputo fractional derivatives of order $q \in (0, 1)$ via the equalities

$$(I_{a+}^q g)(t) = \frac{1}{\Gamma(q)} \int_a^t (t-v)^{q-1} g(v) dv, \quad {}^{RL}D_{a+}^q g(t) = \frac{d}{dt} (I_{a+}^{1-q} g)(t),$$

$${}^CD_{a+}^q g(t) = {}^{RL}D_{a+}^q [g(v) - g(a)](t), \quad t > a$$

$$\begin{aligned}
(I_{b-}^q g)(t) &= \frac{1}{\Gamma(q)} \int_t^b (v-t)^{q-1} g(v) dv, \quad {}^{RL}D_{b-}^q g(t) = -\frac{d}{dt}((I_{b-}^{1-q})g(t), \\
{}^C D_{b-}^q g(t) &= {}^{RL}D_{b-}^q [g(v) - g(b)](t), t < b \\
D_{a+}^0 g(t) &= {}^C D_{b-}^0 g(t) = g(t).
\end{aligned}$$

The notations $\mathbb{R}_+ = (0, \infty)$, $J_a = [a, \infty)$, $q = (q_1, \dots, q_n)$, $q_k \in (0, 1]$, $k \in \langle n \rangle = \{1, 2, \dots, n\}$ will be used in the following; also, for any $n \in \mathbb{N}$ the real linear space of the square matrices is denoted by $\mathbb{R}^{n \times n}$, where $I, \Theta \in \mathbb{R}^{n \times n}$ are the identity and zero matrix, respectively, and $\mathbf{0} \in \mathbb{R}^n$ denotes the zero vector-column and $\mathbb{R}^{n*} = (\mathbb{R}^n)^T$ (the symbol T means transposition).

For $W(t, \theta) = \{w_{kj}(t, \theta)\}_{k,j=1}^n : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$, we denote by $|W(t, \theta)| = \sum_{k,j=1}^n |w_{kj}(t, \theta)|$ a Euclidean norm in $\mathbb{R}^{n \times n}$ for any fixed $(t, \theta) \in \mathbb{R}^2$, and by $BV^{loc}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^{n \times n}) = BV^{loc}$ we denote the linear space of the matrix-valued functions $W(t, \theta)$ with bounded variation in θ on every compact interval $[c, d] \subset \mathbb{R}$ for any $t \in \mathbb{R}$; i.e., $\text{Var}_{[c,d]} W(t, \cdot) = \sum_{k,j=1}^n \text{Var}_{[c,d]} w_{kj}(t, \cdot)$.

The notation $I_\beta(Y(t)) = \text{diag}((y_1(t))^{\beta_1}, \dots, (y_n(t))^{\beta_n})$ is used for any

$$y(t) = (y^1(t), \dots, y^n(t))^T : \mathbb{R} \rightarrow \mathbb{R}^n, \quad (y(t) = (y^1(t), \dots, y^n(t)) : \mathbb{R} \rightarrow \mathbb{R}^{n*}),$$

$$\beta = (\beta_1, \dots, \beta_n), \beta_k \in [-1, 1], k \in \langle n \rangle.$$

Let $h > 0$ be arbitrary. The Banach space **BL** of vector functions

$$\varphi = (\varphi_1, \dots, \varphi_n)^T : [-h, 0] \rightarrow \mathbb{R}^n$$

consists of all vector functions that are bounded and Lebesgue-measurable on the interval $[-h, 0]$, with norm $\|\varphi\| = \sum_{k=1}^n \sup_{s \in [-h, 0]} |\varphi_k(s)| < \infty$. With PC^{loc} (PAC^{loc}) we denote the real

linear space of vector functions $\varphi = (\varphi_1, \dots, \varphi_n)^T : \mathbb{R} \rightarrow \mathbb{R}^n$, which are piecewise continuous (piecewise absolutely continuous) over any compact interval $[a, b] \subset \mathbb{R}$. This means that any function can be broken down into a finite number of separate continuous (absolutely continuous) pieces. In addition, $\mathbf{PAC} = PAC^{loc}([-h, 0], \mathbb{R}^n)$ and $\mathbf{PC} = PC^{loc}([-h, 0], \mathbb{R}^n)$.

With $\mathbf{C}_0^*$, we denote the Banach space of row-valued functions $\psi : \mathbb{R} \rightarrow \mathbb{R}^{n*}$, where $\psi : BV([-h, 0], \mathbb{R}^{n*})$ and norm $\|\psi\|_V = \text{Var}_{[-h, 0]} \psi(\cdot)$, which satisfy the following conditions:

- (i) ψ is left-continuous on $(-h, 0)$;
- (ii) $\psi(s) = \psi(-h)$, $s \in (-\infty, -h]$ and $\psi(\theta) = \mathbf{0}^T$, $\theta \in \mathbb{R}_+$.

Then, for arbitrary fixed $t \in \mathbb{R}$, with $\mathbf{C}_t^*$ we denote the real linear space of all vector functions:

$$y(s; t) = (y_1(s; t), \dots, y_n(s; t)) : \mathbb{R} \rightarrow \mathbb{R}^{n*}, y(s; t) \in C((-\infty, t], \mathbb{R}^{n*})$$

with $y_t(-\theta) := y(t - \theta) = \psi(\theta)$ for some $\psi(\theta) \in \mathbf{C}_0^*$, $\theta \in [-h, 0]$, $y(s) = \mathbf{0}^T$ for $s \in (t + h, \infty)$.

Let $a \in \mathbb{R}$, $h > 0$ be arbitrary fixed numbers and consider for $t > a$ the following inhomogeneous fractional linear delayed system with Caputo-type derivatives of incommensurate orders and distributed delays,

$$D_{a+}^q x(t) = \int_{-h}^0 [d_\theta U(t, \theta)] x(t + \theta) + f(t), \quad (1)$$

with an initial condition for each $\varphi \in \mathbf{BL}$ as follows:

$$x_a(\theta) = x(a + \theta) = \varphi(\theta) \quad \text{for} \quad -h \leq \theta \leq 0, \quad (2)$$

where $D_{a+}^q = \text{diag}({}_C D_{a+}^{q_1}, \dots, {}_C D_{a+}^{q_n})$, $D_{a+}^q x(t) = (D_{a+}^{q_1} x_1(t), \dots, D_{a+}^{q_n} x_n(t))^T$ (the symbol T means transposition), $D_{a+}^{q_k} x_k(t) = {}_C D_{a+}^{q_k} x_k(t)$, $k \in \langle n \rangle = \{1, 2, \dots, n\}$; $x, f: \mathbb{R} \rightarrow \mathbb{R}^n$; $x(t) = (x_1(t), \dots, x_n(t))^T$, $f(t) = (f_1(t), \dots, f_n(t))^T$, $f \in L_1^{loc}(\mathbb{R}, \mathbb{R}^n)$, and $x_t(\theta) = x(t + \theta)$, $-h \leq \theta \leq 0$.

Everywhere below, we will assume that the kernel $U(t, \theta) \in BL^{loc}$ and its Lebesgue decomposition in θ has the form

$$U(t, \theta) = U^{ju}(t, \theta) + U^{ac}(t, \theta) + U^s(t, \theta)$$

with $U^{ju}(t, \theta) = \sum_{i \in \langle m \rangle_0} U_i^{ju}(t, \theta)$, $i \in \langle m \rangle_0 = \{0, 1, \dots, m\}$, $m \in \mathbb{N}$, where $U_i^{ju}(t, \theta)$, $U^{ac}(t, \theta)$, and $U^s(t, \theta)$ are the jump, absolutely continuous, and singular parts, respectively, in the decomposition. Note that the integral in (1) is understood in the Lebesgue–Stieltjes sense for any $\varphi \in \mathbf{BL}$.

Remark 1. The initial condition (2) means that the function $t \rightarrow x(t)$, $t \in J_a$ is considered as a prolongation from right of the function $t \rightarrow \phi(t - a)$ for $t \in J_a$.

The corresponding homogeneous system of (1) (i.e., when $f(t) \equiv 0$, $t \in \mathbb{R}$) written in detail for any $k \in \langle n \rangle$ has the form

$$D_{a+}^q x_k(t) = \sum_{i \in \langle m \rangle_0} \left(\sum_{j \in \langle n \rangle} \int_{-h}^0 x_k(t + \theta) d_\theta u_{k,j}^i(t, \theta) \right). \quad (3)$$

Definition 1 ([14–16]). We say that for the kernel $U: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$, conditions (S) are fulfilled if for each $(t, \theta) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ and for any partition $0 = \sigma_0 < \sigma_1 < \sigma_2 < \dots < \sigma_m = h$ of $[0, h]$ the following conditions hold:

(S1) The functions $U_i^{ju}(t, \theta)$, $U^{ac}(t, \theta)$, $U^s(t, \theta)$, $i \in \langle m \rangle_0$, $m \in \mathbb{N}$ are measurable in $(t, \theta) \in \mathbb{R} \times \mathbb{R}$, continuous from left in θ on $(-h, 0)$, normalized so that for $\theta \geq 0$ the following equality is satisfied: $U_i^{ju}(t, \theta) = U_{ac}(t, \theta) = U_s(t, \theta) = \Theta$, $U_i^{ju}(t, \theta) = U_i^{ju}(t, -\sigma_i)$ for $\theta \leq -\sigma_i$, where $\sigma_i(t) \in C(\mathbb{R}, [0, \sigma_i])$, $\sigma_i = \max_{t \in \mathbb{R}} \sigma_i(t)$, and $U^{ac}(t, \theta) = U^{ac}(t, -h)$ and $U^s(t, \theta) = U^s(t, -h)$ when $\theta \leq -h$.

(S2) The functions $u^{ju}(t, \theta)$, $u^{ac}(t, \theta)$, $u^s(t, \theta) \in BL_1^{loc}(J_a, [0, \infty))$, and

$$u(t) = u^{ju}(t) + u^{ac}(t) + u^s(t),$$

where

$$u^{ju}(t) = \sum_{i \in \langle m \rangle} u_i^{ju}(t) = \sum_{i \in \langle m \rangle} \sup_{\eta \in [a, t]} |Var_{[-h, 0]} U_i^{ju}(\eta, \cdot)|,$$

$$u^{ac}(t) = \sup_{\eta \in [a, t]} |Var_{[-h, 0]} U^{ac}(\eta, \cdot)|,$$

$$u^s(t) = \sup_{\eta \in [a, t]} |Var_{[-h, 0]} U^s(\eta, \cdot)|.$$

(S3) $U_i^{ju}(t, \theta) = \{a_i^{kj}(t) H(\theta + \sigma_i(t))\}_{k,j=1}^n$, $A^i(t) = \{a_i^{kj}(t)\}_{k,j=1}^n \in BL_1^{loc}(\mathbb{R}, \mathbb{R}^{n \times n})$, where $H(t)$ is the Heaviside function and the matrix $\frac{\partial U^{ac}(\cdot, \theta)}{\partial \theta} = \partial U^{ac}(\cdot, \theta) = A_*(\cdot, \theta) \in BL_1^{loc}(\mathbb{R}, \mathbb{R}^{n \times n})$.

(S4) [18] The following relations hold:

$$\lim_{t \rightarrow t^*} \int_{-h}^0 |U_i(t, \theta) - U_i(t^*, \theta)| d\theta = 0, \quad \lim_{t \rightarrow t^*} \int_{-h}^0 |U^{ac}(t, \theta) - U^{ac}(t^*, \theta)| d\theta = 0,$$

$$\lim_{t \rightarrow t^*} \int_{-h}^0 |U^s(t, \theta) - U^s(t^*, \theta)| d\theta = 0$$

for each $t^* \in J_a$. For any $\varphi \in \mathbf{BL}$ we denote the set of all its jump points by S_φ and assume that the sets $S_\varphi^i = \{t \in J_a | t - \sigma_i(t) \in S_\varphi\}$ do not have limit points.

Note that for any $\varphi \in \mathbf{PC}(\mathbf{PAC})$ the set S_φ is at most finite, and from Condition **(S4)** it follows that the sets $S_\varphi^i(b) = \{t \in [a, a + b] | t - \sigma_i(t) \in S_\varphi\}$ are also finite or empty for all $i \in \langle m \rangle$.

Remark 2. In many works, instead of condition **(S4)**, the authors suppose that the kernels $U_j^i(t, \theta)$ are continuous in t , which excludes from consideration the case of systems with variable concentrated delays (see [18]), which is important for applications.

Definition 2. The vector function $x(t; a, \varphi, f) = (x_1(t), \dots, x_n(t))^T$ is a solution of the initial problem (IP) (1), (2) in J_a for some $\varphi \in \mathbf{BL}$ if $x(t; a, \varphi, f) \in C(J_a, \mathbb{R}^n)$ satisfies the system (1) for all $t \in (a, \infty)$ and the initial condition (2) too. (When misunderstanding is not possible, we will write only $x(t)$.)

Applying to both sides of (1) the left-sided fractional integral operator I_{a+}^q , we obtain the following auxiliary system:

$$x(t) = \varphi(0) + I_{-1}(\Gamma(q)) \left(\int_a^t I_{q-1}(t-s) \int_{-h}^0 [d_\theta U(s, \theta)] x(s+\theta) ds + \int_a^t I_{q-1}(t-s) f(s) ds \right), \quad (4)$$

where $I_{-1}(\Gamma(q)) = \text{diag}(\Gamma^{-1}(q_1), \dots, \Gamma^{-1}(q_n))$.

Definition 3. The vector function $x(t) = (x_1(t), \dots, x_n(t))^T$ is a solution of IP (4), (2) in J_a if $x(t) \in C(J_a, \mathbb{R}^n)$ satisfies system (4) for all $t \in (a, \infty)$ and the initial condition (2) too.

In our exposition below, we need some auxiliary results when the initial function φ is discontinuous, i.e., $\varphi \in \mathbf{BL}$, stated in the theorem below:

Theorem 1 (see [19]). Let the following conditions hold:

1. Conditions **(S)** hold.
2. The function $f \in BL_1^{loc}(J_a, \mathbb{R}^n)$.

Then, for an arbitrary fixed initial point $a \in \mathbb{R}$ the following statements hold:

- (i) Every solution $x(t; a, \varphi, f)$ of IP (1), (2) is a solution of IP (4), (2) and vice versa.
- (ii) For any initial function $\varphi \in \mathbf{BL}$ and $f \in BL_1^{loc}(J_a, \mathbb{R}^n)$, IP (4), (2) has a unique solution in J_a .

For any $l \in \langle n \rangle$, define the following initial functions: $\varphi_0^l \in \mathbf{PC}$, $\varphi_0^l(\theta) : [-h, 0] \rightarrow \mathbb{R}^n$ with $\varphi_0^l(0) = I^l$ (where I^l is the l -th column of the identity matrix $I \in \mathbb{R}^{n \times n}$ and $\varphi_0^l(\theta) = \mathbf{0}$ for $\theta \in [-h, 0)$).

Let $s \in \mathbb{R}$ be an arbitrary fixed point and be the low terminal and initial point. Consider for $t \geq s$ IP (4), (2) in the homogeneous case of (4) (i.e., when $f(t) \equiv \mathbf{0}$, $t \in \mathbb{R}$) with initial function $\varphi_0^l(\theta) = \varphi_0^l(t-s) \in \mathbf{PC}$, $\theta = t-s \in [-h, 0]$, $l \in \langle n \rangle$ as follows:

$$\begin{aligned} C^l(t, s) &= C^l(t; s, \varphi^l, \mathbf{0}) = \\ &= \varphi^l(0) + I_{-1}(\Gamma(\alpha)) \left(\int_s^t I_{\alpha-1}(t-\eta) \int_{-h}^0 [d_\theta U(\eta, \theta)] C^l(\eta + \theta, s) d\eta \right), t \in (s, \infty) \end{aligned} \quad (5)$$

and

$$C_s^l(\theta, s) = C^l(s + \theta, s) = \varphi_0^l(\theta), \quad \theta = s - t \in [-h, 0]. \quad (6)$$

Corollary 1. *Let conditions (S) hold.*

Then, IP (5), (6) has a unique solution $C^l(t, s)$ for any $l \in \langle n \rangle$ and arbitrary fixed number $s \in \mathbb{R}$.

Proof. Since the statement of Theorem 1 holds for any fixed initial point $a \in \mathbb{R}$, then the assertion of Corollary 1 follows immediately from Theorem 1 for any $s \geq a$. $\square$

Definition 4 (see [20]). *The matrix-valued function*

$$C(t, s) = (C^1(t, s), \dots, C^n(t, s)) = \{C_k^j(t, s)\}_{k,j=1}^n : J_a \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$$

is called a fundamental (or Cauchy) matrix if for any $l \in \langle n \rangle$ its l -th column is a solution of IP (5), (6).

As usual, everywhere below we will assume that $C(t, s)$ is prolonged as Θ when $t \in (-\infty, s)$ for arbitrary fixed $s \in \mathbb{R}$.

Remark 3. *Below, we formulate as a corollary the important properties of the fundamental matrix $C(t, s)$ needed in our exposition.*

Corollary 2 (see [20]). *Let the conditions of Theorem 1 hold and $f(t) \equiv 0, t \in J_a$.*

Then, the following assertions hold:

1. *The system (3) and (4) has a unique fundamental matrix $C(t, s)$, which is absolutely continuous in t on every compact sub-interval of J_a , i.e., for any fixed $s \in \mathbb{R}$, i.e., $C(\cdot, s) \in AC_{loc}((a, \infty), \mathbb{R}^{n \times n})$.*
2. *For any arbitrary fixed initial point $a \in \mathbb{R}$ and each fixed $t > a$, $C(t, \cdot) : J_a \rightarrow \mathbb{R}^n$ is continuous at $s \in (a, t) \cup (t, \infty)$, right continuous at $s = a$, left continuous at $s = t$, and has jumps of the first kind for $s \in \{a, t\}$.*
3. *The fundamental matrix $C(t, s)$ has bounded variation in s on $[a, T]$ for arbitrary $T \in J_a$ and its total variation in s on $[a, t]$ is bounded in $t, t \in [a, T]$.*

Everywhere below in our exposition, we will assume that the conditions of Theorem 1 hold, and when we speak about the integer case we will understand the same form of systems in which the differentiation orders are $q_1 = q_2 = \dots = q_n = 1$, i.e., the same systems with first-order derivatives.

Remark 4. *From the results in [19] and [20] it follows that for any initial function $\varphi \in \mathbf{PC}$ with $\varphi \in BV([-h, 0], \mathbb{R}^n)$, or with $\varphi \in \mathbf{PAC} = PAC([-h, 0], \mathbb{R}^n)$, IP (1), (2) has a forward noncontinuable unique solution $x(t; a, \varphi, f) \in BV^{loc}(J_a, \mathbb{R}^n)$ or, respectively, $x(t; a, \varphi, f) \in AC^{loc}(J_a, \mathbb{R}^n)$.*

3. Formal Adjoint System and Adjoint Fundamental Matrix

This section aims to analyse the applicability of the notion of a formal adjoint system introduced in [17] for the homogeneous system (3). In addition, we weaken the sufficient conditions for the existence of the adjoint fundamental matrix, which enlarges the area of its applicability. We introduce an extension of the approach introduced in [17] that partially reduces the disadvantages caused by the use of fractional derivatives by losing the symmetry of differentiation. This approach enables us to establish a relationship between the formal adjoint fundamental matrix and the resolvent kernel of the adjoint system. It must be noted that for $q_i = 1, i \in \langle n \rangle$, the obtained relation coincides with the well-known classical relation in the case of delayed systems with first-order derivatives.

It is well known that, as in the integer case, the predominant type of the practically applicable mathematical models based on fractional-order derivatives and delays are

considered under the assumption that Lebesgue decomposition of the kernel $U(t, \theta)$ has no singular part, i.e., $U^s(t, \theta) \equiv \Theta$ for $(t, \theta) \in \mathbb{R}^2$. The main reason for this assumption is the lack of a physical interpretation of the singular part. Therefore, the kernel $U(t, \theta)$ obtains the form

$$U(t, \theta) = \sum_{i \in \langle m_0 \rangle} U_i^{ju}(t, \theta) + U^{ac}(t, \theta) = \sum_{i \in \langle m_0 \rangle} A_i(t)H(\theta + \sigma_i) + \int_0^\theta A_*(t, \xi) d\xi, \quad t \in J_a. \quad (7)$$

Since $H(t)$ is a scalar function, and the Dirac delta function $\delta(\theta)$ can be treated as a measure, then the following equality holds,

$$\begin{aligned} \int_{-h}^0 \delta_{-\sigma_i}(\theta) x(t + \theta) d\theta &= \int_{-h}^0 [d_\theta H(\theta + \sigma_i)] x(t + \theta) = x(t - \sigma_i) = \\ &= \int_{-h}^0 x(t + \theta) d_\theta H(\theta + \sigma_i) = \int_{-h}^0 x(t + \theta) \delta_{-\sigma_i}(\theta) d\theta \end{aligned} \quad (8)$$

and hence the partial derivative concerning the second argument of $\frac{\partial U(t, \theta)}{\partial \theta} = \partial U(t, \theta)$ has the form

$$\partial U(t, \theta) = \sum_{i \in \langle m_0 \rangle} A_i(t) \delta_{-\sigma_i}(\theta) + A_*(t, \theta),$$

where $\delta(\theta)$ is understood as measure.

Thus, for $t \in J_a$, taking into account (8), when the kernel $U(t, \theta)$ is defined via (7), the system (1) obtains the form

$$\begin{aligned} D_{a+}^q x(t) &= \int_{-h}^0 [d_\theta U(t, \theta)] x(t + \theta) + f(t) \\ &= \int_{-h}^0 \left[d_\theta \left(\sum_{i \in \langle m_0 \rangle} A_i(t) H(\theta + \sigma_i) + \int_0^\theta A_*(t, \xi) d\xi \right) \right] x(t + \theta) + f(t) \\ &= \sum_{i \in \langle m_0 \rangle} A_i(t) \int_{-h}^0 [d_\theta H(\theta + \sigma_i)] x(t + \theta) + \int_{-h}^0 \left[d_\theta \left(\int_0^\theta A_*(t, \xi) d\xi \right) \right] x(t + \theta) + f(t) \\ &= \int_{-h}^0 \partial_\theta U(t, \theta) x(t + \theta) d\theta + f(t) \\ &= \sum_{i \in \langle m_0 \rangle} \int_{-h}^0 A_i(t) \delta_{-\sigma_i}(\theta) x(t + \theta) d\theta + \int_{-h}^0 A_*(t, \theta) x(t + \theta) d\theta + f(t) \\ &= \sum_{i \in \langle m_0 \rangle} A_i(t) x(t - \sigma_i) + \int_{-h}^0 A_*(t, \theta) x(t + \theta) d\theta + f(t) \\ &\stackrel{t+\theta=\alpha}{=} \sum_{i \in \langle m_0 \rangle} A_i(t) x(t - \sigma_i) + \int_{t-h}^t A_*(t, \alpha - t) x(\alpha) d\theta + f(t) \end{aligned} \quad (9)$$

For any fixed $b \in \mathbb{R}$ and each $t \leq b$, we define the formal (Lagrange) adjoint kernel of the kernel (7) via the equality

$$U_*(t, \theta) = U(t - \theta, \theta) \quad (10)$$

and since $U^s(t, \theta) \equiv \Theta$ for $(t, \theta) \in \mathbb{R}^2$, then the kernel $U(t, \theta)$, according to conditions (S), is PAC in θ and hence $U_*(t, \theta)$ defined via (10) obtains the form

$$U_*(t, \theta) = \sum_{i \in \langle m_0 \rangle} H(\theta - \sigma_i) A_i(t + \sigma_i) + \int_0^\theta (A_*(t - \xi, \xi)) d\xi. \quad (11)$$

For arbitrary fixed $b \in \mathbb{R}$, $f(t) \equiv \mathbf{0}$, $t \leq b$ and $y(t) = (y^1(t), \dots, y^n(t)) \in \mathbf{C}_b^*$ using (10), we define the formal (Lagrange) adjoint system of system (3) as the following fractional system:

$$\begin{aligned} D_{b-}^q y(t) &= - \int_{-h}^0 y(t-\theta) d_\theta U_*(t, \theta) = - \int_{-h}^0 y(t-\theta) d_\theta U(t-\theta, \theta) \\ &= - \int_{-h}^0 y(t-\theta) \partial_\theta U(t-\theta, \theta) d\theta \\ &= - \sum_{i \in \langle m_0 \rangle} y(t+\sigma_i) A_i(t+\sigma_i) - \int_{-h}^0 y(t-\theta) A_*(t-\theta, \theta) d\theta \\ &\stackrel{\alpha=t-\theta}{=} - \sum_{i \in \langle m_0 \rangle} y(t+\sigma_i) A_i(s+\sigma_i) - \int_t^{t+h} y(\alpha) A_*(\alpha, t-\alpha) d\alpha \end{aligned} \quad (12)$$

where $D_{b-}^q y(t) = \text{diag}(D_{b-}^{q_1} y_1(t), \dots, D_{b-}^{q_n} y_n(t))$.

Applying the operator I_{b-}^q to both sides of (12), we obtain the formal (Lagrange) adjoint system of system (3) in the form of a Volterra-type integral system as follows:

$$y(t) = C_* - I_{-1}(\Gamma(q)) \int_t^b I_{q-1}(\tau-t) \left(\sum_{i \in \langle m_0 \rangle} y(\tau+\sigma_i) A_i(\tau+\sigma_i) + \int_\tau^{\tau+h} y(\alpha) A_*(\alpha, \tau-\alpha) d\alpha \right) d\tau, \quad (13)$$

where $C_* = (c_1^*, \dots, c_n^*) \in \mathbb{R}^{n*}$ is an arbitrary constant vector.

For any $\psi \in \mathbf{C}_0^*$, we consider the final condition

$$y_b(-\theta) := y(b-\theta) = \psi(\theta) \quad \text{for} \quad -h \leq \theta \leq 0 \quad (14)$$

and hence for any solution $y(t) \in \mathbf{C}_b^*$ of FP (12), (14) we have that $y(b) = \psi(0) = C_*$.

Remark 5. We emphasize that if the kernel $U(t, \theta)$ satisfies the conditions (S), then the adjoint kernel $U_*(t, \theta) = U(t-\theta, \theta)$ satisfies the conditions (S) too, and obviously $U_*^s(t, \theta) \equiv \Theta$ when $U^s(t, \theta) \equiv \Theta$ for $(t, \theta) \in \mathbb{R}^2$.

Definition 5. For arbitrary fixed $t^* < b$ with $t^* < t \leq b$ (the case when $t^* = -\infty$ is not excluded), the vector function $y(t) \in \mathbf{C}_b^*$ is a solution on the interval $(t^*, b]$ of the final problem FP (12), (14) (FP (13), (14)) for some $\psi \in \mathbf{C}_0^*$ if $y_b(\theta; b) = y(b-\theta; b) = \psi(\theta)$, $\theta \in [-h, 0]$ and $y(t)$ satisfies system (12) or (13) for any $t \in (t^*, b)$. These solutions will be denoted with $y(t; b, \psi, \mathbf{0}^T)$ and their corresponding inhomogeneous variants with a forced term $g(t) \neq \mathbf{0}^T$ by $y(t; b, \psi, g)$.

Lemma 1. Let the conditions (S) hold.

Then, for $t \leq b$ and the arbitrary final function $\psi \in \mathbf{C}_0^*$, every solution $y(t; b, \psi, \mathbf{0}^T)$ of FP (12), (14) is a solution of FP (13), (14) for $t \in (-\infty, b)$ and vice versa if and only if $C_* = \psi(0)$.

Proof. Let $\psi \in \mathbf{C}_0^*$ be arbitrary and $y(t) \in \mathbf{C}_b^*$ be the corresponding solution of (12) that satisfies (14). Then, we have $y_b(0) = C_* = \psi(0)$. For the right-sided fractional derivatives, applying the operator I_{b-}^q to both sides of (12), as in the case for the left-sided fractional derivatives, and since $y_b(0) = C_* = \psi(0)$, we can conclude that each solution in the sense of Definition 5 of FP (12), (14) is a solution of FP (13), (14) in the same sense.

For the vice versa assertion, applying the operator D_{b-}^q to both sides of (13) obtains that any solution of FP (13), (14) is a solution of FP (12), (14), since from (14) it follows that $y_b(0) = C_* = \psi(0)$. $\square$

Theorem 2. Let the conditions (S) hold.

Then, for any final function $\psi \in \mathbf{C}_0^*$ there exists a number $t^* < b$ such that FP (13), (14) has a unique solution in $(t^*, b]$.

Proof. For $t \leq b$ and any final function $\psi \in \mathbf{C}_0^*$, we introduce the set

$$\Omega(\psi) = \{y(t) \in C_{b-}^* | y_b(-\theta) = y(b - \theta) = \psi(\theta), \theta \in [-h, 0]\}.$$

For any $t^* < b$, $t \in [t^*, b]$ consider FP (13), (14) and define the following set:

$$\Omega(\psi, t^*) = \{y^*(t) | y^*(t) = y(t)|_{t \in [t^*, b]}; y(t) \in \Omega(\psi)\}.$$

Define, for any $y_*, \bar{y}_* \in \Omega(\psi, t^*)$, the metric function $d(y_*, \bar{y}_*) = \sum_{k=1}^n \sup_{t \in [t^*, b]} |y_*^k(t) - \bar{y}_*^k(t)|$, and then it is clear that the set $\Omega(\psi, t^*)$ is a complete metric space with respect to this metric function.

For any $\psi \in \mathbf{C}_0^*$ and $y^* \in \Omega(\psi, t^*)$, we define the operator

$$(\mathfrak{R}y)(t) = (\mathfrak{R}_1 y_1(t), \dots, \mathfrak{R}_n y_n(t))$$

via (13) and Lemma 1 as follows:

$$\begin{aligned} (\mathfrak{R}y^*)(t) &= \\ &= \psi(0) - I_{-1}(\Gamma(q)) \int_t^b I_{q-1}(\tau - t) \left(\sum_{i \in \langle m_0 \rangle} y^*(\tau + \sigma_i) A_i(\tau + \sigma_i) + \int_{\tau}^{\tau+h} y^*(\alpha) A_*(\alpha, \tau - \alpha) d\alpha \right) d\tau, \end{aligned} \quad (15)$$

where $t \leq b$ is arbitrary, with the final condition

$$(\mathfrak{R}y^*)_b(-\theta) = (\mathfrak{R}y^*)(b - \theta) = (\mathfrak{R}_1 y_1^*(b - \theta), \dots, \mathfrak{R}_n y_n^*(b - \theta)) = \psi(\theta) \quad (16)$$

for $t \in [b, b + h]$, $t - b = -\theta$.

First, we will prove that the operator $\mathfrak{R}$ maps $\Omega(\psi, t^*)$ into $\Omega(\psi, t^*)$.

Since $y^*(t) \in BV([b, b + h], \mathbb{R}^{n^*})$, then it is not necessary for $y^*(t)$ to be continuous at $t = b$. Thus, we must prove only that $y^*(t)$ has at most a jump of the first kind at $t = b$. From conditions (S), it follows that for each $y^* \in \Omega(\psi, t^*)$ and for any $\tau, \alpha \in [t^*, b]$ with $\tau - \alpha \in [-h, 0]$, the vector function $\tau \rightarrow \int_{\tau}^{\tau+h} y^*(\alpha) A_*(\alpha, \tau - \alpha) d\alpha$ is locally bounded and at least Lebesgue-integrable. When $\tau - \alpha \leq -h$, then for any $\psi \in \mathbf{C}_0^*$ the function $\tau \rightarrow \int_{\tau}^{\tau+h} y^*(\alpha) A_*(\alpha, \tau - \alpha) d\alpha$ is also locally bounded and at least Lebesgue-integrable. Then, from (15) it follows that $\mathfrak{R}y^*(s)$ is absolutely continuous and hence we have

$$\lim_{t \rightarrow b-0} \mathfrak{R}y^*(t) = \psi(\theta).$$

For any $t \in [b, b + h]$ since $\psi \in \mathbf{C}_0^*$, from (16) it follows that

$$\lim_{\theta \rightarrow 0-} (\mathfrak{R}y^*)_b(-\theta) = \lim_{\theta \rightarrow 0-} \psi(\theta) < \infty,$$

and hence $(\mathfrak{R}y^*)(t)$ has at most a jump of the first kind. Thus, we conclude that

$$\mathfrak{R}\Omega(\psi, t^*) \subset \Omega(\psi, t^*).$$

Note that if in addition $y^* \in \Omega(\psi, t^*)$ is continuous, then according to Lemma 1 in [17], $\mathfrak{R}y^*(t)$ will be continuously differentiable.

Denote $\Gamma_n = \sum_{k \in \langle n \rangle} \Gamma^{-1}(1 + q_k)$ and

$$A_{t^*} = \max \left(\sup_{t \in [t^*, b+h]} \left(\sup_{\theta \in [-h, 0]} |A(t, \theta)| \right), \sup_{t \in [t^*, b]} \sum_{i \in \langle m \rangle_0} |A_i(t)| \right).$$

Note that condition S3 implies that $A_{t^*} < \infty$ for any compact sub-interval of $\mathbb{R}$. Then, for any $y, \bar{y} \in \Omega(\psi, s^*)$, from (15) it follows that for each $t \in [t^*, b]$ we have

$$\begin{aligned} |\Re y^*(t) - \Re \bar{y}^*(t)| &= \sum_{k \in \langle n \rangle} |\Re_k y_k(t) - \Re_k \bar{y}_k(t)| \\ &\leq \sum_{k \in \langle n \rangle} \frac{1}{\Gamma(q_k)} \int_t^b (\tau - t)^{q_k-1} \left(\sum_{j \in \langle n \rangle} \sum_{i \in \langle m \rangle_0} |y_k^j(\tau + \sigma_i) - \bar{y}_k^j(\tau + \sigma_i)| |A_i(\tau + \sigma_i)| + \int_{\tau}^{\tau+h} |y_k^j(\alpha) - \bar{y}_k^j(\alpha)| |A_*(\alpha, \tau - \alpha)| d\alpha \right) d\tau \\ &\leq 2mn A_{t^*} d(y^*, \bar{y}^*) \sum_{k \in \langle n \rangle} \frac{1}{\Gamma(q_k)} \int_t^b (\tau - t)^{q_k-1} d\tau \leq 2mn A_{t^*} d(y^*, \bar{y}^*) \sum_{k \in \langle n \rangle} \frac{(t-t^*)^{q_k}}{q_k \Gamma(q_k)} \\ &\leq 2mn^2 A_{t^*} \Gamma_n d(y^*, \bar{y}^*) \sum_{k \in \langle n \rangle} (t-t^*)^{q_k} = 2mn^3 A_{t^*} \Gamma_n \max_{k \in \langle n \rangle} (t-t^*)^{q_k} d(y^*, \bar{y}^*). \end{aligned} \quad (17)$$

Then, choosing $t - t^*$ small enough so that $\max_{k \in \langle n \rangle} (t - t^*)^{q_k} \leq \frac{1}{4mn^3 A_{t^*} \Gamma_n}$, from (17) we obtain that $d(\Re y^*, \Re \bar{y}^*) \leq \frac{1}{2} d(y^*, \bar{y}^*)$, i.e., the operator is contractive in $\Omega(\psi, s^*)$. Thus, we can conclude that for the fixed point $y_0^*(t) = \Re y_0^*(t)$ we have that

$$y_0(t) \in C([t^*, b], \mathbb{R}^{n^*}) \cap BV([t^*, b], \mathbb{R}^{n^*}).$$

□

Definition 6 ([15]). The solution $y(t; b, \psi, \mathbf{0}^T)$ will be called a continuation of the solution $\bar{y}(t; b, \psi, \mathbf{0}^T)$ if from the relation $(t_y^*, b] \subset (t_y^*, b]$ it follows that $y(t; b, \psi, \mathbf{0}^T) \equiv \bar{y}(t; b, \psi, \mathbf{0}^T)$ for $t \in (t_y^*, b]$. The solution $y_{\max}(t; b, \psi, \mathbf{0}^T)$ of FP (12), (14), with an interval of existence $(t_{\max}^*, t]$, is called maximal if it is a continuation of any other solution of FP (12), (14).

Theorem 3. Let the conditions of Theorem 2 hold.

Then, for any $\psi \in \mathbf{C}_0^*$ the interval of existence of the maximal solution of FP (12), (14) cannot be a finite and closed from left interval.

Proof. Assume the contrary, that there exists $\psi \in \mathbf{C}_0^*$ such that the maximal solution $y_{\max}(t; b, \psi, \mathbf{0}^T)$ of FP (12), (14) has a finite and closed from left interval of existence, i.e., $[t_{\max}^*, b]$. Define a final function as follows, in two variants depending on the length of the interval $[t_{\max}^*, t_{\max}^* + h]$ (part of the interval $[t_{\max}^*, b + h]$):

- (a) $t_{\max}^* + h < b$; then $\bar{\psi}(\theta) \equiv y_{\max}(t^* - \theta; b, \psi, \mathbf{0}^T)$ for $\theta \in [-h, 0]$;
- (b) $t_{\max}^* + h \geq b$; then $\bar{\psi}(\theta) \equiv y_{\max}(t^* - \theta; b, \psi, \mathbf{0}^T)$ for $t - \theta < b$ and $\bar{\psi}(\theta) \equiv \psi(\theta)$ for $t - \theta > b$.

Since $y_{\max}(t; b, \psi, \mathbf{0}^T) \in \Omega(\psi, t_{\max}^*)$, then the defined above final function $\bar{\psi} \in \mathbf{C}_0^*$. Define the set

$$\Omega(\bar{\psi}, \bar{t}^*) = \{\bar{y}(t) | \bar{y}(t) = y(t) |_{[\bar{t}^*, b]}, \bar{t}^* < t_{\max}^*; y(t) \in \Omega(\psi), \bar{y}_{t_{\max}^*}(-\theta) = \bar{y}(t_{\max}^* - \theta) = \bar{\psi}(\theta)\}.$$

As above, introduce for any $y^*(t), y(t) \in \Omega(\bar{\psi}, \bar{t}^*)$ the metric function

$$d(y_*, \bar{y}_*) = \sum_{k=1}^n \sup_{t \in [\bar{t}^*, b]} |y_*^k(t) - \bar{y}_*^k(t)|,$$

and hence $\Omega(\bar{\psi}, \bar{t}^*)$ is a complete metric space concerning this metric function.

Define the operator $(\mathfrak{R}\tilde{y})(t) = (\mathfrak{R}_1 y_1(t), \dots, \mathfrak{R}_n y_n(t))$ via (15) for $t \leq t_{\max}^*$ and for $t \in [t_{\max}^*, t_{\max}^* + h]$ with the final condition

$$(\mathfrak{R}\tilde{y})_{t_{\max}^*}^*(-\theta) = (\mathfrak{R}\tilde{y})(t_{\max}^* - \theta) = \bar{\psi}(\theta).$$

Then, for any $y_*, \bar{y}_* \in \Omega(\bar{\psi}, \bar{t}^*)$, as in (17), for any $t \in [\bar{t}^*, t_{\max}^*]$ we have

$$\begin{aligned} |\mathfrak{R}y_*(t) - \mathfrak{R}\bar{y}_*(t)| &= \sum_{k \in \langle n \rangle} |\mathfrak{R}_k y_k(t) - \mathfrak{R}_k \bar{y}_k(t)| \\ &\leq \frac{1}{\Gamma(q_k)} \int_t^b (\tau - t)^{q_k-1} \left(\sum_{j \in \langle n \rangle} \sum_{i \in \langle m_0 \rangle} |y_k^j(\tau + \sigma_i) - \bar{y}_k^j(\tau + \sigma_i)| |A_i(\tau + \sigma_i)| + \int_{\tau}^{\tau+h} |y_k^j(\alpha) - \bar{y}_k^j(\alpha)| |A_*(\alpha, \tau - \alpha)| d\alpha \right) d\tau \\ &\leq 2mn A_{\bar{t}^*} d(y_*, \bar{y}_*) \sum_{k \in \langle n \rangle} \frac{1}{\Gamma(q_k)} \int_t^{t_{\max}^*} (\tau - t)^{q_k-1} d\tau \leq 2mn A_{\bar{t}^*} d(y_*, \bar{y}_*) \sum_{k \in \langle n \rangle} \frac{(t_{\max}^* - \bar{t}^*)^{q_k}}{q_k \Gamma(q_k)} \\ &\leq 2mn^2 \Gamma_n A_{\bar{t}^*} d(y_*, \bar{y}_*) \sum_{k \in \langle n \rangle} (t_{\max}^* - \bar{t}^*)^{q_k} = 2mn^3 \Gamma_{\max} A_{\bar{t}^*} d(y_*, \bar{y}_*) \max_{k \in \langle n \rangle} (t_{\max}^* - \bar{t}^*)^{q_k} \end{aligned} \quad (18)$$

and then, as above, we can choose $t_{\max}^* - \bar{t}^*$ small enough such that from (18) it follows that $d(\mathfrak{R}y_*, \mathfrak{R}\bar{y}_*) \leq \frac{1}{2}d(y_*, \bar{y}_*)$. Thus, there exists a unique fixed point $y_0^*(t) \in \Omega(\bar{\psi}, \bar{t}^*)$ for the operator $\mathfrak{R}$ and hence we have constructed a prolongation of $y_{\max}(t; b, \psi, \mathbf{0}^T)$ that contradicts our assumption. $\square$

Corollary 3. Let the conditions of Theorem 2 hold.

Then, for any $\psi \in \mathbf{C}_0^*$ the interval of existence of the maximal solution of FP (12), (14) is $(-\infty, b]$.

Proof. Let $\psi \in \mathbf{C}_0^*$ and then, according to Theorem 3, the interval of existence of the maximal solution $y_{\max}(t; b, \psi, \mathbf{0}^T)$ has the form $(t_{\max}^*, b] \neq \emptyset$. Assume the contrary, that $t_{\max}^* > -\infty$, and then $y_{\max}(t; b, \psi, \mathbf{0}^T)$ satisfies system (13) for $t \in (t_{\max}^*, b]$ with $C_* = \psi(0)$. Taking into account that the right side of (13) is a continuous function for $t \in [t_{\max}^*, b]$, then taking the limit for $t \rightarrow t_{\max}^* + 0$, we obtain that $y_{\max}(t; b, \psi, \mathbf{0}^T)$ satisfies (13) for $t = t_{\max}^*$ too. Thus, we have constructed a continuation of $y_{\max}(t; b, \psi, \mathbf{0}^T)$ that is impossible, and hence $t_{\max}^* = -\infty$. $\square$

Introduce the matrix $\zeta(\theta) = \{\zeta_k^j(\theta)\}_{k,j \in \langle n \rangle}$ with rows $\zeta_k(\theta) = (\zeta_k^1(\theta), \dots, \zeta_k^n(\theta)) \in \mathbf{C}_0^*$, defined as follows: $\zeta_k^j(\theta) = \mathbf{0}^T$, $k \neq j$, $\theta \in [-h, 0]$, $\zeta_k^k(0) = I_k$, $\zeta_k^k(\theta) = 0$, $\theta \in [-h, 0)$ for any $k \in \langle n \rangle$, where I_k denotes the k -th row of the identity matrix I in $\mathbb{R}^n$.

Corollary 4. Let the conditions of Theorem 2 hold.

Then, FP (12), (14) for any $k \in \langle n \rangle$ with final function $\zeta_k(\theta) \in \mathbf{C}_0^*$ has a unique solution $Y_k(t; b, \zeta_k) \in \mathbf{C}_b^*$ with an interval of existence $(-\infty, b]$ and $Y_k(\cdot; (-\infty, b], \zeta_k) \in \text{PAC}_{loc}((-\infty, b], \mathbb{R}^n)$.

Proof. The first assertion follows immediately from Theorem 2, Theorem 3, and Corollary 3. The second one follows from Lemma 1 in [20]. $\square$

Definition 7 ([16,17]). For arbitrary fixed $t \in \mathbb{R}$ with $s \leq t$, the matrix $Y(s; t, \zeta) = (Y_1(s, t), \dots, Y_n(s, t))^T$ with rows $Y_k(s, t)$, which are solutions of FP (12), (14), with final function $\zeta_k(\theta) \in \mathbf{C}_0^*$ for any $k \in \langle n \rangle$ will be called the formal adjoint fundamental matrix of (12).

For arbitrary fixed $t \in \mathbb{R}$ with $s \leq t$ and for any $\psi \in \mathbf{C}_0^*$, introduce the following auxiliary Volterra-type system:

$$y(s; t) = \psi(0) - \int_s^t y(\alpha; t) K(\alpha, s) d\alpha, \quad (19)$$

where the kernel $K(t, s)$ is defined via the equality

$$K(t, s) = I_{-1}(\Gamma(q)) \int_s^t I_{q-1}(\tau - s) \partial U(\alpha, \tau - \alpha) d\tau. \quad (20)$$

Lemma 2. For any fixed $t \leq b$, let the following conditions be fulfilled:

1. The conditions **(S)** hold.
2. For any $\psi \in \mathbf{C}_0^*$ and $s \leq t$, the kernel $K(t, s)$ in (19) is defined via (20).

Then, for each $\psi \in \mathbf{C}_0^*$ the corresponding solution $y(s; t, \psi, 0) \in C_t^*$ of FP (13), (14) is a solution of FP (19), (14) with $C_* = \psi(0)$ and vice versa.

Proof. Let $\psi \in \mathbf{C}_0^*$ be arbitrary and $y(s; t, \psi, 0^T) \in C_t^*$ be the corresponding solution. Then, for any arbitrary fixed $s \leq t$, taking into account Lemma 1 and applying the Fubini theorem, from (13) it follows that

$$\begin{aligned} y(s) &= \psi(0) - I_{-1}(\Gamma(q)) \int_s^t I_{q-1}(\tau - s) \left(\int_s^t y(\alpha) \partial U(\alpha, \tau - \alpha) d\alpha \right) d\tau \\ &= \psi(0) - \int_s^t y(\alpha) \left(I_{-1}(\Gamma(q)) \int_s^t I_{q-1}(\tau - s) \partial U(\alpha, \tau - \alpha) d\tau \right) d\alpha \\ &= \psi(0) - \int_s^t y(\alpha) K(\alpha, s) d\alpha \end{aligned} \quad (21)$$

and hence any FP (13), (14) is a solution of FP (19), (14).

The statement that when $y(s, t, \psi, 0) \in C_t^*$ is a solution of FP (19), (14) then it is also a solution of FP (13), (14) can be proved in the reverse way. $\square$

Consider the matrix system

$$Y(s, t) = 1_{s \leq t} I - \int_s^t Y(\alpha, t) K(\alpha, s) d\alpha. \quad (22)$$

Corollary 5. Let the following conditions hold:

1. The conditions **(S)** hold.
2. $Y(s, t, \xi)$ is the formal adjoint fundamental matrix of (4) (i.e., its rows $Y^k(s, t, \xi)$ satisfy FP (13), (14) with final function $\zeta_k(\theta) \in \mathbf{C}_0^*$ for any $k \in \langle n \rangle$).
3. For any fixed $t, s \in \mathbb{R}$, the kernel $K(t, s)$ is defined via (20).

Then, the matrix $Y(s, t, \xi)$ satisfies system (22) for any $s \in (-\infty, t]$ and it is unique.

Proof. Since for any $k \in \langle n \rangle$ every row $Y_k(s, t) = (y_k^1(s, t), \dots, y_k^n(s, t))$ of the matrix $Y(s, t, \xi)$ is a solution of FP (13), (14) with initial function $\xi_k(\theta)$, then from Lemma 2 it follows that $Y(s, t, \xi)$ is unique and satisfies the matrix system (22) for any $s < t$ and vice versa. From (22) it follows that $\lim_{s \rightarrow t-0} Y(s, t, \xi) = I$, and hence, according to Lemma 1, we have that $Y(t, t, \xi) = I$. Thus, we conclude that $Y(s, t, \xi)$ satisfies the matrix system (22) for $s = t$ too. Since $Y(s, t, \xi) = \Theta$ when $s > t$, then from (22) it follows that $Y(s, t, \xi)$ is a solution of (22) for any $s \leq t$. In addition, Lemma 2 implies that the statements of Corollary 5 hold for the matrix variant of system (13) too. $\square$

Definition 8 ([21]). A measurable function $K(t, s) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ will be called a Volterra kernel of type L_1^{loc} on $\mathbb{R} \times \mathbb{R}$ if for any compact sub-interval $[a, b] \subset \mathbb{R}$ the following conditions hold:

1. The kernel $K(t, s) = \Theta$ for any $(t, s) \in \mathbb{R} \times \mathbb{R}$ with $s > t$.

2. $\|K\|_{L_1} < \infty$ for any $g(s) \in L_1^{loc}(\mathbb{R}, \mathbb{R})$ with $\|g\|_1 \stackrel{\text{def}}{=} \int_a^b |g(s)| ds \leq 1$, where the norm $\|\cdot\|_{L_1}$ is defined as

$$\|K\|_{L_1} \stackrel{\text{def}}{=} \sup_{\|g\|_1 \leq 1} \int_a^b \left(\int_a^b |K(t, s)g(s)| ds \right) dt = \text{ess sup}_{s \in [a, b]} \int_a^b |K(t, s)| dt.$$

Let conditions **(S)** hold. Then, it is simple to check that the kernel $K(t, s)$ defined via (20) for any fixed interval $[a, b] \subset \mathbb{R}$ satisfies the conditions in Definition 8, and hence is a Volterra kernel of type L_1^{loc} on $\mathbb{R} \times \mathbb{R}$. We will use the notation $K(t, s) \in VK_{loc}(\mathbb{R} \times \mathbb{R})$.

Following [22], introduce the corresponding resolvent system for system (19) for the kernel $K(t, s) \in VK_{loc}(\mathbb{R} \times \mathbb{R})$ in the form

$$R(s, t) = K(t, s) - \int_s^t R(\alpha, t)K(\alpha, s)d\alpha = K(t, s) - \int_s^t K(t, \alpha)R(s, \alpha)d\alpha, \quad (23)$$

and the solution $R(s, t) \in VK_{loc}(\mathbb{R} \times \mathbb{R})$ of (23) will be called a $VK_{loc}(\mathbb{R} \times \mathbb{R})$ -resolvent kernel for the kernel $K(t, s)$. Note that $R(t, t) = K(t, t) = \Theta$. Concerning the existence of a unique resolvent kernel for the kernel $K(t, s)$ when conditions **(S)** hold, see [22].

Introduce the following relation:

$$Y_*(s, t) = \mathbf{I}_{s \leq t} + \int_s^t R(s, \tau)d\tau. \quad (24)$$

Theorem 4. For any fixed $t, s \in \mathbb{R}$ with $s \leq t$, let the following conditions hold:

1. The kernel $U : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ satisfies conditions **(S)**.
2. The kernel $K(t, s)$ is defined via (20).

Then, the matrix $Y_*(s, t)$ defined via the relation (24) is a solution of the resolvent Equation (23) if and only if the matrix $Y_*(s, t)$ and the formal adjoint fundamental matrix $Y(s, t, \xi)$ coincide.

Proof. Necessity. Let the matrix $R(s, t)$ be a solution of the resolvent Equation (23). We will prove that the function $Y_*(s, t) = \mathbf{1}_{s \leq t}I + \int_s^t R(\tau, s)d\tau$ satisfies Equation (22). Indeed, we have

$$\begin{aligned} \mathbf{1}_{s \leq t}I + \int_s^t R(\tau, s)d\tau &= \mathbf{1}_{s \leq t}I + \int_s^t \left[-K(\tau, s) - \int_s^\tau R(\alpha, \tau)K(\alpha, s)d\alpha \right] d\tau \\ &= \mathbf{1}_{s \leq t}I - \left(\int_s^t K(\alpha, s)d\alpha + \int_s^t \left[\int_\alpha^t R(\alpha, \tau)d\tau \right] K(\alpha, s)d\alpha \right) \\ &= \mathbf{1}_{s \leq t}I - \int_s^t \left(\mathbf{1}_{s \leq t}I + \int_\alpha^t R(\alpha, \tau)d\tau \right) K(\alpha, s)d\alpha. \end{aligned} \quad (25)$$

Thus, $Y_*(s, t) = \mathbf{1}_{s \leq t}I + \int_s^t R(\tau, s)d\tau$ satisfies the same equation and initial condition as $Y(s, t)$ and hence, from the uniqueness of the adjoint fundamental matrix, it follows that $Y_*(s, t) \equiv Y(s, t, \xi)$.

Sufficiency. Let the formal adjoint fundamental matrix $Y(s, t, \xi)$ have the representation (24). Then, in virtue of (24), we have that $\frac{\partial Y(s, t)}{\partial t} = \partial_t Y(s, t) = R(s, t)$, and substituting $R(s, t) = \partial_t Y(s, t)$ in (23) we obtain that

$$\begin{aligned} R(s, t) &= K(t, s) - \int_s^t K(t, \alpha) \partial_t Y(s, \alpha) d\alpha = K(t, s) - \int_s^t K(\alpha, s) d_\alpha Y(s, \alpha) \\ &= K(t, s) - K(t, s) + \int_s^t Y(\alpha, t) d_\alpha K(\alpha, s) = \int_s^t \left(I + \int_s^t R(\alpha, \tau) d\tau \right) d_\alpha K(\alpha, s) \\ &= K(t, s) - \int_s^t K(\alpha, s) d_\alpha \left(I + \int_s^t R(\alpha, \tau) d\tau \right) = K(t, s) - \int_s^t K(\alpha, s) R(\alpha, \tau) d\alpha \end{aligned}$$

and hence the kernel $R(s, t)$ is a solution of the resolvent system (23). From Corollary 5, it follows that $R(s, t)$ is the unique solution. $\square$

4. Generalized Halanay Relation. Explicit Relation Between the Fundamental and the Adjoint Fundamental Matrices

In classical integer-order theory, the Halanay inequality is used as a critical tool for investigating the asymptotic stability of delayed systems. This inequality was generalized for fractional delayed systems by H. Ye, J. Gao, Y. Ding in [23] and plays the same role for the stability of fractional delayed systems.

Here, we generalize another classical result—the relation introduced by A. Halanay, also in [14], which is a bilinear form with the following property: after substituting the arbitrary solution $x(t)$ of the studied system and arbitrary solution $y(t)$ of its Lagrange adjoint system in the Halanay form, this form becomes a constant for all t . This form is a key tool in the classical integer-order case for establishing an important kind of integral representation of the arbitrary solution $x(t)$ via the adjoint fundamental matrix. The main problem with the classical Halanay form is that it is not directly useful for systems with fractional derivatives. This leads to the necessity to reformulate the Halanay form as a bilinear form which is appropriate for application to the fractional delayed systems while retaining its main property.

So, in this section we introduce and prove a generalization for the fractional case of the famous Halanay relation (see [14]), which gives a basic relation between the solutions of both systems (3) and (12).

Furthermore, we establish the main relation between the fundamental matrix $C(t, s)$ and its formal adjoint fundamental matrix $Y(s, t)$.

Let $[a, b] \subset \mathbb{R}$ with $a < b$ be arbitrary fixed and for any $t \leq b$ introduce the bilinear form

$$\begin{aligned} \langle y, x \rangle &= \int_a^t \left(D_{b-}^q y(\alpha) x(\alpha) + y(\alpha) D_{a+}^q x(\alpha) \right) d\alpha + \sum_{i \in \langle m \rangle} \int_t^{t+\sigma_i} y(\alpha) A^i(\alpha) x(\alpha - \sigma_i) d\alpha \\ &\quad - \int_{-h}^0 \left(\int_{t-\tau}^t (y(\theta) A(\theta, \tau) x(\theta + \tau)) d\theta \right) d\tau, \quad t \in [a, b] \end{aligned} \quad (26)$$

We will prove that for arbitrary solutions $x(t)$ of IP (3), (2) and $y(t)$ of FP (12), (14), for any $t \in [a, b]$ the following relation holds:

$$\begin{aligned} \langle y, x \rangle &= \int_a^t \left(D_{b-}^q y(\alpha) x(\alpha) + y(\alpha) D_{a+}^q x(\alpha) \right) d\alpha + \sum_{i \in \langle m \rangle} \int_t^{t+\sigma_i} y(\alpha) A^i(\alpha) x(\alpha - \sigma_i) d\alpha \\ &\quad + \int_{-h}^0 \left(\int_{t-\tau}^t (y(\theta) A(\theta, \tau) x(\theta + \tau)) d\theta \right) d\tau \equiv \text{const}. \end{aligned} \quad (27)$$

Theorem 5. For any compact sub-interval $[a, b] \subset \mathbb{R}$, let the following conditions hold:

1. The kernel $U : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ satisfies conditions **(S)**.
2. The kernel $K(t, s)$ is defined via (20).

Then, for arbitrary solutions $x(t)$ of IP (3), (2) and $y(t)$ of FP (12), (14), the relation (27) holds for any $t \in [a, b]$.

Proof. Let $x(t)$ be an arbitrary solution of IP (3), (2). Then, for any solution $y(t)$ of FP (12), (14) and $t \in [a, b]$ from (26), it follows that

$$\begin{aligned}
 \frac{d}{dt} \langle y, x \rangle &= D_{b-}^q y(t)x(t) + y(t)D_{a+}^q x(t) + \sum_{i \in \langle m \rangle} y(t + \sigma_i)A^i(t + \sigma_i)x(t) \\
 &- \sum_{i \in \langle m \rangle} y(t)A^i(t)x(t - \sigma_i) - \frac{d}{dt} \left(\int_{-h}^0 \left(\int_{t-\tau}^t (y(\theta)A(\theta, \tau)x(\theta + \tau))d\theta \right) d\tau \right) \\
 &= - \int_{-h}^0 y(t - \theta)A_*(t - \theta)x(t)d\theta - \sum_{i \in \langle m \rangle} y(t + \sigma_i)A^i(t + \sigma_i)x(t) \\
 &+ \int_{-h}^0 y(t)A_*(t)x(t + \theta)d\theta + \sum_{i \in \langle m \rangle} y(t)A^i(t)x(t - \sigma_i) + \sum_{i \in \langle m \rangle} y(t + \sigma_i)A^i(t + \sigma_i)x(t) \\
 &- \sum_{i \in \langle m \rangle} y(t)A^i(t)x(t - \sigma_i) + \int_{-h}^0 (y(t - \tau)A_*(t - \tau)x(t) - y(t)A_*(t)x(t + \tau))d\tau \\
 &= - \int_{-h}^0 y(t - \theta)A_*(t - \theta)x(t)d\theta + \int_{-h}^0 y(t)A_*(t)x(t + \theta)d\theta \\
 &+ \int_{-h}^0 y(t - \theta)A_*(t - \theta)x(t)d\theta - \int_{-h}^0 y(t)A_*(t)x(t + \theta)d\theta = 0,
 \end{aligned} \tag{28}$$

and hence from (28) it follows that (27) holds. $\square$

Remark 6. Note that if all $q_k = 1$, $k \in \langle n \rangle$, we have that

$$\begin{aligned}
 &\int_a^t \left(D_{b-}^q y(\alpha)x(\alpha) + y(\alpha)D_{a+}^q x(\alpha) \right) d\alpha \\
 &= \int_a^t \left({}^{RL}D_{b-}^q (y(\alpha) - y(b))x(\alpha) + y(\alpha){}^{RL}D_{a+}^q (x(\alpha) - x(a)) \right) d\alpha \\
 &\text{and} \\
 &\int_a^t \left(I_{b-}^{1-q} y'(\alpha)x(\alpha) + y(\alpha)I_{a+}^{1-q} x'(\alpha) \right) d\alpha = \int_a^t (y'(\alpha)x(\alpha) + y(\alpha)x'(\alpha))d\alpha \\
 &= \int_a^t (y(\alpha)x(\alpha))' d\alpha = y(t)x(t) - y(a)x(a)
 \end{aligned}$$

and hence the relation (27) coincides with the classical one (see [14]) in the integer case ($q_k = 1$, $k \in \langle n \rangle$) and can be called a generalized Halanay relation.

Theorem 6. Let the following conditions be fulfilled:

1. The conditions **(S)** hold.
2. The function $f \in BL_1^{loc}(J_a, \mathbb{R}^n)$ and $f(0) = \mathbf{0}$.

Then, $D_{a+}^{1-q} Y(s, \cdot)(t) = D_{t-}^{1-q} C(t, \cdot)(s)$ for almost everywhere, $s \in [a, t]$.

Proof. Let $x(t)$ be an arbitrary solution of IP (3), (2) and $Y(s; t)$ be the adjoint fundamental matrix. Multiplying from left with $Y(s; t)$ system (9) and integrating in s from a to arbitrary $t \in [a, b]$, we obtain that

$$\begin{aligned}
\int_a^t Y(\alpha; t) D_{a+}^q x(\alpha) d\alpha &= \int_a^t Y(\alpha; t) \sum_{i \in \langle m_0 \rangle} A_i(\alpha) x(\alpha - \sigma_i) d\alpha \\
&+ \int_a^t Y(\alpha; t) \left(\int_{-h}^0 A_*(\alpha, \theta) x(\alpha + \theta) d\theta \right) d\alpha + \int_a^b Y(\alpha; b) f(\alpha) d\alpha \\
&\stackrel{s=\alpha+\theta}{=} \int_a^b \sum_{i \in \langle m_0 \rangle} Y(\alpha; t) A_i(\alpha) x(\alpha - \sigma_i) d\alpha + \int_a^t \left(\int_{\alpha-h}^{\alpha} Y(\alpha; t) A_*(\alpha, s-\alpha) x(s) ds \right) d\alpha \\
&+ \int_a^t Y(\alpha; t) f(\alpha) d\alpha.
\end{aligned} \tag{29}$$

Since $x(t)$ is continuous at any $t \in [a, b]$ and hence bounded on $[a, b]$, then we have that $I_{a+}^{1-q} x(\alpha) = \int_a^\alpha (\alpha - s)^{-q} x(s) ds|_{\alpha=a} = 0$. Then, we have

$$\begin{aligned}
\int_a^t Y(\alpha; t) D_{a+}^q x(\alpha) d\alpha &= \int_a^t Y(\alpha; t)^{RL} D_{a+}^q (x(\alpha) - x(a)) d\alpha = \int_a^t Y(\alpha; t) \frac{dI_{a+}^{1-q}(x(\alpha) - x(a))}{d\alpha} d\alpha \\
&= \int_a^t Y(\alpha; t) d_\alpha I_{a+}^q (x(\alpha) - x(a)) = Y(t; t) I_{a+}^{1-q} (x(\alpha) - x(a))_{\alpha=t} - \underbrace{Y(a; t) I_{a+}^{1-q} (x(\alpha) - x(a))_{\alpha=a}}_{=0} \\
&= I_{a+}^{1-q} x(t) - \int_a^t I_{a+}^{1-q} x(\alpha) \frac{dY(\alpha; t)}{d\alpha} d\alpha = I_{a+}^{1-q} x(t) - \int_a^t \underbrace{I_{t-}^{1-q} \frac{dY(\alpha; t)}{d\alpha}}_{D_{t-}^q} x(\alpha) d\alpha \\
&= I_{a+}^{1-q} x(t) - \int_a^t D_{t-}^q Y(\alpha; t) x(\alpha) d\alpha,
\end{aligned} \tag{30}$$

where formula 2.20 in p.2.3 in [24] is used in the calculation. Then, from (29) and (30) we obtain

$$\begin{aligned}
I_{a+}^{1-q} x(t) &= \int_a^t \sum_{i \in \langle m \rangle_0} Y(\alpha; t) A_i(\alpha) x(\alpha - \sigma_i) d\alpha + \int_a^t \left(\int_{-h}^0 Y(\alpha; t) A_*(\alpha, \theta) x(\alpha + \theta) d\theta \right) d\alpha \\
&+ \int_a^t Y(\alpha; t) f(\alpha) d\alpha + \int_a^t D_{t-}^q Y(\alpha; t) x(\alpha) d\alpha \\
&= \int_a^t \sum_{i \in \langle m \rangle_0} Y(\alpha; t) A_i(\alpha) x(\alpha - \sigma_i) d\alpha + \int_a^t Y(\alpha; t) f(\alpha) d\alpha + \int_a^t D_{t-}^q Y(\alpha; t) x(\alpha) d\alpha \\
&\stackrel{\alpha+\theta=s}{+} \int_a^t \left(\int_{s-h}^s Y(\alpha; t) A_*(\alpha, s-\alpha) x(s) ds \right) d\alpha \\
&\stackrel{\alpha-\sigma_i=\bar{\alpha}}{=} \sum_{i \in \langle m \rangle_0} \int_{a-\sigma_i}^a Y(\bar{\alpha} + \sigma_i; t) A_i(\bar{\alpha} + \sigma_i, t) x(\bar{\alpha}) d\bar{\alpha} + \int_a^t Y(\alpha; t) f(\alpha) d\alpha + \int_a^t D_{t-}^q Y(\alpha; t) x(\alpha) d\alpha \\
&+ \sum_{i \in \langle m \rangle_0} \int_a^t Y(\bar{\alpha} + \sigma_i; t) A_i(\bar{\alpha} + \sigma_i, t) x(\bar{\alpha}) d\bar{\alpha} + \int_a^t \left(\int_{s-h}^s Y(\alpha; t) A_*(\alpha, s-\alpha) x(s) ds \right) d\alpha \\
&\stackrel{\alpha-\sigma_i=\bar{\alpha}}{=} \sum_{i \in \langle m \rangle_0} \int_{a-\sigma_i}^a Y(\alpha + \sigma_i; t) A_i(\alpha + \sigma_i, t) x(\alpha) d\alpha + \int_a^t Y(\alpha; t) f(\alpha) d\alpha \\
&+ \int_a^t D_{t-}^q Y(\alpha; t) x(\alpha) d\alpha + \sum_{i \in \langle m \rangle_0} \int_a^t Y(\alpha + \sigma_i; t) A_i(\alpha + \sigma_i, t) x(\alpha) d\alpha \\
&+ \int_a^t \left(\int_s^{s+h} Y(\alpha; t) A_*(\alpha, s-\alpha) x(s) ds \right) d\alpha + \int_{a-h}^a \left(\int_a^{s+h} Y(\alpha; t) A_*(\alpha, s-\alpha) x(s) ds \right) d\alpha \\
&+ \underbrace{\int_{t-h}^t \left(\int_s^{s+h} Y(\alpha; b) A_*(\alpha, s-\alpha) x(s) ds \right) d\alpha}_{=0} \\
&= \int_{a-h}^a \left(\int_a^{s+h} Y(\alpha; t) A_*(\alpha, s-\alpha) x(s) ds \right) d\alpha + \int_a^t Y(\alpha; t) f(\alpha) d\alpha.
\end{aligned} \tag{31}$$

From (31), taking into account that any row of $Y(s; t)$ satisfies system (12), we obtain for $x(t)$ via $Y(s; t)$ the following integral representation:

$$I_{a+}^{1-q}x(t) = \int_{a-h}^a \left(\int_a^{s+h} Y(\alpha; t) A_*(\alpha, s - \alpha) x(s) ds \right) d\alpha + \int_a^t Y(\alpha; t) f(\alpha) d\alpha,$$

or in equivalent form

$$x(t) = D_{a+}^{1-q} \left(\int_{a-h}^a \left(\int_a^{s+h} Y(\alpha; t) A_*(\alpha, s - \alpha) x(s) ds \right) d\alpha \right) (t) + D_{a+}^{1-q} \left(\int_a^t Y(\alpha; t) f(\alpha) d\alpha \right) (t). \quad (32)$$

Let $x(t)$ be the unique solution of IP (1), (2) with arbitrary forcing term $f(t) \in BL_1^{loc}(J_a, \mathbb{R})$ and initial condition $x_a(\theta) = x(a + \theta) = \mathbf{0}, \theta \in [-h, 0]$. From (32), it follows that $x(t)$ has the integral representation via the adjoint fundamental matrix $Y(s; t)$ in the form

$$x(t) = D_{a+}^{1-q} \left(\int_a^t Y(\alpha; t) f(\alpha) d\alpha \right) (t). \quad (33)$$

Taking into account that $Y(s; t) = \Theta$ for any $t < s$ from (33), it follows that

$$\begin{aligned} D_{a+}^{1-q} I_{a+}^{1-q} x(t) &= x(t) = {}^{RL}D_{a+}^{1-q} \left(\underbrace{\int_a^t Y(s, t) f(s) ds}_{=K(t,s)} \right) (t) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\eta)^{-\alpha} \left(\underbrace{\int_a^\eta Y(s, \eta) f(s) ds}_{=K(\eta,s)} \right) d\eta = \frac{d}{dt} \int_a^t \left(\underbrace{\frac{1}{\Gamma(1-\alpha)} \int_s^t (t-\eta)^{-\alpha} K(\eta, s) d\eta}_{=\bar{Y}(t,s)} \right) ds \\ &= \frac{d}{dt} \int_a^t \left(\underbrace{\frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\eta)^{-\alpha} K(\eta, s) d\eta}_{=\bar{Y}(t,s)} \right) ds = \frac{d}{dt} \int_a^t \bar{Y}(t, s) ds = \int_a^t \frac{\partial \bar{Y}(t, s)}{\partial t} ds + \lim_{s \rightarrow t-0} \bar{Y}(t, s) \\ &= \int_a^t D_{a+}^{1-q} Y(s, t) f(s) ds + \lim_{s \rightarrow t-0} D_{a+}^{1-q} \underbrace{\int_a^t Y(s, t) f(s) ds}_{=K(t,s)} \\ &= \int_a^t D_{a+}^{1-q} Y(s, t) f(s) ds + \lim_{s \rightarrow t-0} (t-a)^{1-\alpha} \int_a^t Y(t, t) f(s) ds = \int_a^t \left(D_{a+}^{1-q} Y(s, \cdot) \right) (t) f(s) ds \end{aligned} \quad (34)$$

and thus we have that

$$x(t) = \int_a^t \left(D_{a+}^{1-q} Y(s, \cdot) \right) (t) f(s) ds. \quad (35)$$

In virtue of Theorem 8 in [20], the same solution has the integral representation via the fundamental matrix $C(t, s)$ as follows:

$$x(t) = \int_a^t C(t, s) {}^{RL}D_{a+}^{1-q} f(s) ds.$$

Applying to the right side of (35) formula 2.64 in p.2.3 in [24] and taking into account that $f(0) = \mathbf{0}$, we obtain

$$x(t) = \int_a^t C(t,s)^{RL} D_{a+}^{1-q} f(s) ds = \int_a^t {}^{RL} D_{t-}^{1-q} C(t,s) f(s) ds = \int_a^t D_{t-}^{1-q} C(t,s) f(s) ds.$$

Then, since for any $f \in BL_1^{loc}(J_a, \mathbb{R}^n)$, we have that

$$\int_a^t \left(D_{a+}^{1-q} Y(s, \cdot) \right)(t) f(s) ds = \int_a^t D_{t-}^{1-q} C(t, \cdot) f(s) ds$$

and hence the statement of the theorem follows from the relation above. $\square$

Remark 7. Note that if all $q_k = 1$, $k \in \langle n \rangle$, we have that

$$D_{a+}^0 Y(s, \cdot)(t) = Y(s, t), \quad D_{t-}^0 C(t, \cdot)(s) = C(t, s)$$

and hence the proved relation coincides with the classical one $C(t, s) = Y(s, t)$ obtained in the integer case.

5. Examples

The next mathematical example considers a class of delayed systems for which the applicability and the feasibility of Theorem 5 can be checked immediately.

Example 1. Let $[a, b] \subset \mathbb{R}$ with $a < b$ and consider the system $D_{a+}^q x(t) = Ax(t) + Bx(t-h)$, with its Lagrange adjoint system $D_{b-}^q y(t) = -Ay(t) - By(t-h)$, $t \in [a, b]$, $A, B \in \mathbb{R}^{n \times n}$, $x(t) : [a, \infty) \rightarrow \mathbb{R}^n$, $y(t) : (-\infty, b] \rightarrow \mathbb{R}^{n*}$, and $h > 0$ is an arbitrary number. Then, from (27) we obtain that the generalized Halanay relation has the form

$$\langle y, x \rangle = \int_a^t (D_{b-}^q y(\alpha) x(\alpha) + y(\alpha) D_{a+}^q x(\alpha)) d\alpha + \int_t^{t+h} y(\alpha) Bx(\alpha-h) d\alpha.$$

Then, substituting arbitrary solutions of both systems, with simple calculations we have

$$\begin{aligned} \frac{d}{dt} \langle y, x \rangle &= D_{b-}^q y(t) x(t) + y(t) D_{a+}^q x(t) + y(t+h) Bx(t) + y(t) Bx(t-h) \\ &= -y(t) Ax(t) - y(t+h) Bx(t) + y(t) Ax(t) \\ &\quad + y(t) Bx(t-h) + y(t+h) Bx(t) - y(t) Bx(t-h) = 0 \end{aligned}$$

and hence $\langle y, x \rangle = \text{const}$, which illustrates the applicability of Theorem 5.

Below, we provide an actual example of how our system can be applied in an engineering context.

Example 2. The presented example below is a generalization of the J.J. Hopfield model, introduced in 1984, for a network of n voltage amplifiers (neurons), which describes the evolution of the voltage on the input of each neuron. Since it was established that the updating and propagation do not occur instantaneously in the model, it was upgraded as follows (see [25,26]):

$${}^C D_{0+}^q x_i(t) = -(C_i R_i)^{-1} x_i(t) + C_i^{-1} \sum_{j \in \langle n \rangle} T_{ij} f_j(x_j(t-h_{ij})), \quad i \in \langle n \rangle, \quad (36)$$

where $x_i(t) : [0, \infty) \rightarrow \mathbb{R}$ represents the voltage on the input of the i -th neuron. For any $i \in \langle n \rangle$, the i -th neuron is characterized with input capacitance of its cell membranes C_i , the delays $h_{i1}, \dots, h_{in}$,

and the transfer functions $f_j(x_j)$ are called the input–output relation; each of them is nonlinear and sigmoidal, saturating at ± 1 with the maximal slope at $x_j = 0, j \in \langle n \rangle$. T_{ij} is the synapse efficiency between neuron i and j , with $T_{ij} = R_{ij}^{-1}$ when the noninverting output of neuron j is connected to the input of neuron i via a resistance R_{ij} and $T_{ij} = -R_{ij}^{-1}$ when the inverting output of neuron j is connected to the input of neuron i through a resistance R_{ij} . R_{ij} is the total parallel transmembrane resistance of the i -th neuron and $R_i^{-1} = r_i^{-1} + \sum_{j \in \langle n \rangle} R_{ij}^{-1}, i \in \langle n \rangle$, where r_i denotes the input resistance of the i -th neuron, corresponding to the connection to the outside of the network. Linearizing $f_j(x_j)$ from (36), it follows that

$${}^C D_{0+}^q x_i(t) = -(C_i R_i)^{-1} x_i(t) + C_i^{-1} \sum_{j \in \langle n \rangle} \alpha J_{ij} x_j(t - h_{ij}), \quad i \in \langle n \rangle, \quad (37)$$

where the gain α is defined as the slope of $f(x) = (f_1(x_1), \dots, f_n(x_n))$ at $x(t) = (x_1, \dots, x_n)^T = \mathbf{0}$ and $\{J_{ij}\}_{i,j \in \langle n \rangle} \in \mathbb{R}^{n \times n}$ is the connection matrix. Note that the system (37) is a partial case of our studied system and in [27] it is established that even in the integer case the delays may cause sustained oscillations in the neural network.

6. Comments and Conclusions

In the present work, we have considered linear systems with derivatives in Caputo's sense of incommensurate order and distributed delays. The following main results were obtained in the presented article:

1. An explicit formula connecting the adjoint fundamental matrix with the measure-resolvent kernel $R(s, t)$ has been proven for Volterra kernels $K(t, s)$ of type L_1^{loc} .
2. A bilinear form is introduced, which is proven to become equal to a constant after the substitution of an arbitrary solution of IP (3), (2) and an arbitrary solution of FP (10), (12) and generalizes the classical result of Halanay [14] for systems with first-order derivatives.
3. An explicit formula connecting the adjoint fundamental matrix with the fundamental matrix has been proven.

Below, we make some notes related to the obtained main results listed above.

To our knowledge, all the results obtained in the article are new for fractional equations and/or systems with any kind of delays (constant, variable, distributed).

The results described above in points 2 and 3 for the fractional adaptation of the Halanay form introduced by us coincide, in the particular case when all fractional derivative orders are $q_1 = q_2 = \dots = q_n = 1$, with the classical results of Halanay [14] for systems with first-order derivatives.

For the introduced notion of a formal (Lagrange) adjoint system of the studied homogeneous system (3), it must be noted that in [17] the existence of a unique adjoint fundamental matrix is obtained as a consequence of the possibility of backward continuation for the solutions. This work makes an extension of the approach introduced in [17] that partially decreases the disadvantages caused by the usage of the fractional derivatives by losing the symmetry of differentiation. In contrast with [17], as an application of the extended approach we have obtained the adjoint fundamental matrix as a solution of an appropriately formulated final problem, which allowed us to weaken the conditions in this work for its existence. We emphasize that the final problems in the work are considered as problem of the renewal of a process with after effects under a given final observation. This formulation is equivalent, from the mathematical point of view, to the problem of forward prolongation of the solutions of fractional differential equations with deviating argument of advanced type.

One of the main obstacles (at least technically but not only) in formulating and proving an analogue of the famous Halanay relation was the absence, for fractional derivatives, of the Leibniz rule for differentiating a product of two functions, which demanded a new formulation of this relation. Also, the absence of key properties in the fractional derivatives such as locality and symmetry are not only technical problems but are related to the inner nature of the described processes. So, for the fractional case, taking into account the geometric and physical interpretation of the fractional integration and differentiation (see [28,29]), we think that fractional models are closer to the point of view developed in the relativistic theory for real-world phenomena. Note that, as far as we know, at this time an analysis of the consequences from this point of view does not exist.

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References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
2. Podlubny, I. *Fractional Differential Equation*; Academic Press: San Diego, CA, USA, 1999.
3. Hilfer, R. Threefold Introduction to Fractional Derivatives. In *Anomalous Transport: Foundations and Applications*; Klages, R., Radons, G., Sokolov, I.M., Eds.; Wiley-VCH: Weinheim, Germany, 2008; p. 17.
4. Jiao, Z.; Chen, Y.Q.; Podlubny, I. *Distributed-Order Dynamic Systems: Stability, Simulation, Applications and Perspectives*; Springer: Berlin/Heidelberg, Germany, 2012.
5. Stamova, I.; Stamov, G. *Functional and Impulsive Differential Equations of Fractional Order: Qualitative Analysis and Applications*; CRC Press: Boca Raton, MA, USA, 2017.
6. Kochubei, A.; Luchko, Y. (Eds.) *Handbook of Fractional Calculus with Applications*; Volume 1: Basic Theory; Series edited by J. A. Tenreiro Machado; De Gruyter: Berlin, Germany, 2019. [[CrossRef](#)]
7. Kochubei, A.; Luchko, Y. (Eds.) *Handbook of Fractional Calculus with Applications*; Volume 2: Fractional Differential Equations; Series edited by J. A. Tenreiro Machado; De Gruyter: Berlin, Germany, 2019. [[CrossRef](#)]
8. Tarasov, V.E. General Fractional Dynamics. *Mathematics* **2021**, *9*, 1464. [[CrossRef](#)]
9. Fernandez, E.; Huilcapi, V.; Birs, I.; Cajo, R. The Role of Fractional Calculus in Modern Optimization: A Survey of Algorithms, Applications, and Open Challenges. *Mathematics* **2025**, *13*, 3172. [[CrossRef](#)]
10. Barbero, G.; Evangelista, L.R.; Zola, R.S.; Lenzi, E.K.; Scarfone, A.M. A Brief Review of Fractional Calculus as a Tool for Applications in Physics: Adsorption Phenomena and Electrical Impedance in Complex Fluids. *Fractal Fract.* **2024**, *8*, 369. [[CrossRef](#)]
11. Godinho, C.F.d.L.; Vanea, I.V. Fractional Calculus in Physics: A Brief Review of Fundamental Formalisms. *Mathematics* **2025**, *13*, 3643. [[CrossRef](#)]
12. Wang, C.; Li, Y.; Yang, G.; Deng, Q. A Review of Fractional-Order Chaotic Systems of Memristive Neural Networks. *Mathematics* **2025**, *13*, 1600. [[CrossRef](#)]
13. Myshkis, A.D. General theory of differential equations with a retarded argument. *Uspekhi Mat. Nauk.* **1949**, *4*, 99–141.
14. Halanay, A. *Differential Equations, Stability, Oscillations, Time Lags*; Academic Press: New York, NY, USA, 1966.
15. Hale, J.; Lunel, S. *Introduction to Functional Differential Equations*; Springer: New York, NY, USA, 1993.

16. Kolmanovskii, V.; Myshkis, A. *Introduction to the Theory and Applications of Functional Differential Equations*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1999.
17. Kiskinov, H.; Milev, M.; Petkova, M.; Zahariev, A. Backward Continuation of the Solutions of the Cauchy Problem for Linear Fractional System with Deviating Argument. *Mathematics* **2025**, *13*, 76. [[CrossRef](#)]
18. Myshkis, A. *Linear Differential Equations with Retarded Argument*; Nauka: Moscow, Russia, 1972. (In Russian)
19. Zahariev, A.; Kiskinov, H.; Angelova, E. Linear fractional system of incommensurate type with distributed delay and bounded Lebesgue measurable initial conditions. *Dyn. Syst. Appl.* **2019**, *28*, 491–506.
20. Kiskinov, H.; Madamlieva, E.; Veselinova, M.; Zahariev, A. Existence of Absolutely Continuous Fundamental Matrix of Linear Fractional System with Distributed Delays. *Mathematics* **2021**, *9*, 150. [[CrossRef](#)]
21. Gripenberg, G.; Londen, S.O.; Staffans, O. *Volterra Integral and Functional Equations*; Encyclopedia of Mathematics and its Applications; Cambridge University Press: Cambridge, UK; New York, NY, USA; Sydney, Australia, 1990; Volume 34.
22. Kiskinov, H.; Milev, M.; Petkova, M.; Zahariev, A. Fundamental Matrix, Measure Resolvent Kernel and Stability Properties of Fractional Linear Delayed System with Discontinuous Initial Conditions. *Mathematics* **2025**, *13*, 1408. [[CrossRef](#)]
23. Ye, H.; Gao, J.; Ding, Y. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **2007**, *328*, 1075–1081. [[CrossRef](#)]
24. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon and Breach Science Publishers: Yverdon, Switzerland, 1993.
25. Belair, J. Stability in a model of a delayed neural network. *J. Diff. Eqs.* **1993**, *5*, 607–623. [[CrossRef](#)]
26. Marcus, C.M.; Westervelt, R.M. Stability of analog networks with delay. *Phys. Rev. A* **1989**, *39*, 347–359. [[CrossRef](#)] [[PubMed](#)]
27. Wu, J.; Zou, X. Patterns of sustained oscillations in neural networks with delayed interactions. *Appl. Math. Comput.* **1995**, *73*, 55–75. [[CrossRef](#)]
28. Podlubny, I. Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calc. Appl. Anal.* **2002**, *5*, 367–386.
29. Podlubny, I.; Heymans, N. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheol. Acta* **2006**, *45*, 765–771.

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