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Impulsive Tempered Ψ-Fractional Differential Equations with Boundary and Integral Conditions

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Abstract

This paper studies mixed impulsive boundary value problems involving tempered Ψ-fractional derivatives of Caputo type. By introducing exponential tempering into the fractional framework, the proposed model effectively captures systems with fading memory—an improvement over conventional power-law kernels that assume long-range dependence. The generalized tempered Ψ-operator unifies several existing fractional derivatives, offering enhanced flexibility for modeling complex dynamical phenomena. Impulsive effects and integral boundary conditions are incorporated to describe processes subject to sudden changes and historical dependence. The problem is reformulated as a Volterra integral equation, and fixed-point theory is employed to establish analytical results. Existence and uniqueness of solutions are proven using the Banach Contraction Mapping Principle, while the Leray–Schauder nonlinear alternative ensures existence in non-contractive cases. The proposed framework provides a rigorous analytical basis for modeling phenomena characterized by both fading memory and sudden perturbations, with potential applications in physics, control theory, population dynamics, and epidemiology. A numerical example is presented to illustrate the validity and applicability of the main theoretical results.

Keywords: boundary value problems; fractional differential equations; tempered fractional derivatives; existence; uniqueness; existence; uniqueness

1. Introduction

Fractional Calculus (FC) is a branch of mathematical analysis that extends the classical notions of differentiation and integration to arbitrary, non-integer (real or complex) orders. Although its conceptual roots date back to a 1695 correspondence between Leibniz and L'Hôpital, the field has evolved through the contributions of eminent mathematicians such as Euler, Laplace, Liouville, and Riemann. Over the past few decades, interest in FC has grown rapidly due to its ability to provide more accurate mathematical descriptions of various natural and engineering processes characterized by memory and hereditary properties—features that conventional integer-order models fail to capture. As a result, FC has established itself as a versatile and powerful framework for both theoretical investigation and practical modeling across diverse disciplines, including physics, engineering, biology, economics, and finance (see [1–3]).

While classical fractional operators—such as those of Riemann–Liouville, Caputo, and Hilfer—have proven highly effective in modeling a broad spectrum of dynamical systems,

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they typically rely on power-law kernels to represent memory effects. However, such kernels often imply long-range memory with slow decay, which may not adequately capture the finite-memory behavior observed in many real-world phenomena. To overcome this limitation, the framework of tempered fractional calculus was introduced, incorporating an exponential tempering factor into the kernel. This modification enables the modeling of processes in which memory effects gradually diminish over time, offering a more realistic representation of systems with bounded or fading memory.

Among the various tempered operators [4–6], the tempered Ψ-fractional derivative is particularly noteworthy due to its generalized formulation. It bridges the gap between the tempered Riemann–Liouville and tempered Caputo derivatives by introducing parameters that govern both the fractional order and the derivative type (left- or right-sided). This inherent flexibility renders the tempered Ψ-operator a powerful and versatile mathematical tool for accurately characterizing complex phenomena in which both memory and tempering effects play a crucial role (see [7–9]).

The study of differential equations involving fractional operators—particularly those that capture memory effects—plays a crucial role in modeling a wide range of real-world phenomena. However, many natural and engineered processes are not entirely continuous; they often undergo abrupt changes or sudden disturbances at specific moments in time. Such instantaneous variations are mathematically described using impulsive conditions, which can profoundly influence the qualitative behavior of system solutions. When these impulsive effects are incorporated alongside boundary or initial conditions to define a well-posed framework, the resulting formulation is commonly known as a mixed problem (see [10–14]).

The incorporation of tempered fractional derivatives into mixed problems, especially in conjunction with the flexible Hilfer operator, provides a refined analytical framework for investigating systems characterized by both fading long-range memory and instantaneous, short-lived perturbations. This integrated approach facilitates a more realistic and comprehensive modeling of complex dynamical systems encountered in fields such as control theory, population dynamics, and epidemiological modeling—where both historical dependencies and sudden shocks are fundamental to system behavior.

Motivated by the aforementioned considerations, we undertake a comprehensive analysis to establish the necessary conditions ensuring the existence and uniqueness of solutions for mixed impulsive tempered Ψ-fractional boundary value problems of the form:

$$\begin{cases} ({}_{\mathfrak{s}_i}\mathfrak{D}^{\alpha_i,\lambda_i}_{\Psi_i}\mathfrak{u})(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t})), \quad \mathfrak{t}\in[\mathfrak{s}_i,\mathfrak{t}_{i+1}), \quad i=0,1,2,\ldots,m, \\ ({}_{\mathfrak{t}_j}\mathfrak{D}^{\beta_j,\gamma_j}_{\Omega_j}\mathfrak{u})(\mathfrak{t}) = \mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t})), \quad \mathfrak{t}\in[\mathfrak{t}_j,\mathfrak{s}_j), \quad j=1,2,3,\ldots,m, \\ \mathfrak{u}(\mathfrak{t}_j^+) = \omega_j\mathfrak{u}(\mathfrak{t}_j^-), \quad \mathfrak{u}(\mathfrak{s}_i^+) = \eta_i\mathfrak{u}(\mathfrak{s}_i^-), \quad i=1,2,3,\ldots,m, \quad j=1,2,3,\ldots,m, \\ \xi_1\mathfrak{u}(0)+\xi_2\mathfrak{u}(T) = \xi_3\Big[({}_{\mathfrak{t}_m}\mathfrak{I}^{\beta_m,\gamma_m}_{\Omega_m}\mathfrak{u})(\mathfrak{s}_\mathfrak{m}^-) - ({}_{\mathfrak{s}_m}\mathfrak{I}^{\alpha_m,\lambda_m}_{\Psi_m}\mathfrak{u})(\mathfrak{t}_{\mathfrak{m}+1}^-)\Big], \end{cases} \tag{1}$$

where

1. $0<\alpha_i,\beta_j\le 1$, $i=0,1,\ldots,m$ and $j=1,2,\ldots,m$.
2. The operators ${}_{\mathfrak{s}_i}\mathfrak{D}^{\alpha_i,\lambda_i}_{\Psi_i}$, ${}_{\mathfrak{t}_j}\mathfrak{D}^{\beta_j,\gamma_j}_{\Omega_j}$ denote the tempered Hilfer Ψ-fractional derivatives of Caputo type, of orders $\alpha_i,\beta_j\in(0,1]$.
3. The operators ${}_{\mathfrak{t}_m}\mathfrak{I}^{\beta_m,\gamma_m}_{\Omega_m}$, ${}_{\mathfrak{s}_m}\mathfrak{I}^{\alpha_m,\lambda_m}_{\Psi_m}$ represent the tempered Ψ-fractional integral of Riemann–Liouville type, of orders $\beta_m,\alpha_m\in(0,1]$.
4. λ_i, γ_j, ω_j, η_i, ξ_1, ξ_2 and ξ_3 are real constants.
5. The sets $\mathbb{J}_1=\bigcup_{i=0}^{m}[\mathfrak{s}_i,\mathfrak{t}_{i+1})$, $\mathbb{J}_2=\bigcup_{j=1}^{m}[\mathfrak{t}_j,\mathfrak{s}_j)$ and $\mathbb{J}=\mathbb{J}_1\cup\mathbb{J}_2\cup\{T\}$ define the specified subintervals under consideration.
6. The fixed points $0=\mathfrak{s}_0<\mathfrak{t}_1<\mathfrak{s}_1<\mathfrak{t}_2<\ldots<\mathfrak{t}_m<\mathfrak{s}_m<\mathfrak{t}_{m+1}=T<\infty$.

7. The functions $\mathfrak{f} : \mathbb{J}_1 \times \mathbb{R} \to \mathbb{R}$ and $\mathfrak{g} : \mathbb{J}_2 \times \mathbb{R} \to \mathbb{R}$ are continuous.

Special cases

(i). When $\lambda = 0$, the operator ${}_{\mathfrak{s}_i}\mathfrak{D}^{\alpha_i,\lambda_i}_{\Psi_i}$ is reduced to Caputo Ψ-fractional derivative.

(ii). When $\Psi(\mathfrak{t}) = \mathfrak{t}$, the operator ${}_{\mathfrak{s}_i}\mathfrak{D}^{\alpha_i,\lambda_i}_{\Psi_i}$ is reduced to tempered Caputo fractional derivative.

(iii). When $\lambda = 0$ and $\Psi(\mathfrak{t}) = \mathfrak{t}$, the operator ${}_{\mathfrak{s}_i}\mathfrak{D}^{\alpha_i,\lambda_i}_{\Psi_i}$ is reduced to original Caputo fractional derivative.

Remark 1. *The boundary condition in Equation (1) plays a role that goes beyond the standard measurement of the unknown function at the initial and terminal points of the interval. Indeed, the term*

$$\xi_3\left[({}_{\mathfrak{t}_m}\mathfrak{I}^{\beta_m,\gamma_m}_{\Omega_m}\mathfrak{u})(\mathfrak{s}^-_{\mathfrak{m}}) - ({}_{\mathfrak{s}_m}\mathfrak{I}^{\alpha_m,\lambda_m}_{\Psi_m}\mathfrak{u})(\mathfrak{t}^-_{\mathfrak{m}+1})\right]$$

captures the jump in the accumulated (integral-type) values of the solution across the last impulsive point $\mathfrak{s}_m$. This means that the boundary condition simultaneously governs

(i). the values of $u(0)$ and $u(T)$; and

(ii). the discrepancy between the fractional integrals of the solution immediately before and immediately after the final impulse.

Such a structure highlights the mixed nature of the problem, where both pointwise information and history-dependent integral information—interrupted by an impulse—jointly determine the admissible solution.

In this study, the boundary value problem is reformulated into an equivalent Volterra integral equation, enabling the application of Fixed Point Theory as the primary analytical framework. The Banach Contraction Mapping Principle [14] is employed to establish the existence and uniqueness of solutions under suitable Lipschitz conditions, while the Leray–Schauder nonlinear alternative [15] is utilized to demonstrate the existence of at least one solution in cases where the contraction criteria are not satisfied.

The intricacy of mixed impulsive boundary value problems arises from the coexistence of multiple challenging features within a single formulation. Firstly, the inclusion of tempered fractional derivatives allows the modeling of processes exhibiting long-term memory with gradual decay, thereby enhancing the accuracy of descriptions for various natural and engineering phenomena. Secondly, the introduction of impulsive conditions accounts for systems subject to abrupt or instantaneous changes—such as vaccination interventions or external shock forces—as discussed in [16].

The remainder of this paper is organized as follows. Section 2 presents several fundamental preliminaries and definitions. In Section 3, we establish the existence and uniqueness of solutions by employing Banach's contraction mapping principle and the Leray–Schauder nonlinear alternative. Section 4 provides a concrete example that illustrates the applicability and validity of the main results.

2. Preliminaries

This section of the paper is devoted to the presentation of fundamental concepts in fractional calculus, with particular emphasis on tempered Ψ-fractional derivatives and integrals. For further details and extended discussions, readers are referred to [3]. Below, we provide the formulations of the Riemann–Liouville fractional Ψ-integral, the Ψ-derivative, and the Caputo fractional Ψ-derivative defined over the interval $[a, b]$ as follows:

Definition 1 ([7]). *Let $\Psi \in C^1[a,b]$ be a positive increasing function satisfying $\Psi'(t) \neq 0$ for all $t \in [a,b]$. The tempered Riemann–Liouville Ψ-fractional integral of a given function $f \in L_1[a,b]$ of order $\alpha > 0$ and parameter $\lambda \in \mathbb{R}^+$ is defined by*

$$({}_a\mathfrak{I}_\Psi^{\alpha,\lambda}\mathfrak{f})(\mathfrak{t}) = \frac{1}{\Gamma(\alpha)}\int_a^{\mathfrak{t}} (\Psi(\mathfrak{t}) - \Psi(s))^{\alpha-1} e^{-\lambda(\Psi(\mathfrak{t})-\Psi(s))}\Psi'(s)\mathfrak{f}(s)ds.$$

We define the symbol $\mathfrak{d} := \dfrac{1}{\Psi'}\dfrac{d}{d\mathfrak{t}} + \lambda$ in this work.

Definition 2 ([7]). *Let $\Psi \in C^1[a,b]$ be a positive increasing function satisfy $\Psi'(t) > 0$ for all $t \in [a,b]$. The tempered Riemann–Liouville Ψ-fractional derivative of order $\alpha, \alpha \in (n-1,n), n \in \mathbb{N}$, with parameter $\lambda \in \mathbb{R}^+$, applied to the function $f \in L_1[a,b]$ is defined by*

$$\begin{aligned} {}_a\mathcal{D}_\Psi^{\alpha,\lambda}\mathfrak{f}(\mathfrak{t}) &= \mathfrak{d}^n\, {}_a\mathcal{I}_\Psi^{n-\alpha,\lambda}\mathfrak{f}(\mathfrak{t}), \\ &= \frac{1}{\Gamma(\alpha)}\left[\frac{1}{\Psi'(\mathfrak{t})}\frac{d}{d\mathfrak{t}}\right]^n \int_a^{\mathfrak{t}} [\Psi(\mathfrak{t}) - \Psi(s)]^{\alpha-1}\Psi'(s)e^{-\lambda[\Psi(\mathfrak{t})-\Psi(s)]}f(s)ds. \end{aligned}$$

Definition 3 ([7]). *Let $\Psi \in C^1[a,b]$ be a positive increasing function satisfy $\Psi'(t) > 0$ for all $t \in [a,b]$. The tempered Caputo Ψ-fractional derivative of order $\alpha, \alpha \in (n-1,n), n \in \mathbb{N}$, and with parameter $\lambda \in \mathbb{R}^+$, applied to the function $f \in AC[a,b]$ is defined by*

$$\begin{aligned} {}_a\mathfrak{D}_\Psi^{\alpha,\lambda}\mathfrak{f}(\mathfrak{t}) &= {}_a\mathcal{I}_\Psi^{n-\alpha,\lambda}\mathfrak{d}^n\mathfrak{f}(\mathfrak{t}) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^{\mathfrak{t}} [\Psi(\mathfrak{t}) - \Psi(s)]^{\alpha-1}\Psi'(s)e^{-\lambda[\Psi(\mathfrak{t})-\Psi(s)]}\left[\frac{1}{\Psi'(s)}\frac{d}{ds}\right]^n f(s)ds. \end{aligned}$$

Example 1. *Let us consider an example of a Ψ-tempered fractional derivative applied to a rapidly increasing function, where $\alpha = 1/2, \mathfrak{f}(\mathfrak{t}) = e^{\mathfrak{t}^2}, \Psi(\mathfrak{t}) = \mathfrak{t}$, $\mathfrak{t} \in [0,3]$. For various curves corresponding to different values of a parameter λ ranging from 0 to 10, the derivative is expressed as*

$${}_0\mathfrak{D}_\mathfrak{t}^{1/2,\lambda}\left(e^{\mathfrak{t}^2}\right) = \frac{e^{-\lambda\mathfrak{t}}}{\Gamma(0.5)}\int_0^{\mathfrak{t}} (\mathfrak{t}-s)^{-0.5}\frac{d}{ds}\left[e^{\lambda s}e^{s^2}\right]ds.$$

A key observation from the corresponding plot (see Figure 1) is the significant influence of the parameter λ on both the magnitude and the qualitative behavior of the fractional derivative.

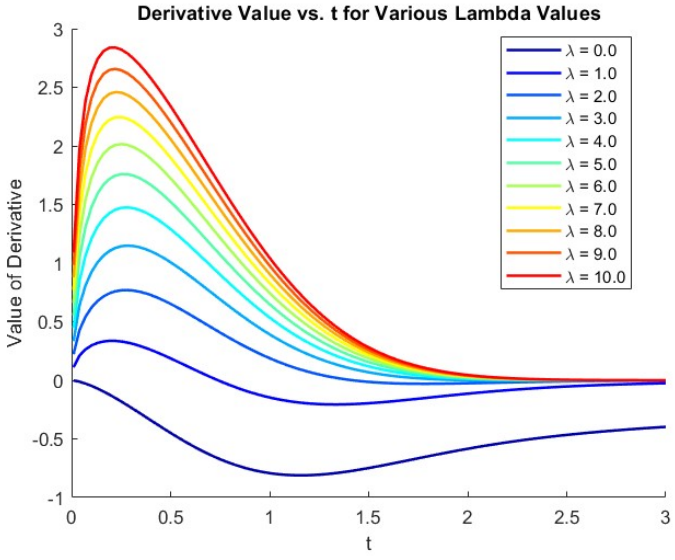

Figure 1. The Caputo-type Ψ-tempered fractional derivative ${}_0\mathfrak{D}_\mathfrak{t}^{1/2,\lambda}\left(e^{\mathfrak{t}^2}\right)$ for different values of the tempering parameter λ (with $\Psi(\mathfrak{t}) = \mathfrak{t}, \mathfrak{t} \in [0,3]$ and $\lambda = 0, 1, \ldots, 10$).

When $\lambda = 0$, the curve exhibits a distinctive pattern—starting from a negative value and gradually increasing toward zero as t increases. This indicates that, for this specific case, the fractional derivative's initial response differs fundamentally from those associated with nonzero values of λ.

For $\lambda > 0$, a consistent trend is observed: each curve originates from the origin ($t = 0$, where the derivative of $\mathfrak{f}(\mathfrak{t})$ is zero), rises to a maximum value, and subsequently decays toward zero as t approaches 3. This behavior demonstrates a damping effect, in which the fractional derivative initially grows before gradually diminishing, reflecting the tempered memory influence inherent in the model.

In summary, the parameter λ plays a crucial role in govering the behavior of the Φ-tempered fractional derivative. It not only modulates the amplitude of the derivative's peak but also shifts the point at which this peak occurs. This analysis underscores the importance of λ as a tuning parameter in fractional-order systems, particularly in modeling physical processes where transient dynamics and decay rates are of interest. The clear distinction between the $\lambda = 0$ and $\lambda > 0$ cases further highlights its pivotal influence on the system's fundamental characteristics.

Example 2. *As an illustrative example of the Ψ-tempered fractional integral, consider the function $\mathfrak{f}(\mathfrak{t}) = 1 - e^{\mathfrak{t}^2}$ with $\Psi(\mathfrak{t}) = \mathfrak{t}$, fractional order $\alpha = \frac{1}{2}$, and $\mathfrak{t} \in [0, 3]$. For different values of the tempering parameter $\lambda \in [0, 10]$, the integral is given by*

$$ {}_0\mathfrak{I}_{\mathfrak{t}}^{1/2,\lambda}\left(1 - e^{\mathfrak{t}^2}\right) = \frac{e^{-\lambda \mathfrak{t}}}{\Gamma(0.5)} \int_0^{\mathfrak{t}} (\mathfrak{t} - s)^{-0.5} e^{\lambda s}\left(1 - e^{s^2}\right) ds. $$

The resulting plot demonstrates a distinct and systematic dependence of the integral's magnitude on the parameter λ. A prominent feature is the inverse relationship between λ and the overall value of the integral: as λ increases, the final value at $\mathfrak{t} = 3$ decreases notably. This indicates that the tempering parameter acts as a damping or attenuation factor, effectively moderating the integral's growth over time, as illustrated in Figure 2.

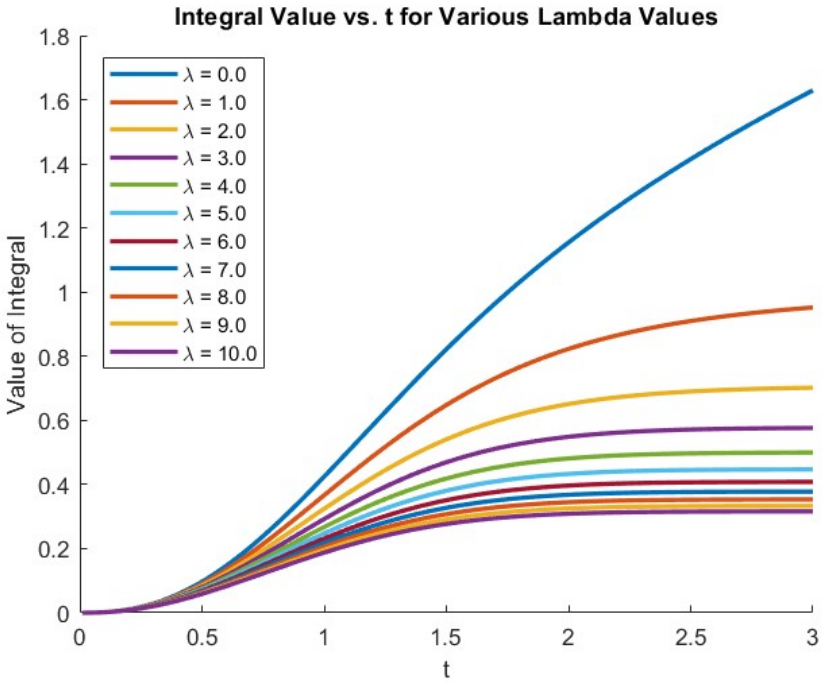

Figure 2. The Ψ-tempered fractional integral ${}_0\mathfrak{I}_{\mathfrak{t}}^{1/2,\lambda}\left(1 - e^{\mathfrak{t}^2}\right)$ for different values of the tempering parameter λ (with $\Psi(\mathfrak{t}) = \mathfrak{t}, \mathfrak{t} \in [0, 3]$ and $\lambda = 0, 1, \ldots, 10$).

For all considered values of λ, the integral starts from the origin—($\mathfrak{t} = 0$, integral value = 0)—and increases monotonically with $\mathfrak{t}$. This behavior aligns with the general property of integrals, where cumulative effects grow over the integration domain. However, the rate of this accumulation is significantly affected by the tempering parameter λ.

In conclusion, the parameter λ in the Ψ-tempered fractional integral functions as a regulatory factor that governs the integral's rate of growth. Larger values of λ introduce stronger tempering effects, suppressing the integral's accumulation and modifying its asymptotic behavior. This emphasizes the critical role of λ in modeling physical and natural processes where memory effects or cumulative responses diminish gradually over time.

Theorem 1 ([7]). *Let $\alpha > 0$ and $\lambda \in \mathbb{R}$.*

(a) If $n-1 < \alpha < n \in \mathbb{N}$ and $\mathfrak{f} \in C^n[0,b]$, then ${}_a\mathfrak{D}_\Psi^{\alpha,\lambda}\,{}_a\mathfrak{I}_\Psi^{\alpha,\lambda}\mathfrak{f}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t})$.

(b) If $n-1 < \alpha < n \in \mathbb{N}$ and $\mathfrak{f} \in AC^n[0,b]$, then

$$ {}_a\mathfrak{I}_\Psi^{\alpha,\lambda}\,{}_a\mathfrak{D}_\Psi^{\alpha,\lambda}\mathfrak{f}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}) - e^{-\lambda\Psi(\mathfrak{t})}\sum_{k=0}^{n-1}\frac{(\psi(\mathfrak{t})-\psi(a))^k}{\Gamma(k+1)}\Big[\mathfrak{d}^k e^{\lambda\Psi(\mathfrak{t})}\mathfrak{f}(\mathfrak{t})\Big]. $$

Lemma 1 ([7]). *(The semigroup property and integration of power function formula) Let $\alpha > 0$ and $\delta > 1$ are fixed. Then we have:*

$$ \begin{aligned} (i)\quad {}_a\mathfrak{I}_\Psi^{\alpha,\lambda}\Big(e^{-\lambda\Psi(\mathfrak{t})}(\psi(\mathfrak{t})-\psi(a))^{\delta-1}\Big) &= \frac{\Gamma(\delta)}{\Gamma(\delta+\alpha)}e^{-\lambda\Psi(\mathfrak{t})}(\psi(\mathfrak{t})-\psi(a))^{\delta+\alpha-1}, \\ (ii)\quad {}_a\mathfrak{D}_\Psi^{\alpha,\lambda}\Big(e^{-\lambda\Psi(\mathfrak{t})}(\psi(\mathfrak{t})-\psi(a))^{\delta-1}\Big) &= \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)}e^{-\lambda\Psi(\mathfrak{t})}(\psi(\mathfrak{t})-\psi(a))^{\delta-\alpha-1}. \end{aligned} $$

The following theorem, known as the Leray–Schauder nonlinear alternative, will be employed to establish our main existence result.

Theorem 2 ([15]). *Let $\mathcal{U}$ be a Banach space, and let $\mathcal{V}$ be a closed, convex subset of $\mathcal{U}$. Furthermore, let $\mathcal{W}$ be an open subset of $\mathcal{V}$ such that $0 \in \mathcal{W}$. Suppose that the operator $\mathcal{A} : \overline{\mathcal{W}} \to B$ is a continuous and compact (i.e., $\mathcal{A}(\overline{\mathcal{W}})$ is relatively compact in $\mathcal{U}$). Then one of the following alternatives holds:*

(i) The operator $\mathcal{A}$ admits a fixed point in $\overline{\mathcal{W}}$, or

(ii) There exists a point $u \in \partial\mathcal{W}$ (the boundary of $\mathcal{W}$ in $\mathcal{V}$) and a scalar $\rho \in (0,1)$ with $u = \rho\mathcal{A}(u)$.

Additionally, we define the following notation

$$ \begin{aligned} \Lambda_1(i,j) &= \prod_{n=j}^{i}\eta_n\omega_n, \qquad \Lambda_2(i,j) = \prod_{n=j}^{i}\omega_n\prod_{n=j}^{i-1}\eta_n, \\ \Lambda_3(i,j) &= \prod_{n=j}^{i-1}\eta_n\omega_{n+1}, \qquad \Lambda_4(i,j) = \prod_{n=j}^{i}\eta_n\prod_{n=j}^{i-1}\omega_{n+1}. \end{aligned} $$

For context, we show some examples of the use of this notation. Set $j = 2$ and $i = 6$. Then we get

$$ \begin{aligned} \Lambda_1(2,6) &= \omega_2\eta_2\omega_3\eta_3\omega_4\eta_4\omega_5\eta_5\omega_6\eta_6, \quad \Lambda_2(2,6) = \omega_2\eta_2\omega_3\eta_3\omega_4\eta_4\omega_5\eta_5\omega_6, \\ \Lambda_3(2,6) &= \eta_2\omega_3\eta_3\omega_4\eta_4\omega_5\eta_5\omega_6, \qquad \Lambda_4(2,6) = \eta_2\omega_3\eta_3\omega_4\eta_4\omega_5\eta_5\omega_6\eta_6. \end{aligned} $$

In what follows, we employ the following notation:

$$
\begin{aligned}
\Phi &= \xi_1 + \xi_2\Lambda_1(1,m) \\
&\quad -\xi_3\left[\Lambda_2(1,m)\frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)} - \Lambda_1(m,1)\frac{\left(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\right)^{\alpha_m}}{\Gamma(\alpha_m+1)}\right], \\
\Phi_1 &= \frac{|\Lambda_1(1,m)|}{|\Phi|}\left[|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\right. \\
&\quad \times\sum_{k=1}^{m}\left(|\Lambda_2(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right) \\
&\quad +|\xi_3|\frac{\left(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\right)^{\alpha_m}}{\Gamma(\alpha_m+1)} \\
&\quad \times\sum_{k=1}^{m+1}\left(|\Lambda_1(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right) \\
&\quad \left.+\sum_{k=1}^{m+1}\left(|\Lambda_1(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right)\right] \\
&\quad +\sum_{k=1}^{m+1}\left(|\Lambda_1(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right), \\
\Phi_2 &= \frac{|\Lambda_1(1,m)|}{|\Phi|}\left[|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\sum_{k=1}^{m}\left(|\Lambda_3(k,m)|\frac{(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+))^{\beta_k}}{\Gamma(\beta_k+1)}\right)\right. \\
&\quad +|\xi_3|\frac{\left(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\right)^{\alpha_m}}{\Gamma(\alpha_m+1)}\sum_{k=1}^{m}\left(|\Lambda_4(k,m)|\frac{(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+))^{\beta_k}}{\Gamma(\beta_k+1)}\right) \\
&\quad \left.+\sum_{k=1}^{m}\left(|\Lambda_4(k,m)|\frac{(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+))^{\beta_k}}{\Gamma(\beta_k+1)}\right)\right] \\
&\quad +\sum_{k=1}^{m}\left(|\Lambda_4(k,m)|\frac{(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+))^{\beta_k}}{\Gamma(\beta_k+1)}\right).
\end{aligned}
\tag{2}
$$

For the convenience of the reader, the main symbols and operators used throughout the manuscript are summarized in Table 1.

Table 1. Summary of main symbols and operators used in the manuscript.

Symbol	Description
α_i, β_j	Fractional orders of the tempered derivatives
λ	Exponential tempering parameter ($\lambda \geq 0$).
$\Psi(\cdot)$	Increasing function defining the Ψ-fractional framework.
${}^{C}D_{a+;\Psi}^{\alpha,\lambda}$	Caputo-type tempered Ψ-fractional derivative operator.
$I_{a+;\Psi}^{\alpha,\lambda}$	Tempered Ψ-fractional integral operator.
t_i	Points of instantaneous impulsive perturbations ($i = 1, \ldots, m$).
Φ, Φ_1, Φ_2	Complex symbolic constants derived from boundary/integral conditions.
$PC(J, \mathbb{R})$	Space of bounded piecewise continuous functions on interval J.
f, g	Nonlinear functions satisfying Lipschitz or growth conditions.

3. Main Results

Let us define the function spaces and operators associated with the problem under consideration, as described in (1).

Let $\mathbb{J}$ be as described in (1) and let $\mathcal{PC}(\mathbb{J},\mathbb{R})$ be the space of piecewise continuous functions, where $\mathcal{PC}(\mathbb{J},\mathbb{R}) = \{u : \mathbb{J} \to \mathbb{R}; u(t)$ is continuous everywhere except at finitely many points t_k, at which $u(t_k^+)$ and $u(t_k^-)$ exist and $u(t_k^-) = u(t_k)$, $k = 1,2,\ldots,m\}$.

Evidently, $\mathcal{PC}(\mathbb{J},\mathbb{R})$ is a Banach space with norm $\|u\| = \sup\{|u(t)|, t \in \mathbb{J}\}$.

Throughout this paper we suppose that the continuous functions $\mathfrak{f}$ and $\mathfrak{g}$ satisfy the following hypotheses:

(H_1) There exist constants $L_1, L_2 > 0$ such that, for all $u, v \in \mathbb{R}$,
$|\mathfrak{f}(t,u) - \mathfrak{f}(t,v)| \le L_1|u-v|$, $t \in \mathbb{J}_1$, and $|\mathfrak{g}(t,u) - \mathfrak{g}(t,v)| \le L_2|u-v|$, $t \in \mathbb{J}_2$.

(H_2) There exist two continuous, nondecreasing functions $\mathcal{Q}_1, \mathcal{Q}_2 : [0,\infty) \to (0,\infty)$, and continuous functions $\mathfrak{p}_1, \mathfrak{p}_2 : \mathbb{J} \to \mathbb{R}^+$, such that

$$|\mathfrak{f}(t,u)| \le \mathfrak{p}_1(t)\mathcal{Q}_1(|u|) + \mathfrak{p}_2(t), \quad \text{for each } (t,u) \in \mathbb{J}_1 \times \mathbb{R},$$

and

$$|\mathfrak{g}(t,u)| \le \mathfrak{q}_1(t)\mathcal{Q}_2(|u|) + \mathfrak{q}_1(t), \quad \text{for each } (t,u) \in \mathbb{J}_2 \times \mathbb{R}.$$

(H_3) There exists a positive constant $\mathcal{N}$ such that

$$\frac{\mathcal{N}}{[\|\mathfrak{p}_1\|\mathcal{Q}_1(\mathcal{N}) + \|\mathfrak{p}_2\|]\Phi_1 + [\|\mathfrak{q}_1\|\mathcal{Q}_2(\mathcal{N}) + \|\mathfrak{q}_2\|]\Phi_2} > 1.$$

We begin by deriving the equivalent integral form of the following linear fractional differential equation.

Lemma 2. *Let $\mathfrak{f}^* : \mathbb{J}_1 \to \mathbb{R}$ and $\mathfrak{g}^* : \mathbb{J}_2 \to \mathbb{R}$ be given functions and let $\Lambda \neq 0$. Moreover, let the parameters α_i, β_j, γ_i, λ_j, $\omega_j, \eta_i, \xi_1, \xi_2, \xi_3, i = 0,1,\ldots,m$ and $j = 1,2,\ldots,m$ be prescribed constants satisfying the conditions of problem (1). Then, the corresponding linear boundary value problem involving mixed types of impulsive tempered Ψ-fractional derivatives of the form:*

$$\begin{cases} ({}_{\mathfrak{s}_i}\mathfrak{D}^{\alpha_i,\lambda_i}_{\Psi_i} u)(t) = \mathfrak{f}^*(t), \quad t \in [\mathfrak{s}_i, \mathfrak{t}_{i+1}), \quad i = 0,1,2,\ldots,m, \\ ({}_{\mathfrak{t}_j}\mathfrak{D}^{\beta_j,\gamma_j}_{\Omega_j} u)(t) = \mathfrak{g}^*(t), \quad t \in [\mathfrak{t}_j, \mathfrak{s}_j), \quad j = 1,2,3,\ldots,m, \\ u(\mathfrak{t}_j^+) = \omega_j u(\mathfrak{t}_j^-), \quad u(\mathfrak{s}_i^+) = \eta_i u(\mathfrak{s}_i^-), \quad i = 1,2,3,\ldots,m, \\ \xi_1 u(0) + \xi_2 u(T) = \xi_3\Big[({}_{\mathfrak{t}_m}\mathfrak{J}^{\beta_m,\gamma_m}_{\Omega_m} u)(\mathfrak{s}_{\mathfrak{m}}^+) - ({}_{\mathfrak{s}_m}\mathfrak{J}^{\alpha_m,\lambda_m}_{\Psi_m} u)(\mathfrak{t}_{\mathfrak{m}+1}^+)\Big], \end{cases} \tag{3}$$

has a unique solution u on $\mathbb{J}$ presented by

$$
u(t)=\begin{cases}
\dfrac{\Lambda_1(1,i)}{\Phi}\Bigg[\xi_3\dfrac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\Big[\displaystyle\sum_{k=1}^{m}\Big(\Lambda_2(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\Big)(\mathfrak{t}_k^-)\Big)\\
+\displaystyle\sum_{k=1}^{m}\Big(\Lambda_3(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
-\xi_3\dfrac{(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)}\Big[\displaystyle\sum_{k=1}^{m+1}\Big(\Lambda_1(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\Big)(\mathfrak{t}_k^-)\Big)\\
+\displaystyle\sum_{k=1}^{m}\Big(\Lambda_4(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\Big)(\mathfrak{s}_k^-)\Big)\Big]-\displaystyle\sum_{k=1}^{m+1}\Big(\Lambda_1(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\Big)(\mathfrak{t}_k^-)\Big)\\
-\displaystyle\sum_{k=1}^{m}\Big(\Lambda_4(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\Big)(\mathfrak{s}_k^-)\Big)\Bigg]+\displaystyle\sum_{k=1}^{i}\Big(\Lambda_1(k,i)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\Big)(\mathfrak{t}_k^-)\Big)\\
+\displaystyle\sum_{k=1}^{i}\Big(\Lambda_4(k,i)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\Big)(\mathfrak{s}_k^-)\Big)+\Big({}_{\mathfrak{s}_i}\mathfrak{I}^{\alpha_i,\lambda_i}_{\Psi_i}\mathfrak{f}^*\Big)(t),\quad t\in[\mathfrak{s}_i,\mathfrak{t}_{i+1}),\ i=0,1,2,\ldots,m,\\
\dfrac{\Lambda_2(1,j)}{\Phi}\Bigg[\xi_3\dfrac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\Big[\displaystyle\sum_{k=1}^{m}\Big(\Lambda_2(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\Big)(\mathfrak{t}_k^-)\Big)\\
+\displaystyle\sum_{k=1}^{m}\Big(\Lambda_3(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
-\xi_3\dfrac{(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)}\Big[\displaystyle\sum_{k=1}^{m+1}\Big(\Lambda_1(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\Big)(\mathfrak{t}_k^-)\Big)\\
+\displaystyle\sum_{k=1}^{m}\Big(\Lambda_4(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\Big)(\mathfrak{s}_k^-)\Big)\Big]-\displaystyle\sum_{k=1}^{m+1}\Big(\Lambda_1(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\Big)(\mathfrak{t}_k^-)\Big)\\
-\displaystyle\sum_{k=1}^{m}\Big(\Lambda_4(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\Big)(\mathfrak{s}_k^-)\Big)\Bigg]+\displaystyle\sum_{k=1}^{j}\Big(\Lambda_2(k,j)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\Big)(\mathfrak{t}_k^-)\Big)\\
+\displaystyle\sum_{k=1}^{j-1}\Big(\Lambda_3(k,j)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\Big)(\mathfrak{s}_k^-)\Big)+\Big({}_{\mathfrak{t}_j}\mathfrak{I}^{\beta_j,\gamma_j}_{\Omega_j}\mathfrak{g}^*\Big)(t),\quad t\in[\mathfrak{t}_j,\mathfrak{s}_j),\ j=1,2,3,\ldots,m,
\end{cases}
\tag{4}
$$

with $\prod_a^b(\cdot)=1$, $\sum_a^b(\cdot)=0$, *if* $b<a$.

Proof. To begin, consider the first equation in (3). By applying the fractional tempered Ψ-integral of order α_0 over the interval $[\mathfrak{s}_0,\mathfrak{t}_1)$ with $t\in[\mathfrak{s}_0,\mathfrak{t}_1)$, we obtain

$$
{}_{\mathfrak{s}_0}\mathfrak{I}^{\alpha_0,\lambda_0}_{\Psi_0}\Big({}_{\mathfrak{s}_0}\mathfrak{D}^{\alpha_0,\lambda_0}_{\Psi_0}u\Big)(t)={}_{\mathfrak{s}_0}\mathfrak{I}^{\alpha_0,\lambda_0}_{\Psi_0}\mathfrak{f}^*(t),
$$

which leads to

$$
u(t)=c_0+\Big({}_{\mathfrak{s}_0}\mathfrak{I}^{\alpha_0,\lambda_0}_{\Psi_0}\mathfrak{f}^*\Big)(t),
\tag{5}
$$

where the initial condition is given by $c_0=u(0)$.

During the second interval $[\mathfrak{t}_1,\mathfrak{s}_1)$, the function $u(t)$, can be determined by applying the fractional tempered Ω-integral of order $\beta_1>0$, yielding

$$
u(t)=u(\mathfrak{t}_1^+)+\Big({}_{\mathfrak{t}_1}\mathfrak{I}^{\beta_1,\gamma_1}_{\Omega_1}\mathfrak{g}^*\Big)(t).
$$

Since

$$
u(\mathfrak{t}_1^+)=\omega_1u(\mathfrak{t}_1^-)\ \text{ and }\ u(\mathfrak{t}_1^-)=\omega_1\Big[c_0+\Big({}_{\mathfrak{s}_0}\mathfrak{I}^{\alpha_0,\lambda_0}_{\Psi_0}\mathfrak{f}^*\Big)(\mathfrak{t}_1^-)\Big],
$$

it follows that

$$
u(t)=\omega_1c_0+\omega_1\Big({}_{\mathfrak{s}_0}\mathfrak{I}^{\alpha_0,\lambda_0}_{\Psi_0}\mathfrak{f}^*\Big)(\mathfrak{t}_1^-)+\Big({}_{\mathfrak{t}_1}\mathfrak{I}^{\beta_1,\gamma_1}_{\Omega_1}\mathfrak{g}^*\Big)(t),
\tag{6}
$$

for all $t\in[\mathfrak{t}_1,\mathfrak{s}_1)$.

Next, in the interval $[\mathfrak{s}_1, \mathfrak{t}_2)$, we consider the tempered Ψ-derivative of the unknown function $u(t)$. By applying the fractional tempered Ψ-integral of order α_1 we have

$$u(t) = u(\mathfrak{s}_1^+) + \left({}_{\mathfrak{s}_1}\mathfrak{I}^{\alpha_1,\lambda_1}_{\Psi_1} f^*\right)(t).$$

According to the impulsive condition, this can be expressed as

$$u(t) = \omega_1\eta_1 c_0 + \omega_1\eta_1\left({}_{\mathfrak{s}_0}\mathfrak{I}^{\alpha_0,\lambda_0}_{\Psi_0} f^*\right)(\mathfrak{t}_1^-) + \eta_1\left({}_{\mathfrak{t}_1}\mathfrak{I}^{\beta_1,\gamma_1}_{\Omega_1} g^*\right)(\mathfrak{s}_1^-) + \left({}_{\mathfrak{s}_1}\mathfrak{I}^{\alpha_1,\lambda_1}_{\Psi_1} f^*\right)(t).$$

In the fourth interval $[\mathfrak{t}_2, \mathfrak{s}_2)$, we can, by direct computation, obtain that

$$\begin{aligned} u(t) &= \omega_1\eta_1\omega_2 c_0 + \omega_1\eta_1\omega_2\left({}_{\mathfrak{s}_0}\mathfrak{I}^{\alpha_0,\lambda_0}_{\Psi_0} f^*\right)(\mathfrak{t}_1^-) + \eta_1\omega_2\left({}_{\mathfrak{t}_1}\mathfrak{I}^{\beta_1,\gamma_1}_{\Omega_1} g^*\right)(\mathfrak{s}_1^-) \\ &\quad + \omega_2\left({}_{\mathfrak{s}_1}\mathfrak{I}^{\alpha_1,\lambda_1}_{\Psi_1} f^*\right)(\mathfrak{t}_2^-) + \left({}_{\mathfrak{t}_2}\mathfrak{I}^{\beta_2,\gamma_2}_{\Omega_2} g^*\right)(t). \end{aligned}$$

Therefore, the solution $u(t)$ of (3) can be represented as:

$$u(t) = \begin{cases} c_0\Lambda_1(1,i) + \sum_{k=1}^{i} \Lambda_1(k,i)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}} f^*\right)(\mathfrak{t}_k^-) \\ + \sum_{k=1}^{i} \Lambda_4(k,i)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k} g^*\right)(\mathfrak{s}_k^-) + \left({}_{\mathfrak{s}_i}\mathfrak{I}^{\alpha_i,\lambda_i}_{\Psi_i} f^*\right)(t), \quad t \in [\mathfrak{s}_i, \mathfrak{t}_{i+1}),\ i = 0,1,2,\ldots,m, \\ c_0\Lambda_2(1,j) + \sum_{k=1}^{j} \Lambda_2(k,j)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}} f^*\right)(\mathfrak{t}_k^-) \\ + \sum_{k=1}^{j-1} \Lambda_3(k,j)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k} g^*\right)(\mathfrak{s}_k^-) + \left({}_{\mathfrak{t}_j}\mathfrak{I}^{\beta_j,\gamma_j}_{\Omega_j} g^*\right)(t), \quad t \in [\mathfrak{t}_j, \mathfrak{s}_j),\ j = 1,2,3,\ldots,m. \end{cases} \tag{7}$$

To verify the correctness of Formula (7), we employ the principle of mathematical induction. We begin by substituting $i = 0$ and $i = 1$ into the preceding and subsequent stages of (7), respectively. The base cases are thus confirmed by Equations (5) and (6). For the inductive step, assume that the first part of (7) holds for $i = n$ where $t \in [\mathfrak{s}_n, \mathfrak{t}_{n+1})$, that is,

$$\begin{aligned} u(t) &= c_0\Lambda_1(1,i) + \sum_{k=1}^{n} \Lambda_1(k,n)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}} f^*\right)(\mathfrak{t}_k^-) \\ &\quad + \sum_{k=1}^{n} \Lambda_4(k,n)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k} g^*\right)(\mathfrak{s}_k^-) + \left({}_{\mathfrak{s}_n}\mathfrak{I}^{\alpha_n,\lambda_n}_{\Psi_n} f^*\right)(t). \end{aligned}$$

Then, for the next interval $[\mathfrak{t}_{n+1}, \mathfrak{s}_{n+1})$, we have

$$\begin{aligned} u(t) &= u(\mathfrak{t}_{n+1}^+) + \left({}_{\mathfrak{t}_{n+1}}\mathfrak{I}^{\beta_{n+1},\gamma_{n+1}}_{\Omega_{n+1}} g^*\right)(t) \\ &= \omega_{n+1}u(\mathfrak{t}_{n+1}^-) + \left({}_{\mathfrak{t}_{n+1}}\mathfrak{I}^{\beta_{n+1},\gamma_{n+1}}_{\Omega_{n+1}} g^*\right)(t) \\ &= \omega_{n+1}\Bigg[c_0\Lambda_1(1,n) + \sum_{k=1}^{n} \Lambda_1(k,i)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}} f^*\right)(\mathfrak{t}_k^-) \\ &\quad + \sum_{k=1}^{n} \Lambda_4(k,n)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k} g^*\right)(\mathfrak{s}_k^-) + \left({}_{\mathfrak{s}_n}\mathfrak{I}^{\alpha_n,\lambda_n}_{\Psi_n} f^*\right)(\mathfrak{t}_{n+1}^-)\Bigg] + \left({}_{\mathfrak{t}_{n+1}}\mathfrak{I}^{\beta_{n+1},\gamma_{n+1}}_{\Omega_{n+1}} g^*\right)(t) \\ &= \omega_{n+1}\Bigg[c_0\Lambda_1(1,n) + \sum_{k=1}^{n+1} \Lambda_1(k,i)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}} f^*\right)(\mathfrak{t}_k^-) \end{aligned}$$

$$+\sum_{k=1}^{n}\Lambda_4(k,n)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Big]+\left({}_{\mathfrak{t}_{n+1}}\mathfrak{I}^{\beta_{n+1},\gamma_{n+1}}_{\Omega_{n+1}}\mathfrak{g}^*\right)(\mathfrak{t}).$$

This confirms that the second component of (7) is valid for $\mathfrak{t}\in[\mathfrak{t}_{n+1},\mathfrak{s}_{n+1})$.

Furthermore, consider the case where the latter part of (7) is applied for $j=n$ with $\mathfrak{t}\in[\mathfrak{t}_n,\mathfrak{s}_n)$, that is,

$$\begin{aligned}\mathfrak{u}(\mathfrak{t}) &= c_0\Lambda_2(1,n)+\sum_{k=1}^{n}\Lambda_2(k,n)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)\\&\quad+\sum_{k=1}^{n-1}\Lambda_3(k,n)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)+\left({}_{\mathfrak{t}_n}\mathfrak{I}^{\beta_n,\gamma_n}_{\Omega_n}\mathfrak{g}^*\right)(\mathfrak{t}).\end{aligned}$$

Hence, in the subsequent interval $[\mathfrak{s}_n,\mathfrak{t}_{n+1})$, we obtain

$$\begin{aligned}\mathfrak{u}(\mathfrak{t}) &= \mathfrak{u}(\mathfrak{s}_n^+)+\left({}_{\mathfrak{s}_n}\mathfrak{I}^{\alpha_n,\lambda_n}_{\Psi_n}\mathfrak{f}^*\right)(\mathfrak{t})=\eta_n\mathfrak{u}(\mathfrak{s}_n^-)+\left({}_{\mathfrak{s}_n}\mathfrak{I}^{\alpha_n,\lambda_n}_{\Psi_n}\mathfrak{f}^*\right)(\mathfrak{t})\\&=\eta_n\Big[c_0\Lambda_2(1,n)+\sum_{k=1}^{n}\Lambda_2(k,n)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)\\&\quad+\sum_{k=1}^{n-1}\Lambda_3(k,n)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)+\left({}_{\mathfrak{t}_n}\mathfrak{I}^{\beta_n,\gamma_n}_{\Omega_n}\mathfrak{g}^*\right)(\mathfrak{s}_n^-)\Big]+\left({}_{\mathfrak{s}_n}\mathfrak{I}^{\alpha_n,\lambda_n}_{\Psi_n}\mathfrak{f}^*\right)(\mathfrak{t})\\&=\eta_n\Big[c_0\Lambda_2(1,n)+\sum_{k=1}^{n}\Lambda_2(k,n)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)\\&\quad+\sum_{k=1}^{n}\Lambda_3(k,n)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Big]+\left({}_{\mathfrak{s}_n}\mathfrak{I}^{\alpha_n,\lambda_n}_{\Psi_n}\mathfrak{f}^*\right)(\mathfrak{t}).\end{aligned}$$

Accordingly, the initial part of (7) is verified to be valid. Hence, the Formula (7) holds for all $\mathfrak{t}\in[0,T]$.

To apply the boundary condition specified in (3), we substitute $\mathfrak{t}=T$ into (7) with $i=m$. Consequently, we obtain

$$\begin{aligned}\mathfrak{u}(T) &= c_0\Lambda_1(1,m)+\sum_{k=1}^{m}\Lambda_1(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)\\&\quad+\sum_{k=1}^{m}\Lambda_4(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)+\left({}_{\mathfrak{s}_m}\mathfrak{I}^{\alpha_m,\lambda_m}_{\Psi_m}\mathfrak{f}^*\right)(T),\\\xi_2 u(T) &= \xi_2\Big[c_0\Lambda_1(1,m)+\sum_{k=1}^{m+1}\Lambda_1(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)\\&\quad+\sum_{k=1}^{m}\Lambda_4(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Big].\end{aligned}$$

Taking tempered Ω-integral of order β_m for $\mathfrak{t}\in[\mathfrak{t}_m,\mathfrak{s}_m)$ in second part (7), we get

$$\begin{aligned}&\left({}_{\mathfrak{t}_m}\mathfrak{I}^{\beta_m,\gamma_m}_{\Omega_m}\mathfrak{u}\right)(\mathfrak{s}_m^-)\\&={}_{\mathfrak{t}_m}\mathfrak{I}^{\beta_m,\gamma_m}_{\Omega_m}\Big[c_0\Lambda_2(1,m)+\sum_{k=1}^{m}\Lambda_2(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)\\&\quad+\sum_{k=1}^{m-1}\Lambda_3(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)+\left({}_{\mathfrak{t}_m}\mathfrak{I}^{\beta_m,\gamma_m}_{\Omega_m}\mathfrak{g}^*\right)(\mathfrak{s}_m^-)\Big](\mathfrak{s}_m^-)\\&={}_{\mathfrak{t}_m}\mathfrak{I}^{\beta_m,\gamma_m}_{\Omega_m}\Big[c_0\Lambda_2(1,m)+\sum_{k=1}^{m}\Lambda_2(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)\end{aligned}$$

$$+\sum_{k=1}^{m}\Lambda_3(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Bigg](\mathfrak{s}_m^-)$$

$$= \frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\Bigg[c_0\Lambda_2(1,m)$$

$$+\sum_{k=1}^{m}\Lambda_2(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)+\sum_{k=1}^{m}\Lambda_3(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Bigg].$$

Taking tempered Ψ-integral of order α_m for $\mathfrak{t}\in[\mathfrak{s}_m,\mathfrak{t}_{m+1})$ in first part of (7), we get

$$({}_{\mathfrak{s}_m}\mathfrak{I}^{\alpha_m,\lambda_m}_{\Psi_m}\mathfrak{u})(\mathfrak{t}_{m+1}^-)$$

$$= {}_{\mathfrak{s}_m}\mathfrak{I}^{\alpha_m,\lambda_m}_{\Psi_m}\Bigg[c_0\Lambda_1(1,m)+\sum_{k=1}^{m}\Lambda_1(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)$$

$$+\sum_{k=1}^{m}\Lambda_4(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)+\left({}_{\mathfrak{s}_m}\mathfrak{I}^{\alpha_m,\lambda_m}_{\Psi_m}\mathfrak{f}^*\right)(\mathfrak{t}_{m+1}^-)\Bigg](\mathfrak{t}_{m+1}^-)$$

$$= {}_{\mathfrak{s}_m}\mathfrak{I}^{\alpha_m,\lambda_m}_{\Psi_m}\Bigg[c_0\Lambda_1(1,m)+\sum_{k=1}^{m+1}\Lambda_1(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)$$

$$+\sum_{k=1}^{m}\Lambda_4(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Bigg](\mathfrak{t}_{m+1}^-)$$

$$= \frac{(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)}\Bigg[c_0\Lambda_1(1,m)+\sum_{k=1}^{m+1}\Lambda_1(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)$$

$$+\sum_{k=1}^{m}\Lambda_4(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Bigg].$$

Substituting $\xi_2\mathfrak{u}(T)$, $({}_{\mathfrak{t}_m}\mathfrak{I}^{\beta_m,\gamma_m}_{\Omega_m}\mathfrak{u})(\mathfrak{s}_m^-)$ and $({}_{\mathfrak{s}_m}\mathfrak{I}^{\alpha_m,\lambda_m}_{\Psi_m}\mathfrak{u})(\mathfrak{t}_{m+1}^-)$ into (3) gives

$$\xi_1c_0+\xi_2\Bigg[c_0\Lambda_1(1,m)+\sum_{k=1}^{m+1}\Lambda_1(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)$$

$$+\sum_{k=1}^{m}\Lambda_4(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Bigg]$$

$$= \xi_3\Bigg\{\frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\Bigg[c_0\Lambda_2(1,m)+\sum_{k=1}^{m}\Lambda_2(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)$$

$$+\sum_{k=1}^{m}\Lambda_3(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Bigg]-\frac{(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)}\Bigg[c_0\Lambda_1(1,m)$$

$$+\sum_{k=1}^{m+1}\Lambda_1(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)+\sum_{k=1}^{m}\Lambda_4(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Bigg]\Bigg\},$$

which leads to

$$c_0 = \frac{1}{\Phi}\Bigg\{\xi_3\frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\Bigg[\sum_{k=1}^{m}\Lambda_2(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)$$

$$+\sum_{k=1}^{m}\Lambda_3(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Bigg]$$

$$-\xi_3\frac{\left(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\right)^{\alpha_m}}{\Gamma(\alpha_m+1)}\Bigg[\sum_{k=1}^{m+1}\Lambda_1(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)$$
$$+\sum_{k=1}^{m}\Lambda_4(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Bigg]$$
$$-\sum_{k=1}^{m+1}\Lambda_1(k,m)\left({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}^*\right)(\mathfrak{t}_k^-)-\sum_{k=1}^{m}\Lambda_4(k,m)\left({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}^*\right)(\mathfrak{s}_k^-)\Bigg\}. \tag{8}$$

Substituting the constant c_0, given by (8), into (7) yields the unique solution of the linear boundary value problem (3). Conversely, assume that $\mathfrak{u}(\mathfrak{t})$ satisfies the impulsive fractional integral equation (4); then, by direct computation, it can be seen that the solution given by (4) satisfies (3). □

Now, we define an operator $\mathcal{H}:\mathcal{PC}(\mathbb{J},\mathbb{R})\to\mathcal{PC}(\mathbb{J},\mathbb{R})$ by

$$\mathcal{H}(\mathfrak{t})=\begin{cases}
\frac{\Lambda_1(1,i)}{\Phi}\Bigg[\xi_3\frac{\left(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+)\right)^{\beta_m}}{\Gamma(\beta_m+1)}\Big[\sum_{k=1}^{m}\Big(\Lambda_2(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}_{\mathfrak{u}}\Big)(\mathfrak{t}_k^-)\Big)\\
+\sum_{k=1}^{m}\Big(\Lambda_3(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}_{\mathfrak{u}}\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
-\xi_3\frac{\left(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\right)^{\alpha_m}}{\Gamma(\alpha_m+1)}\Big[\sum_{k=1}^{m+1}\Big(\Lambda_1(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}_{\mathfrak{u}}\Big)(\mathfrak{t}_k^-)\Big)\\
+\sum_{k=1}^{m}\Big(\Lambda_4(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}_{\mathfrak{u}}\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
-\sum_{k=1}^{m+1}\Big(\Lambda_1(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}_{\mathfrak{u}}\Big)(\mathfrak{t}_k^-)\Big)\\
-\sum_{k=1}^{m}\Big(\Lambda_4(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}_{\mathfrak{u}}\Big)(\mathfrak{s}_k^-)\Big)\Bigg]\\
+\sum_{k=1}^{i}\Big(\Lambda_1(k,i)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}_{\mathfrak{u}}\Big)(\mathfrak{t}_k^-)\Big)+\sum_{k=1}^{i}\Big(\Lambda_4(k,i)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}_{\mathfrak{u}}\Big)(\mathfrak{s}_k^-)\Big)\\
+\Big({}_{\mathfrak{s}_i}\mathfrak{I}^{\alpha_i,\lambda_i}_{\Psi_i}\mathfrak{f}_{\mathfrak{u}}\Big)(\mathfrak{t}), \qquad \mathfrak{t}\in[\mathfrak{s}_i,\mathfrak{t}_{i+1}),\ i=0,1,2,\ldots,m,\\
\frac{\Lambda_2(1,j)}{\Phi}\Bigg[\xi_3\frac{\left(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+)\right)^{\beta_m}}{\Gamma(\beta_m+1)}\Big[\sum_{k=1}^{m}\Big(\Lambda_2(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}_{\mathfrak{u}}\Big)(\mathfrak{t}_k^-)\Big)\\
+\sum_{k=1}^{m}\Big(\Lambda_3(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}_{\mathfrak{u}}\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
-\xi_3\frac{\left(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\right)^{\alpha_m}}{\Gamma(\alpha_m+1)}\Big[\sum_{k=1}^{m+1}\Big(\Lambda_1(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}_{\mathfrak{u}}\Big)(\mathfrak{t}_k^-)\Big)\\
+\sum_{k=1}^{m}\Big(\Lambda_4(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}_{\mathfrak{u}}\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
-\sum_{k=1}^{m+1}\Big(\Lambda_1(k,m)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}_{\mathfrak{u}}\Big)(\mathfrak{t}_k^-)\Big)\\
-\sum_{k=1}^{m}\Big(\Lambda_4(k,m)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}_{\mathfrak{u}}\Big)(\mathfrak{s}_k^-)\Big)\Bigg]\\
+\sum_{k=1}^{j}\Big(\Lambda_2(k,j)\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}\mathfrak{f}_{\mathfrak{u}}\Big)(\mathfrak{t}_k^-)\Big)+\sum_{k=1}^{j-1}\Big(\Lambda_3(k,j)\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}\mathfrak{g}_{\mathfrak{u}}\Big)(\mathfrak{s}_k^-)\Big)\\
+\Big({}_{\mathfrak{t}_j}\mathfrak{I}^{\beta_j,\gamma_j}_{\Omega_j}\mathfrak{g}_{\mathfrak{u}}\Big)(\mathfrak{t}), \qquad \mathfrak{t}\in[\mathfrak{t}_j,\mathfrak{s}_j),\ j=1,2,3,\ldots,m,
\end{cases}$$

where the abbreviations $\mathfrak{f}_{\mathfrak{u}}(\mathfrak{t}),\mathfrak{g}_{\mathfrak{u}}(\mathfrak{t})$ mean $\mathfrak{f}_{\mathfrak{u}}(\mathfrak{t})=\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))$ and $\mathfrak{g}_{\mathfrak{u}}(\mathfrak{t})=\mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))$, respectively.

By applying the Banach contraction mapping principle and the Leray–Schauder nonlinear alternative, we aim to establish the existence and uniqueness of solutions to the problem (1).

Theorem 3. *Suppose that the nonlinear functions* $\mathfrak{f} : \mathbb{J}_1 \times \mathbb{R} \to \mathbb{R}$ *and* $\mathfrak{g} : \mathbb{J}_2 \times \mathbb{R} \to \mathbb{R}$ *satisfy condition* (H_1). *If* $L_1\Phi_1 + L_2\Phi_2 < 1$, *then the mixed-type* ψ*-Caputo fractional derivative impulsive boundary value problem* (1) *admits a unique solution on* $\mathbb{J}$.

Proof. Let B_r be a neighborhood of radius $r > 0$, defined by $B_r = \{\mathfrak{u} \in \mathcal{PC}(\mathbb{J}, \mathbb{R}) : \|\mathfrak{u}\| \leq r\}$, with

$$r \geq \frac{M_1\Phi_1 + M_2\Phi_2}{1 - (L_1\Phi_1 + L_2\Phi_2)}. \tag{9}$$

We will show that $\mathcal{H}B_r \subset B_r$. Put $M = \sup\{|\mathfrak{f}(\mathfrak{t}, 0)|, \mathfrak{t} \in \mathbb{J}_1\}$ and $N = \sup\{|\mathfrak{g}(\mathfrak{t}, 0)|, \mathfrak{t} \in \mathbb{J}_2\}$. For $\mathfrak{t} \in \mathbb{J}_1$ and $\mathfrak{t} \in \mathbb{J}_2$, in the given sequence, we have

$$\begin{aligned}
|\mathcal{H}\mathfrak{u}(\mathfrak{t})| \leq\ & \frac{|\Lambda_1(1,i)|}{|\Phi|}\Bigg[|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-) - \Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\Big[\sum_{k=1}^{m}\Big(|\Lambda_2(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_3(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_\mathfrak{u}|\Big)(\mathfrak{s}_k^-)\Big)\Big] \\
&+|\xi_3|\frac{(\Psi_m(\mathfrak{t}_{m+1}^-) - \Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)}\Big[\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_\mathfrak{u}|\Big)(\mathfrak{s}_k^-)\Big)\Big] \\
&+\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_\mathfrak{u}|\Big)(\mathfrak{s}_k^-)\Big)\Bigg] \\
&+\sum_{k=1}^{i}\Big(|\Lambda_1(k,i)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{i}\Big(|\Lambda_4(k,i)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_\mathfrak{u}|\Big)(\mathfrak{s}_k^-)\Big) + \Big({}_{\mathfrak{s}_i}\mathfrak{I}^{\alpha_i,\lambda_i}_{\Psi_i}|\mathfrak{f}_\mathfrak{u}|\Big)(\mathfrak{t}),
\end{aligned}$$

for $\mathfrak{t} \in [\mathfrak{s}_i, \mathfrak{t}_{i+1}), i = 0, 1, 2, \ldots, m$, and

$$\begin{aligned}
|\mathcal{H}u(\mathfrak{t})| \leq\ & \frac{|\Lambda_2(1,j)|}{|\Phi|}\Bigg[|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-) - \Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)} \\
&\times\Big[\sum_{k=1}^{m}\Big(|\Lambda_2(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_3(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_\mathfrak{u}|\Big)(\mathfrak{s}_k^-)\Big)\Big] \\
&+|\xi_3|\frac{(\Psi_m(\mathfrak{t}_{m+1}^-) - \Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)}\Big[\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_\mathfrak{u}|\Big)(\mathfrak{s}_k^-)\Big)\Big] \\
&+\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u}|\Big)(\mathfrak{t}_k^-)\Big)
\end{aligned}$$

$$+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_u|\Big)(\mathfrak{s}_k^-)\Big)\Big]$$
$$+\sum_{k=1}^{j}\Big(|\Lambda_2(k,j)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_u|\Big)(\mathfrak{t}_k^-)\Big)$$
$$+\sum_{k=1}^{j-1}\Big(|\Lambda_3(k,j)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_u|\Big)(\mathfrak{s}_k^-)\Big)+\Big({}_{\mathfrak{t}_j}\mathfrak{I}^{\beta_j,\gamma_j}_{\Omega_j}|\mathfrak{g}_u|\Big)(\mathfrak{t}),$$

for $\mathfrak{t}\in[\mathfrak{t}_j,\mathfrak{s}_j)$, $j=1,2,3,\ldots,m$. From the triangle inequality and (H_1), we obtain $|\mathfrak{f}_u(\mathfrak{t})|=|\mathfrak{f}(\mathfrak{t},u)|\le|\mathfrak{f}(\mathfrak{t},u)-\mathfrak{f}(\mathfrak{t},0)|+|\mathfrak{f}(\mathfrak{t},0)|\le L_1r+M$ and $|\mathfrak{g}_u(\mathfrak{t})|=|\mathfrak{g}(\mathfrak{t},u)|\le|\mathfrak{g}(\mathfrak{t},u)-\mathfrak{g}(\mathfrak{t},0)|+|\mathfrak{g}(\mathfrak{t},0)|\le L_2r+N$. As a result, we have

$$\begin{aligned}
\sup_{\mathfrak{t}\in J}|\mathcal{H}u(\mathfrak{t})| &\le \frac{|\Lambda_1(1,m)|}{|\Phi|}\Bigg[|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\\
&\times\Big[(L_1r+M)\sum_{k=1}^{m}\Big(|\Lambda_2(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big)\\
&+(L_2r+N)\sum_{k=1}^{m}\Big(|\Lambda_3(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+|\xi_3|\frac{(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)}\\
&\times\Big[(L_1r+M)\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big)\\
&+(L_2r+N)\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+(L_1r+M)\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big)\\
&+(L_2r+N)\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+(L_1r+M)\sum_{k=1}^{m}\Big(|\Lambda_1(k,i)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big)\\
&+(L_2r+N)\sum_{k=1}^{m}\Big(|\Lambda_4(k,i)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\\
&+(L_1r+M)\Big({}_{\mathfrak{s}_i}\mathfrak{I}^{\alpha_m,\lambda_m}_{\Psi_m}(1)\Big)(\mathfrak{t}_{m+1})\\
&\le \frac{|\Lambda_1(1,m)|}{|\Phi|}\Bigg[|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\\
&\times\Big[(L_1r+M)\sum_{k=1}^{m}\left(|\Lambda_2(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right)\\
&+(L_2r+N)\sum_{k=1}^{m}\left(|\Lambda_3(k,m)|\frac{(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+))^{\beta_k}}{\Gamma(\beta_k+1)}\right)\Big]\\
&+|\xi_3|\frac{(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)}\\
&\times\Big[(L_1r+M)\sum_{k=1}^{m+1}\left(|\Lambda_1(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+(L_2 r+N)\sum_{k=1}^{m}\left(|\Lambda_4(k,m)|\frac{\left(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+)\right)^{\beta_k}}{\Gamma(\beta_k+1)}\right)\Bigg]\\
&+(L_1 r+M)\sum_{k=1}^{m+1}\left(|\Lambda_1(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right)\\
&+(L_2 r+N)\sum_{k=1}^{m}\left(|\Lambda_4(k,m)|\frac{\left(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+)\right)^{\beta_k}}{\Gamma(\beta_k+1)}\right)\Bigg]\\
&+(L_1 r+M)\sum_{k=1}^{m+1}\left(|\Lambda_1(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right)\\
&+(L_2 r+N)\sum_{k=1}^{m}\left(|\Lambda_4(k,m)|\frac{\left(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+)\right)^{\beta_k}}{\Gamma(\beta_k+1)}\right)\\
=\;&(L_1 r+M)\Bigg\{\frac{|\Lambda_1(1,m)|}{\Phi}\Bigg[|\xi_3|\frac{\left(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+)\right)^{\beta_m}}{\Gamma(\beta_m+1)}\\
&\times\sum_{k=1}^{m}\left(|\Lambda_2(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right)\\
&+|\xi_3|\frac{\left(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\right)^{\alpha_m}}{\Gamma(\alpha_m+1)}\\
&\times\sum_{k=1}^{m+1}\left(|\Lambda_1(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right)\\
&+\sum_{k=1}^{m+1}\left(|\Lambda_1(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right)\Bigg]\\
&+\sum_{k=1}^{m+1}\left(|\Lambda_1(k,m)|\frac{\left(\Psi_{k-1}(\mathfrak{t}_k^-)-\Psi_{k-1}(\mathfrak{s}_{k-1}^+)\right)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\right)\Bigg\}\\
&+(L_2 r+N)\Bigg\{\frac{|\Lambda_1(1,m)|}{\Phi}\Bigg[|\xi_3|\frac{\left(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+)\right)^{\beta_m}}{\Gamma(\beta_m+1)}\\
&\times\sum_{k=1}^{m}\left(|\Lambda_3(k,m)|\frac{\left(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+)\right)^{\beta_k}}{\Gamma(\beta_k+1)}\right)\\
&+|\xi_3|\frac{\left(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\right)^{\alpha_m}}{\Gamma(\alpha_m+1)}\\
&\times\sum_{k=1}^{m}\left(|\Lambda_4(k,m)|\frac{\left(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+)\right)^{\beta_k}}{\Gamma(\beta_k+1)}\right)\\
&+\sum_{k=1}^{m}\left(|\Lambda_4(k,m)|\frac{\left(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+)\right)^{\beta_k}}{\Gamma(\beta_k+1)}\right)\Bigg]\\
&+\sum_{k=1}^{m}\left(|\Lambda_4(k,m)|\frac{\left(\Omega_k(\mathfrak{s}_k^-)-\Omega_k(\mathfrak{t}_k^+)\right)^{\beta_k}}{\Gamma(\beta_k+1)}\right)\Bigg\}\\
=\;&(L_1 r+M)\Phi_1+(L_2 r+N)\Phi_2\\
\leq\;&r.
\end{aligned}
$$

Thus $\|\mathcal{H}\mathfrak{u}\| \leq r$, where r satisfies the condition (9). Therefore, $\mathcal{H}B_r \subset B_r$ is valid.

Next, we will prove that $\mathcal{H}$ is a contraction operator. Consider any $\mathfrak{u}, \mathfrak{v} \in B_r$. Then, we have

$$\begin{aligned}
|\mathcal{H}\mathfrak{u}(\mathfrak{t}) - \mathcal{H}\mathfrak{v}(\mathfrak{t})| \leq\ & \frac{|\Lambda_1(1,i)|}{|\Phi|}\Bigg\{|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-) - \Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)} \\
&\times\Big[\sum_{k=1}^{m}\Big(|\Lambda_2(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_3(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{s}_k^-)\Big)\Big] \\
&+|\xi_3|\frac{(\Psi_m(\mathfrak{t}_{m+1}^-) - \Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)} \\
&\times\Big[\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{s}_k^-)\Big)\Big] \\
&+\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{s}_k^-)\Big)\Bigg\} \\
&+\sum_{k=1}^{i}\Big(|\Lambda_1(k,i)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{i}\Big(|\Lambda_4(k,i)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{s}_k^-)\Big) + \Big({}_{\mathfrak{s}_i}\mathfrak{I}^{\alpha_i,\lambda_i}_{\Psi_i}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{t}),
\end{aligned}$$

for $\mathfrak{t} \in [\mathfrak{s}_i, \mathfrak{t}_{i+1})$, $i = 0, 1, 2, \ldots, m$, and

$$\begin{aligned}
|\mathcal{H}\mathfrak{u}(\mathfrak{t}) - \mathcal{H}\mathfrak{v}(\mathfrak{t})| \leq\ & \frac{|\Lambda_2(1,j)|}{|\Phi|}\Bigg[|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-) - \Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)} \\
&\times\Big[\sum_{k=1}^{m}\Big(|\Lambda_2(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_3(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{s}_k^-)\Big)\Big] \\
&+|\xi_3|\frac{(\Psi_m(\mathfrak{t}_{m+1}^-) - \Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)} \\
&\times\Big[\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{s}_k^-)\Big)\Big] \\
&+\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{t}_k^-)\Big) \\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{s}_k^-)\Big)\Bigg] \\
&+\sum_{k=1}^{j}\Big(|\Lambda_2(k,j)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_\mathfrak{u} - \mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{t}_k^-)\Big)
\end{aligned}$$

$$+\sum_{k=1}^{j-1}\Big(|\Lambda_3(k,j)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{f}_\mathfrak{u}-\mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{s}_k^-)\Big)+\Big({}_{\mathfrak{t}_j}\mathfrak{I}^{\beta_j,\gamma_j}_{\Omega_j}|\mathfrak{f}_\mathfrak{u}-\mathfrak{f}_\mathfrak{v}|\Big)(\mathfrak{t}),$$

for $\mathfrak{t}\in[\mathfrak{t}_j,\mathfrak{s}_j), j=1,2,3,\ldots,m,$. This leads to

$$\begin{aligned}
\sup_{\mathfrak{t}\in J}|\mathcal{H}\mathfrak{u}(\mathfrak{t})-\mathcal{H}\mathfrak{v}(\mathfrak{t})| \;\leq\;& \frac{|\Lambda_1(1,i)|}{|\Phi|}\Bigg\{|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\\
&\times\Big[L_1\|\mathfrak{u}-\mathfrak{v}\|\sum_{k=1}^{m}\Big(|\Lambda_2(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big)\\
&+L_2\|\mathfrak{u}-\mathfrak{v}\|\sum_{k=1}^{m}\Big(|\Lambda_3(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+|\xi_3|\frac{(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+))^{\alpha_m}}{\Gamma(\alpha_m+1)}\\
&\times\Big[L_1\|\mathfrak{u}-\mathfrak{v}\|\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big)\\
&+L_2\|\mathfrak{u}-\mathfrak{v}\|\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+L_1\|\mathfrak{u}-\mathfrak{v}\|\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big)\\
&+L_2\|\mathfrak{u}-\mathfrak{v}\|\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\Bigg\}\\
&+L_1\|\mathfrak{u}-\mathfrak{v}\|\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big)\\
&+L_2\|\mathfrak{u}-\mathfrak{v}\|\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\\
=\;& (L_1\Phi_1+L_2\Phi_2)\|\mathfrak{u}-\mathfrak{v}\|,
\end{aligned}$$

which leads to $\|\mathcal{H}\mathfrak{u}-\mathcal{H}\mathfrak{v}\|\leq(L_1\Phi_1+L_2\Phi_2)\|\mathfrak{u}-\mathfrak{v}\|$. Since $(L_1\Phi_1+L_2\Phi_2)<1$, then $\mathcal{H}$ is a contraction. Consequently, by applying Banach's contraction mapping principle, we conclude that the operator $\mathcal{H}$ possesses a fixed point, which corresponds to the solution of the mixed-type impulsive fractional boundary value problem (1). Hence, the proof is complete. □

Theorem 4. *Assume that* (H_2) *and* (H_3) *are satisfied. Then, the mixed types of impulsive fractional boundary value problems* (1) *has at least one solution on* $\mathbb{J}$.

Proof. Define a ball $B_\kappa=\{\mathfrak{u}\in\mathcal{PC}(\mathbb{J},\mathbb{R}):\|\mathfrak{u}\|\leq\kappa\}$. It is clear that B_κ is a closed, convex subset of $\mathcal{B}$. There exist $Q_1,Q_2>0$ such that $\|\mathfrak{f}(\mathfrak{t},\mathfrak{u})\|\leq\|\mathfrak{p}_1\|\mathcal{Q}_1(\kappa)+\|\mathfrak{p}_2\|:=Q_1$ and $\|\mathfrak{g}(\mathfrak{t},\mathfrak{u})\|\leq\|\mathfrak{q}_1\|\mathcal{Q}_2(\kappa)+\|\mathfrak{q}_2\|:=Q_2$. To utilize the result in Theorem 2, we begin by proving that $\mathcal{H}$ is a continuous operator. Define a convergent sequence $\{\mathfrak{u}_n\}$ such that $\{\mathfrak{u}_n\}\to\mathfrak{u}$. As a result, we conclude

$$\begin{aligned}
|\mathcal{H}\mathfrak{u}_n(\mathfrak{t})-\mathcal{H}\mathfrak{u}(\mathfrak{t})| \;\leq\;& \frac{|\Lambda_1(1,i)|}{|\Phi|}\Bigg[|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)}\\
&\times\Big[\sum_{k=1}^{m}\Big(|\Lambda_2(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_{\mathfrak{u}_n}-\mathfrak{f}_\mathfrak{u}|\Big)(\mathfrak{t}_k^-)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{m}\Big(|\Lambda_3(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_{\mathfrak{u}_n}-\mathfrak{g}_{\mathfrak{u}}|\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+|\xi_3|\frac{\big(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\big)^{\alpha_m}}{\Gamma(\alpha_m+1)}\\
&\times\Big[\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_{\mathfrak{u}_n}-\mathfrak{f}_{\mathfrak{u}}|\Big)(\mathfrak{t}_k^-)\Big)\\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_{\mathfrak{u}_n}-\mathfrak{g}_{\mathfrak{u}}|\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_{\mathfrak{u}_n}-\mathfrak{f}_{\mathfrak{u}}|\Big)(\mathfrak{t}_k^-)\Big)\\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_{\mathfrak{u}_n}-\mathfrak{g}_{\mathfrak{u}}|\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+\sum_{k=1}^{i}\Big(|\Lambda_1(k,i)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_{\mathfrak{u}_n}-\mathfrak{f}_{\mathfrak{u}}|\Big)(\mathfrak{t}_k^-)\Big)\\
&+\sum_{k=1}^{i}\Big(|\Lambda_4(k,i)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_{\mathfrak{u}_n}-\mathfrak{g}_{\mathfrak{u}}|\Big)(\mathfrak{s}_k^-)\Big)\\
&+\Big({}_{\mathfrak{s}_i}\mathfrak{I}^{\alpha_i,\lambda_i}_{\Psi_i}|\mathfrak{f}_{\mathfrak{u}_n}-\mathfrak{f}_{\mathfrak{u}}|\Big)(\mathfrak{t})\to 0,\quad \text{as}\quad n\to\infty,
\end{aligned}
$$

for $\mathfrak{t}\in[\mathfrak{s}_i,\mathfrak{t}_{i+1}), i=0,1,2,\ldots,m$, and

$$
\begin{aligned}
|\mathcal{H}\mathfrak{u}_n(\mathfrak{t})-\mathcal{H}\mathfrak{u}(\mathfrak{t})| \;\leq\; &\frac{|\Lambda_2(1,j)|}{|\Phi|}\Bigg[|\xi_3|\frac{\big(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+)\big)^{\beta_m}}{\Gamma(\beta_m+1)}\\
&\times\Big[\sum_{k=1}^{m}\Big(|\Lambda_2(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_{\mathfrak{u}_n}-\mathfrak{f}_{\mathfrak{u}}|\Big)(\mathfrak{t}_k^-)\Big)\\
&+\sum_{k=1}^{m}\Big(|\Lambda_3(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_{\mathfrak{u}_n}-\mathfrak{g}_{\mathfrak{u}}|\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+|\xi_3|\frac{\big(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\big)^{\alpha_m}}{\Gamma(\alpha_m+1)}\\
&\times\Big[\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_{\mathfrak{u}_n}-\mathfrak{f}_{\mathfrak{u}}|\Big)(\mathfrak{t}_k^-)\Big)\\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_{\mathfrak{u}_n}-\mathfrak{g}_{\mathfrak{u}}|\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_{\mathfrak{u}_n}-\mathfrak{f}_{\mathfrak{u}}|\Big)(\mathfrak{t}_k^-)\Big)\\
&+\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_{\mathfrak{u}_n}-\mathfrak{g}_{\mathfrak{u}}|\Big)(\mathfrak{s}_k^-)\Big)\Big]\\
&+\sum_{k=1}^{j}\Big(|\Lambda_2(k,j)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}|\mathfrak{f}_{\mathfrak{u}_n}-\mathfrak{f}_{\mathfrak{u}}|\Big)(\mathfrak{t}_k^-)\Big)\\
&+\sum_{k=1}^{j-1}\Big(|\Lambda_3(k,j)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}|\mathfrak{g}_{\mathfrak{u}_n}-\mathfrak{g}_{\mathfrak{u}}|\Big)(\mathfrak{s}_k^-)\Big)\\
&+\Big({}_{\mathfrak{t}_j}\mathfrak{I}^{\beta_j,\gamma_j}_{\Omega_j}|\mathfrak{g}_{\mathfrak{u}_n}-\mathfrak{g}_{\mathfrak{u}}|\Big)(\mathfrak{t})\to 0,\quad \text{as}\quad n\to\infty,
\end{aligned}
$$

for $\mathfrak{t}\in[\mathfrak{t}_j,\mathfrak{s}_j), j=1,2,3,\ldots,m$. Thus, from the preceding inequalities, it follows that the operator $\mathcal{H}$ is continuous.

We proceed to prove the compactness of the operator $\mathcal{H}$. For $\mathfrak{u} \in B_\kappa$ we have

$$\begin{aligned}
|\mathcal{H}u(\mathfrak{t})| \leq\ & \frac{|\Lambda_1(1,i)|}{|\Phi|}\Bigg[|\xi_3|\frac{(\Omega_m(\mathfrak{s}_m^-)-\Omega_m(\mathfrak{t}_m^+))^{\beta_m}}{\Gamma(\beta_m+1)} \\
& \times\Big[Q_1\sum_{k=1}^{m}\Big(|\Lambda_2(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big) \\
& +Q_2\sum_{k=1}^{m}\Big(|\Lambda_3(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\Big] \\
& +|\xi_3|\frac{\big(\Psi_m(\mathfrak{t}_{m+1}^-)-\Psi_m(\mathfrak{s}_m^+)\big)^{\alpha_m}}{\Gamma(\alpha_m+1)} \\
& \times\Big[Q_1\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big) \\
& +Q_2(\kappa)\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\Big] \\
& +Q_1\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big) \\
& +Q_2\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big)\Bigg] \\
& +Q_1\sum_{k=1}^{m+1}\Big(|\Lambda_1(k,m)|\Big({}_{\mathfrak{s}_{k-1}}\mathfrak{I}^{\alpha_{k-1},\lambda_{k-1}}_{\Psi_{k-1}}(1)\Big)(\mathfrak{t}_k^-)\Big) \\
& +Q_2\sum_{k=1}^{m}\Big(|\Lambda_4(k,m)|\Big({}_{\mathfrak{t}_k}\mathfrak{I}^{\beta_k,\gamma_k}_{\Omega_k}(1)\Big)(\mathfrak{s}_k^-)\Big) \\
=\ & Q_1\Phi_1+Q_2\Phi_2 := \Phi_3, \qquad (10)
\end{aligned}$$

suggesting that $\|\mathcal{H}\mathfrak{u}\| \leq \Phi_3$. Hence, $\mathcal{H}(B_\kappa)$ is a uniformly bounded set.

To indicate the equicontinuity of the set $\mathcal{H}(B_\kappa)$, let $\mathfrak{t}_1, \mathfrak{t}_2 \in [0,T]$ under the condition that $\mathfrak{t}_1 < \mathfrak{t}_2$. Thus, for any $\mathfrak{u} \in B_\kappa$, we obtain

$$\begin{aligned}
|\mathcal{H}\mathfrak{u}(\mathfrak{t}_2)-\mathcal{H}\mathfrak{u}(\mathfrak{t}_1)| =\ & \Big|\Big({}_{\mathfrak{s}_i}\mathfrak{I}^{\alpha_i,\lambda_i}_{\Psi_i}|\mathfrak{f}|\Big)(\mathfrak{t}_1)-\Big({}_{\mathfrak{s}_i}\mathfrak{I}^{\alpha_i,\lambda_i}_{\Psi_i}|\mathfrak{f}|\Big)(\mathfrak{t}_2)\Big| \\
=\ & \Bigg|\frac{1}{\Gamma(\alpha_i)}\int_{\mathfrak{s}_i}^{\mathfrak{t}_2}(\Psi_i(\mathfrak{t}_2)-\Psi_i(s))^{\alpha_i-1}e^{-\lambda_i(\Psi_i(\mathfrak{t}_2)-\Psi_i(s))}\Psi'(s)\mathfrak{f}(s)ds \\
& -\frac{1}{\Gamma(\alpha_i)}\int_{\mathfrak{s}_i}^{\mathfrak{t}_1}(\Psi_i(\mathfrak{t}_1)-\Psi_i(s))^{\alpha_i-1}e^{-\lambda_i(\Psi_i(\mathfrak{t}_1)-\Psi_i(s))}\Psi'(s)\mathfrak{f}(s)ds\Bigg| \\
=\ & \Bigg|\frac{1}{\Gamma(\alpha_i)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\Psi_i(\mathfrak{t}_2)-\Psi_i(s))^{\alpha_i-1}e^{-\lambda_i(\Psi_i(\mathfrak{t}_2)-\Psi_i(s))}\Psi'(s)\mathfrak{f}(s)ds \\
& +\frac{1}{\Gamma(\alpha_i)}\int_{\mathfrak{s}_i}^{\mathfrak{t}_1}\Big[(\Psi_i(\mathfrak{t}_2)-\Psi_i(s))^{\alpha_i-1}e^{-\lambda_i(\Psi_i(\mathfrak{t}_2)-\Psi_i(s))} \\
& -(\Psi_i(\mathfrak{t}_1)-\Psi_i(s))^{\alpha_i-1}e^{-\lambda_i(\Psi_i(\mathfrak{t}_1)-\Psi_i(s))}\Big]\Psi'(s)\mathfrak{f}(s)ds\Bigg| \\
\leq\ & Q_1\Bigg|\frac{1}{\Gamma(\alpha_i)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\Psi_i(\mathfrak{t}_2)-\Psi_i(s))^{\alpha_i-1}e^{-\lambda_i(\Psi_i(\mathfrak{t}_2)-\Psi_i(s))}\Psi'(s)ds \\
& +\frac{1}{\Gamma(\alpha_i)}\int_{\mathfrak{s}_i}^{\mathfrak{t}_1}\Big[(\Psi_i(\mathfrak{t}_2)-\Psi_i(s))^{\alpha_i-1}e^{-\lambda_i(\Psi_i(\mathfrak{t}_2)-\Psi_i(s))}
\end{aligned}$$

$$
\begin{aligned}
&\quad -(\Psi_i(\mathfrak{t}_1)-\Psi_i(s))^{\alpha_i-1}e^{-\lambda_i(\Psi_i(\mathfrak{t}_1)-\Psi_i(s))}\Big]\Psi'(s)ds\Big| \\
&\leq \frac{Q_1}{\Gamma(\alpha_i+1)}\Big[2(\Psi_i(\mathfrak{t}_2)-\Psi_i(\mathfrak{t}_1))^{\alpha_i} \\
&\quad +\Big|(\Psi_i(\mathfrak{t}_2)-\Psi_i(\mathfrak{s}_i))^{\alpha_i}-(\Psi_i(\mathfrak{t}_1)-\Psi_i(\mathfrak{s}_i))^{\alpha_i}\Big|\Big] \to 0
\end{aligned}
$$

as $\mathfrak{t}_1 \to \mathfrak{t}_2$ when $\mathfrak{t}_1, \mathfrak{t}_2 \in [\mathfrak{s}_i, \mathfrak{t}_{i+1}), i = 0, 1, 2, \ldots, m$, Similarly, we can show that $|\mathcal{H}\mathfrak{u}(\mathfrak{t}_2) - \mathcal{H}\mathfrak{u}(\mathfrak{t}_1)| \to 0$ as $\mathfrak{t}_1 \to \mathfrak{t}_2$ for $\mathfrak{t}_1, \mathfrak{t}_2 \in [\mathfrak{t}_j, \mathfrak{s}_j), j = 1, 2, \ldots, m$. Given that the above two inequalities converge to zero independently of $\mathfrak{u}$, $\mathcal{H}(B_\kappa)$ is an equicontinuous set. We conclude that $\mathcal{H}B_\kappa$ is relatively compact. By the Arzelá–Ascoli theorem, the operator $\mathcal{H}$ is completely continuous.

In the final stage of the proof, we verify that the second condition of Theorem 2 does not hold. Assume that $\mathfrak{u}$ is a solution of the problem (1). Then, for a fixed $\rho \in (0,1)$, we have $\mathfrak{u} = \rho\mathcal{H}\mathfrak{u}$, which implies as in (10) that

$$
\frac{\|\mathfrak{u}\|}{[\|\mathfrak{p}_1\|\mathcal{Q}_1(\|\mathfrak{u}\|)+\|\mathfrak{p}_2\|]\Phi_1+[\|\mathfrak{q}_1\|\mathcal{Q}_2(\|\mathfrak{u}\|)+\|\mathfrak{q}_2\|]\Phi_2} \leq 1.
$$

From hypothesis (H_3), there exists a positive constant $\mathcal{N}$ such that $\|u\| \neq \mathcal{N}$. We define the open subset of B_κ by $\mathcal{W} = \{\mathfrak{u} \in B_\kappa : \|\mathfrak{u}\| < \mathcal{N}\}$. We remark that $0 \in \mathcal{W}$ and $\mathcal{H}(\overline{\mathcal{W}})$ is a relatively compact subset of B_κ. Furthermore, there does not exist $\mathfrak{u} \in \partial\mathcal{W}$ such that $\mathfrak{u} = \rho\mathcal{H}\mathfrak{u}$ for some $\rho \in (0,1)$. Hence, by applying part (i) of Theorem 2, we deduce that the operator $\mathcal{H}$ possesses a fixed point $\mathfrak{u} \in \overline{\mathcal{W}}$ satisfying the conditions of the problem (1) on $\mathbb{J}$. Therefore, the proof is complete. □

4. Example

In the current section, we outline several illustrative cases that demonstrate its practical utility and theoretical importance.

Example 3. *Consider the following given mixed types of impulsive fractional boundary value problems in the form:*

$$
\begin{cases}
\left({}_{\frac{2i}{9}}\mathfrak{D}_{\mathfrak{t}^i}^{\frac{i+2}{5},i+1}\mathfrak{u}\right)(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t})), \quad \mathfrak{t} \in \left[\frac{2i}{9},\frac{2i+1}{9}\right), \quad i = 0,1,2 \\
\left({}_{\frac{2j-1}{9}}\mathfrak{D}_{e^{\mathfrak{t}^j}}^{\frac{2j-1}{6},j^j}\mathfrak{u}\right)(\mathfrak{t}) = \mathfrak{g}(\mathfrak{t},\mathfrak{u}(\mathfrak{t})), \quad \mathfrak{t} \in \left[\frac{2j-1}{9},\frac{2j}{9}\right), \quad j = 1,2, \\
\mathfrak{u}\left(\frac{2j-1}{9}^+\right) = \omega_j\mathfrak{u}\left(\frac{2j-1}{9}^-\right), \quad \mathfrak{u}\left(\frac{2i}{9}^+\right) = \eta_i\mathfrak{u}\left(\frac{2i}{9}^-\right), \quad i = 1,2, \\
\xi_1\mathfrak{u}(0)+\xi_2\mathfrak{u}\left(\frac{5}{9}\right) = \xi_3\left[({}_{1/3}\mathfrak{I}_{\ln\mathfrak{t}^2}^{1/2,4}\mathfrak{u})\left(\frac{4}{9}\right)-({}_{4/9}\mathfrak{I}_{\mathfrak{t}^2}^{4/5,3}\mathfrak{u})\left(\frac{5}{9}\right)\right],
\end{cases}
\tag{11}
$$

Defining $\alpha_i = \frac{i+2}{5}, \lambda_i = i+1, i = 0,1,2, \beta_j = \frac{2j-1}{6}, \gamma_j = j^j, j = 1,2, \omega_1 = 2/3$, $\omega_2 = 6/5, \eta_1 = 4/5, \eta_2 = 6/5, \xi_1 = 3/4, \xi_2 = 2/3, \xi_3 = 5/6$ and $\Psi_i(\mathfrak{t}) = \mathfrak{t}^i$, $\Omega_j(\mathfrak{t}) = e^{\mathfrak{t}^j}$ Based on the given information, we calculate that $\Phi = 1.1648108665$, $\Phi_1 = 1.7448535150$, $\Phi_2 = 2.8397567074$. where $\mathfrak{t} \in [2i/9, (2i+1)/9), \quad i = 0,1,2, \mathfrak{t} \in [(2i-1)/9, 2i/9), \quad j = 1,2$, respectively.

(i.) Assuming the following properties of the functions $\mathfrak{f}$ and $\mathfrak{g}$ are prescribed for $\mathfrak{t} \in [0, 5/9]$ as indicated below:

$$
\mathfrak{f}(\mathfrak{t},\mathfrak{u}) = \frac{2|\mathfrak{u}|e^{-2\mathfrak{t}}}{|\mathfrak{u}|+9}+\frac{1}{8}\mathfrak{t}^2+\frac{3}{7},
\tag{12}
$$

$$\mathfrak{g}(\mathfrak{t},\mathfrak{u}) = \frac{\sin^2|\mathfrak{u}|}{5} + \frac{2}{5}\mathfrak{t}^3 + \frac{3}{8}. \tag{13}$$

From Equations (12) and (13), we derive the inequalities $|\mathfrak{f}(\mathfrak{t},\mathfrak{u}) - \mathfrak{f}(\mathfrak{t},\mathfrak{v})| \leq (2/9)|\mathfrak{u}-\mathfrak{v}|$ and $|\mathfrak{g}(\mathfrak{t},\mathfrak{u}) - \mathfrak{g}(\mathfrak{t},\mathfrak{v})| \leq (1/5)|\mathfrak{u}-\mathfrak{v}|$ for $\mathfrak{u},\mathfrak{v} \in \mathbb{R}$. By setting $L_1 = 2/9$ and $L_2 = 1/5$, we obtain the relation $L_1\Phi_1 + L_2\Phi_2 = 0.9556965670 < 1$. Therefore, by applying Theorem 3, we can conclude that the mixed types of impulsive fractional boundary value problems (11) with $\mathfrak{f}$ and $\mathfrak{g}$ as given by (12) and (13), have a unique solution on the interval $[0,1]$.

(ii.) Assuming the following properties of the functions $\mathfrak{f}$ and $\mathfrak{g}$ are defined for $\mathfrak{t} \in [0,5/9]$ as follows:

$$\mathfrak{f}(\mathfrak{t},\mathfrak{u}) = \frac{\sin^2\mathfrak{t}}{\mathfrak{t}+9}\left(\frac{|\mathfrak{u}|^{13}}{|\mathfrak{u}|^{11}+8} + \frac{1}{8}\right) + \frac{\mathfrak{t}^2}{4}, \tag{14}$$

$$\mathfrak{g}(\mathfrak{t},\mathfrak{u}) = \frac{e^{-\mathfrak{t}^2}}{\mathfrak{t}+8}\left(\frac{\mathfrak{u}^2}{2} + \frac{1}{7}\right) + \frac{\mathfrak{t}e^{-\mathfrak{t}}}{6}. \tag{15}$$

It is apparent that

$$\mathfrak{f}(\mathfrak{t},\mathfrak{u}) \leq \frac{1}{\mathfrak{t}+9}\left(\mathfrak{u}^2 + \frac{1}{8}\right) + \frac{\mathfrak{t}^2}{4}, \tag{16}$$

$$\mathfrak{g}(\mathfrak{t},\mathfrak{u}) \leq \frac{1}{\mathfrak{t}+8}\left(\frac{\mathfrak{u}^2}{2} + \frac{1}{7}\right) + \frac{\mathfrak{t}}{6}. \tag{17}$$

Subsequently, we define $\mathfrak{p}_1(\mathfrak{t}) = 1/(\mathfrak{t}+9), \mathfrak{p}_2(\mathfrak{t}) = \mathfrak{t}^2/4, Q_1(\mathfrak{u}) = \mathfrak{u}^2 + 1/8, \mathfrak{q}_1(\mathfrak{t}) = 1/(\mathfrak{t}+8), \mathfrak{q}_2(\mathfrak{t}) = \mathfrak{t}/10$ and $Q_2(\mathfrak{u}) = \mathfrak{u}^2/2 + 1/7$. From this, we can deduce that $\|\mathfrak{p}_1\| = 1/9, \|\mathfrak{p}_2\| = 25/324, \|\mathfrak{q}_1\| = 1/8$ and $\|\mathfrak{q}_2\| = 5/54$ and there exists a constant $\mathcal{N} \in (1.2225630178, 4.1630842384)$ that satisfies the condition (H_3). Therefore, by applying the results established in Theorem 4, it follows that the mixed types of impulsive fractional boundary value problems (11) with $\mathfrak{f}$ and $\mathfrak{g}$ as given by (12) and (13), possess at least one solution on the interval $[0,5/9]$.

5. Conclusions

This paper examined mixed impulsive boundary value problems involving tempered Ψ-fractional derivatives of Caputo type. By incorporating exponential tempering, the proposed model effectively describes systems with fading memory and sudden perturbations. The reformulation of the boundary value problem into an equivalent Volterra integral equation allowed the application of fixed point theory to derive rigorous analytical results. Using the Banach Contraction Mapping Principle, we proved the existence and uniqueness of solutions under appropriate Lipschitz conditions, while the Leray–Schauder nonlinear alternative was employed to ensure existence in more general, non-contractive cases. These results collectively provide a solid theoretical foundation for analyzing impulsive systems governed by tempered fractional dynamics. A numerical example further confirms the applicability of the theoretical findings.

Future research may focus on extending these analytical methods to systems with stochastic, distributed, or variable-order tempered fractional operators, as well as developing numerical algorithms optimized for high-precision simulation of mixed impulsive fractional systems.

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