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Normalized Solutions and Critical Growth in Fractional Nonlinear Schrödinger Equations with Potential

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Abstract

We investigate the existence of positive normalized (mass-constrained) solutions for the fractional nonlinear Schrödinger equation $(-\Delta)^s v + V(x)v = \lambda v + \mu|v|^{p-2}v + |v|^{2_s^*-2}v$ in $\mathbb{R}^N$, $\|v\|_2^2 = b^2$, where $N > 2s$, $s \in (0,1)$, $\mu > 0$, $p \in (2, 2_s^*)$, and $2_s^* = \frac{2N}{N-2s}$. Here, $\lambda \in \mathbb{R}$ denotes the Lagrange multiplier associated with the prescribed mass $b > 0$. The potential $V \in C^1(\mathbb{R}^N)$ is allowed to be nonconstant and satisfies $V(x) \to V_\infty$ as $|x| \to \infty$; moreover, the perturbations induced by $V - V_\infty$ and $x\cdot\nabla V$ are assumed to be small in the quadratic-form sense compared with the fractional Dirichlet form $\|(-\Delta)^{s/2}v\|_2^2$. Using the Caffarelli–Silvestre extension, we establish a Pohozaev identity adapted to the presence of $V(x)$ and introduce a Pohozaev manifold on the L^2-sphere. Combining Jeanjean's augmented functional approach with a splitting analysis at the Sobolev-critical level, we construct compact Palais–Smale sequences below a suitable critical energy level. As a consequence, we prove the existence of positive normalized solutions for small masses $b \in (0, b_0)$ in the L^2-critical and L^2-supercritical regimes (with respect to the lower-order power p).

Keywords: positive normalized solutions; fractional nonlinear Schrödinger equation; critical exponent

1. Introduction

In this paper, we mainly study the fractional nonlinear Schrödinger equations with both critical growth and nontrivial potential

$$\begin{cases} (-\Delta)^s v + V(x)v = \lambda v + \mu|v|^{p-2}v + |v|^{2_s^*-2}v, & x \in \mathbb{R}^N, \\ \|v\|_2^2 = b^2, \end{cases} \tag{1}$$

where $N > 2s$, $s \in (0,1)$, $\mu > 0$, $b > 0$, $p \in (2, 2_s^*)$, and $2_s^* = \frac{2N}{N-2s}$. Here, $\lambda \in \mathbb{R}$ denotes the Lagrange multiplier associated with the mass constraint $\|v\|_2 = b$. The fractional Laplacian operator $(-\Delta)^s$ may be defined as

$$(-\Delta)^s v(x) = C(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x-y|^{N+2s}}\,\mathrm{d}y, \quad x \in \mathbb{R}^N, \tag{2}$$

for $v \in C_0^\infty(\mathbb{R}^N)$, where $C(N,s)$ denotes a suitable positive normalizing constant and P.V. stands for the principal value that may be regarded as the infinitesimal generators of Lévy stable diffusion processes (see [1–3]). In particular, these works analyze the

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existence/stability and dynamical properties of solitary waves in (fractional/pseudo-relativistic) Schrödinger-type models. For background on the fractional Laplacian and on fractional Sobolev spaces, we refer, for instance, to [4–7]. Here, references [4,6] are monographs on nonlocal diffusion/variational methods, reference [5] surveys fractional Sobolev spaces, and [7] provides probabilistic/physical intuition for the fractional Laplacian. We mainly study Equation (1), which is the stationary equation satisfied by the standing wave solutions to the following time-dependent fractional Schrödinger equation:

$$i\Psi_t = (-\Delta)^s\Psi + V(x)\Psi - \mu|\Psi|^{p-2}\Psi - |\Psi|^{2_s^*-2}\Psi, \tag{3}$$

where Ψ has the form

$$\Psi(t,x) = e^{-i\lambda t}v(x), \quad (t,x) \in (0,\infty)\times\mathbb{R}^N,$$

where $v : \mathbb{R}^N \to \mathbb{C}$ is a time-independent function leading to the study of Equation (3), i is the imaginary unit, and $\Psi(t,x)$ represents the quantum mechanical probability amplitude for a given unit mass particle to have position x at time t (the corresponding probability density is $|\Psi|^2$), under a confinement induced by the potential V. Equation (3) has special significance in fractional quantum mechanics, where it is used to study particles on random fields modeled by Lévy processes. Laskin [8] formalized the path integral of Lévy processes and the fractional Schrödinger equation of fractional quantum mechanics based on the path integral ideas of Feynman and Hibbs. For further physical background on Equation (3), we recommend that the readers refer to [9].

To search for standing waves in Equation (3), there are two different methods: one treats the frequency λ as a fixed value, while the other treats the frequency λ as an unknown quantity and specifies the mass. The issue of fixed-frequency λ has been widely studied in the literature (see [10–13] and the references cited therein). The fixed-frequency approach has been developed for the existence of bound/ground states and concentration phenomena, including critical growth and potential effects. In the latter case, it appears particularly interesting from a physical perspective, as λ appears as a Lagrange multiplier under mass constraints—see [14–16].

Here, we would like to mention Jin and Zhang [17]. When $s = 1$, Equation (1) becomes the classical Schrödinger equation with nonlinear terms and a potential, as shown below:

$$\begin{cases} -\Delta v + V(x)v = \lambda v + \mu|v|^{p-2}v + |v|^{2^*-2}v, \\ \|v\|_2^2 = b^2 > 0. \end{cases} \tag{4}$$

Jin and Zhang [17] utilized the fibering map method and the Pohozaev identity to analyze the solutions of Equation (4) in the critical and supercritical cases.

When $s \in (0,1)$, Peng and Xia [18] have studied the following equation:

$$\begin{cases} (-\Delta)^s v + V(x)v + \omega v = |v|^{p-2}v, \\ \|v\|_2^2 = b^2 > 0, \end{cases} \tag{5}$$

where $(v,\omega) \in H^s(\mathbb{R}^N)\times\mathbb{R}^+$. When the potential $V \in C^1(\mathbb{R}^N)$, $V(x) > 0$, $\lim_{|x|\to\infty} V(x) = 0$, and $|x\nabla V(x)| \le \gamma V(x)$ with $\gamma > 0$, the existence of a solution to Equation (5) depends on the potential V that is obtained. Moreover, Peng and Xia [18] proposed a novel minimax structure, constructed auxiliary functionals and scaling transformations, and handled the possible energy loss by means of the splitting lemma to establish the existence of normalized solutions for the fractional-order Schrödinger equation.

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In [17], for the classical case $s = 1$, the normalized solutions of the Schrödinger equation with potential and nonlinear terms were studied. A method was proposed to overcome the lack of compactness of Palais–Smale sequences by proving that both the energy functional and the augmented energy functional possess the Mountain-Pass geometry. Furthermore, it was shown that the Palais–Smale sequence converges to the Pohozaev manifold, which ensures the existence of a normalized solution. In [18], when $s \in (0,1)$, the nonlocal version of the splitting lemma is particularly instructive.

If $f(v) = \mu|v|^{p-2}v + |v|^{2_s^*-2}v$, then $f(\Psi) = e^{-i\lambda t} f(v)$; Equation (1) can be reduced to

$$(-\Delta)^s v + V(x)v = f(v) + \lambda v, \tag{6}$$

with v satisfying the following sphere

$$S_b := \{v \in H^s(\mathbb{R}^N) : \|v\|_2^2 = b^2\}. \tag{7}$$

Therefore, a solution v satisfying Equation (6) is called a normalized solution. Based on Equations (6) and (7), we define

$$E_{\mu,V}(v) = \frac{1}{2}\|v\|^2_{D^s(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} V(x)v^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(v)\mathrm{d}x, \tag{8}$$

where $F(v) = \int_0^v f(t)\mathrm{d}t = \frac{\mu}{p}|v|^p + \frac{1}{2_s^*}|v|^{2_s^*}$, and define

$$\|(-\Delta)^{\frac{s}{2}} v\|_2^2 = \|v\|^2_{D^s(\mathbb{R}^N)}.$$

1.1. Zero-Potential Case ($V \equiv 0$)

First, we introduce the power-type nonlinearity $f(v) = |v|^{p-2}v$ and consider

$$(-\Delta)^s V_{\lambda,s} + \lambda V_{\lambda,s} = |V_{\lambda,s}|^{p-2} V_{\lambda,s}, \qquad x \in \mathbb{R}^N,\ \lambda > 0. \tag{9}$$

Let $V_{\lambda,s} \in H^s_{\mathrm{rad}}(\mathbb{R}^N)$ be the positive radial ground state (unique in the radial class, see [19]). By translation invariance, we may assume it is centred at the origin. For any scaling factor $\sigma > 0$, define

$$V_{\lambda,s,\sigma}(x) = \sigma^{\frac{1}{p-2}}\, V_{\lambda,s}(\sigma^{\frac{1}{2s}} x), \qquad x \in \mathbb{R}^N.$$

Because $(-\Delta)^s(v(\sigma^{\frac{1}{2s}}\cdot)) = \sigma(-\Delta)^s v(\sigma^{\frac{1}{2s}}\cdot)$, $V_{\lambda,s,\sigma}$ solves

$$(-\Delta)^s V_{\lambda,s,\sigma} + (\sigma\lambda)V_{\lambda,s,\sigma} = |V_{\lambda,s,\sigma}|^{p-2} V_{\lambda,s,\sigma}.$$

Changing variables $y = \sigma^{\frac{1}{2s}} x$ yields

$$\begin{aligned}
\|V_{\lambda,s,\sigma}\|_2^2 &= \int_{\mathbb{R}^N} \sigma^{\frac{2}{p-2}} \left|V_{\lambda,s}(\sigma^{\frac{1}{2s}} x)\right|^2 \mathrm{d}x \\
&= = \sigma^{\frac{2}{p-2} - \frac{N}{2s}} \int_{\mathbb{R}^N} \left|V_{\lambda,s}(y)\right|^2 \mathrm{d}y \\
&= \sigma^{\frac{4s-(p-2)N}{2s(p-2)}} \|V_{\lambda,s}\|_2^2.
\end{aligned}$$

Denote the L^2-critical exponent by $\bar{p} := 2 + \dfrac{4s}{N}$:

- If $p \neq \bar{p}$, then for each $b > 0$ there exists a unique scaling $\sigma(b) > 0$ such that $\|V_{\lambda,s,\sigma(b)}\|_2 = b$. Hence, Equation (9) admits a unique positive radial normalised solution $v \in S_b$ for all $p \in (2,\bar{p}) \cup (\bar{p}, 2_s)$.

- If $p = \bar{p}$, the exponent above vanishes; thus, $\|V_{\lambda,s,\sigma}\|_2 \equiv \|V_{\lambda,s}\|_2$, and a normalized solution exists iff $\|V_{\lambda,s}\|_2^2 = b^2$.

  Moreover, Equation (8) is intimately related to dilation, preserving the L^2-norm; we define

$$v_h(x) := h^{\frac{N}{2}} v(hx), h > 0. \tag{10}$$

This scaling leads to the following relations:

$$\|v_h\|^2_{D^s(\mathbb{R}^N)} = h^{2s}\|v\|^2_{D^s(\mathbb{R}^N)}, \quad \|v_h\|_p^p = h^{\frac{N(p-2)}{2}}\|v\|_p^p.$$

In particular, recalling $\gamma_{p,s} := \frac{N(p-2)}{2ps}$, we have $\frac{N(p-2)}{2} = sp\gamma_{p,s}$ and hence $\|v_h\|_p^p = h^{sp\gamma_{p,s}}\|v\|_p^p$. Moreover, since $2_s^* = \frac{2N}{N-2s}$, one checks that $\|v_h\|_{2_s^*}^{2_s^*} = h^{s2_s^*}\|v\|_{2_s^*}^{2_s^*}$.

Thus, to have the balance $h^{2s} = h^{\frac{N(p-2)}{2}}$ for all $h \neq 1$, it is necessary that $p = \bar{p}$. For a fixed $v \in S_b$, the sign of $p - \bar{p}$ governs the behavior of the fibering map $h \mapsto E_\mu(v_h)$.

It is useful to introduce the following fractional Gagliardo–Nirenberg–Sobolev inequality [19]:

$$\|v\|_{L^p} \le C_{N,p,s}\,\|v\|_{D^s}^{\gamma_{p,s}}\,\|v\|_{L^2}^{1-\gamma_{p,s}}, \qquad \forall\, v \in H^s(\mathbb{R}^N). \tag{11}$$

It is straightforward that

$$p\,\gamma_{p,s} = \begin{cases} = 2, & p = \bar{p} := 2 + \frac{4s}{N}, \\ > 2, & \bar{p} < p < 2_s := \frac{2N}{N-2s}, \end{cases}$$

and, from $2_s := \frac{2N}{N-2s}$, we obtain $\gamma_{2_s^*} = 1$. For $p \in (\bar{p}, 2_s^*)$, the functional E_μ is unbounded on S_b.

For $p = 2_s^*$, Equation (11) reduces to the following fractional Sobolev inequality [20]:

$$S_s\,\|v\|_{2_s^*}^2 \le \|v\|^2_{D^s(\mathbb{R}^N)}, \tag{12}$$

where

$$S_s := \inf_{v \in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\|v\|^2_{D^s(\mathbb{R}^N)}}{\left(\int_{\mathbb{R}^N} |v|^{2_s^*}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}}} \tag{13}$$

is defined as the sharp constant for the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$.

It is well known from [20] that equality in the fractional Sobolev inequality Equation (12) is achieved by a family of functions, up to translations and dilations, of the form

$$\widetilde{V}_{\varepsilon,y,s}(x) = M_{N,s}\left(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2}\right)^{\frac{N-2s}{2}}, \qquad \varepsilon > 0,\ y \in \mathbb{R}^N, \tag{14}$$

where $M_{N,s} > 0$ is chosen so that

$$\|(-\Delta)^{s/2}\widetilde{V}_{\varepsilon,y,s}\|_2^2 = S_s\,\|\widetilde{V}_{\varepsilon,y,s}\|_{2_s^*}^2.$$

Moreover, all positive solutions of $(-\Delta)^s u = u^{2_s^*-1}$ in $\mathbb{R}^N$ are given by the family Equation (14), up to translations and dilations.

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Recently, Zhen and Zhang [21] investigated the following elliptic problems characterized by combined nonlinear terms containing the fractional Sobolev critical exponent:

$$\begin{cases} (-\Delta)^s v = \lambda v + \mu |v|^{p-2} v + |v|^{2_s^*-2} v; \\ \|v\|_2^2 = b^2, \quad v \in H^s(\mathbb{R}^N). \end{cases} \tag{15}$$

Define the associated Pohozaev functional as

$$P_\mu(v) = s\|v\|^2_{D^s(\mathbb{R}^N)} - \mu\gamma_{p,s} s\|v\|_p^p - s\|v\|_{2_s^*}^{2_s^*}.$$

Additionally, let

$$m_b = \inf_{v \in P_\mu} E_\mu, \tag{16}$$

where

$$E_\mu(v) = \frac{1}{2}\|v\|^2_{D^s(\mathbb{R}^N)} - \frac{\mu}{p}\|v\|_p^p - \frac{1}{2_s^*}\|v\|_{2_s^*}^{2_s^*}, \qquad v \in H^s(\mathbb{R}^N). \tag{17}$$

Zhen and Zhang [21] proved that, when $p \in (2, 2_s^*)$, the estimate in Equation (34) holds provided that $N > 2s$, $\mu > 0$, $b > 0$, and $\mu b^{p(1-\gamma_{p,s})} < \alpha_{N,p,s}$. Let v_μ be the corresponding positive ground-state solution obtained in ([21], Theorems 1.1–1.3); then, the following conclusions hold:

(1) $2 < p < \bar{p} = 2 + \frac{4s}{N}, m_b \to 0, \|v_\mu\|^2_{D^s(\mathbb{R}^N)} \to 0$ as $\mu \to 0^+$;

(2) $\bar{p} \le p < 2_s^* = \frac{2N}{N-2s}, m_b \to \frac{s}{N} S_s^{\frac{N}{2s}}, \|v_\mu\|^2_{D^s(\mathbb{R}^N)} \to S_s^{\frac{N}{2s}}$ as $\mu \to 0^+$.

Next, we present some theorems about situations where there is no potential from [21].

Theorem 1. *Let $N > 2s$, $b > 0$, and $\mu > 0$, and suppose $2 < p = \bar{p}$. If*

$$\mu b^{\frac{4s}{N}} < \bar{p}(2C^{\bar{p}}_{N,\bar{p},s})^{-1},$$

then $E_\mu|_{S_b}$ has a ground state v with the following properties: v is positive and radially symmetric, and solves Equation (15) for some $\lambda < 0$. Moreover, $0 < m_b < \frac{s}{N} S_s^{\frac{N}{2s}}$, and v is a Mountain-Pass-type point.

Theorem 2. *Let $N > 2s$, $b, \mu > 0$, and $2 < p < 2_s^*$. If one of the following conditions holds:*

(1) $N > 4s$ *and* $\mu b^{p(1-\gamma_{p,s})} < \frac{S_s^{\frac{Np(1-\gamma_{p,s})}{4s}}}{\gamma_{p,s}}$;

(2) $N = \frac{p}{p-1} 2s$ *and* $\mu b^{p(1-\gamma_{p,s})} < \frac{S_s^{\frac{Np(1-\gamma_{p,s})}{4s}}}{\gamma_{p,s}}$;

(3) $N = 4s$ *or* $\frac{p}{p-1} 2s < N < 4s$ *or* $2s < N < \frac{p}{p-1} 2s$,

then $E_\mu|_{S_b}$ has a ground state v with the following properties: v is a positive, radially symmetric function and solves Equation (15) for some $\lambda < 0$. Moreover,

$$0 < m_b < \frac{s}{N} S_s^{N/(2s)},$$

and m_b is a critical point of Mountain-Pass type.

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In contrast to the case $V \equiv 0$ discussed in Section 1.1, the presence of a nontrivial potential $V(x)$ introduces additional analytical difficulties, particularly in verifying the compactness of Palais–Smale sequences.

1.2. Nonzero-Potential Case ($V \neq 0$)

In this case, the variational method is typically employed to obtain solutions of Equations (6) and (7); for instance, see [18,21,22]—these references establish variational frameworks for normalized solutions with nonzero potentials and discuss the compactness difficulties arising from the constraint.

1.2.1. L^2-Subcritical Growth

Suppose that f has L^2-subcritical growth of the form

$$f(v) = \sum_{1\le j\le k} a_j |v|^{\sigma_j - 2} v,$$

where $k \geq 1$, $N \geq 2$, $a_j > 0$, and $2 \leq \sigma_j \leq \bar{p} = 2 + \frac{4s}{N}$. E_V (see Equation (8)) admits a lower bound on S_b. The main challenge lies in deriving suitable subadditive inequalities for the associated constrained minimization problem.

In [22], when $N \geq 2$, Liu and Zhou introduced the following assumptions on the potential $V(x)$:

(W0) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) > -\infty$;
(W1) $V(x) \leq \lim_{|x|\to\infty} V(x) := V_\infty \in (0, +\infty)$.

They further employed iterative techniques to deduce a subadditive inequality for the L^2-constrained minimization problem, and proved the existence of normalized solutions under the mass-subcritical growth condition.

1.2.2. Mass-Supercritical Growth and Below-Sobolev-Critical Growth

Suppose that f has L^2-supercritical growth, such as,

$$f(v) = \sum_{1\le j\le k} a_j |v|^{\sigma_j - 2} v,$$

where $k \geq 1$, $a_j > 0$, and for each j,

$$\bar{p} < \sigma_j < 2_s^* \quad \text{whenever } N > 2s,$$

while

$$\bar{p} < \sigma_j \quad \text{whenever } N = s \text{ or } N = 2s.$$

The energy functional $E_{\mu,V}$ is unbounded from below on S_b. The existence of solutions to Equations (6) and (7) is often obtained by means of the Mountain-Pass geometry.

It is necessary to verify that $E_{\mu,V}$ is unbounded from below on S_b. A major difficulty arises here: the lack of compactness of the working space $H^s(\mathbb{R}^N)$, which leads to the Palais–Smale (P–S) sequence, may diverge, making it impossible to obtain a solution directly using standard variational methods (such as Mountain Pass). To elaborate, we may consider the following typical conditions, which were used by Peng and Xia [18]:

(M0) $V(x) \in C^1(\mathbb{R}^N)$, $V(x) \geq 0$, $\lim_{|x|\to\infty} V(x) = 0$, $|x\nabla V(x)| \leq \Upsilon V(x)$, for some constants $\Upsilon > 0$.
(M1) $V(x) \in L^\infty(\mathbb{R}^N)$, and $0 < \|V\|_{L^\infty(\mathbb{R}^N)} < 2\kappa$, where $\kappa = \min\left\{1, \frac{p(2s-N)+2N}{(p-2)(\Upsilon+2s)} \cdot \frac{m_\rho}{\rho^2}\right\}$.

(M2) $\|V\|_{L^{\frac{N}{2}}(\mathbb{R}^N)} < \min\Big\{ \frac{2N(p-2)}{N(p-2)-4s}\Big(2^{\frac{N(p-2)-4s}{2N(p-2)}}-1\Big)\cdot\frac{m_\rho}{\|U_\rho\|_{2_s^*}^2}, \frac{N(p-2)-4s}{2\mathrm{Y}A^2},$
$\frac{(p(2s-N)+2N)(N(p-2)-4s)}{2N(p-2)^2A^2(\mathrm{Y}+2s)+2pA^2(p(2s-N)+2N)}\Big\}.$

Above,

$$A = \frac{\Gamma\left(\frac{N-2s}{2}\right)}{2^{2s}\pi^s\Gamma\left(\frac{N+2s}{2}\right)}\left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right)^{\frac{2s}{N}},$$

where Γ denotes the Gamma function and A is the Aubin–Talenti constant (see [20]), that is, the best constant in the Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$. Under the above assumptions, Peng and Xia [18] showed that the energy functional remains compact on a certain level set by imposing the above restrictions on $V(x)$.

1.3. Existence Results for Fractional Schrödinger Equations with Sobolev-Critical Growth

From the above-mentioned literature, most of them do not directly address nonlinearities of Sobolev-critical type. Inspired by [21], which treats the non-potential case, we now turn our attention to Equation (1) in the presence of L^2-critical and L^2-supercritical. Accordingly, we introduce the following assumptions, which are used in the subsequent proof:

(V0) $\lim_{|x|\to\infty} V(x) = \sup_{x\in\mathbb{R}^N} V(x) =: V_\infty < \infty$ and there exists $\sigma_1 > 0$ such that

$$\int_{\mathbb{R}^N} |V - V_\infty| v^2 \, \mathrm{d}x \le \sigma_1 \|v\|^2_{D^s(\mathbb{R}^N)}, \quad \forall v \in H^s(\mathbb{R}^N); \tag{18}$$

(V1) Let $W(x) := \frac{1}{2}(\nabla V(x)\cdot x)$,

$$\Big|\int_{\mathbb{R}^N}\Big(\frac{1}{2}(V - V_\infty) + \frac{1}{p\gamma_{p,s}s}W\Big)v^2 \, \mathrm{d}x\Big| \le \sigma_2 \|v\|^2_{D^s(\mathbb{R}^N)}, \quad \forall v \in H^s(\mathbb{R}^N); \tag{19}$$

(V2) There exists $\sigma_3 > 0$ such that

$$\Big|\int_{\mathbb{R}^N} W v^2 \, \mathrm{d}x\Big| \le \sigma_3 \|v\|^2_{D^s(\mathbb{R}^N)}, \quad \forall v \in H^s(\mathbb{R}^N); \tag{20}$$

(V3) for any $x \in \mathbb{R}^N$, $s \in (0,1)$, there holds

$$(sV(x) + W(x)) < V_\infty. \tag{21}$$

Remark 1. *Assumptions* (V0)–(V2) *are fractional form-boundedness conditions: they require that the quadratic forms generated by* $V - V_\infty$ *and* $W(x) = \frac{1}{2}x\cdot\nabla V(x)$ *be small perturbations of the fractional Dirichlet form* $\|v\|^2_{D^s(\mathbb{R}^N)}$*. A convenient sufficient condition is the following: if* $V - V_\infty \in L^{\frac{N}{2s}}(\mathbb{R}^N)$ *and* $\|V - V_\infty\|_{L^{\frac{N}{2s}}}$ *is small enough, then, by Hölder's inequality and the Sobolev embedding* $H^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$*, one obtains Equation* (18)*. Similarly, if* $W \in L^{\frac{N}{2s}}(\mathbb{R}^N)$ *with small norm, then Equations* (19) *and* (20) *follow. Typical examples include*

$$V(x) = V_\infty + \frac{c}{(1+|x|^2)^{\frac{\alpha}{2}}} \quad \textit{with } \alpha > 2s \textit{ and } |c| \textit{ small.}$$

Assumption (V3) *is a virial-type strict inequality ensuring that the potential part does not destroy the Pohozaev geometry; it is satisfied, for instance, by radial potentials for which* $r \mapsto V(r)$ *is nonincreasing and* $sV(r) + \frac{r}{2}V'(r) < V_\infty$ *for all* $r > 0$.

In assumption (V0), when $V_\infty \neq 0$, we may perform the change of variables $(V, \lambda) \mapsto (V - V_\infty, \lambda - V_\infty)$. Therefore, up to this transformation, we may assume that $V_\infty = 0$.

Under the assumption $V_\infty = 0$, Equation (1) can be viewed as a perturbation of the problem studied in [21,23], whose associated energy functional is given by

$$E_{\mu,V}(v) = \frac{1}{2} \|v\|^2_{D^s(\mathbb{R}^N)} + \frac{1}{2} \int_{\mathbb{R}^N} V(x)\, v^2 \, \mathrm{d}x - \frac{\mu}{p} \|v\|_p^p - \frac{1}{2_s^*} \|v\|_{2_s^*}^{2_s^*}, \qquad v \in H^s(\mathbb{R}^N).$$

When $p \in [\bar{p}, 2_s^*)$, the normalization of problem (([21], Theorems 1.1–1.3) and ([23], Theorems 1.1 and 1.2)) was obtained via Jeanjean's Mountain-Pass approach [24]. Furthermore, the methods described in ([14] Lemma 2.4) and ([18] Lemma 2.1) are used to overcome the lack of compactness in the nonlocal setting. In this paper, we first aim to preserve the Mountain-Pass geometry induced by V in the energy functional $E_{\mu,V}$ on S_b, and then resolve the compactness deficiency in the critical and supercritical cases of p. Finally, under assumptions (V0)–(V3) on V, we derive the corresponding Pohozaev identity for Equation (1).

If $V \in C^1(\mathbb{R}^N)$, Equations (18) and (20) hold; any solution to Equation (6) will satisfy

$$P_{\mu,V}(v) := s\|v\|^2_{D^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} W(x) v^2 \, \mathrm{d}x - \mu \gamma_{p,s} s \|v\|_p^p - s\|v\|_{2_s^*}^{2_s^*} = 0. \tag{22}$$

Through Equation (22), we now state our main results.

Theorem 3. *Suppose that $N > 2s$, and $V \in C^1(\mathbb{R}^N)$ satisfies assumptions (V0), (V1), and (V3). If $p = \bar{p}$, $\sigma_1 < 1$ and $\sigma_2 < \frac{s}{N}$, then there exists a positive constant $b_0 = b_0(\mu, p, N)$ such that, for every $b \in (0, b_0)$, Equation (1) possesses a positive solution.*

Theorem 4. *Suppose that $N > 2s$, and $V \in C^1(\mathbb{R}^N)$ satisfies (V0), (V1), and (V3). If $p \in (\bar{p}, 2_s^*)$, $\sigma_1 < 1$, and $\sigma_2 < \frac{1}{2} - \frac{1}{p\gamma_{p,s}}$, then there exists a positive constant $b_0 = b_0(\mu, p, N)$ such that, for every $b \in (0, b_0)$, Equation (1) possesses a positive solution.*

Remark 2. *The restriction $b \in (0, b_0)$ in Theorems 3 and 4 is used to keep the Mountain-Pass level strictly below the fractional critical threshold $\frac{s}{N} S_s^{\frac{N}{2s}}$, which is the energy of the Sobolev optimizer. This prevents the appearance of critical bubbles and guarantees the compactness of constrained Palais–Smale sequences. In the L^2-critical case $p = \bar{p}$, the smallness of b is additionally related to the L^2-scaling invariance. The existence theory for large masses and multiplicity/uniqueness questions remain interesting open problems in the presence of a nonconstant potential and critical growth.*

Main Novelties

Compared with the critical fractional problem without potential [21] and with the local normalized problem with potential [17], our analysis addresses simultaneously the nonlocal operator $(-\Delta)^s$, a nonconstant potential $V(x)$, and the presence of the Sobolev-critical term $|v|^{2_s^*-2}v$ under the L^2-constraint. The main new ingredients are the following:

- A full-space Pohozaev identity for Equation (1) with nonconstant $V(x)$ obtained through the Caffarelli–Silvestre extension and a careful treatment of the boundary terms at infinity;
- A Pohozaev manifold on the L^2-sphere tailored to the L^2-critical/-supercritical scaling, which restores the missing compactness and yields bounded constrained Palais–Smale sequences;

- A fractional splitting analysis at the critical level, combined with a sharp energy threshold, which rules out the loss of compactness caused by translation and critical bubbling.

Scope. We stress that our results concern stationary normalized states of Equation (1) on $\mathbb{R}^N$. Topics such as time-dependent stabilization, dispersive dynamics, or energy decay for fractional-wave/Schrödinger models require different tools and are not addressed here; we added a brief discussion and point to the literature (see [25]).

The structure of this paper is organized as follows. In Section 2, we present some preliminaries to facilitate the subsequent proofs. In Section 3, we establish this strict upper bound by constructing a specific path, thereby showing that the Mountain-Pass level of $E_{\mu,V}$ satisfies $m_b < \frac{s}{N} S_s^{N/(2s)}$ (see Lemmas 1 and 9). In Section 4, we utilize an augmented energy functional technique to construct a bounded Palais–Smale sequence whose energy converges to m_b and which asymptotically satisfies the Pohozaev identity. In Section 5, we prove that this bounded Palais–Smale sequence $\{v_n\}$ possesses a strongly convergent subsequence. The limit of this subsequence yields the desired solution, thereby completing the proof of Theorems 3 and 4.

2. Preliminaries

Let S_s be the best fractional Sobolev constant given in Equation (13), and let $C_{N,p,s}$ be the optimal constant in the fractional Gagliardo–Nirenberg inequality Equation (11). We denote by v_h the L^2-preserving scaling defined in Equation (10).

We denote by $H^s(\mathbb{R}^N)$ the fractional Sobolev space endowed with the norm

$$\|v\|_{H^s} = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 \, dx + \int_{\mathbb{R}^N} |v|^2 \, dx\right)^{\frac{1}{2}},$$

and the inner product

$$\langle u, v\rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u \, (-\Delta)^{\frac{s}{2}} v \, dx + \int_{\mathbb{R}^N} uv \, dx.$$

Let $D^s(\mathbb{R}^N)$ be the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the inner product

$$\langle u, v\rangle_{D^s(\mathbb{R}^N)} = C(N,s) \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy,$$

and the corresponding norm

$$\|v\|^2_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 \, dx = C(N,s) \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2s}} \, dx \, dy.$$

For $2 < p < 2_s^*$, the Pohozaev identity associated with the equation

$$(-\Delta)^s v = \lambda v + \mu |v|^{p-2} v + |v|^{2_s^*-2} v, \quad \text{in } \mathbb{R}^N. \tag{23}$$

If $v \in H^s(\mathbb{R}^N)$ is a solution of Equation (23), then, by ([21], Lemma 2.1), it satisfies the Pohozaev identity

$$P_\mu(v) := s\|v\|^2_{D^s} - \mu \gamma_{p,s} s \int_{\mathbb{R}^N} |v|^p \, dx - s \int_{\mathbb{R}^N} |v|^{2_s^*} \, dx = 0. \tag{24}$$

Next, we present some preliminary results. We give the Pohozaev identity for $v \in H^s(\mathbb{R}^N)$, solving

$$(-\Delta)^s v + V(x)v = \lambda v + \mu|v|^{p-2}v + |v|^{2_s^*-2}v, \quad \text{in } \mathbb{R}^N. \tag{25}$$

Let $h > 0$, $v \in S_b$, denoting the energy functional along the fiber generated by v_h; we define

$$E_{\mu,V}(v_h) = \frac{h^{2s}}{2}\|v\|^2_{D^s(\mathbb{R}^N)} + \frac{1}{2}\int_{\mathbb{R}^N} V(h^{-1}x)v^2 \,\mathrm{d}x - \mu h^{sp\gamma_{p,s}}\frac{\|v\|_p^p}{p} - \frac{1}{2_s^*}h^{s2_s^*}\|v\|_{2_s^*}^{2_s^*}. \tag{26}$$

We now prove that solutions of Equation (25) satisfy the Pohozaev identity $P_{\mu,V}(v) = 0$ defined in Equation (22).

Lemma 1. *Assume $N > 2s$, $V \in C^1(\mathbb{R}^N)$, and (V2) holds. Let $W(x) := \frac{1}{2}\, x\cdot\nabla V(x)$. Then, for any $v \in H^s(\mathbb{R}^N)$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} V\Big(\frac{x}{t}\Big)v^2(x)\,\mathrm{d}x\bigg|_{t=1} = -2\int_{\mathbb{R}^N} W(x)\,v^2(x)\,\mathrm{d}x. \tag{27}$$

Proof. First, consider $v \in C_c^\infty(\mathbb{R}^N)$ and define

$$\varphi(t) = \int_{\mathbb{R}^N} V\Big(\frac{x}{t}\Big)\,v^2(x)\,\mathrm{d}x.$$

Making the change of variables $y = x/t$ gives

$$\varphi(t) = t^N\int_{\mathbb{R}^N} V(y)\,v^2(ty)\,\mathrm{d}y.$$

Differentiating,

$$\varphi'(t) = Nt^{N-1}\int_{\mathbb{R}^N} V(y)\,v^2(ty)\,\mathrm{d}y + 2t^N\int_{\mathbb{R}^N} V(y)\,v(ty)\,\nabla v(ty)\cdot y\,\mathrm{d}y.$$

At $t = 1$,

$$\varphi'(1) = N\int_{\mathbb{R}^N} V(x)\,v^2(x)\,\mathrm{d}x + 2\int_{\mathbb{R}^N} V(x)\,v(x)\,\nabla v(x)\cdot x\,\mathrm{d}x. \tag{28}$$

Consider $F(x) = V(x)v^2(x)x$. Since v has compact support, by the divergence theorem,

$$0 = \int_{\mathbb{R}^N}\nabla\cdot F\,\mathrm{d}x = \int_{\mathbb{R}^N}\Big(\langle\nabla V(x), x\rangle v^2(x) + 2V(x)v(x)\nabla v(x)\cdot x + NV(x)v^2(x)\Big)\,\mathrm{d}x,$$

hence,

$$2\int_{\mathbb{R}^N} Vv\nabla v\cdot x\,\mathrm{d}x = -\int_{\mathbb{R}^N}\langle\nabla V, x\rangle v^2\,\mathrm{d}x - N\int_{\mathbb{R}^N} Vv^2\,\mathrm{d}x.$$

Substituting into Equation (28) yields

$$\varphi'(1) = -\int_{\mathbb{R}^N}\langle\nabla V(x), x\rangle v^2(x)\,\mathrm{d}x = -2\int_{\mathbb{R}^N} W(x)v^2(x)\,\mathrm{d}x.$$

For general $v \in H^s(\mathbb{R}^N)$, take $v_n \in C_c^\infty(\mathbb{R}^N)$ with $v_n \to v$ in H^s. Using the form-boundedness assumption Equation (20), $v \mapsto \int_{\mathbb{R}^N} Wv^2(x)\,\mathrm{d}x$ is continuous with respect to $\|\cdot\|_{D^s}$; we pass to the limit and obtain Equation (27). □

Proposition 1. *Assume $N > 2s$ and $V \in C^1(\mathbb{R}^N)$. If $v \in H^s(\mathbb{R}^N)$ solves Equation (25), then v satisfies the Pohozaev identity Equation (22).*

Proof. Assume $N > 2s$ and $V \in C^1(\mathbb{R}^N)$. Let $W(x) := \frac{1}{2}\, x \cdot \nabla V(x)$. Assume that Equations (18) and (20) hold. If $v \in H^s(\mathbb{R}^N)$ solves Equation (25), then v satisfies the Pohozaev identity Equation (22).

Here, $B_r^+ := \{(x,t) \in \mathbb{R}^{N+1}_+ : |(x,t)| < r\}$, $S_r^+ := \partial B_r^+$, and $B'_r := \{x \in \mathbb{R}^N : |x| < r\}$.

We follow the Caffarelli–Silvestre extension approach as in [26]. Let $w = w(x,t)$ be the extension of v, solving

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla w) = 0, & (x,t) \in \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\,\partial_t w(x,t) = \kappa_s\big((\lambda - V(x))v(x) + f(v(x))\big), & x \in \mathbb{R}^N, \end{cases} \tag{29}$$

where $v(x) = w(x,0)$ and $f(v) := \mu|v|^{p-2}v + |v|^{2_s^*-2}v$. Then,

$$(-\Delta)^s v(x) = -\kappa_s \lim_{t\to 0^+} t^{1-2s}\,\partial_t w(x,t).$$

Let $\xi(z) = z \cdot \nabla w(z)$ with $z = (x,t)$. Testing $\operatorname{div}(t^{1-2s}\nabla w) = 0$ against ξ on B_r^+ and integrating by parts (as in [26]) yields an identity of the form

$$-\frac{N-2s}{2}\int_{B_r^+} t^{1-2s}|\nabla w|^2\,dz + \frac{r}{2}\int_{S_r^+} t^{1-2s}|\nabla w|^2\,dS \tag{30}$$

$$= r\int_{S_r^+} t^{1-2s}(\partial_\nu w)^2\,dS - \int_{B'_r} \kappa_s\big((\lambda - V)v + f(v)\big)(x\cdot\nabla v)\,dx \tag{31}$$

$$-\frac{r}{2}\int_{\partial B'_r} \kappa_s\big((\lambda - V)v^2 + 2F(v)\big)\,dS. \tag{32}$$

where $F(v) = \frac{\mu}{p}|v|^p + \frac{1}{2_s^*}|v|^{2_s^*}$.

Using the integrability of $t^{1-2s}|\nabla w|^2$ in $\mathbb{R}^{N+1}_+$ and of $F(v)$ in $\mathbb{R}^N$, one can choose a sequence $r_n \to \infty$ such that the boundary terms on $S_{r_n}^+$ and $\partial B'_{r_n}$ vanish as $n \to \infty$. Passing to the limit $r = r_n \to \infty$ in Equation (30), we obtain the Pohozaev identity

$$\begin{aligned} &\frac{N-2s}{2}\|v\|^2_{D^s} + \frac{1}{2}\int_{\mathbb{R}^N}(x\cdot\nabla V(x))\,v^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} V(x)\,v^2\,dx \\ &= \frac{N\lambda}{2}\|v\|_2^2 + \frac{N\mu}{p}\|v\|_p^p + \frac{N}{2_s^*}\|v\|_{2_s^*}^{2_s^*}. \end{aligned} \tag{33}$$

Finally, combining Equation (33) with Equation (25) tested against v and with Lemma 1, we obtain Equation (22). □

Remark 3. *Two points in the above derivation deserve justification:*

(i) **Local vs. nonlocal.** *The Caffarelli–Silvestre extension provides a* local *realization in $\mathbb{R}^{N+1}_+$ of the* nonlocal *operator $(-\Delta)^s$ acting on the trace $v = w(\cdot,0)$. When $s = \frac{1}{2}$, the weight t^{1-2s} becomes constant and Equation (29) reduces to the harmonic extension $\Delta w = 0$ in $\mathbb{R}^{N+1}_+$; moreover, the Dirichlet-to-Neumann constant satisfies $\kappa_{1/2} = 2^{1-2(1/2)}\Gamma(1-1/2)/\Gamma(1/2) = 1$. This simplification affects only weights/constants and does not change the argument, which works for all $s \in (0,1)$.*

(ii) **Vanishing of boundary terms.** *Set*

$$A(r) := \int_{S_r^+} t^{1-2s}|\nabla w|^2\,dS, \qquad B(r) := \int_{\partial B'_r}\big(|V|v^2 + |F(v)|\big)\,dS.$$

Since $t^{1-2s}|\nabla w|^2 \in L^1(\mathbb{R}^{N+1}_+)$, the co-area formula yields

$$\int_0^\infty A(r)\,dr = \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla w|^2\,dz < \infty.$$

Likewise, $F(v) \in L^1(\mathbb{R}^N)$ implies

$$\int_0^\infty \int_{\partial B_r'} |F(v)| \, \mathrm{d}S \, \mathrm{d}r = \int_{\mathbb{R}^N} |F(v(x))| \, \mathrm{d}x < \infty,$$

and the same holds for $\int_{\partial B_r'} |V|v^2 \, \mathrm{d}x$ under the assumptions of the proposition. Therefore, there exists a sequence $r_n \to \infty$ such that $A(r_n) + B(r_n) \to 0$, which justifies passing to the limit $r = r_n \to \infty$ in Equation (30) (cf. [26]).

The following result is a direct consequence of ([21], Theorems 1.2 and 1.3 and Lemma 9.2); it collects monotonicity and variational characterizations of m_b.

Lemma 2. *Assume that $N > 2s$ and $\bar{p} \le p < 2_s^*$. Then, there exists a constant $\alpha_{N,p,s} > 0$ such that m_b is strictly decreasing for $0 < \mu < b^{p\gamma_{p,s}-p}\alpha_{N,p,s}$ and is nonincreasing for $\mu \ge b^{p\gamma_{p,s}-p}\alpha_{N,p,s}$. Moreover,*

$$0 < m_b < \frac{s}{N} S_s^{\frac{N}{2s}} \quad \text{whenever } \bar{p} < p < 2_s^*, \qquad \text{and} \qquad m_b = 0 \text{ for } \mu \ge b^{p\gamma_{p,s}-p}\alpha_{N,p,s} \text{ if } p = \bar{p}.$$

In particular, when $p = \bar{p}$, one can take

$$\alpha_{N,\bar{p},s} = \frac{\bar{p}}{2\big(C_{N,\bar{p},s}\big)^{\bar{p}}}.$$

Finally, if either $\bar{p} < p < 2_s^*$ or $p = \bar{p}$ and $0 < \mu b^{\frac{4s}{N}} < \dfrac{\bar{p}}{2\big(C_{N,\bar{p},s}\big)^{\bar{p}}}$, then $E_\mu|_{S_b}$ has a Mountain-Pass geometry and, for any $v \in S_b$, the fiber map $h \mapsto E_\mu(v_h)$ admits a unique maximizer on $\mathbb{R}_+$; consequently,

$$m_b = \inf_{v \in S_b} \max_{h \in \mathbb{R}^+} E_\mu(v_h). \tag{34}$$

Lemma 3. *Assume that $N > 2s$ and $\bar{p} \le p < 2_s^*$. Under the assumptions of Lemma 2, the map $b \mapsto m_b$ is nonincreasing on $(0, \infty)$.*

Proof. This is proven in ([21], Lemma 9.2); we omit the details. □

Next, we state the following lemma concerning Equation (24).

Lemma 4. *Suppose that $N > 2s$ and $\bar{p} \le p < 2_s^*$, and let $\{v_n\} \subset S_b$ be a bounded sequence in $H^s(\mathbb{R}^N)$ such that $P_\mu(v_n) \to 0$. Assume that there exists $c > 0$ such that $\|v_n\|^2_{D^s(\mathbb{R}^N)} \ge c$ for all n. Then,*

$$m_b \le \liminf_{n \to \infty} E_\mu(v_n). \tag{35}$$

Proof. For clarity, we denote the Pohozaev manifold on the sphere by

$$\mathcal{P}_\mu := \{v \in S_b : \; P_\mu(v) = 0\},$$

which coincides with the constraint set used in Equation (16). Since $\bar{p} \le p < 2_s^*$, by Lemma 2 (fiber uniqueness), for each n, there exists a unique $h_n > 0$ such that $(v_n)_{h_n} \in \mathcal{P}_\mu$, that is,

$$P_\mu\big((v_n)_{h_n}\big) = 0. \tag{36}$$

On the other hand, $P_\mu(v_n) = o_n(1)$ reads

$$s\|v_n\|^2_{D^s} - \mu\gamma_{p,s}s\|v_n\|_p^p - s\|v_n\|_{2_s^*}^{2_s^*} = o_n(1). \tag{37}$$

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Subtracting Equation (36) from Equation (37) and using the scaling laws in Equation (10), we obtain

$$\mu\gamma_{p,s}\big(h_n^{ps\gamma_{p,s}}-1\big)\|v_n\|_p^p+\big(h_n^{s2_s^*}-1\big)\|v_n\|_{2_s^*}^{2_s^*}=\big(h_n^{2s}-1\big)\|v_n\|_{D^s}^2+o_n(1).$$

Since $\|v_n\|_{D^s}^2\ge c>0$ and $\{v_n\}$ is bounded in H^s, the above identity forces $h_n\to 1$ (otherwise the left-hand side would stay separated from 0 while the right-hand side cannot, due to the uniform bounds and $P_\mu(v_n)\to 0$). Consequently, by continuity of $E_\mu(v_h)$ with respect to h on bounded sets,

$$E_\mu(v_n)=E_\mu\big((v_n)_{h_n}\big)+o_n(1).$$

Finally, since $(v_n)_{h_n}\in\mathcal{P}_\mu$, by the definition of m_b, we have

$$E_\mu\big((v_n)_{h_n}\big)\ge m_b,$$

hence, $E_\mu(v_n)\ge m_b+o_n(1)$, and Equation (35) follows. □

Lemma 5. *Suppose that $N>2s$ and $\bar{p}\le p<2_s^*$, and $V\in C^1(\mathbb{R}^N)$ satisfies Equation (18), Equation (20) and (V3). If $v\in H^s(\mathbb{R}^N)\setminus\{0\}$ solves Equation (25), then the associated Lagrange multiplier satisfies $\lambda<0$.*

Proof. Assume that v is a nontrivial solution to Equation (25). Multiplying Equation (25) by v and integrating over $\mathbb{R}^N$, and then multiplying the resulting identity by s, we obtain

$$s\|v\|_{D^s(\mathbb{R}^N)}^2+s\int_{\mathbb{R}^N}V(x)v^2\,\mathrm{d}x=s\lambda\|v\|_2^2+s\mu\|v\|_p^p+s\|v\|_{2_s^*}^{2_s^*}. \tag{38}$$

By Proposition 1, the Pohozaev identity Equation (22) holds. Combining Equation (38) with Equation (22), we have

$$\int_{\mathbb{R}^N}(sV(x)+W(x))|v|^2\,\mathrm{d}x=s\mu(1-\gamma_{p,s})\|v\|_p^p+s\lambda\|v\|_2^2.$$

Since $p\ge\bar{p}$ implies $1-\gamma_{p,s}\ge 0$, the first term on the right-hand side is nonnegative (and it is strictly positive when $p>\bar{p}$). On the other hand, (V3) gives $sV(x)+W(x)<V_\infty$ for all x, and after the reduction $V_\infty=0$ we have $sV+W<0$. Therefore, the left-hand side is strictly negative, which forces $s\lambda\|v\|_2^2<0$ and hence $\lambda<0$. □

We prove that, under appropriate conditions, the energy functional $E_{\mu,V}$ is nonnegative for any solution to Equation (25).

Lemma 6. *Suppose that $N>2s$, $\bar{p}\le p<2_s^*$, $s\in(0,1)$, and $V\in C^1(\mathbb{R}^N)$ satisfies* (V0)–(V1). *Assume one of the following:*

(i) $\bar{p}<p<2_s^*$ *and*

$$\sigma_2\le\frac{1}{2}-\frac{1}{p\gamma_{p,s}}; \tag{39}$$

(ii) $p=\bar{p}$ *and*

$$\sigma_2\le\frac{s}{N}\Big(1-\frac{2\mu}{\bar{p}}\,C_{N,\bar{p},s}^{\bar{p}}\,b^{\frac{4s}{N}}\Big). \tag{40}$$

If v is a solution of Equation (25), then $E_{\mu,V}(v)\ge 0$.

Proof. Let v solve Equation (25). By Proposition 1, v satisfies the Pohozaev identity Equation (22).

Case (i): $\bar{p} < p < 2_s^*$. Using Equation (22) to eliminate the L^p–term in $E_{\mu,V}(v)$, we obtain

$$\begin{aligned}
E_{\mu,V}(v) &= \frac{1}{2}\|v\|_{D^s}^2 + \frac{1}{2}\int_{\mathbb{R}^N} Vv^2\,\mathrm{d}x - \frac{\mu}{p}\|v\|_p^p - \frac{1}{2_s^*}\|v\|_{2_s^*}^{2_s^*} \\
&= \frac{1}{2}\|v\|_{D^s}^2 + \frac{1}{2}\int_{\mathbb{R}^N} Vv^2\,\mathrm{d}x - \frac{1}{p\gamma_{p,s}s}\Big(s\|v\|_{D^s}^2 - \int_{\mathbb{R}^N} Wv^2\,\mathrm{d}x - s\|v\|_{2_s^*}^{2_s^*}\Big) - \frac{1}{2_s^*}\|v\|_{2_s^*}^{2_s^*} \\
&= \Big(\frac{1}{2} - \frac{1}{p\gamma_{p,s}}\Big)\|v\|_{D^s}^2 + \frac{1}{2}\int_{\mathbb{R}^N} Vv^2\,\mathrm{d}x + \frac{1}{p\gamma_{p,s}s}\int_{\mathbb{R}^N} Wv^2\,\mathrm{d}x + \Big(\frac{1}{p\gamma_{p,s}} - \frac{1}{2_s^*}\Big)\|v\|_{2_s^*}^{2_s^*}.
\end{aligned}$$

Since $\bar{p} < p < 2_s^*$, we have $\frac{1}{p\gamma_{p,s}} > \frac{1}{2_s^*}$, hence the last term is nonnegative and can be dropped in a lower bound. By (V1) (after the reduction $V_\infty = 0$),

$$\frac{1}{2}\int_{\mathbb{R}^N} Vv^2\,\mathrm{d}x + \frac{1}{p\gamma_{p,s}s}\int_{\mathbb{R}^N} Wv^2\,\mathrm{d}x \geq -\sigma_2\|v\|_{D^s}^2,$$

therefore, using Equation (39), we have

$$E_{\mu,V}(v) \geq \Big(\frac{1}{2} - \frac{1}{p\gamma_{p,s}} - \sigma_2\Big)\|v\|_{D^s}^2 \geq 0.$$

Case (ii): $p = \bar{p}$. Starting from the identity

$$E_{\mu,V}(v) = \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)\|v\|_{D^s}^2 + \frac{1}{2}\int_{\mathbb{R}^N} Vv^2\,\mathrm{d}x + \frac{1}{2_s^* s}\int_{\mathbb{R}^N} Wv^2\,\mathrm{d}x - \mu\Big(\frac{1}{p} - \frac{\gamma_{p,s}}{2_s^*}\Big)\|v\|_p^p,$$

we use that $p = \bar{p}$ implies $p\gamma_{p,s} = 2$ and hence $\|v\|_p^p \leq C_{N,\bar{p},s}^{\bar{p}}\|v\|_{D^s}^2\, b^{\frac{4s}{N}}$ by Equation (11). Moreover, $\frac{1}{2} - \frac{1}{2_s^*} = \frac{s}{N}$ and, by (V1), we also have

$$\frac{1}{2}\int_{\mathbb{R}^N} Vv^2\,\mathrm{d}x + \frac{1}{2_s^* s}\int_{\mathbb{R}^N} Wv^2\,\mathrm{d}x \geq -\sigma_2\|v\|_{D^s}^2.$$

Combining these inequalities and Equation (40) yields $E_{\mu,V}(v) \geq 0$. □

To construct Palais–Smale sequences at a minimax level, we use the following deformation lemma (see ([27], Theorem 4.5)).

Proposition 2. *Let X denote a Hilbert manifold and $E \in C^1(X,\mathbb{R})$. Suppose that $K \subset X$ is compact, and let*

$$\mathcal{F} \subset \{D \subset X : D \text{ is compact},\ K \subset D\},$$

be a family invariant under deformations, leaving K fixed. Assume that

$$\max_{v\in K} E(v) < c := \inf_{D\in\mathcal{F}} \max_{v\in D} E(v) \in \mathbb{R}.$$

Let $\sigma_n \in \mathbb{R}$ with $\sigma_n \to 0$ and $D_n \in \mathcal{F}$ be a sequence such that

$$c \leq \max_{v\in D_n} E(v) < c + \sigma_n.$$

Then, there exists a sequence $\{v_n\} \subset X$ and a constant $M' > 0$ such that the following hold:

(i) $c \leq E(v_n) < c + \sigma_n$;
(ii) $\|\nabla_X E(v_n)\| < M'\sqrt{\sigma_n}$;
(iii) $\mathrm{dist}(v_n, D_n) < M'\sqrt{\sigma_n}$.

3. Mountain-Pass Geometry

This section focuses on the construction of the Mountain-Pass geometry associated with $E_{\mu,V}$ over S_b.

Lemma 7. *Suppose that $N > 2s$, $\bar{p} \le p < 2_s^*$, and (V0) holds. Then, for all v belonging to S_b,*

$$\lim_{h\to 0^+} E_{\mu,V}(v_h) = 0 \quad \textit{and} \quad \lim_{h\to\infty} E_{\mu,V}(v_h) = -\infty.$$

Proof. Let $v \in S_b$; we have

$$E_{\mu,V}(v_h) \le \frac{h^{2s}}{2}\|v\|^2_{D^s(\mathbb{R}^N)} - \frac{\mu}{p}\, h^{sp\gamma_{p,s}}\|v\|_p^p - \frac{1}{2_s^*}\, h^{s2_s^*}\|v\|_{2_s^*}^{2_s^*}.$$

For $\bar{p} \le p < 2_s^*$, we can obtain that

$$\lim_{h\to\infty} E_{\mu,V}(v_h) = -\infty.$$

On the other hand, writing $V = V_\infty + (V - V_\infty)$ and applying (V0) to v_h, we have

$$\Big|\int_{\mathbb{R}^N} (V(x) - V_\infty)\, v_h^2 \, \mathrm{d}x\Big| \le \sigma_1\|v_h\|^2_{D^s(\mathbb{R}^N)} = \sigma_1 h^{2s}\|v\|^2_{D^s(\mathbb{R}^N)}.$$

Therefore,

$$\int_{\mathbb{R}^N} V(x)\, v_h^2 \, \mathrm{d}x = V_\infty\|v_h\|_2^2 + O(h^{2s}) = V_\infty b^2 + O(h^{2s}),$$

and hence

$$\big|E_{\mu,V}(v_h)\big| \le \frac{h^{2s}}{2}(1+\sigma_1)\|v\|^2_{D^s(\mathbb{R}^N)} + \frac{V_\infty}{2}b^2 + \mu h^{p\gamma_{p,s}s}\frac{\|v\|_p^p}{p} + h^{s2_s^*}\frac{\|v\|_{2_s^*}^{2_s^*}}{2_s^*}.$$

In particular, after the harmless shift discussed in the Introduction (so that $V_\infty = 0$), we conclude that

$$\lim_{h\to 0^+} E_{\mu,V}(v_h) = 0.$$

□

Now, we consider, for $k > 0$,

$$B_k := \{v \in S_b : \|v\|_{D^s(\mathbb{R}^N)} < k\}, \qquad \partial B_k := \{v \in S_b : \|v\|_{D^s(\mathbb{R}^N)} = k\}.$$

A key observation is that, under (V0), $E_{\mu,V}$ is bounded below by a positive number on ∂B_k for sufficiently small $k > 0$.

Lemma 8. *Suppose that $N > 2s$, $\bar{p} \le p < 2_s^*$, and* (V0) *holds with*

$$\sigma_1 < 1 \quad \textit{if } \bar{p} < p < 2_s^*, \qquad \textit{or} \qquad \sigma_1 < 1 - \frac{2\mu}{\bar{p}}\big(C_{N,\bar{p},s}\big)^{\bar{p}} b^{\frac{4s}{N}} \quad \textit{if } p = \bar{p}. \tag{41}$$

Then, there exist constants $k_0 > 0$ and $\beta > 0$ such that

$$E_{\mu,V}(v) \ge \beta \qquad \textit{for all } v \in \partial B_{k_0}.$$

Proof. After the reduction $V_\infty = 0$ (hence $\sup_{\mathbb{R}^N} V = 0$), by Equation (18), we have $V \le 0$ and

$$\int_{\mathbb{R}^N} V(x)v^2 \, \mathrm{d}x \ge -\Big|\int_{\mathbb{R}^N} V(x)v^2 \, \mathrm{d}x\Big| \ge -\sigma_1\|v\|^2_{D^s}.$$

Moreover, by the fractional Gagliardo–Nirenberg inequality in Equation (11) and the Sobolev inequality in Equation (12), for any $v \in S_b$, we have

$$\|v\|_p^p \leq (C_{N,p,s})^p \, b^{p(1-\gamma_{p,s})} \, \|v\|_{D^s}^{p\gamma_{p,s}}, \qquad \|v\|_{2_s^*}^{2_s^*} \leq S_s^{-\frac{2_s^*}{2}} \, \|v\|_{D^s}^{2_s^*}.$$

Therefore, setting $t := \|v\|_{D^s}$, we obtain

$$E_{\mu,V}(v) \geq \frac{1-\sigma_1}{2}\, t^2 - \frac{\mu}{p}(C_{N,p,s})^p \, b^{p(1-\gamma_{p,s})} \, t^{p\gamma_{p,s}} - \frac{1}{2_s^*} S_s^{-\frac{2_s^*}{2}} \, t^{2_s^*}.$$

If $\bar{p} < p < 2_s^*$, then $p\gamma_{p,s} > 2$ and $2_s^* > 2$. Hence, the right-hand side is positive for $t > 0$ small enough. Choose $k_0 > 0$ so that

$$\frac{\mu}{p}(C_{N,p,s})^p b^{p(1-\gamma_{p,s})} \, k_0^{p\gamma_{p,s}-2} + \frac{1}{2_s^*} S_s^{-\frac{2_s^*}{2}} \, k_0^{2_s^*-2} \leq \frac{1-\sigma_1}{4},$$

and set $\beta := \frac{1-\sigma_1}{4} k_0^2 > 0$. Then, $E_{\mu,V}(v) \geq \beta$ on ∂B_{k_0}.

If $p = \bar{p}$, then $p\gamma_{p,s} = 2$ and the L^p-term is also of order t^2. Using $\|v\|_{\bar{p}}^{\bar{p}} \leq (C_{N,\bar{p},s})^{\bar{p}} b^{\frac{4s}{N}} t^2$, we get

$$E_{\mu,V}(v) \geq \left(\frac{1-\sigma_1}{2} - \frac{\mu}{\bar{p}}(C_{N,\bar{p},s})^{\bar{p}} b^{\frac{4s}{N}}\right) t^2 - \frac{1}{2_s^*} S_s^{-\frac{2_s^*}{2}} \, t^{2_s^*}.$$

The assumption in Equation (41) ensures that the coefficient of t^2 is positive; choosing $k_0 > 0$ small yields again $E_{\mu,V} \geq \beta > 0$ on ∂B_{k_0}. $\square$

It follows from Lemmas 7 and 8 that Equation (41) guarantees the Mountain-Pass geometry of $E_{\mu,V}$ on S_b. Let $v^b \in S_b$ be a positive ground state of the zero-potential problem in Equation (15), so that

$$E_\mu(v^b) = m_b. \tag{42}$$

By Lemma 7, we have

$$\lim_{h\to 0^+} E_{\mu,V}(v_h^b) = 0 \quad \text{and} \quad \lim_{h\to\infty} E_{\mu,V}(v_h^b) = -\infty.$$

Hence, there exist $h_1 > h_0 > 0$ such that

$$e_0 := v_{h_0}^b \in B_{k_0}, \qquad e_1 := v_{h_1}^b \in S_b \setminus B_{k_0}, \qquad E_{\mu,V}(e_0) < \beta, \qquad E_{\mu,V}(e_1) < 0,$$

where k_0 and β are as in Lemma 8. Define the class of Mountain-Pass paths

$$\Gamma := \{\, \xi \in C([0,1], S_b) \mid \xi(0) = e_0,\ \xi(1) = e_1 \,\},$$

and the Mountain-Pass level

$$m_{V,b} := \inf_{\xi\in\Gamma} \max_{t\in[0,1]} E_{\mu,V}(\xi(t)).$$

It is straightforward that

$$m_{V,b} \geq \beta > 0. \tag{43}$$

Lemma 9. *Suppose that $N > 2s$, $\bar{p} \leq p < 2_s^*$, and the assumption (V0) is satisfied by Equation (41). If $V \not\equiv 0$, then $m_b > m_{V,b}$.*

Proof. By Equation (42) and the fiber map Equation (34), we have

$$m_b = \max_{h>0} E_\mu\big((v^b)_h\big).$$

Moreover, after the reduction $V_\infty = 0$, we have $V \le 0$ and $V \not\equiv 0$; hence, $V < 0$ on a set of positive measure. Since $v^b > 0$ in $\mathbb{R}^N$, it follows that, for every $h > 0$,

$$\int_{\mathbb{R}^N} V(h^{-1}x)\,(v^b)^2(x)\;\mathrm{d}x < 0,$$

and therefore

$$E_{\mu,V}\big((v^b)_h\big) = E_\mu\big((v^b)_h\big) + \frac{1}{2}\int_{\mathbb{R}^N} V(h^{-1}x)\,(v^b)^2(x)\;\mathrm{d}x < E_\mu\big((v^b)_h\big).$$

Taking the maximum over $h > 0$ and using the definition of $m_{V,b}$, we obtain

$$m_{V,b} \le \max_{h>0} E_{\mu,V}\big((v^b)_h\big) < \max_{h>0} E_\mu\big((v^b)_h\big) = m_b.$$

□

To derive a bounded Palais–Smale sequence for $E_{\mu,V}$ at the Mountain-Pass level $m_{V,b}$, we adopt the methodological framework from [24] to construct an augmented functional. This functional is defined as follows:

$$\tilde{E}_{\mu,V}(v,h) := E_{\mu,V}(v_h), \quad (v,h) \in H^s(\mathbb{R}^N) \times \mathbb{R}^+, \tag{44}$$

where v_h is the function specified in Equation (10). In fact, this augmented functional $\tilde{E}_{\mu,V}$ also exhibits a Mountain-Pass structure on the product space $S_b \times \mathbb{R}^+$. For the space $S_b \times \mathbb{R}^+$, we define the set of new Mountain-Pass paths as

$$\tilde{\Gamma} := \big\{\, \xi \in C([0,1], S_b \times \mathbb{R}^+) \,\big|\, \xi(0) = (e_0, 1),\, \xi(1) = (e_1, 1) \big\}.$$

Any path $\tilde{\xi} \in \tilde{\Gamma}$ can be expressed in the form:

$$\tilde{\xi}(t) = (\xi(t), l(t)),$$

where $\xi \in \Gamma$ (consistent with the earlier definition of Γ), $l \in C([0,1], \mathbb{R}^+)$, and the boundary condition $l(0) = l(1) = 1$ is satisfied. Notably, every such path ξ must intersect the set

$$\partial\tilde{B}_{k_0} := \{\, (v,h) \in S_b \times \mathbb{R}^+ \mid v_h \in \partial B_{k_0} \,\}.$$

Furthermore, it holds that $E_{\mu,V}(v) \ge \beta$ for all $v \in \partial B_{k_0}$. Corresponding to the new path set $\tilde{\Gamma}$, the updated Mountain-Pass level is

$$\tilde{m}_{V,b} := \inf_{\xi \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{E}_{\mu,V}(\xi(t)).$$

Lemma 10. *Suppose that $N > 2s$, $\bar{p} \le p < 2_s^*$, and (V0) holds in conjunction with Equation (41); then, $m_{V,b} = \tilde{m}_{V,b}$.*

Proof. For any $\xi \in \Gamma$, by the definition of $\tilde{\Gamma}$, we have $(\xi, 1) \in \tilde{\Gamma}$ and, for each $t \in (0,1)$,

$$\tilde{m}_{V,b} \le \inf_{\xi \in \Gamma} \max_t \tilde{E}_{\mu,V}(\xi(t), 1) = \inf_{\xi \in \Gamma} \max_t E_{\mu,V}(\xi(t)) = m_{V,b}.$$

Conversely, take any $\tilde{\xi} = (\xi, l) \in \tilde{\Gamma}$, where $\xi \in \Gamma$, $l \in C([0,1], \mathbb{R}^+)$, and $l(0) = l(1) = 1$. Let $\xi_l(t) := \xi(t)_{l(t)}$ for $t \in [0,1]$. Then, $\xi_l \in \Gamma$, and we obtain

$$m_{V,b} \leq \max_t E_{\mu,V}(\xi_l(t)) = \max_t \tilde{E}_{\mu,V}(\tilde{\xi}(t)).$$

Since $\tilde{\xi}$ is arbitrary in $\tilde{\Gamma}$, taking the infimum over all $\tilde{\xi}$ yields

$$m_{V,b} \leq \inf_{\tilde{\xi} \in \tilde{\Gamma}} \max_t \tilde{E}_{\mu,V}(\tilde{\xi}(t)) = \tilde{m}_{V,b}.$$

From $\tilde{m}_{V,b} \leq m_{V,b}$ and $m_{V,b} \leq \tilde{m}_{V,b}$, we have

$$m_{V,b} = \tilde{m}_{V,b}.$$

□

4. Palais–Smale Sequence

In this section, we construct a bounded Palais–Smale sequence $\{v_n\} \subset S_b$ at the Mountain-Pass level $m_{V,b}$, which asymptotically satisfies the Pohozaev identity $P_{\mu,V}(v_n) = o_n(1)$. Recall that $\{v_n\} \subset S_b$ is a Palais–Smale sequence for $E_{\mu,V}|_{S_b}$ at level c if

$$E_{\mu,V}(v_n) = c + o_n(1) \quad \text{and} \quad \|(E_{\mu,V}|_{S_b})'(v_n)\|_{(T_{v_n} S_b)^*} = o_n(1),$$

where

$$T_{v_n} S_b := \Big\{ w \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} v_n w \, \mathrm{d}x = 0 \Big\}.$$

With the help of Proposition 2, we obtain the following lemmas.

Lemma 11. *Suppose $N > 2s$, $\bar{p} \leq p < 2_s^*$, and $V \in C^1(\mathbb{R}^N)$ satisfies* (V0), (V1), ***and*** (V2). *Then, there exists a Palais–Smale sequence $\{v_n\} \subset S_b$ for $E_{\mu,V}|_{S_b}$ at the level $m_{V,b}$ such that*

$$P_{\mu,V}(v_n) \to 0 \quad \text{and} \quad \|v_n^-\|_2 \to 0 \qquad \text{as } n \to \infty. \tag{45}$$

Proof. By the definition of $m_{V,b}$ and Lemma 10, there exists a minimizing sequence $\{\xi_n\} \subset \Gamma$ such that

$$m_{V,b} \leq \max_{t \in [0,1]} E_{\mu,V}(\xi_n(t)) < m_{V,b} + \frac{1}{n}.$$

Since $E_{\mu,V}(|v|) \leq E_{\mu,V}(v)$ for all $v \in S_b$, we may replace ξ_n by $|\xi_n|$ and assume $\xi_n(t) \geq 0$ for all $t \in [0,1]$.

Consider the augmented functional $\tilde{E}_{\mu,V} : S_b \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\tilde{E}_{\mu,V}(v, h) := E_{\mu,V}(v_h),$$

and apply Proposition 2 with

$$X = S_b \times \mathbb{R}^+, \qquad K = \{(e_0, 1), (e_1, 1)\}, \qquad \mathcal{F} = \{\mathrm{Im}(\tilde{\xi}) : \tilde{\xi} \in \tilde{\Gamma}\},$$

$$D_n = \{(\xi_n(t), 1) : t \in [0,1]\}, \qquad \sigma_n = \frac{1}{n}.$$

Then, there exist $\{(u_n, h_n)\} \subset S_b \times \mathbb{R}^+$ and a constant $c > 0$ such that

$$\begin{cases} m_{V,b} \le \tilde{E}_{\mu,V}(u_n, h_n) < m_{V,b} + \frac{1}{n}, \\ \|d(\tilde{E}_{\mu,V}|_{S_b \times \mathbb{R}^+})(u_n, h_n)\|_{(T_{(u_n,h_n)}(S_b \times \mathbb{R}^+))^*} < c\, n^{-1/2}, \\ \mathrm{dist}_{H^s \times \mathbb{R}}\big((u_n, h_n), D_n\big) < c\, n^{-1/2}. \end{cases} \tag{46}$$

In particular, from the last line, we have $h_n \to 1$ and $\mathrm{dist}_{H^s}(u_n, \xi_n([0,1])) \to 0$; hence, $\|u_n^-\|_2 \to 0$ since $\xi_n(t) \ge 0$.

Next, by Lemma 1 and the definition of $P_{\mu,V}$ in Equation (22), the scaling derivative satisfies

$$\frac{\partial}{\partial h} \tilde{E}_{\mu,V}(u, h) = \frac{1}{h} P_{\mu,V}(u_h), \qquad (u, h) \in S_b \times \mathbb{R}^+.$$

Therefore, the second line of Equation (46) yields

$$\left| \frac{\partial \tilde{E}_{\mu,V}}{\partial h}(u_n, h_n) \right| = \frac{1}{h_n} \left| P_{\mu,V}\big((u_n)_{h_n}\big) \right| \xrightarrow[n \to \infty]{} 0. \tag{47}$$

Finally, set

$$v_n := (u_n)_{h_n} \in S_b.$$

Then $E_{\mu,V}(v_n) = \tilde{E}_{\mu,V}(u_n, h_n) = m_{V,b} + o_n(1)$ by Equation (46), and $P_{\mu,V}(v_n) \to 0$ by Equation (47) and $h_n \to 1$. Moreover, since $u_n \to v_n$ in H^s (because $h_n \to 1$ and the scaling map is continuous on H^s), the smallness of the constrained differential in Equation (46) implies

$$\|(E_{\mu,V}|_{S_b})'(v_n)\|_{(T_{v_n} S_b)^*} = o_n(1),$$

so $\{v_n\}$ is a Palais–Smale sequence for $E_{\mu,V}|_{S_b}$ at level $m_{V,b}$. Finally, $\|v_n^-\|_2 \to 0$ follows from $\|u_n^-\|_2 \to 0$ and the fact that the scaling $u \mapsto u_h$ preserves the sign decomposition. □

By (V0)–(V1), we now prove that the above Palais–Smale sequence is bounded in $H^s(\mathbb{R}^N)$.

Lemma 12. *Let $N > 2s$ and $\bar{p} \le p < 2_s^*$. Suppose that $V \in C^1(\mathbb{R}^N)$ and that* (V0)–(V1) *hold with σ_1 satisfying Equation* (41)*, and with σ_2 satisfying the strict version of Equation* (39) *(if $\bar{p} < p < 2_s^*$) or of Equation* (40) *(if $p = \bar{p}$). In addition, if $p = \bar{p}$, we assume that*

$$\left| \int_{\mathbb{R}^N} \Big(\tfrac{1}{2}(V(x) - V_\infty) + \tfrac{1}{2_s^* s} W(x) \Big) v^2 \, \mathrm{d}x \right| \le \sigma_2 \|v\|_{D^s}^2, \qquad \forall\, v \in H^s(\mathbb{R}^N).$$

Let $\{v_n\} \subset S_b$ be a Palais–Smale sequence for $E_{\mu,V}$ at the level $m_{V,b}$ such that $P_{\mu,V}(v_n) \to 0$. Then, $\{v_n\}$ is bounded in $H^s(\mathbb{R}^N)$.

Proof. Since $\|v_n\|_2 = b$ for all n, it suffices to prove that $\|v_n\|_{D^s}$ is bounded. For convenience, set

$$a_n := \|v_n\|_{D^s(\mathbb{R}^N)}^2, \qquad b_n := \int_{\mathbb{R}^N} V(x)\, v_n^2 \, \mathrm{d}x, \qquad c_n := \|v_n\|_p^p,$$

$$d_n := \|v_n\|_{2_s^*}^{2_s^*}, \qquad e_n := \int_{\mathbb{R}^N} W(x)\, v_n^2 \, \mathrm{d}x.$$

From $E_{\mu,V}(v_n) = m_{V,b} + o_n(1)$, we have

$$a_n + b_n - \frac{2\mu}{p} c_n - \frac{2}{2_s^*} d_n = 2m_{V,b} + o_n(1), \tag{48}$$

and, from $P_{\mu,V}(v_n) = o_n(1)$—namely, Equation (22)—we obtain

$$sa_n - e_n - s\mu\gamma_{p,s}c_n - sd_n = o_n(1). \tag{49}$$

Case 1: $\bar{p} < p < 2_s^*$. Since $p \le 2_s^*$ implies $p\gamma_{p,s} \le 2_s^*$, we have $\frac{1}{2_s^*} \le \frac{1}{p\gamma_{p,s}}$; hence,

$$\frac{2\mu}{p} c_n + \frac{2}{2_s^*} d_n \le \frac{2}{p\gamma_{p,s}s} (s\mu\gamma_{p,s}c_n + sd_n) = \frac{2}{p\gamma_{p,s}s} (sa_n - e_n) + o_n(1), \tag{50}$$

where we use Equation (49) in the last equality. Combining Equation (48) with Equation (50) gives

$$2m_{V,b} + o_n(1) \ge \left(1 - \frac{2}{p\gamma_{p,s}}\right) a_n + b_n + \frac{2}{sp\gamma_{p,s}} e_n.$$

By (V0) with $V_\infty = 0$, we have $|b_n| \le \sigma_1 a_n$, and, by (V1) (equivalently Equation (19) with $V_\infty = 0$), we have

$$\left| b_n + \frac{2}{sp\gamma_{p,s}} e_n \right| \le 2\sigma_2 a_n.$$

Therefore,

$$2m_{V,b} + o_n(1) \ge \left(1 - \frac{2}{p\gamma_{p,s}} - 2\sigma_2\right) a_n.$$

Under the strict assumption $\sigma_2 < \frac{1}{2} - \frac{1}{p\gamma_{p,s}}$, the coefficient is positive; hence, $\{a_n\}$ is bounded.

Case 2: $p = \bar{p}$. From Equation (49), we have

$$sd_n = sa_n - e_n - s\mu\gamma_{\bar{p},s}c_n + o_n(1). \tag{51}$$

Substituting Equation (51) into $s\times$ Equation (48) yields

$$2sm_{V,b} + o_n(1) = \left(1 - \frac{2}{2_s^*}\right)(sa_n - s\mu\gamma_{\bar{p},s}c_n) + sb_n + \frac{2}{2_s^*} e_n + o_n(1).$$

Using the fractional Gagliardo–Nirenberg inequality in Equation (11) with $p = \bar{p}$ and $\|v_n\|_2 = b$, we obtain

$$c_n = \|v_n\|_{\bar{p}}^{\bar{p}} \le C_{N,\bar{p},s}^{\bar{p}} \, b^{\frac{4s}{N}} \, a_n, \qquad \text{and} \qquad \gamma_{\bar{p},s} = \frac{2}{\bar{p}}.$$

Hence,

$$sa_n - s\mu\gamma_{\bar{p},s}c_n \ge s\left(1 - \frac{2\mu}{\bar{p}} C_{N,\bar{p},s}^{\bar{p}} b^{\frac{4s}{N}}\right) a_n.$$

Moreover, by the additional assumption in the critical case (with $V_\infty = 0$),

$$\left| \frac{s}{2} b_n + \frac{1}{2_s^*} e_n \right| \le s\sigma_2 a_n, \qquad \text{hence} \qquad \left| sb_n + \frac{2}{2_s^*} e_n \right| \le 2s\sigma_2 a_n.$$

Therefore,

$$2sm_{V,b} + o_n(1) \ge \left[\left(1 - \frac{2}{2_s^*}\right) s \left(1 - \frac{2\mu}{\bar{p}} C_{N,\bar{p},s}^{\bar{p}} b^{\frac{4s}{N}}\right) - 2s\sigma_2 \right] a_n.$$

Under the strict assumption in Equation (40), the bracket is positive; hence, $\{a_n\}$ is also bounded in this case.

In both cases, $\{a_n\}$ is bounded, and, since $\|v_n\|_2 = b$, we conclude that $\{v_n\}$ is bounded in $H^s(\mathbb{R}^N)$. $\square$

5. Proof of Theorems 3 and 4

In the previous sections, we have obtained a bounded Palais–Smale sequence $\{v_n\} \subset S_b$ for $E_{\mu,V}|_{S_b}$ at the Mountain-Pass level $m_{V,b}$ such that

$$E_{\mu,V}(v_n) = m_{V,b} + o_n(1), \qquad \|(E_{\mu,V}|_{S_b})'(v_n)\|_{(T_{v_n}S_b)^*} = o_n(1), \qquad P_{\mu,V}(v_n) = o_n(1).$$

We denote the associated Pohozaev manifold by

$$\mathcal{P}_{\mu,V} := \{v \in S_b : \, P_{\mu,V}(v) = 0\}.$$

To conclude the proofs of Theorems 3 and 4, it remains to show that $\{v_n\}$ admits a strongly convergent subsequence in $H^s(\mathbb{R}^N)$. Since V is not assumed to be radial, compactness cannot be recovered by restricting to $H^s_{\text{rad}}(\mathbb{R}^N)$; moreover, the presence of the critical exponent $2^*_s = \frac{2N}{N-2s}$ yields a loss of compactness in $H^s(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$. We use Lions' concentration–compactness principle and a profile decomposition. A key input is the sharp Sobolev inequality Equation (12) with best constant $S_s > 0$, together with the energy bound

$$m_{V,b} < m_b < \frac{s}{N} S_s^{\frac{N}{2s}},$$

which follows from Lemmas 2 and 9.

Lemma 13. *Let $N > 2s$ and $\bar{p} \le p < 2^*_s$. Assume that $V \in C^1(\mathbb{R}^N)$ and* (V0), (V1), (V2), (V3) *hold, with σ_1 satisfying Equation (41), and with σ_2 satisfying Equation (39) when $\bar{p} < p < 2^*_s$ or Equation (40) when $p = \bar{p}$. Let $\{v_n\} \subset S_b$ be a Palais–Smale sequence for $E_{\mu,V}|_{S_b}$ at level $m_{V,b}$ such that $P_{\mu,V}(v_n) \to 0$. Then, up to a subsequence, there exist $v \in H^s(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ such that*

$$v_n \rightharpoonup v \quad \text{weakly in } H^s(\mathbb{R}^N), \qquad \lambda_n \to \lambda,$$

where $\{\lambda_n\}$ is the Lagrange multiplier sequence associated with $\{v_n\}$. If $v_n \not\to v$ strongly in $H^s(\mathbb{R}^N)$, then there exist an integer $k \in \mathbb{N}^+$, nontrivial functions $w^1, \dots, w^k \in H^s(\mathbb{R}^N) \setminus \{0\}$ solving Equation (23) with the same μ and with $\lambda < 0$, and sequences $\{y^j_n\} \subset \mathbb{R}^N$ $(1 \le j \le k)$ such that $|y^j_n| \to \infty$ and $|y^j_n - y^i_n| \to \infty$ for $i \ne j$, and

$$v_n = v + \sum_{j=1}^{k} w^j(\cdot + y^j_n) + o_n(1) \qquad \text{in } H^s(\mathbb{R}^N). \tag{52}$$

Moreover,

$$\|v_n\|_2^2 = \|v\|_2^2 + \sum_{j=1}^{k} \|w^j\|_2^2, \qquad E_{\mu,V}(v_n) = E_{\mu,V}(v) + \sum_{j=1}^{k} E_\mu(w^j) + o_n(1). \tag{53}$$

Proof. Up to a subsequence, we may assume that $v_n \rightharpoonup v$ in $H^s(\mathbb{R}^N)$ and $v_n \to v$ a.e. in $\mathbb{R}^N$. Since $\{v_n\}$ is a Palais–Smale sequence for the constrained functional $E_{\mu,V}|_{S_b}$, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$(-\Delta)^s v_n + V(x)v_n - \mu|v_n|^{p-2}v_n - |v_n|^{2^*_s-2}v_n - \lambda_n v_n \to 0 \qquad \text{in } H^{-s}(\mathbb{R}^N). \tag{54}$$

Testing Equation (54) by v_n and using $\|v_n\|_2^2 = b^2$, we obtain

$$\|v_n\|_{D^s}^2 + \int_{\mathbb{R}^N} V(x)v_n^2 \, \mathrm{d}x - \mu\|v_n\|_p^p - \|v_n\|_{2_s^*}^{2_s^*} - \lambda_n b^2 = o_n(1). \tag{55}$$

By Lemma 12, $\{v_n\}$ is bounded in $H^s(\mathbb{R}^N)$; hence, up to a subsequence, $\lambda_n \to \lambda \in \mathbb{R}$.

Passing to the limit in Equation (54), we infer that v is a weak solution of Equation (25) with the parameter λ. In particular, by the sign information obtained in Lemma 5 (see also its proof), any nontrivial solution corresponds to $\lambda < 0$; hence, either $v = 0$ or $\lambda < 0$.

Step 1: Excluding the case $v = 0$. Assume $v = 0$. Since $\{v_n^2\}$ is bounded in $L^{\frac{N}{N-2s}}(\mathbb{R}^N)$ and $v_n \to 0$ almost everywhere, we have $v_n^2 \rightharpoonup 0$ weakly in $L^{\frac{N}{N-2s}}(\mathbb{R}^N)$. Fix $\varepsilon > 0$. After the harmless shift, $V_\infty = 0$. Using (V2) (in particular, the $L^{\frac{N}{2s}}$-smallness at infinity), choose $R > 0$ such that

$$\|V\|_{L^{\frac{N}{2s}}(\mathbb{R}^N \setminus B_R)} + \|W\|_{L^{\frac{N}{2s}}(\mathbb{R}^N \setminus B_R)} \le \varepsilon.$$

Then,

$$\int_{\mathbb{R}^N} V(x)v_n^2 \, \mathrm{d}x = \int_{B_R} V(x)v_n^2 \, \mathrm{d}x + \int_{\mathbb{R}^N \setminus B_R} V(x)v_n^2 \, \mathrm{d}x. \tag{56}$$

Since $v_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ and the embedding $H^s(B_R) \hookrightarrow L^2(B_R)$ is compact, we have $v_n \to 0$ strongly in $L^2(B_R)$; hence,

$$\int_{B_R} V(x)v_n^2 \, \mathrm{d}x = o_n(1). \tag{57}$$

Moreover,

$$\left|\int_{\mathbb{R}^N \setminus B_R} V(x)v_n^2 \, \mathrm{d}x\right| \le \|V\|_{L^{\frac{N}{2s}}(\mathbb{R}^N \setminus B_R)} \|v_n^2\|_{L^{\frac{N}{N-2s}}(\mathbb{R}^N)} \le C\varepsilon,$$

Since ε is arbitrary, Equations (56) and (57) yield $\int_{\mathbb{R}^N} V(x)v_n^2 \, \mathrm{d}x = o_n(1)$.

Similarly,

$$\int_{\mathbb{R}^N} W(x)v_n^2 \, \mathrm{d}x = o_n(1). \tag{58}$$

Therefore,

$$E_{\mu,V}(v_n) = E_\mu(v_n) + o_n(1), \qquad P_{\mu,V}(v_n) = P_\mu(v_n) + o_n(1).$$

Since $P_{\mu,V}(v_n) \to 0$, we obtain $P_\mu(v_n) \to 0$. Moreover, $m_{V,b} = E_{\mu,V}(v_n) + o_n(1) = E_\mu(v_n) + o_n(1)$. If $\|v_n\|_{D^s}^2 \to 0$, then, by Equation (55) (and $\|v_n\|_2 = b$), we deduce $\|v_n\|_p^p \to 0$ and $\|v_n\|_{2_s^*}^{2_s^*} \to 0$; hence, $m_{V,b} = 0$, contradicting Equation (43). Thus, up to a subsequence, there exists $c_0 > 0$ such that $\|v_n\|_{D^s}^2 \ge c_0$. Applying Lemma 4 to $\{v_n\} \subset S_b$, we infer

$$m_b \le \liminf_{n\to\infty} E_\mu(v_n) = m_{V,b},$$

which contradicts Lemma 9. Hence, $v \ne 0$ and therefore $\lambda < 0$.

Step 2: Profile decomposition when the convergence is not strong. Set $w_n^1 := v_n - v$. Then, $w_n^1 \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ and $w_n^1 \to 0$ in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for every $2 \le r < 2_s^*$. Define

$$L := \liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |w_n^1|^2 \, \mathrm{d}x.$$

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If $L = 0$, then the fractional Lions' lemma (see ([28], Lemma 3.1)) yields $\|w_n^1\|_r \to 0$ for all $2 < r < 2_s^*$ (in particular, $\|w_n^1\|_p \to 0$). Using Brezis–Lieb-type splittings (see ([29], Lemma 1.32)), we have

$$\|v_n\|_{2_s^*}^{2_s^*} = \|v\|_{2_s^*}^{2_s^*} + \|w_n^1\|_{2_s^*}^{2_s^*} + o_n(1), \qquad \|v_n\|_p^p = \|v\|_p^p + \|w_n^1\|_p^p + o_n(1).$$

Moreover, if $L = 0$, then, for every fixed $R > 0$, one has $\|w_n^1\|_{L^2(B_R)} \to 0$. Indeed, B_R can be covered by finitely many unit balls and the definition of L implies $\int_{B_1(y)} |w_n^1|^2 \to 0$ uniformly in y; hence, $\int_{B_R} |w_n^1|^2 \to 0$. Arguing as in Step 1 and using $V(x) \to 0$, $W(x) \to 0$ as $|x| \to \infty$, we obtain

$$\int_{\mathbb{R}^N} V(x)\,(w_n^1)^2 \,\mathrm{d}x = o_n(1), \qquad \int_{\mathbb{R}^N} W(x)\,(w_n^1)^2 \,\mathrm{d}x = o_n(1),$$

and similarly the mixed terms satisfy

$$\int_{\mathbb{R}^N} V(x)\,v\,w_n^1 \,\mathrm{d}x = o_n(1), \qquad \int_{\mathbb{R}^N} W(x)\,v\,w_n^1 \,\mathrm{d}x = o_n(1).$$

Since v is a weak solution of Equation (25), by Proposition 1, we have $P_{\mu,V}(v) = 0$. Together with $P_{\mu,V}(v_n) \to 0$, the above splitting yields

$$P_\mu(w_n^1) = o_n(1), \quad \text{that is} \quad s\|w_n^1\|_{D^s}^2 - s\|w_n^1\|_{2_s^*}^{2_s^*} = o_n(1),$$

because $\|w_n^1\|_p \to 0$ when $L = 0$.

Set $a_n := \|w_n^1\|_{D^s}^2$. By the sharp Sobolev inequality Equation (12),

$$\|w_n^1\|_{2_s^*}^{2_s^*} \le S_s^{-\frac{2_s^*}{2}}\, a_n^{\frac{2_s^*}{2}}.$$

Hence,

$$a_n \le S_s^{-\frac{2_s^*}{2}}\, a_n^{\frac{2_s^*}{2}} + o_n(1),$$

so either $a_n \to 0$ or $\liminf_{n\to\infty} a_n \ge S_s^{\frac{N}{2s}}$. If $\liminf a_n \ge S_s^{\frac{N}{2s}}$, then, using $\|w_n^1\|_p \to 0$, we have

$$E_\mu(w_n^1) = \frac{1}{2}a_n - \frac{1}{2_s^*}\|w_n^1\|_{2_s^*}^{2_s^*} + o_n(1) = \frac{s}{N}a_n + o_n(1) \ge \frac{s}{N}S_s^{\frac{N}{2s}} + o_n(1),$$

and, since $E_{\mu,V}(v) \ge 0$ by Lemma 6, we obtain

$$m_{V,b} = \lim_{n\to\infty} E_{\mu,V}(v_n) \ge \frac{s}{N}S_s^{\frac{N}{2s}},$$

contradicting $m_{V,b} < \frac{s}{N}S_s^{\frac{N}{2s}}$.

Now, subtract Equation (25) (solved by v) from Equation (54) and test the difference by w_n^1. Using $\lambda_n \to \lambda < 0$, $\|w_n^1\|_{D^s} \to 0$, and the standard continuity of the nonlinear terms, we obtain

$$-\lambda\|w_n^1\|_2^2 \le o_n(1),$$

hence, $\|w_n^1\|_2 \to 0$. Therefore, $w_n^1 \to 0$ strongly in $H^s(\mathbb{R}^N)$, contradicting the assumption that v_n does not converge strongly. Consequently, $L > 0$.

Since $L > 0$, we can choose $\{y_n^1\} \subset \mathbb{R}^N$ such that $|y_n^1| \to \infty$ and

$$\int_{B_1(y_n^1)} |w_n^1|^2 \,\mathrm{d}x \ge \frac{L}{2}.$$

Define the translated sequence $\tilde{w}_n^1(x) := w_n^1(x + y_n^1)$. Then, $\{\tilde{w}_n^1\}$ is bounded in $H^s(\mathbb{R}^N)$ and, up to a subsequence,

$$\tilde{w}_n^1 \rightharpoonup w^1 \quad \text{in } H^s(\mathbb{R}^N), \qquad w^1 \not\equiv 0.$$

Translating Equation (54) and using $V(\cdot + y_n^1) \to 0$ in the sense required by (V0)–(V1), we pass to the limit and obtain that w^1 solves Equation (23) with the same μ and with $\lambda < 0$.

Setting $w_n^2 := w_n^1 - w^1(\cdot + y_n^1)$ and iterating the above procedure yields finitely many profiles $w^1, \ldots, w^k$ and translations y_n^j with mutual divergence, such that Equation (52) holds. The L^2-splitting in Equation (53) follows from the Brezis–Lieb lemma and the orthogonality of translations. Finally, Equation (53) for the energy follows from the same splitting together with Equations (57) and (58) and the fact that $V(\cdot + y_n^j) \to 0$. □

Proof of Theorems 3 and 4. Assume the hypotheses of Theorems 3 and 4 hold. By Lemma 11, there exists a Palais–Smale sequence $\{v_n\} \subset S_b$ for $E_{\mu,V}|_{S_b}$ at level $m_{V,b}$ satisfying Equation (45); by Lemma 12, it is bounded in $H^s(\mathbb{R}^N)$. Up to a subsequence, $v_n \rightharpoonup v$ in $H^s(\mathbb{R}^N)$. If the convergence were not strong, then, by Lemma 13, we would have $k \geq 1$ and

$$m_{V,b} = E_{\mu,V}(v) + \sum_{j=1}^{k} E_\mu(w^j) + o_n(1).$$

Since v is a solution of Equation (25), Lemma 6 yields $E_{\mu,V}(v) \geq 0$. Moreover, each $w^j \not\equiv 0$ solves Equation (23); hence, $P_\mu(w^j) = 0$ and $w^j \in S_{\|w^j\|_2}$. Therefore,

$$E_\mu(w^j) \geq m_{\|w^j\|_2}.$$

Since $\|w^j\|_2 \leq b$ and $b \mapsto m_b$ is nonincreasing by Lemma 3, we have $m_{\|w^j\|_2} \geq m_b$; hence,

$$m_{V,b} \geq \sum_{j=1}^{k} E_\mu(w^j) \geq k\, m_b \geq m_b,$$

which contradicts Lemma 9. Hence, $v_n \to v$ strongly in $H^s(\mathbb{R}^N)$.

Finally, since $\|v_n^-\|_2 \to 0$ in Equation (45), we deduce $v \geq 0$ a.e. in $\mathbb{R}^N$. By standard regularity and the strong maximum principle for fractional Schrödinger-type equations (see ([30], B.2 Theorem and B.3 Lemma)), v is a classical solution and satisfies $v > 0$ in $\mathbb{R}^N$. □

6. Discussion and Outlook

The present work addresses the existence of stationary normalized (mass-constrained) solutions to the fractional Schrödinger Equation (1) in $\mathbb{R}^N$, in the presence of a nonconstant potential and Sobolev-critical growth. Under assumptions (V0)–(V3), we set up a Pohozaev-manifold scheme compatible with the mass constraint $\|v\|_2 = b$, which allows one to construct constrained Palais–Smale sequences at the Mountain-Pass level. A crucial feature is that, in the regimes covered by our hypotheses, the resulting minimax level can be kept below the fractional critical threshold $\frac{s}{N} S_s^{\frac{N}{2s}}$; this rules out Sobolev-critical bubbling in the subsequent concentration–compactness argument and yields positive normalized solutions.

In the L^2-critical case $p = \bar{p}$, the above mechanism is implemented by imposing a small-mass condition $b \in (0, b_0)$. In the L^2-supercritical range $\bar{p} < p < 2_s^*$, compactness follows once the Mountain-Pass level lies below the same critical threshold, as ensured by our construction under the stated assumptions (including the smallness conditions on the potential terms appearing in (V0)–(V1)).

6.1. Limitations

Our compactness analysis hinges on the strict inequality $m_{V,b} < \frac{s}{N} S_s^{\frac{N}{2s}}$, which excludes Sobolev–critical bubbling. In the L^2–critical case, this is achieved through the small–mass restriction $b \in (0, b_0)$; it remains an open problem to remove such a constraint under the general assumptions (V0)–(V3). Moreover, we do not address uniqueness, multiplicity, or finer qualitative properties (such as symmetry breaking, orbital stability for the time–dependent dynamics, or sharp decay profiles), which would require tools beyond the variational compactness framework developed here.

6.2. Perspectives

Several directions appear natural. First, it would be of interest to weaken (V0)–(V3) by identifying verifiable classes of potentials (for instance, radial, periodic, or possibly singular Coulomb-type potentials) for which a Pohozaev-manifold approach still yields compactness. Second, one may study multiplicity and concentration phenomena by combining the present constrained setting with topological methods, such as Lusternik–Schnirelmann theory (see [31]) under suitable symmetry or localization assumptions on V. Third, extensions to bounded domains (with Dirichlet exterior conditions) or to models with magnetic fields would require a different treatment of boundary contributions and lie beyond the scope of this paper. Finally, while our arguments apply to arbitrary $s \in (0, 1)$ and do not exploit any special feature of the case $s = \frac{1}{2}$, it would be interesting to connect the stationary theory developed here with dispersive properties of the associated time-dependent fractional Schrödinger equation, including stabilization mechanisms and energy decay estimates.

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